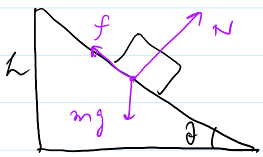
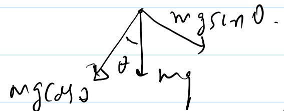
Solutions

Saturday, May 2, 2020 11:55 AM

1.1



Let us consider only vertical motion



$N = mg\cos\theta$

$f = \mu mg\cos\theta$

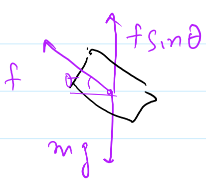
$ma = mg\sin\theta - f$

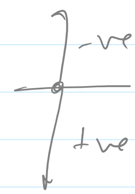
$\Rightarrow a_d = g(\sin\theta - \mu\cos\theta)$ down

When going up, $a_u = g(\sin\theta + \mu\cos\theta)$

Downward motion

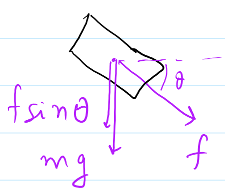
$$ma_d = mg - f\sin\theta$$
$$= mg - \mu mg\cos\theta\sin\theta$$
$$\Rightarrow a_d = g(1 - \mu\sin\theta\cos\theta)$$





Upward motion

$$ma_u = mg + f\sin\theta$$
$$\Rightarrow a_u = g(1 + \mu\sin\theta\cos\theta)$$



$v = u + at$

$x = ut + \frac{1}{2}at^2$

$v^2 = u^2 + 2ax$

So, using $x = ut + \frac{1}{2}at^2$,

$$h = 0 + \frac{1}{2}a_d t_d^2 \Rightarrow \boxed{t_d = \sqrt{\frac{2h}{a_d}}}$$

For upward motion,

$$v = u + at_u \Rightarrow 0 = u_u + a_u t_u \Rightarrow u_u = -a_u t_u$$

Then,

$$v^2 = u^2 + 2ax \Rightarrow 0 = (-a_u t_u)^2 + 2a_u(-h)$$

$$\Rightarrow t_u^2 = \frac{2a_u h}{a_u^2}$$

$$\Rightarrow \boxed{t_u = \sqrt{\frac{2h}{a_u}}}$$

$a_u > a_d$

$$\Rightarrow \sqrt{\frac{1}{a_d}} > \sqrt{\frac{1}{a_u}} \Rightarrow t_d > t_u$$

Method 2 (Krotov-inspired)

While going down,

$v = u + at$

$\Rightarrow v_d = 0 + a_d t$

$= 0 + (a_g - a_f) t$

$\Rightarrow v_d = a_g t - a_f t$

Now,

$s = \langle v \rangle t_d$

$= (a_g t_d - a_f t_d)/2$

where s is dist. travelled on the incline.

$\Rightarrow u_u = a_g t + a_f t$

Now,

$s = \langle v \rangle t_u$

$= (a_g t_u + a_f t_u)/2$

[see "Lessons" for derivation of $\langle v \rangle t$]

Thus, $t_d \frac{(a_g - a_f)}{2} = t_u (a_g + a_f)$

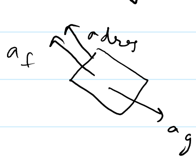
$$\Rightarrow \frac{t_d}{t_u} = \frac{(a_g + a_f)}{(a_g - a_f)} > 1 \Rightarrow t_d > t_u$$

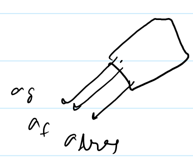
Air drag:

Drag has effects ~ to friction. It will oppose motion

$a_d = a_g - a_f - a_{drag}$

$a_u = a_g + a_f + a_{drag}$





Thus,

$$\frac{t_d}{t_u} = \frac{(a_g + a_f + a_{drag})}{(a_g - a_f - a_{drag})} > 1$$. So, the result is unchanged

Krotov's solution

In absence of friction,

$$t_u = t_d \quad \left(\text{as } \frac{1}{2} m v^{*2} = mgh, \quad s = \langle v \rangle t = \frac{v^* t}{2}\right)$$

In presence of friction,

$$v_d < v^*$$

$$u_u > v^* \Rightarrow v_d < v^* < u_u$$

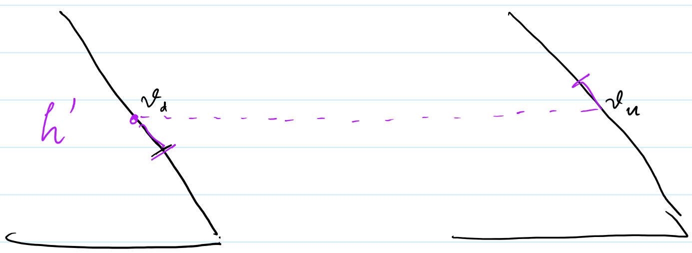
So,

$$s = \frac{\langle v \rangle t}{2} \Rightarrow \langle v_u \rangle t_u = \langle v_d \rangle t_d \quad (\text{as } s \text{ is same})$$

$$\Rightarrow \frac{u_u}{2} t_u = \frac{v_d}{2} t_d$$

$$\Rightarrow \frac{t_d}{t_u} = \frac{u_u}{v_d}. \text{ As } u_u > v_d, \ \frac{t_d}{t_u} > 1$$

Air drag



Let $\bar{\ }$ denote drag

$\bar{v}_d < v_d$

$\bar{v}_u > v_u$ (we need to push initially at a higher velocity b/c of drag. So, every point in the journey will have a higher velocity in case of drag vs case of no drag)

Thus, for dt, assuming $\bar{v}_x(t)$ constant,

$$\delta s = \bar{v}_x \delta t_x$$

So,

$$\delta s = \bar{v}_d \delta t_d = \bar{v}_u \delta t_u$$

Now,

$$\bar{v}_d < \bar{v}_u \quad [\text{why?}]$$

So, $\delta t_d > \delta t_u \Rightarrow \int \delta t_d = \int \delta t_u \Rightarrow t_d > t_u$

1.2

$L = 400$

$v = 54 \times \frac{5}{18} = 15\,m/s$

$t = 60\,s$

$a = -0.3\,m/s^2$

We can compute the distance using $x = ut + \frac{1}{2}at^2$, but that would give us the final displa--cement, including negative motion, if any. So, let us check when the object comes to rest.

$v = u + at$

$\Rightarrow \quad 0 = 15 - 0.3t \Rightarrow t = 50\,s$

Any motion given by the kinematic equations b/w $t=50$ and $t=60s$ will be in the negative direction, which is not possible with a braking system. This could happen on a system which is using negative currents on a motor to stop a robot though.

Now, the distance at the point of halt is our final distance. So,

$$x_{halt} = \frac{v^2 - u^2}{2a} = \frac{0^2 - (15)^2}{2(-0.3)} = \frac{225}{6} = 375\,m$$

So, distance from traffic light $= L - x_{halt} = 400 - 375 = 25\,m$

<u>Incorrect solution</u>

Dist from traffic light $= L - x$

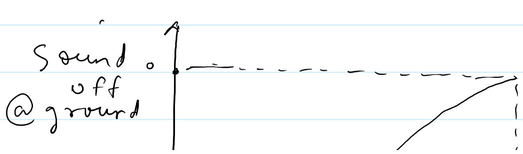
$= L - ut - \frac{1}{2}at^2$

$= 400 - 15(60) - \frac{1}{2}(-0.3)(60)^2$

$= 400 - 900 + 540$

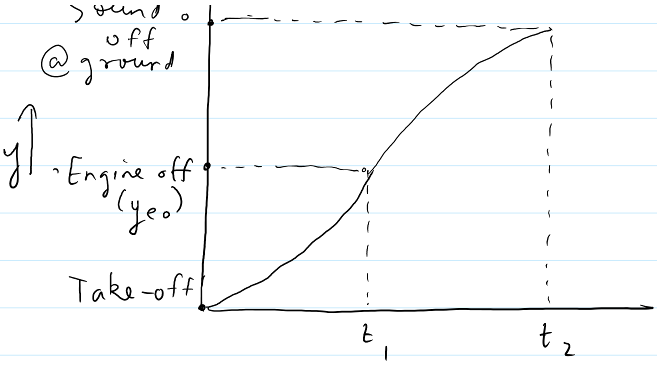
$= 40\,m$

1.3



Given: $v(0) = 0$

$c = 320\,m/s$



Given: $v(0) = 0$

$c = 320\ m/s$

$a = 3\ m/s^2$

$t_2 = 30\ s$

To find: $v(t_1)$

$\Delta t = t_2 - t_1$ = time taken by sound of last instance of engine vibration to travel from copter to ground

$$y_{eo} = c\,\Delta t$$

$$\Rightarrow y_{eo} = c(t_2 - t_1) \quad -(1)$$

Also, $y_{eo} = 0 + \frac{1}{2} a t_1^2 \quad -(2)$

From (1) & (2)

$$c(t_2 - t_1) = \frac{1}{2} a t_1^2$$

$$\Rightarrow a t_1^2 + 2c\,t_1 - 2c\,t_2 = 0$$

So,

$$t_1 = \frac{(-2c) \pm \sqrt{4c^2 + 8act_2}}{2a}$$

$$= \frac{-c \pm \sqrt{c^2 + 2act_2}}{a}$$

Using this, we get

$$v(t_1) = a t_1$$

$$= -c \pm \sqrt{c^2 + 2act_2}$$

As $v(t_1) > 0$,

$$v_s = \sqrt{c^2 + 2act_2} - c$$

$$= \sqrt{c(c + 2at_2)} - c$$

$$= \sqrt{320(320 + 2.3.30)} - 320$$

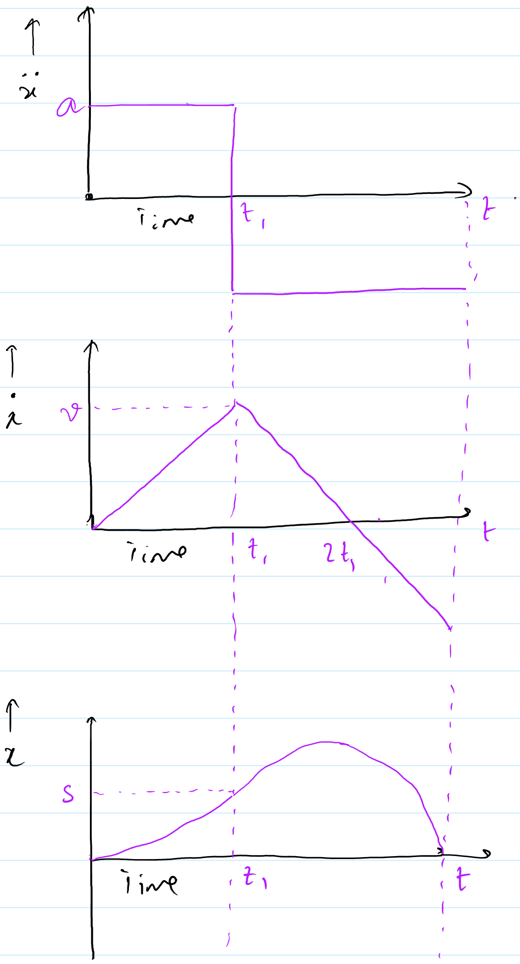
$$= \sqrt{320(500)} - 320$$

$$= \sqrt{16.1000} - 320$$

$$= 400 - 320$$

$= 80\,m/s$

1.4



First leg of motion

$$s = 0 + \frac{1}{2} a t_1^2 \quad -(1)$$

$$v = 0 + a t_1 \quad -(2)$$

Second leg of motion

$$-s = v(t - t_1) - \frac{1}{2} a (t - t_1)^2$$

Plugging in (2), we get

$$s = -(a t_1)(t - t_1) + \frac{1}{2} a (t - t_1)^2 \quad -(3)$$

Using (1), we get

$$\cancel{\frac{1}{2} a t_1^2} = -a t_1 t + a t_1^2 + \frac{1}{2} a t^2 + \cancel{\frac{1}{2} a t_1^2} - a t_1 t$$

$$\Rightarrow \quad 0 = a t_1^2 + \frac{1}{2} a t^2 - 2 a t_1 t$$

$$\Rightarrow \quad t^2 - (4t_1)t + 2t_1^2 = 0$$

So,

$$t = \frac{4t_1 \pm \sqrt{(4t_1)^2 - 4(2t_1^2)}}{2}$$

$$= \frac{4t_1 \pm \sqrt{16t_1^2 - 8t_1^2}}{2}$$

$$= (2 \pm \sqrt{2})t_1$$

$$= (2+\sqrt{2})t_1, \; (2-\sqrt{2})t_1$$

From the figures, we know that time elapsed would be at least $2t_1$. This is the time required for the body to come back to rest, before it starts moving backward.

So,

$$t \neq 2t_1 - \sqrt{2}t_1.$$

Thus,

$$t = (2+\sqrt{2})t_1$$

Method-2 (Krotov)

Replace $(t-t_1)$ in eq^n (3) above by t_2. Then, we have

$$s = -at_1t_2 + \frac{1}{2}at_2^2$$

Use (1).

$$\frac{1}{2}at_1^2 = -at_1t_2 + \frac{1}{2}at_2^2$$

Thus, $$t_2^2 - 2\cancel{t}_1 t_2 - t_1^2 = 0$$

$$t_2 = \frac{(2t_1) \pm \sqrt{4t_1^2 + 4t_1^2}}{2}$$

$$= \frac{2t_1 \pm 2\sqrt{2}\,t_1}{2}$$

$$= t_1(1 \pm \sqrt{2})$$

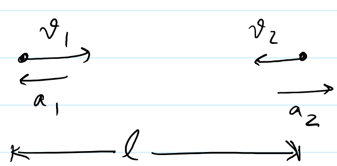
As time cannot be < 0,

$$t_2 = t_1(1+\sqrt{2})$$

So,

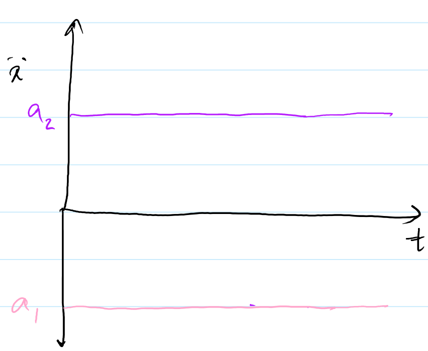
$$t = t_1 + t_2 = (2+\sqrt{2})\,t_1$$

1-5



Let body 1 be at $x = 0$ at $t = 0$

Let body 2 be at $x = l$ at $t = 0$



For 1,

$$x_1 = v_1t_1 - \frac{1}{2}a_1t_1^2$$

$$x_2 = l - v_2t_2 + \frac{1}{2}a_2t_2^2$$

At rendezvous point,

$$x_1 = x_2,\ t_1 = t_2$$

$$\Rightarrow v_1t - \frac{1}{2}a_1t^2 = l - v_2t + \frac{1}{2}a_2t^2$$

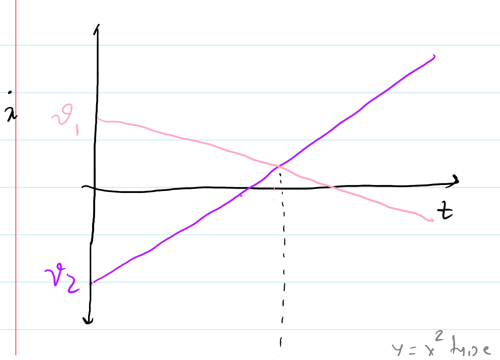
$$\Rightarrow l = (v_2+v_1)t - \frac{1}{2}(a_1+a_2)t^2$$

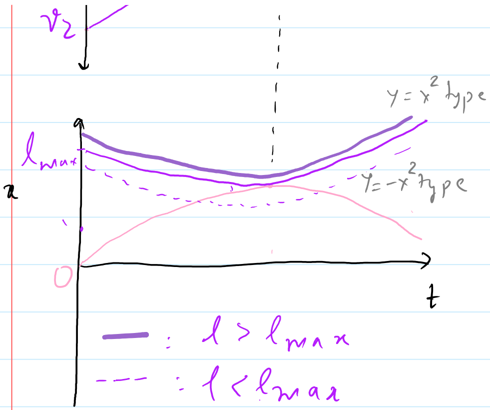
To maximize l, we set $\frac{\partial l}{\partial t} = 0$.

So,

$$(v_2+v_1) - (a_1+a_2)t = 0$$

$$\Rightarrow t = \frac{v_1+v_2}{}$$





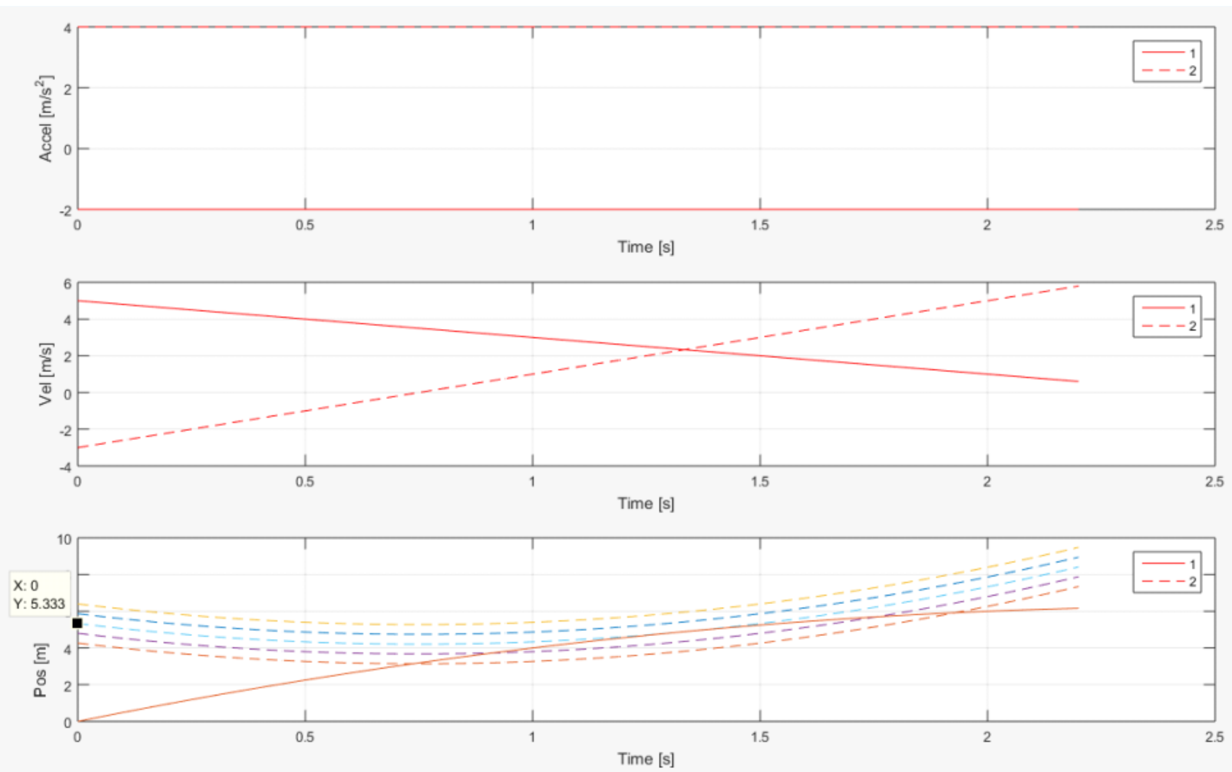
$$(v_2 + v_1) - (a_1 + a_2)t = 0$$

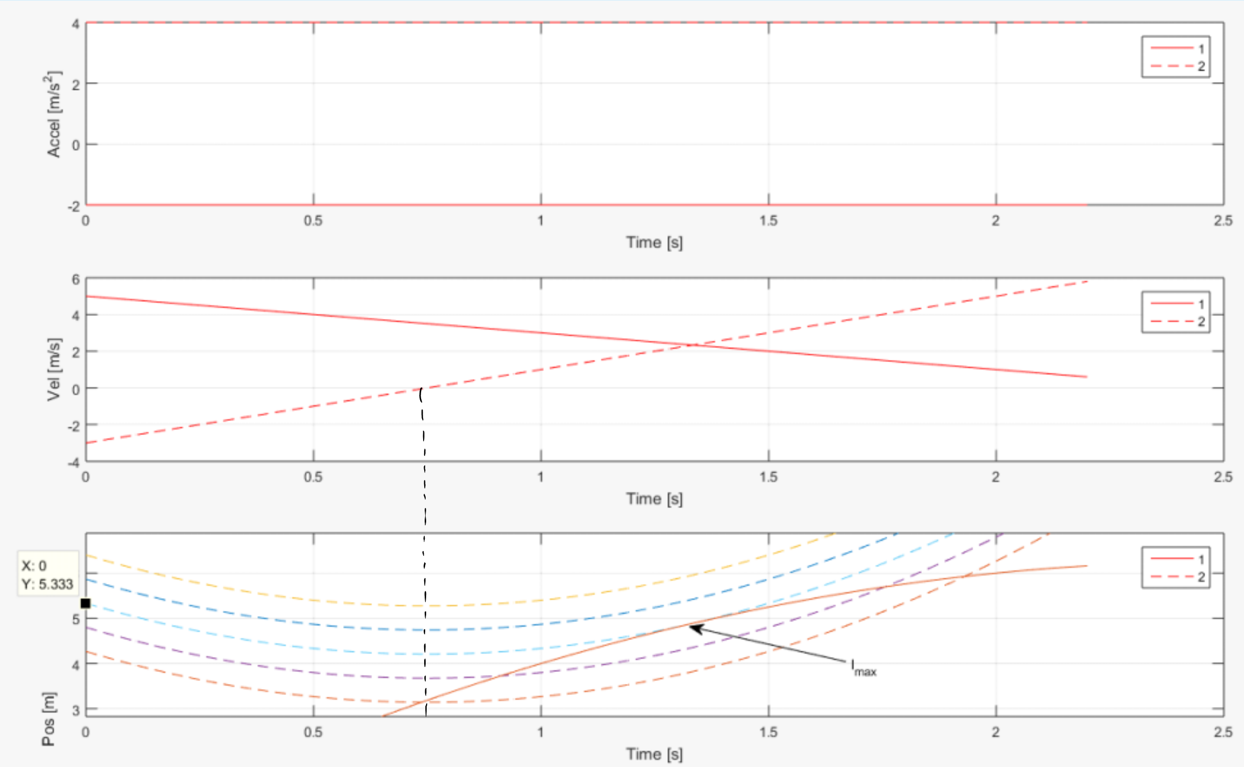
$$\Rightarrow t = \frac{v_1 + v_2}{a_1 + a_2}$$

$$\frac{d^2 l}{dt^2} = -(a_1 + a_2) < 0 \Rightarrow \text{this is a maxima.}$$

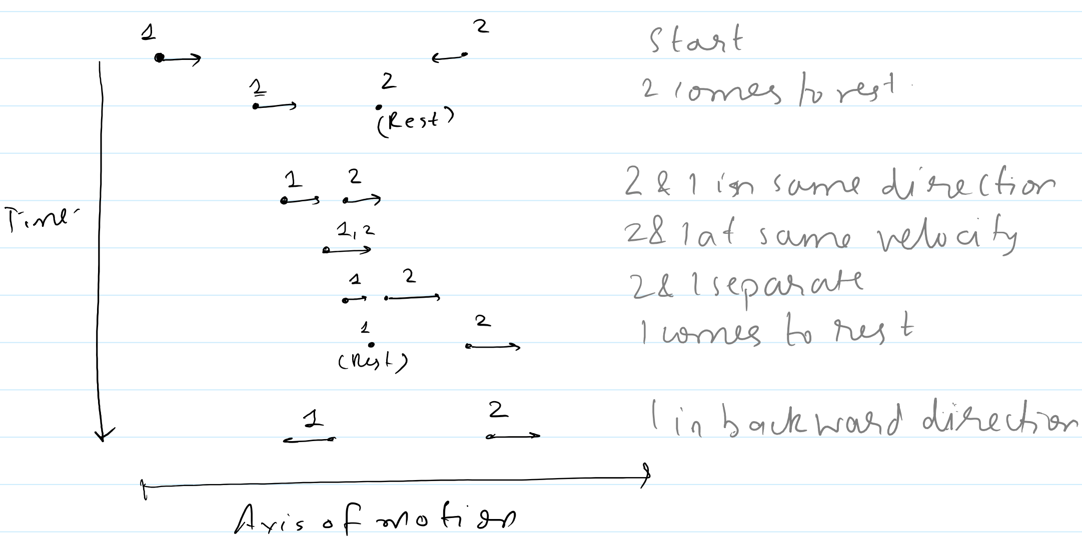
Thus, $l_{max} = (v_1+v_2)\cdot\frac{(v_1+v_2)}{(a_1+a_2)} - \frac{1}{2}(a_1+a_2)\left(\frac{v_1+v_2}{a_1+a_2}\right)^2$

$$\Rightarrow l_{max} = \frac{1}{2}\frac{(v_1+v_2)^2}{(a_1+a_2)}$$





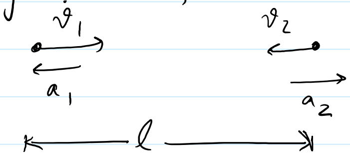
The plots above show one of the many conditions in which this can happen. Here, body 1 moves in a +ve direction (forward) and body 2 moves backward. Body 2 starts reversing its direction before it ever meets body 1 (in case of l_{max} being the initial separation). Then, it picks up speed in the same direction as body 1 and the 2 bodies meet when their forward velocity is identical. Now, these are 2 bodies moving in the same direction but one (body 2) is speeding up and one (body 1) is slowing down. Thus, they separate.



In the digital plots, it can be seen that the only condition in which the 2 bodies meet only once is at $l = l_{max}$.
The code for this plot is present in MATLAB file "prob-1-4.m".

Method-2: No calculus (Krotov)

Let us use the frame of reference of body 1. Then,



$$v_{21} = -v_2 - v_1$$
$$a_{21} = a_2 - (-a_1) = a_2 + a_1$$

Now, as depicted in the graphs above, we want the velocities of two bodies to be the same at the point of contact. This means a relative velocity of zero. So,

$$0^2 = u_{21}^2 + 2a_{21}\,\Delta x_{21}$$
$$\Rightarrow 0^2 = (-v_2 - v_1)^2 + 2(a_2 + a_1)(x_{f,21} - l_{max})$$
$$\Rightarrow 0^2 = (v_1 + v_2)^2 + 2(a_1 + a_2)(0 - l_{max})$$

(relative separation at the instant of meeting $= 0$)

$$\Rightarrow l_{max} = \frac{(v_1 + v_2)^2}{2(a_1 + a_2)}$$

Another way to think of it (as mentioned in Krotov) is that body 2 should approach body 1 for contact to happen. It may pass body 1, but the least we expect is for it to come to rest (in body 1's reference frame) before it reverses direction. Thus, we want final relative velocity & final relative distance to be zero, and then the steps described above follow.