Data Streams

- Imdadullah Khan

Data Stream

Stream Processing: Analytics on a continuous stream of data items

The goal is to draw meaningful analytics from the stream subject to

subject to

- Single Pass: process each item exactly once (common requirement)
- Limited Memory: poly-logarithmic space (in length of stream or domain)
- Constant per item processing: near real time
- Arbitrary arrival order: No assumption on distribution or order of items

Outline

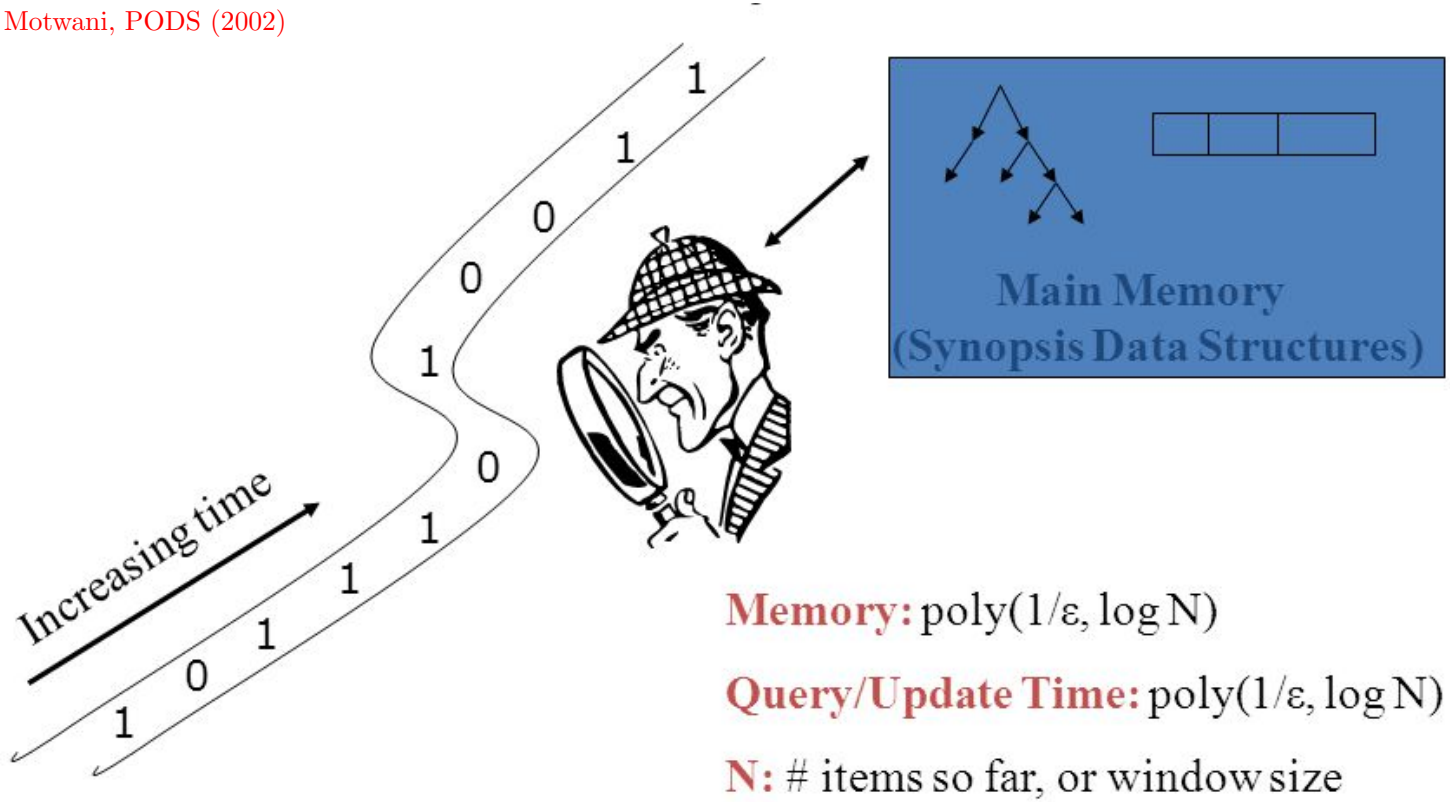
Characteristics of data streams [1]

- Huge volumes of continuous data, possibly infinite
- Fast changing and requires fast, real-time response
- Data stream captures nicely our data processing needs of today
- Random access is expensive
- Single scan algorithm (can only have one look)
- Store only the summary of the data seen thus far
- Most stream data are at pretty low-level or multi-dimensional in nature, needs multi-level and multi-dimensional processing

[1] Based on Han & Kamber, Data Mining Concepts & Techniques, 2nd Ed.

Stream Model of Computation

Motwani, PODS (2002)



Memory: poly(1/ε, log N)

Query/Update Time: poly(1/ε, log N)

N: # items so far, or window size

ε: error parameter

Data Stream

Stream data is fundamentally different than traditional datasets[2]

Traditional Data (DBMS)	Data Stream
Persistent storage	Transient stream(s)
One-time query	Continuous query
Random access	Sequential access
Unbounded disk storage	Bounded main memory
Only current state matters	Arrival-order is critical
No real time services	Real-time requirements
Low update rate	Possibly multi-GB arrival rate (dynamic & fast)
Mixed granularity	Data at fine granularity

[2] R. Motwani, PODS (2002)

Data Stream Processing Model

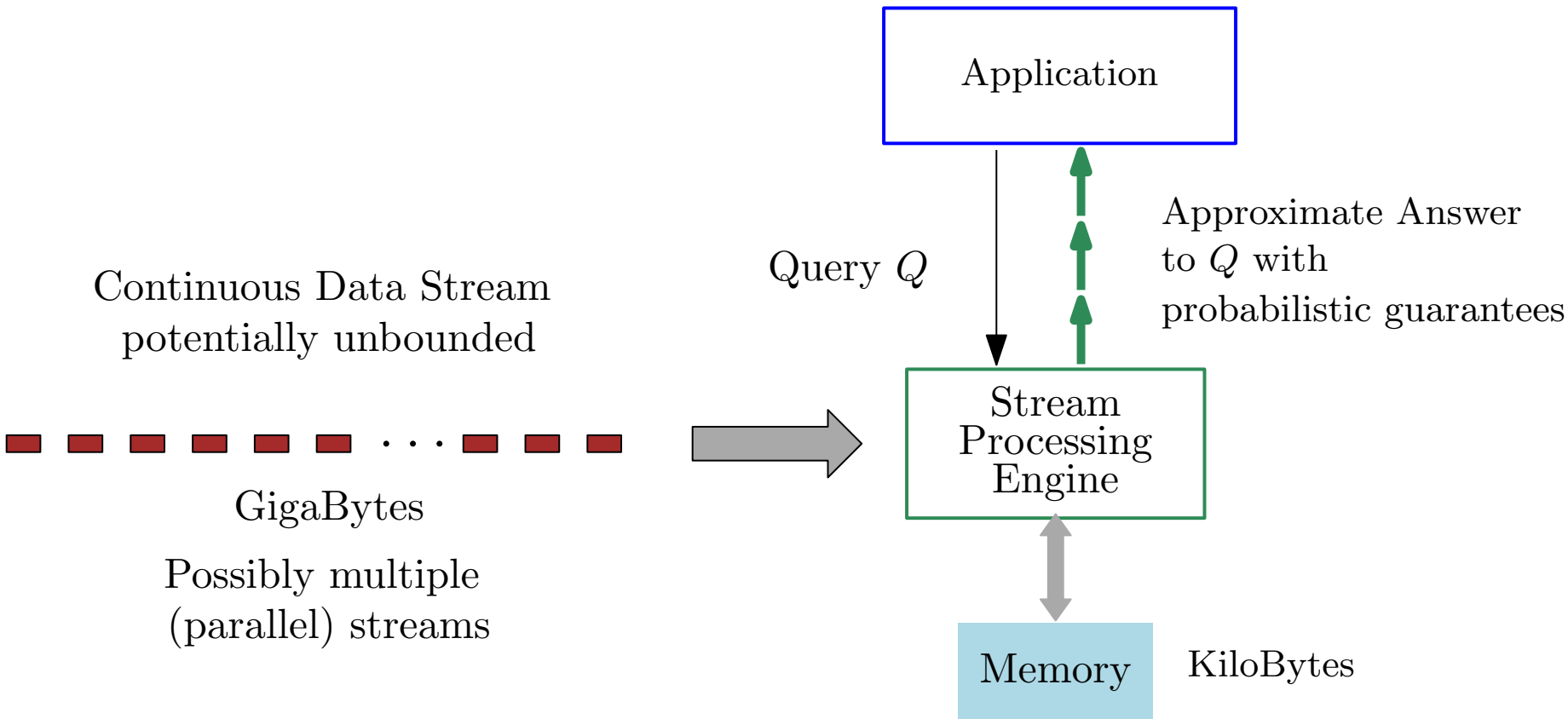
- Since streams are long (potentially unbounded) exact algorithms with limited memory are possible only for a few simple queries
- $\therefore$ we design approximate algorithms (they often suffice)

(ϵ, δ)-approximate algorithm

- $\mathcal{A}$: an algorithm to compute $f(\mathcal{S})$ (a function of stream)
- $\mathcal{A}(\mathcal{S})$: output of $\mathcal{A}$ on $\mathcal{S}$
- For $\epsilon > 0, 0 \leq \delta \leq 1$, $\mathcal{A}$ is an (ϵ, δ)-approximation algorithm if

$$Pr\left[\mathcal{A}(\mathcal{S}) - f(\mathcal{S}) > \epsilon f(\mathcal{S})\right] \leq \delta$$

Data Stream


Application
Query Q
Approximate Answer
to Q with
probabilistic guarantees
Continuous Data Stream
potentially unbounded
Stream
Processing
Engine
GigaBytes
Possibly multiple
(parallel) streams
Memory
KiloBytes

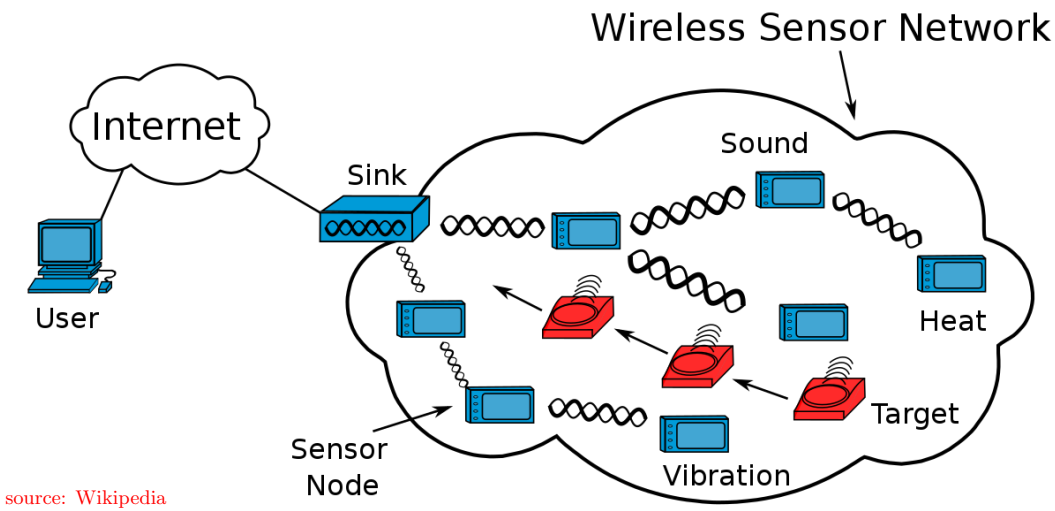
Data Stream Applications

Stream data comes in many domains and has various applications[3]

- Telecommunication calling records
- Business: credit card transaction flows
- Network monitoring and traffic engineering
- Financial market: stock exchange
- Engineering & industrial processes: power supply & manufacturing
- Sensor, monitoring & surveillance: video streams, RFIDs
- Security monitoring
- Web logs and Web page click streams
- Massive data sets (even saved but random access is too expensive)

[3] Based on Han & Kamber, Data Mining Concepts & Techniques, 2nd Ed.

Data Stream Motivation



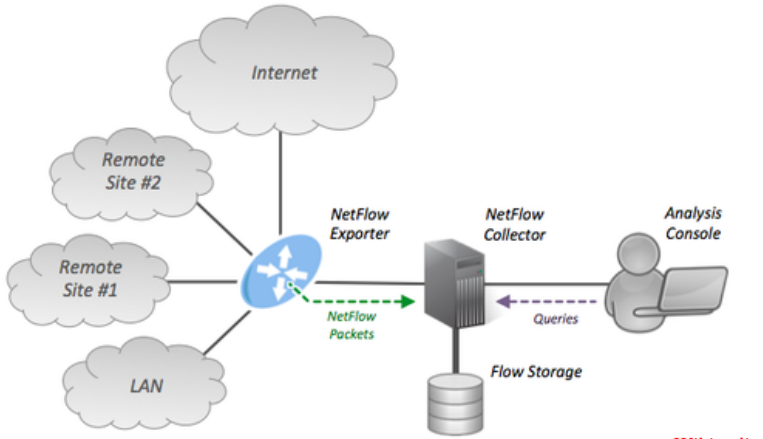
source: Wikipedia

- Sensor nodes collect unlimited amount of data
- have very limited computation power and memory
- Limited battery power constrain communication of all collected data
- 1 bit transmission consumes power $\sim$ to executing 800 instructions[4]
- Streaming algorithm deployed onto nodes are ideally suited for drawing analytics from sensed data

[4] Madden et.al. (2002)

Data Stream Applications

Network Monitoring and Management



source: Wikipedia

NetFlow: A Cisco tool for network administrators (performance metrics, security analysis, detection and forensics). For each Flow it reports (logs)

- Network Interface
- Source/Destination IP Addresses
- IP Protocol
- Source/Destination port
- TCP Flags
- Total packets/bytes in flow,

- AT&T Processes over 567 billion flow records per day[5] ▷ $\sim$ 15 PBytes
- Detects and characterizes approximately 500 anomalies per day

[5] Fred Stinger (AT&T) FloCon (2017) Netflow Collection and Analysis ..

Data Stream Applications

Network Monitoring and Management

Application Areas

- Traffic Engineering
- Traffic Monitoring
- Volume estimation & analysis
- Load Balancing
- Efficient Resource Utilization
- (D)DOS Attack Detection
- SLA Voilation

Quries

- How many bytes sent b/w IP-1 and IP-2?
- How many IP addresses are active?
- Top 100 IP's by traffic volume
- Average duration of IP session?
- Meidan number of bytes in each IP session
- Find sessions that transmitted $> 1k$ bytes
- Find sessions with duration $>$ twice average
- List all IP's with a sudden spike in traffic
- List all IP involved in more than 1k sessions

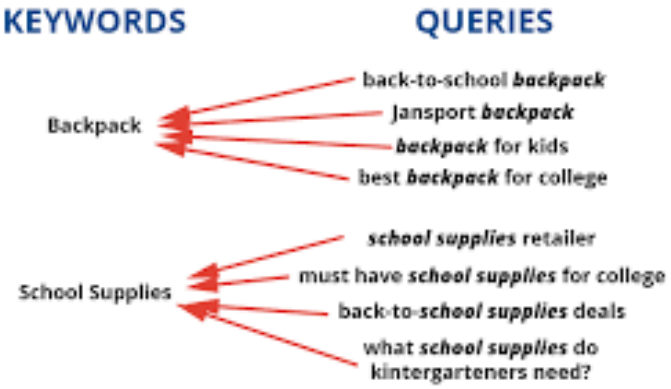
Data Stream Applications

Web Click Stream Analysis: tracking and analysis of websites visits

- A stream of user clicks (and other actions) on a website
- Which webpages are getting large/small number of clicks
- Used to enhance customer experience
- Improving conversion rates
- Optimizing digital marketing
- Up-selling and cross-selling

Data Stream Applications

Search Queries Stream:



- Search Engine: logs of billions of queries every day
- Discover trends and patterns
- Find relevant keywords for your website
- Estimate competition scores (or difficulty) and CPC (cost per click) of keywords
- Improve Good advertisement

Data Stream Applications

Energy consumption Analysis:

- Electricity consumption data from AMI (Automatic Metering Interface)
- Find average hourly load, load surges, anamoly
- Identify faults, drops, failures

Data Stream Applications

Query Execution Plan can be optimized using a synopsis of the database

Suppose we have data of $n = 1M$ people in a database and the query

SELECT * from Table WHERE $25 \leq age \leq 35$ and $54 \leq weight \leq 60$

Runtime of brute force execution is $2n$ comparisons

Suppose we have the following synopsis of distribution of an attribute

Age	Freq
$0-10$	7%
$11-20$	8%
$21-30$	10%
$31-40$	12%
$41-50$	13%
$51-60$	25%
$61-70$	20%
$71+$	5%

First filter on Age, then on weight

Runtime: $1.22n$

Weight	Freq.
$0-20$	20%
$21-40$	25%
$41-60$	10%
$61-80$	15%
$81+$	30%

First filter on Weight, then on age

Runtime: $1.1n$

Stream Model of Computation

Stream $\mathcal{S} := a_1, a_2, a_3, \ldots, a_m$ ▷ m may be unknown

Each $a_i \in [n]$

Goal: Compute a function of the stream $\mathcal{S}$ (e.g. mean, median, number of distinct elements, frequency moments..)

Subject to

- Single pass, read each element of $\mathcal{S}$ only once sequentially
- Per item processing time $O(1)$
- Use memory polynomial in $O(1/\epsilon, 1/\delta, \log n)$
- Return (ϵ, δ)-randomized approximate solution

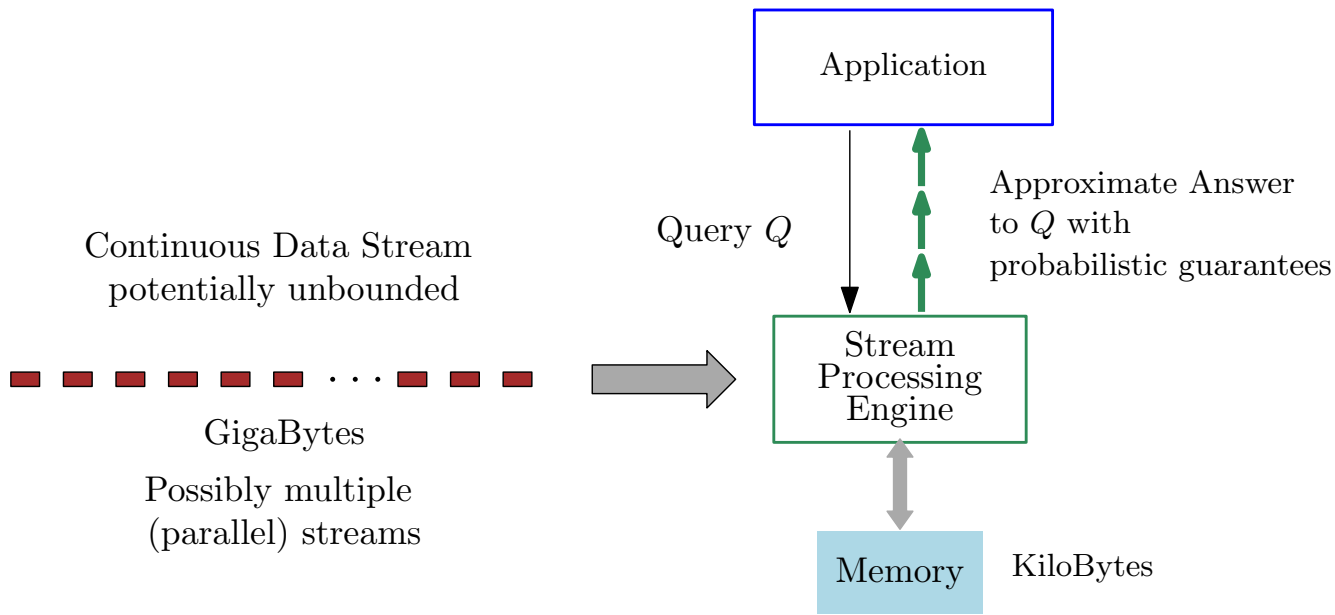
Data Stream: Synopsis

Fundamental Methodology: Keep a synopsis of the stream and answer query based on it. Update synopsis after examining each item in $O(1)$

Synopsis: Succinct summary of the stream (so far) (poly-log bits)

Families of Synopsis

- Sliding Window
- Random Sample
- Histogram
- Wavelets
- Sketch



Synopsis Based Exact Stream Computation

- Length of $\mathcal{S}$ (m): Computed by storing a running counter
- Sum of $\mathcal{S}$: Computed by storing a running sum
- Mean of $\mathcal{S}$: Computed from sum and length of $\mathcal{S}$
- Variance of $\mathcal{S}$: Computed from sum, sum of square, and length of $\mathcal{S}$

$$Var(X) = E(X^2) - (E(X))^2$$

All these use $O(\log n)$ bits memory and constant time per element

Synopsis Based Exact Stream Computation

Missing Element

- $n-1$ unique integers are streamed in from $[n]$
- Find the missing integer?
- Trivial to find it if we use n bits
- A better solution is to save sum S of the stream ▷ $O(\log n)$ bits
- The missing integer is $n(n+1)/2 - S$
- Can do it in exactly $\log n$ bits by storing the parity sum of each bits
- The final parity sum is the missing integer

Synopsis Based Exact Stream Computation

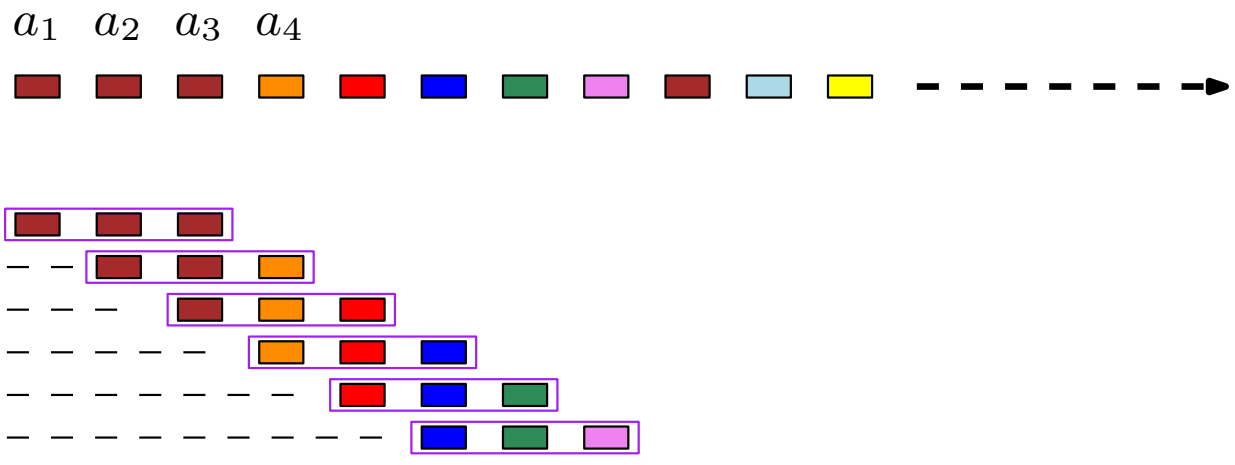
Two Missing Elements

- $n-2$ unique integers are streamed in from $[n]$
- Find the missing integers?
- Trivial to find it if we use n bits
- Save sum of 1st and 2nd powers of stream elements $\triangleright$ $O(\log n)$ bits
- The missing integers are solution to 2 unknowns and two equations
- Readily generalizes to k missing elements

Data Stream: Sliding Window

Synopsis: Sliding Window

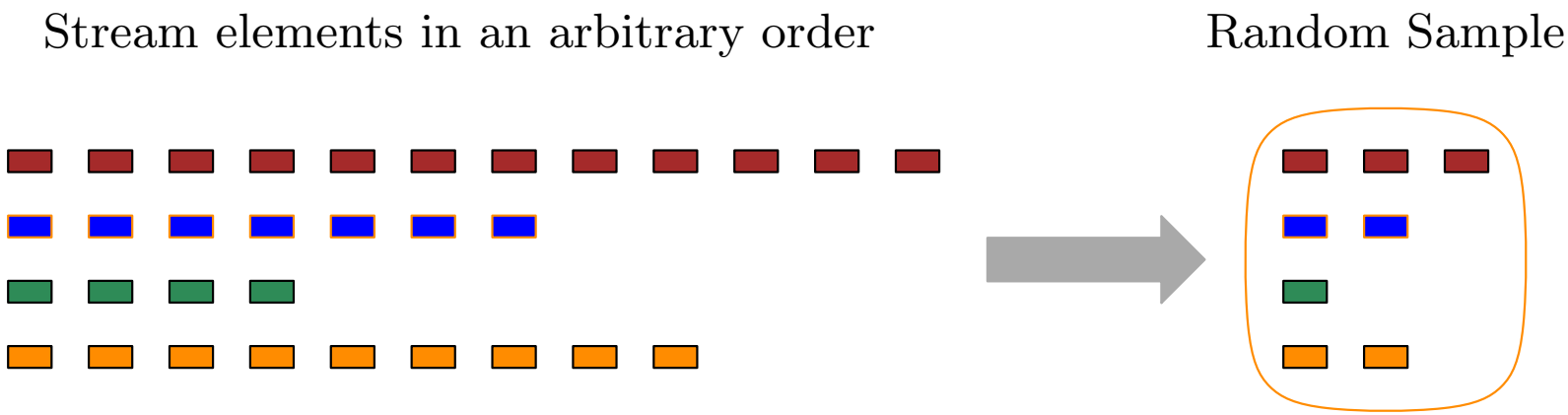
- Keep the last w elements as synopsis (w is length of window)
- On input a_i ($i \geq w$), a_{i-w} expires and a_i added to window
- Can be used for queries like mean, sum, variance, count of pre-specified element(s) (e.g. non-zero, even)
- Extended to compute approximate median, and k-median



Data Stream: Random Sample

Synopsis: Random Sample

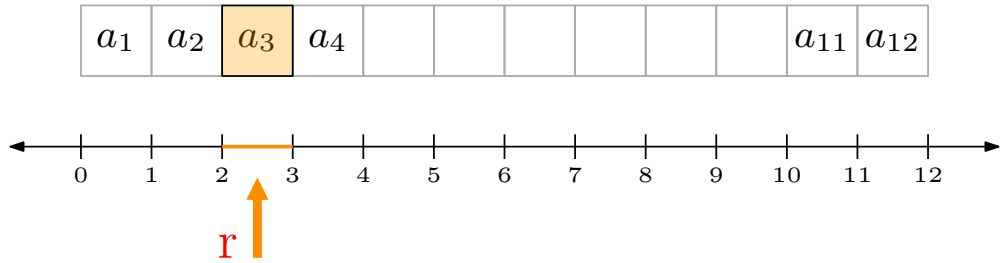
- Keep a "representative" subset of the stream
- Approximately compute query answer on sample (with appropriate scaling etc.)



Data Stream: Random Sample

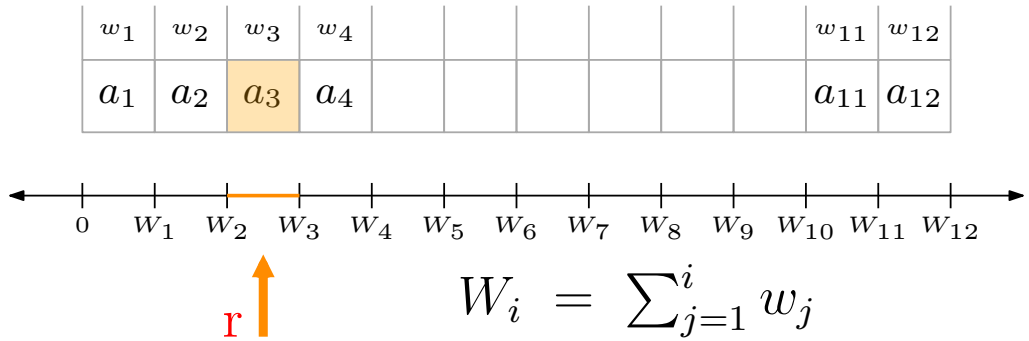
Sample a random element from array A of length n ▷ $A[i]$ with prob $1/n$

- Generate a random number $r \in [0, n]$ ▷ $r \leftarrow \text{RAND}() \times n$
- Return $A[\lceil r \rceil]$



Sample random element (by weight) from array A ▷ $A[i]$ with prob. w_i/W

- Generate a random number $r \in [0, \sum_{j=1}^{n} w_i]$ ▷ $r \leftarrow \text{RAND}() \times W_n$
- Return $A[i]$ if $W_{i-1} \leq r < W_i$



Data Stream: Random Sample

Sample a random element from the stream S ▷ a_i with prob. $1/m$

- If m is known, use algorithm for sampling from array. For unknown m

---

Algorithm : Reservoir Sampling ($\mathcal{S}$)

$R \leftarrow a_1$ ▷ R (reservoir) maintains the sample

for $i \geq 2$ **do**

Pick a_i with probability $1/i$

Replace with current element in R

Prob. that a_i is in the sample R_m (m: stream length or query time)

$$= \underbrace{\Pr \text{ that } a_i \text{ was selected at time } i}_{\frac{1}{i}} \times \underbrace{\Pr \text{ that } a_i \text{ survived in } R \text{ until time } m}_{\prod_{j=i+1}^{m}\left(1-\frac{1}{j}\right)}$$

$$= \frac{1}{\cancel{i}} \times \frac{\cancel{i}}{\cancel{i+1}} \times \frac{\cancel{i+1}}{\cancel{i+2}} \times \frac{\cancel{i+2}}{\cancel{i+3}} \times \ldots \times \frac{\cancel{m-2}}{\cancel{m-1}} \times \frac{\cancel{m-1}}{m} = \frac{1}{m}$$

Data Stream: Random Sample

Sample k random elements from the stream S ▷ a_i with prob. k/m

---

Algorithm : Reservoir Sampling $(\mathcal{S}, k)$

$R \leftarrow a_1, a_2, \ldots, a_k$ ▷ R (reservoir) maintains the sample

for $i \geq k+1$ **do**

Pick a_i with probability k/i

If a_i is picked, replace with it a randomly chosen element in R

Prob. that a_i is in the sample R_m (m: stream length or query time)

$$= \underbrace{\text{Pr that } a_i \text{ was selected at time } i}_{\frac{k}{i}} \times \underbrace{\text{Pr that } a_i \text{ survived in } R \text{ untill time } m}_{\prod_{j=i+1}^{m}\left(1-\left(\frac{k}{j}\times\frac{1}{k}\right)\right)}$$

$$= \frac{k}{\cancel{i}} \times \frac{\cancel{i}}{\cancel{i+1}} \times \frac{\cancel{i+1}}{\cancel{i+2}} \times \frac{\cancel{i+2}}{\cancel{i+3}} \times \ldots \times \frac{\cancel{m-2}}{\cancel{m-1}} \times \frac{\cancel{m-1}}{m} = \frac{k}{m}$$

Data Stream: Histogram and Wavelets

Synopsis: Histogram

- The synopsis is some summary statistics (e.g. frequency, mean) of groups (subsets, buckets) in streams values
  - Equi-width histogram
  - Equidepth histogram
  - V-optimal histogram
  - Multi-dimensional histogram

Synopsis: Wavelets

- Essentially histograms of features (coefficients) in the frequency domain representation of the stream

Data Stream: Linear Sketch

- Sample is a general purpose synopsis
- Process sample only – no advantage from observing the whole stream
- Sketches are specific to a particular purpose (query)
- Sketches (also histograms and wavelets) take advantage from the fact the processor see the whole stream (though can't remember all)

A linear sketch interprets the stream as defining the frequency vector

Often we are interested in functions of the frequency vector from a stream

$\mathcal{S}: \ a_1, a_2, a_3, a_4, \ldots, a_m$

$a_i \in [n]$

$\mathbf{F}:$

1	2	3				n
f_1	f_2	f_3	$\cdots$		$\cdots$	f_n

$f_j = |\{a_i \in \mathcal{S} : a_i = j\}|$ (frequency of j in $\mathcal{S}$)

$\mathcal{S}: \ 2, 5, 6, 7, 8, 2, 1, 2, 7, 5, 5, 4, 2, 8, 8, 9, 5, 6, 4, 4, 2, 5, 5$

$\mathbf{F}:$

1	2	3	4	5	6	7	8	9
1	5	0	3	6	2	2	3	1

Data Stream: Linear Sketch

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$\mathcal{S}: \ 2, 5, 6, 7, 8, 2, 1, 2, 7, 5, 5, 4, 2, 8, 8, 9, 5, 6, 4, 4, 2, 5, 5$

$\mathbf{F}:$

1	2	3	4	5	6	7	8	9
1	5	0	3	6	2	2	3	1

Stream: Frequency Moments

$$\mathcal{S} = < a_1, a_2, a_3, \ldots, a_m > \qquad a_i \in [n]$$

f_i : frequency of i in $\mathcal{S}$ $\qquad$ $\mathbf{F} = \{f_1, f_2, \ldots, f_n\}$

$F_0 := \sum_{i=1}^{n} f_i^0$ ▷ number of distinct elements

$F_1 := \sum_{i=1}^{n} f_i$ ▷ length of stream, m

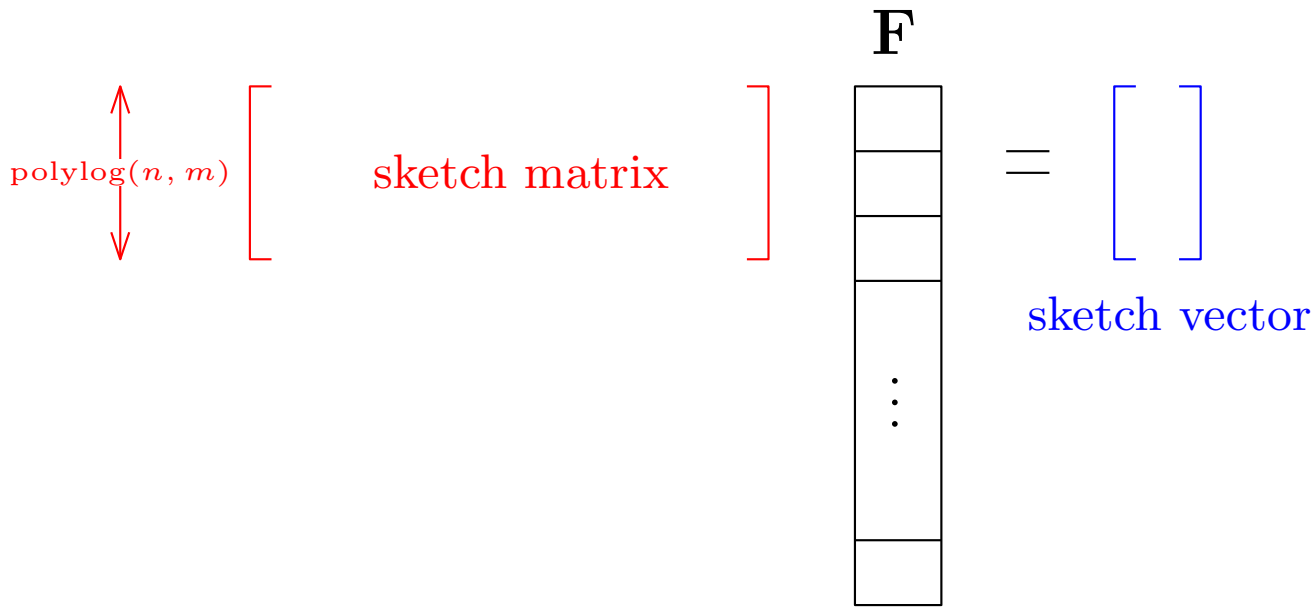
$F_2 := \sum_{i=1}^{n} f_i^2$ ▷ second frequency moment

Data Stream: Linear Sketch

Synopsis: Linear Sketches

Linear sketch is a synopsis that can be computed as a linear transform of **F**

- Best suited for data streams, highly parallelizable
- Very good for our problems of computing norms of **F**
- Can be readily extended to variations of the basic stream model



Data Stream Model: Time Series Model

Time Series Model

Every stream item gives the current frequency of an element ($\mathbf{F}[a_i]$)

Stream items are $a_i = \langle j, c_i \rangle$ and it means $\mathbf{F}[j] \leftarrow c_i$

For stream $\mathcal{S}$: $\langle 7, 3 \rangle, \langle 3, 3 \rangle, \langle 2, 9 \rangle, \langle 7, 2 \rangle, \langle 9, 1 \rangle, \langle 3, 1 \rangle$

The final frequency vector will be

$\mathbf{F} =$

1	2	3	4	5	6	7	8	9
0	9	1	0	0	0	2	0	1

- Used to measure link-bandwidth or energy consumption over time
- Very useful if there are multiple streams (e.g. stock prices for different companies

Data Stream Model: Cash-Register Model

Cash-Register Model aka Arrivals-Only Stream

Every stream item is an increment to a frequency.

Stream items are $a_i = \langle j, c_i \rangle$ and it means $\mathbf{F}[j] \leftarrow \mathbf{F}[j] + c_i \quad c_i \geq 1$

For stream $\mathcal{S}$: $\langle 7,3 \rangle, \langle 3,3 \rangle, \langle 2,9 \rangle, \langle 7,2 \rangle, \langle 9,1 \rangle, \langle 3,1 \rangle$

The final frequency vector will be

$\mathbf{F} =$

1	2	3	4	5	6	7	8	9
0	9	4	0	0	0	5	0	1

Can be used e.g. for packet counts in every flow

Data Stream Model: Turnstile Model

Turnstile Model aka Arrivals and Departures Stream

Every stream item is an update to a frequency

Stream items are $a_i = \langle j, c_i \rangle$ and it means $\mathbf{F}[j] \leftarrow \mathbf{F}[j] + c_i$ ~~$c_i \geq 1$~~

For stream $\mathcal{S} : \langle 7, 3 \rangle, \langle 3, 3 \rangle, \langle 2, 9 \rangle, \langle 7, -2 \rangle, \langle 9, 1 \rangle, \langle 3, -1 \rangle$

The final frequency vector will be

$\mathbf{F} =$

1	2	3	4	5	6	7	8	9
0	9	2	0	0	0	1	0	1

Generally, model has restriction of $\mathbf{F}[\cdot] \geq 0$

Universal hash functions

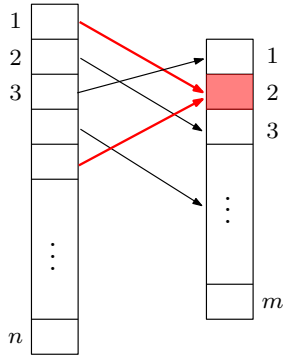
Hash functions/table is an efficient way to implement the Dictionary ADT

Hash functions map keys $A \subset U$ to m buckets labeled $\{0, 1, 2, \ldots, m-1\}$

A is not known in advance and $|A| = n$

Desired properties from hashing

- Fewer collisions
- Small range (m)
- Small space complexity to store hash function
- Easy to evaluate hash value for any key



A family of hash functions $\mathcal{H}$ is 2-universal if

$$\text{for any distinct keys } x, y \in U\ , \qquad \Pr_{h \in_R \mathcal{H}}\left[h(x) = h(y)\right] \leq \frac{1}{m}$$

Source of randomness is picking h (at random) from the family

Universal hash functions

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$$\text{for any distinct keys } x, y \in U \ , \qquad \Pr_{h \in_R \mathcal{H}} \left[h(x) = h(y) \right] \ \leq \ \frac{1}{m}$$

Linear Congruential Generators for $U = \mathbb{Z}$

- Pick a prime number $p > m$
- For any two integers a and b $(1 \leq a \leq p-1)$, $(0 \leq b \leq p-1)$
- A hash function $h_{a,b} : U \mapsto [m]$ is defined as

$$h_{a,b}(x) = (ax + b) \pmod{p} \pmod{m}$$

$\mathcal{H} := \{h_{a,b} \ : \ 1 \leq a \leq p-1 \ , \ 0 \leq b \leq p-1\}$ is 2-universal

Picking a random $h \in \mathcal{H}$ amounts to picking random a and b

Count-Min Sketch

- Count-Min sketch (Cormode & Muthukrishnan 2005) for frequency estimates
- Cannot store frequency of every elements
- Store total frequency of random groups (elements in hash buckets)

Algorithm : Count-Min Sketch (k, ϵ, δ)

$\text{COUNT} \leftarrow \text{ZEROS}(k)$ ▷ sketch consists of k integers

Pick a random $h : [n] \mapsto [k]$ from a 2-universal family $\mathcal{H}$

On input a_i

$\text{COUNT}[h(a_i)] \leftarrow \text{COUNT}[h(a_i)] + 1$ ▷ increment count at index $h(a_i)$

On query j ▷ query: $\mathbf{F}[j] =?$

return $\text{COUNT}[h(j)]$

Count-Min Sketch

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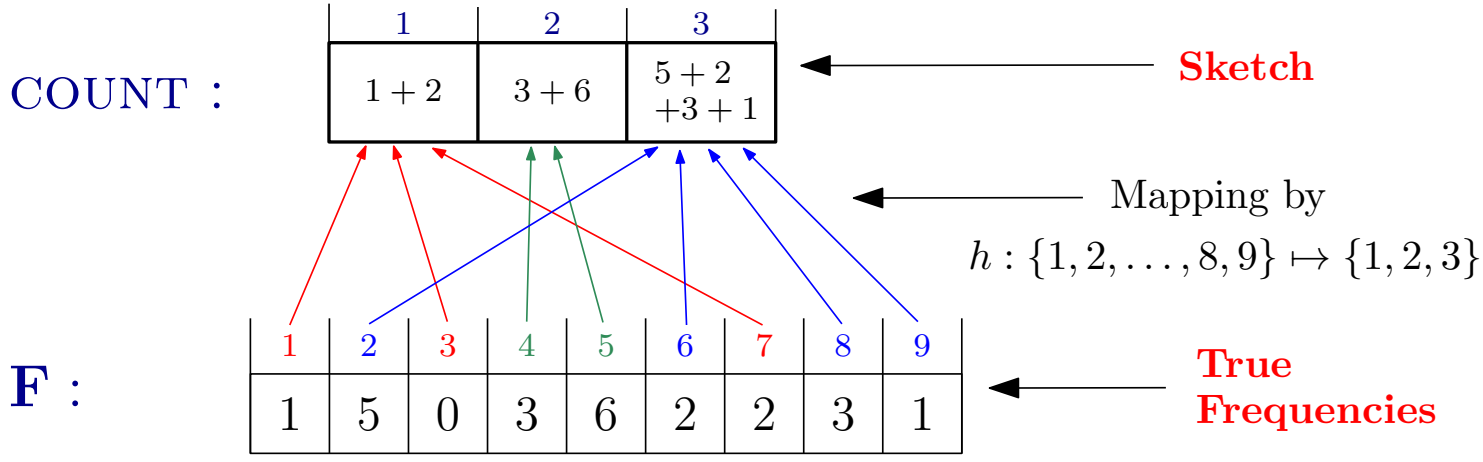
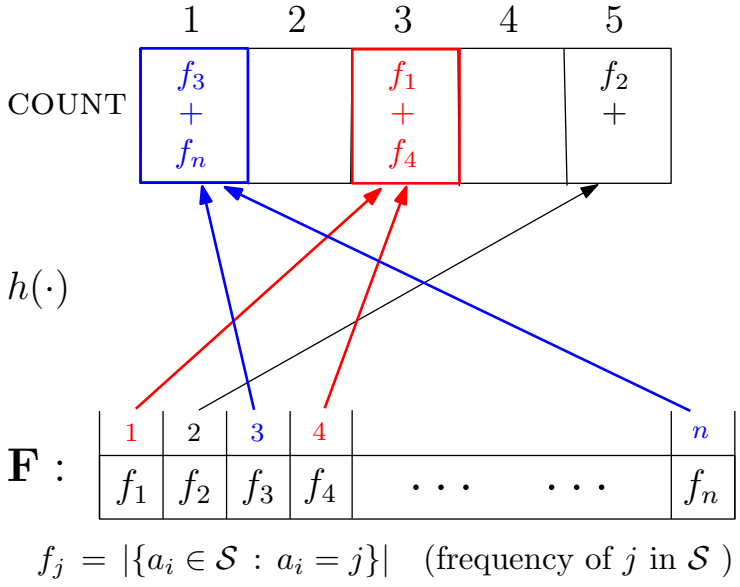
return COUNT$[h(j)]$

$\mathcal{S}$: $2, 5, 6, 7, 8, 2, 1, 2, 7, 5, 5, 4, 2, 8, 8, 9, 5, 6, 4, 4, 2, 5, 5$



Count-Min Sketch



- $k = \frac{2}{\epsilon}$
- Large k means better estimate (smaller groups) but more space
- $\tilde{f}_j$: estimate for f_j – output of algorithm

Count-Min Sketch

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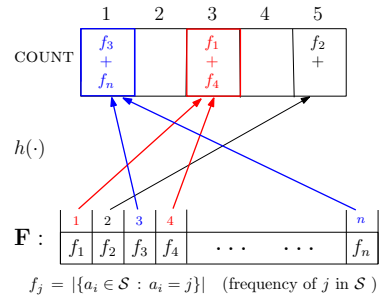
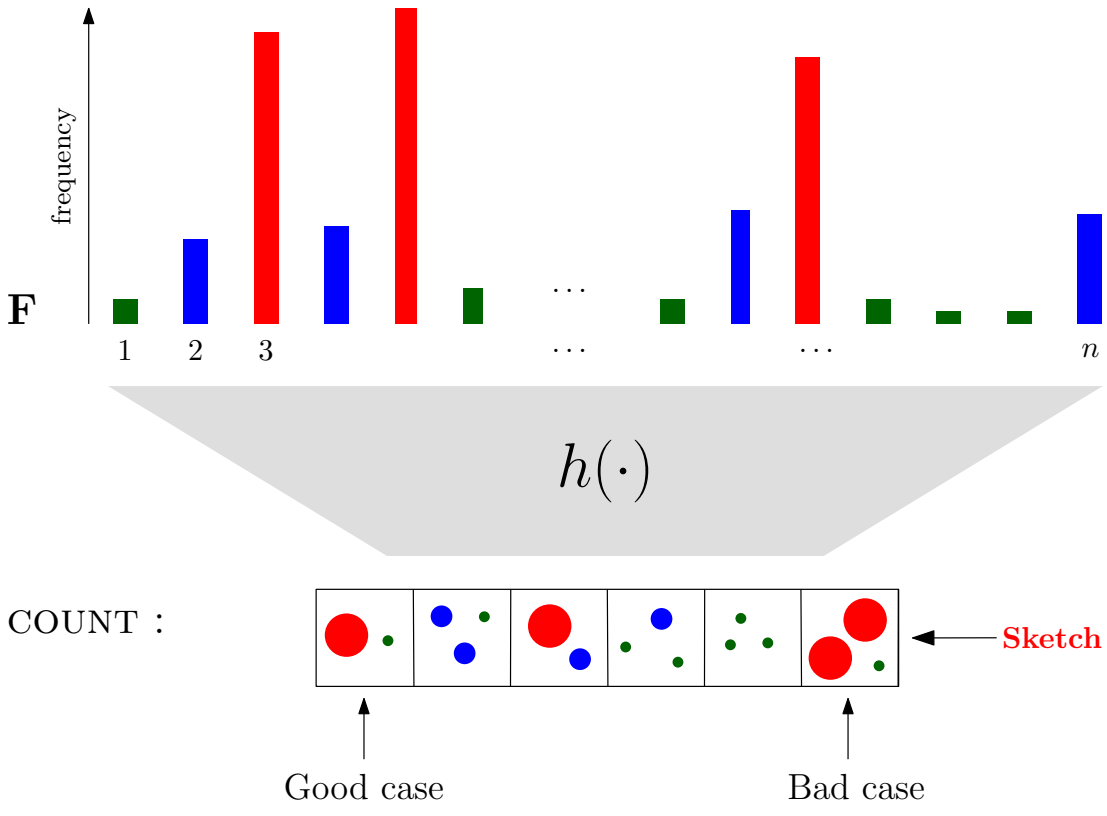
- $\tilde{f}_j$: estimate for f_j – output of algorithm

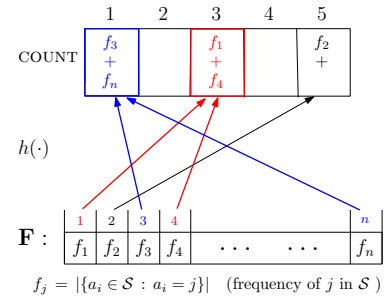


Bounds on $\tilde{f}_j$: (idea)



Count-Min Sketch

- $k = 2/\epsilon$
- Large k means better estimate but more space
- $\tilde{f}_j$: estimate for f_j – output of algorithm



Bounds on $\tilde{f}_j$: (idea)

1. $\tilde{f} \geq f_j$
   - Other elements that hash to $h(j)$ contribute to $\tilde{f}_j$
2. $Pr\left[\tilde{f}_j \leq f_j + \epsilon \|F\|_1\right] \geq \frac{1}{2}$
   - $X_j = \tilde{f}_j - f_j$ ▷ Excess in $\tilde{f}_j$ (error)
   - $X_j = \sum_{i \in [n] \setminus j} f_i \cdot 1_{h(i)=h(j)}$ ▷ $1_{condition}$ is indicator of condition

$$\mathbb{E}(X_j) = \mathbb{E}\left(\sum_{i \in [n] \setminus j} f_i \cdot 1_{h(i)=h(j)}\right) = \sum_{i \in [n] \setminus j} f_i \cdot \frac{1}{k} \leq \sum_{i \in [n] \setminus j} \|F\|_1 \cdot \frac{\epsilon}{2}$$

   - By Markov inequality we get the bound

Count-Min Sketch

Idea: Amplify the probability of the basic count-min sketch

Keep t over-estimates, $t = \log(1/\delta)$, $k = 2/\epsilon$ and return their minimum

Unlikely that all t functions hash j with very frequent elements

---

Algorithm : Count-Min Sketch (k, ϵ, δ)

$\text{COUNT} \leftarrow \text{ZEROS}(t \times k)$ ▷ sketch consists of t rows of k integers

Pick t random functions $h_1, \ldots, h_t : [n] \mapsto [k]$ from a 2-universal family

On input a_i

for $r = 1$ to t **do**

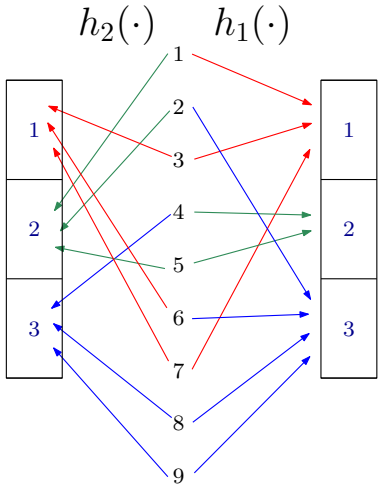
$\text{COUNT}[r][h_r(a_i)] \leftarrow \text{COUNT}[r][h_r(a_i)] + 1$

▷ increment $\text{COUNT}[r]$ at index $h_r(a_i)$

On query j ▷ query: $\mathbf{F}[j] =?$

return $\min_{1 \le r \le t} \text{COUNT}[r][h_r(j)]$

Count-Min Sketch



$\mathcal{S}$: 2, 5, 6, 7, 8, 2, 1, 2, 7, 5, 5, 4, 2, 8, 8, 9, 5, 6, 4, 4, 2, 5, 5

$h_1(\cdot)$

COUNT :

	1	2	3
	$0+1+2$	$3+6$	$5+2$ $+3+1$
	$0+2+2$	$1+5+6$	$3+3+1$

$h_2(\cdot)$

Sketch

F :

1	2	3	4	5	6	7	8	9
1	5	0	3	6	2	2	3	1

True Frequencies

On input a

$h_1(a)$ $h_t(a)$

	1	2	3	...	...	k
COUNT$[1][\cdot]$		$+1$				
COUNT$[2][\cdot]$					$+1$	
COUNT$[3][\cdot]$				$+1$		
$\vdots$						
COUNT$[t][\cdot]$						$+1$

On query a $\quad \text{MIN}_i \ \text{COUNT}[i][h_i(a)]$

Count-Min Sketch

1. $\tilde{f}_j \geq f_j$
   - For every r, other elements that hash to $h_r(j)$ contribute to $\tilde{f}_j$
2. $\tilde{f}_j \leq f_j + \epsilon\|F\|_1$ with probability at least $1 - \delta$
   - X_{jr} : contribution of other elements to $Count[r][h_r(j)]$
   - $\Pr\left[X_{jr} \geq \epsilon\|F\|_1\right] \leq \frac{1}{2}$ for $k = 2/\epsilon$
   - The event $\tilde{f}_j \geq f_j + \epsilon\|F\|_1$ is $\forall\ 1 \leq r \leq t$ $X_{jr} \geq \epsilon\|F\|_1$
   - $\Pr\left[\forall\ r\ X_{jr} \geq \epsilon\|F\|_1\right] \leq \left(\frac{1}{2}\right)^t$
   - $t = \log(\frac{1}{\delta}) \implies \Pr\left[\forall\ r\ X_{jr} \geq \epsilon\|F\|_1\right] \leq \left(\frac{1}{2}\right)^{\log 1/\delta} = \delta$

- Count-Min sketch is an $(\epsilon\|F\|_1, \delta)$-additive approximation algorithm
- Space required is $k \cdot t$ integers $= O(1/\epsilon \log(1/\delta) \log n)$ (plus constant)

The Count Sketch

- In Count-Min sketch error in frequency estimate accumulates (group total)
- The Count Sketch (Charikar, Chen, Farach-Colton, 2002)
- A frequency estimate where errors in a group cancel each other

---

Algorithm : Count Sketch (k, ϵ, δ)

Pick a random $h : [n] \mapsto [k]$ from a 2-universal family $\mathcal{H}$

Pick a random $g : [n] \mapsto \{-1, 1\}$ from a 2-universal family

$\text{COUNT} \leftarrow \text{ZEROS}(k)$ ▷ sketch consists of k integers

On input a_i

$\text{COUNT}[h(a_i)] \leftarrow \text{COUNT}[h(a_i)] + g(a_i)$

▷ increment or decrement, depending on value of $g(a_i)$ COUNT at index $h(a_i)$

On query j ▷ query: $\mathbf{F}[j] =?$

return $g(j) \times \text{COUNT}[h(j)]$

Count Sketch

Algorithm : Count Sketch (k, ϵ, δ)

Pick a random $h : [n] \mapsto [k]$ from a 2-universal family $\mathcal{H}$

Pick a random $g : [n] \mapsto \{-1, 1\}$ from a 2-universal family

$\text{COUNT} \leftarrow \text{ZEROS}(k)$ ▷ sketch consists of k integers

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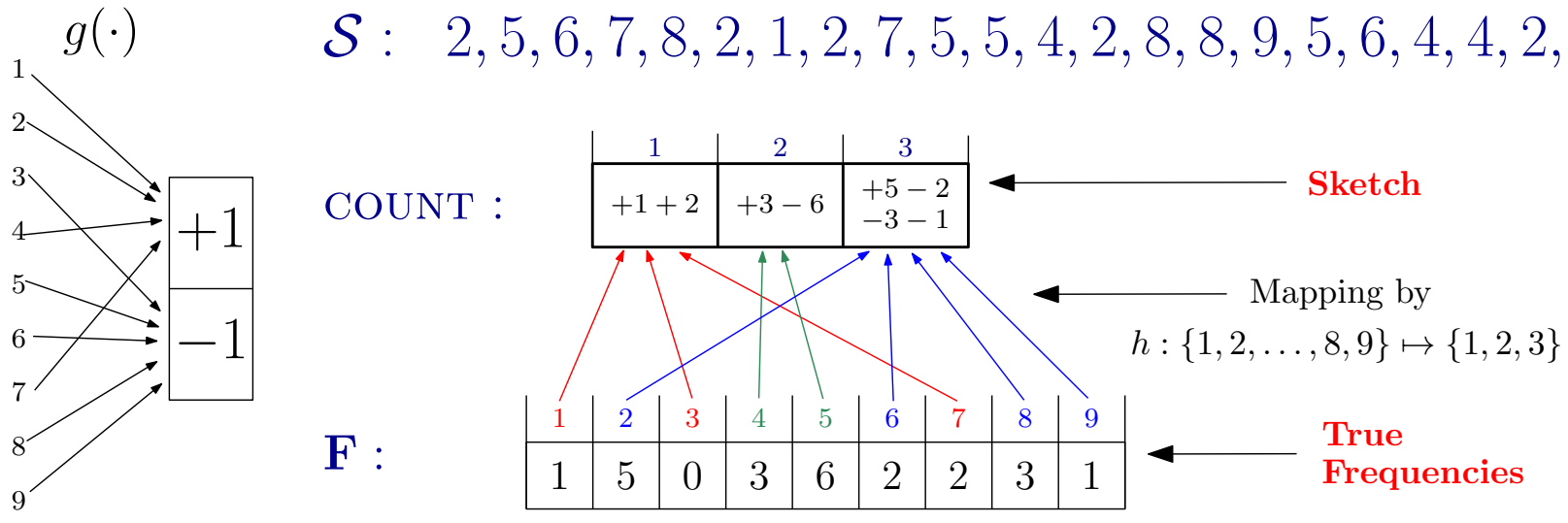
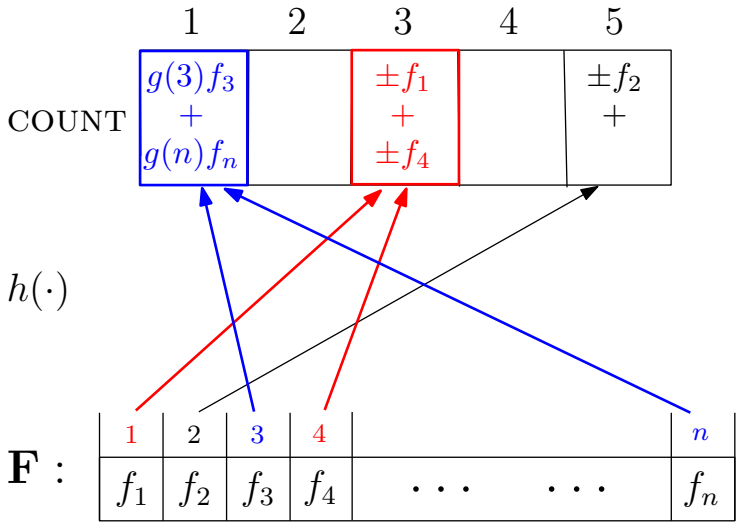
▷ increment or decrement, depending on value of $g(a_i)$ COUNT at index $h(a_i)$

On query j ▷ query: $\mathbf{F}[j] = ?$

return $g(j) \times \text{COUNT}[h(j)]$



The Count Sketch

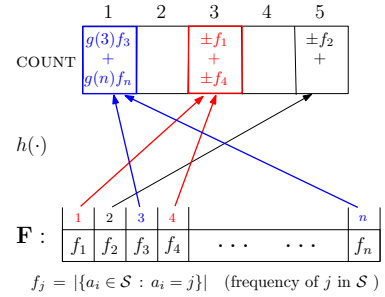


$f_j = |\{a_i \in \mathcal{S} : a_i = j\}|$ (frequency of j in $\mathcal{S}$)

- $k = \dfrac{3}{\epsilon^2}$
- $\tilde{f}_j$: estimate for f_j – output of algorithm

The Count Sketch

- $k = 3/\epsilon^2$
- $\tilde{f_j}$: estimate for f_j – output of algorithm



Bounds on $\tilde{f_j}$:

1. $E(\tilde{f_j}) = f_j$

$$\text{COUNT}[\,h(j)\,] \;=\; \sum_{i\in[n]} f_i \cdot g(i) \cdot 1_{h(i)=h(j)}$$

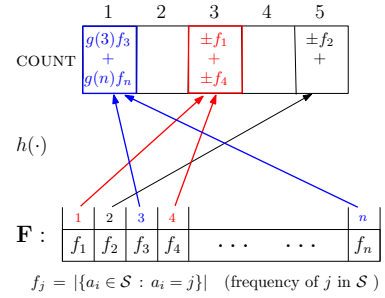
$$\tilde{f_j} \;=\; g(j) \sum_{i\in[n]} f_i \cdot g(i)\, 1_{h(i)=h(j)} = g(j)\left(f(j)g(j) + \sum_{i\in[n]\setminus j} f_i \cdot g(i)\, 1_{h(i)=h(j)} \right)$$

$$= f(j)(g(j))^2 + \sum_{i\in[n]\setminus j} f_i \cdot g(i)g(j) \cdot 1_{h(i)=h(j)} = f(j) + \sum_{i\in[n]\setminus j} f_i \cdot g(i)g(j)\, 1_{h(i)=h(j)}$$

$$\implies \mathbb{E}(\tilde{f_j}) = f_j \qquad \triangleright\ \mathbb{E}(1_{h(i)=h(j)}) = \tfrac{1}{k} \text{ and } \mathbb{E}(g(i)g(j)) = 0$$

The Count Sketch

- $k = 3/\epsilon^2$
- $\tilde{f_j}$: estimate for f_j – output of algorithm



Bounds on $\tilde{f_j}$:

1. $E(\tilde{f_j}) = f_j$
2. $Var(\tilde{f_j}) \leq \frac{1}{k}\|F\|_2$
3. $Pr\left[|\tilde{f_j} - f_j| \geq \epsilon\|F\|_2\right] \leq 1/3$
   - substitute $k = 3/\epsilon^2$ and use Chebychev inequality

Count Sketch

Probability Amplification

Algorithm : Count Sketch (k, ϵ, δ)

$\text{COUNT} \leftarrow \text{ZEROS}(t \times k)$ ▷ sketch consists of t rows of k integers

Pick t random functions $h_1, \ldots, h_t : [n] \mapsto [k]$ from a 2-universal family

Pick t random functions $g_1, \ldots, g_t : [n] \mapsto \{-1, 1\}$ from a 2-uni. family

On input a_i

for $r = 1$ to t **do**

$\text{COUNT}[r][h_r(a_i)] \leftarrow \text{COUNT}[r][h_r(a_i)] + g_r(a_i)$

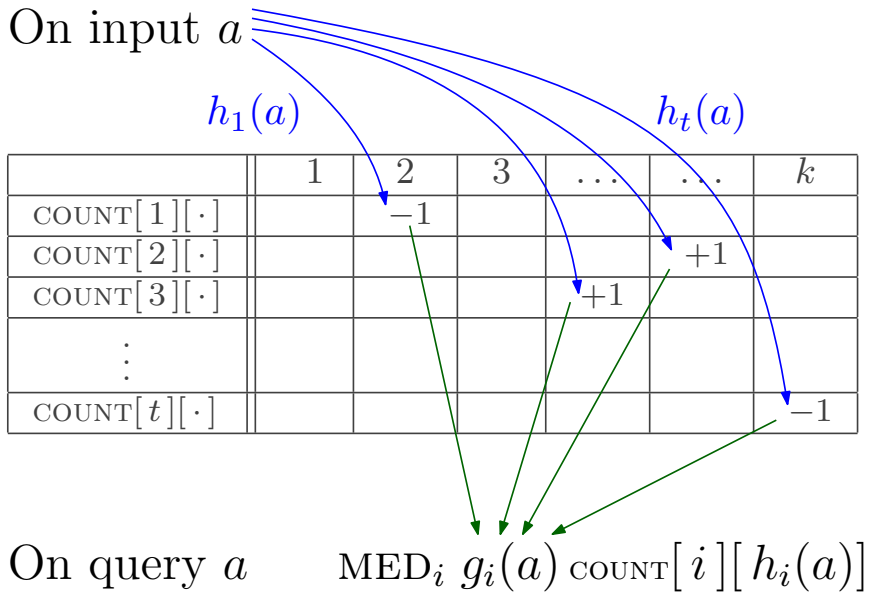
▷ inc/dec $\text{COUNT}[r]$ at index $h_r(a_i)$

On query j ▷ query: $\mathbf{F}[j] =?$

return $\underset{1 \le r \le t}{\text{MEDIAN}} \; g_r(j) \times \text{COUNT}[r][h_r(j)]$

Count Sketch

Keep t unbiassed estimates, $t = \log(1/\delta)$, $k = 3/\epsilon^2$. Their median is a good estimate, unless at least $t/2$ estimates are very bad



1. $E(\tilde{f}_j) = f_j$
2. $|\tilde{f}_j - f_j| \leq \epsilon \|F\|_2$ with probability at least $1 - \delta$ ▷ Uses Chernoff bound

- Count sketch is an $(\epsilon\|F\|_2, \delta)$ additive approximation algorithm
- Space required is $k \cdot t$ integers $= O(1/\epsilon^2 \log(1/\delta) \log n)$ (plus constant)

Estimate F_2 : AMS Algorithm

- The AMS Sketch (Alon, Mathias, Szegedy, 1996)
- A sketch to estimate F_2 (paper has other algorithms for higher moments)

$$\mathcal{S} =< a_1, a_2, a_3, \ldots, a_m > \qquad a_i \in [n]$$

f_i: frequency of i in $\mathcal{S}$ $\quad \mathbf{F} = \{f_1, f_2, \ldots, f_n\}$

F_2: $= \sum_{i=1}^{n} f_i^2$ ▷ second frequency moment

Easy to compute if we store F ▷ $O(n)$ space

Can store $f_1 + f_2 + \ldots + f_n$ ▷ $O(1)$ space

Also easy $(f_1 + f_2 + \ldots + f_n)^2$

Estimate F_2 : AMS Algorithm

$$F_2 := \sum_{i=1}^{n} f_i^2$$

Can store $f_1 + f_2 + \ldots + f_n$ ▷ $O(1)$ space

$(f_1 + f_2 + \ldots + f_n)^2$ can be computed by the following algorithm

Algorithm:

for each $a_i \in \mathcal{S}$

$X \leftarrow X + 1$

return X^2

$X^2 = (f_1 + f_2 + \ldots + f_n)^2$

Estimate F_2 : AMS Algorithm

$F_2 := \sum_{i=1}^{n} f_i^2 = \underline{f_1^2 + f_2^2 + \ldots + f_n^2}$ ▷ We want this

$(f_1 + f_2 + \ldots + f_n)^2$ ▷ Easy but overestimate

$(f_1 + f_2 + f_3 + f_4)^2 = \underline{f_1^2 + f_2^2 + f_3^2 + f_4^2} + 2(f_1 f_2 + f_1 f_3 + f_2 f_3 + f_1 f_4 + f_2 f_4 + f_3 f_4)$

$(f_1 - f_2 + f_3 - f_4)^2 = \underline{f_1^2 + f_2^2 + f_3^2 + f_4^2} + 2(-f_1 f_2 + f_1 f_3 - f_2 f_3 - f_1 f_4 + f_2 f_4 - f_3 f_4)$

Estimate F_2 : AMS Algorithm

Algorithm (AMS):

$g : [n] \to \{-1, +1\}$ ▷ random hash function

for each $a_i \in \mathcal{S}$

$X \leftarrow X + g(a_i)$

return X^2

$X = f_1 g(1) + f_2 g(2) + \ldots + f_n g(n)$

Estimate F_2 : AMS Algorithm

$$X^2 \;=\; \big(f_1 g(1) + f_2 g(2) + \ldots + f_n g(n) \big)^2$$

$$\begin{aligned}
\mathbb{E}\left[X^2\right] &= \mathbb{E}\Big[\sum_i (f_i g(i))^2\Big] \;+\; \mathbb{E}\Big[\sum_{i \neq j} f_i g(i) f_j g(j)\Big] \\
&= \mathbb{E}\Big[\sum_i f_i^2\Big] \;+\; \mathbb{E}\Big[\sum_{i \neq j} f_i f_j g(i) g(j)\Big] \\
&= \mathbb{E}\Big[\sum_i f_i^2\Big] \;+\; \mathbb{E}\Big[\sum_{i \neq j} f_i f_j g(i) g(j)\Big] \\
&= F_2 \;+\; \sum_{i \neq j} f_i f_j \; \mathbb{E}\big[g(i) g(j)\big] \;=\; F_2
\end{aligned}$$

$$\mathbb{E}\left[X^2\right] \;=\; F_2$$

Estimate F_2 : AMS Algorithm

$$X^2 = \big(f_1 g(1) + f_2 g(2) + \ldots + f_n g(n) \big)^2 \qquad \mathbb{E}\big[X^2\big] = F_2$$

$$Var(X^2) \;=\; \mathbb{E}\big[X^4\big] - (\mathbb{E}\big[X^2\big])^2$$

$$\mathbb{E}\big[X^4\big] \;=\; \mathbb{E}\big[\sum_i (f_i g(i))^4 + 6 \sum_{i \neq j} (f_i g(i) f_j g(j))^2\big] + \ldots$$

other terms : $\mathbb{E}\big[g(i)g(j)g(k)g(l)\big] = \mathbb{E}\big[g(i)^2 g(j) g(k)\big] = 0$

$$\mathbb{E}\left[X^4\right] \;=\; \sum_i f_i^4 + 6 \sum_{i \neq j} f_i^2 f_j^2$$

$$Var(X^2) \;=\; \sum_i f_i^4 + 6 \sum_{i \neq j} f_i^2 f_j^2 - (\sum_i f_i^2)^2 \;=\; 4 \sum_{i \neq j} f_i^2 f_j^2 \;\leq\; 2F_2^2$$

Estimate F_2 : AMS Algorithm

Algorithm (AMS):

$g : [n] \mapsto \{-1, +1\}$ ▷ random hash function

for each $a_i \in \mathcal{S}$

$$X \leftarrow X + g(a_i)$$

return X^2

$$\mathbb{E}\left[X^2\right] = F_2 \qquad Var\left(X^2\right) \leq 2F_2$$

Amplify the probability

- Keep $k = {}^{2}/_{\epsilon^2\delta}$ counters
- Return their average $\bar{X} = \frac{1}{k}\sum_i X_i^2$

$$\Pr\left[|\bar{X} - F_2| > \epsilon F_2\right] \leq \delta$$

Estimate F_2 : AMS Algorithm

Algorithm (AMS):

$g : [n] \mapsto \{-1, +1\}$ ▷ random hash function

for each $a_i \in \mathcal{S}$

$$X \leftarrow X + g(a_i)$$

return X^2

$X = f_1 g(1) + f_2 g(2) + \ldots + f_n g(n)$

$\mathbf{g} =$

$g(1)$	$g(2)$	$\ldots$		$g(n)$

F

f_1
f_2
$\vdots$
$\vdots$
f_n

$= X$

Estimate F_2 : AMS Algorithm

Algorithm (AMS):

$g : [n] \mapsto \{-1, +1\}$ ▷ random hash function

for each $a_i \in \mathcal{S}$

$$X \leftarrow X + g(a_i)$$

return X^2

$X = f_1 g(1) + f_2 g(2) + \ldots + f_n g(n)$

$\mathbf{g} =$

$+1$	-1	$\ldots$		$+1$

F

f_1
f_2
$\vdots$
$\vdots$
f_n

$= X$

Estimate F_2 : AMS Algorithm

Algorithm (AMS):

$g : [n] \mapsto \{-1, +1\}$ ▷ random hash function

for each $a_i \in \mathcal{S}$

$$X \leftarrow X + g(a_i)$$

return X^2

$X = f_1 g(1) + f_2 g(2) + \ldots + f_n g(n)$

$\mathbf{G} =$

$+1$	-1	$\ldots$		$+1$
-1	-1	$\ldots$		-1

F

f_1
f_2
$\vdots$
$\vdots$
f_n

X

X_1
X_2

Estimate F_2 : AMS Algorithm

Algorithm (AMS):

$g : [n] \mapsto \{-1, +1\}$ ▷ random hash function

for each $a_i \in \mathcal{S}$

$$X \leftarrow X + g(a_i)$$

return X^2

$X = f_1 g(1) + f_2 g(2) + \ldots + f_n g(n)$

$\mathbf{G} =$

$+1$	-1	$\ldots$		$+1$
-1	-1	$\ldots$		-1
$\vdots$		$\ldots$		$\vdots$
-1	$+1$	$\ldots$		-1

F

f_1
f_2
$\vdots$
$\vdots$
f_n

X

X_1
X_2
$\vdots$
X_k

Estimate F_2 : AMS Algorithm

$\mathbf{G} =$

$+1$	-1	$\dots$		$+1$
-1	-1	$\dots$		-1
$\vdots$		$\dots$		$\vdots$
-1	$+1$	$\dots$		-1

F

f_1
f_2
$\vdots$
$\vdots$
f_n

X

X_1
X_2
$\vdots$
X_k

$$\bar{X} = \frac{1}{k}\sum_{i=1}^{k} X_i^2 \qquad \Pr\left[|\bar{X} - F_2| > \epsilon F_2\right] \leq \delta$$

With probability at leat $1 - \delta$

$$(1-\epsilon)\sum_{i=1}^{n} f_i^2 < \frac{1}{k}\sum_{i=1}^{k} X_i^2 < (1+\epsilon)\sum_{i=1}^{n} f_i^2$$

$$\sqrt{(1-\epsilon)}\|F\|_2 < \frac{1}{\sqrt{k}}\|X\|_2 < \sqrt{(1+\epsilon)}\|F\|_2$$

Estimate F_2 : AMS Algorithm

$\mathbf{G} =$

$+1$	-1	$\ldots$		$+1$
-1	-1	$\ldots$		-1
$\vdots$		$\ldots$		$\vdots$
-1	$+1$	$\ldots$		-1

F

f_1
f_2
$\vdots$
$\vdots$
f_n

X

X_1
X_2
$\vdots$
X_k

$$\sqrt{(1-\epsilon)}\|F\|_2 \;<\; \frac{1}{\sqrt{k}}\|X\|_2 \;<\; \sqrt{(1+\epsilon)}\|F\|_2$$

G is a random linear transformation reduces the dimension of F while preserving its ℓ_2 norm

Since G is linear it is easy to see that given $U, V \in \mathcal{R}^n$

$$\text{w.h.p} \qquad \|\frac{1}{\sqrt{k}}\mathbf{G}U\|_2 - \|\frac{1}{\sqrt{k}}\mathbf{G}V\|_2 \;\sim\; \|U-V\|_2$$

Johnson-Lindenstrauss Lemma

- Given $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathcal{R}^d$
- For any $\epsilon \in (0, 1/2)$, there is a linear map $f : \mathcal{R}^d \mapsto \mathcal{R}^k$
- $k = c \log n / \epsilon^2$, such that for any $\mathbf{u}, \mathbf{v} \in V$

$$(1 - \epsilon)\|\mathbf{u} - \mathbf{v}\|_2 \leq \|f(\mathbf{u}) - f(\mathbf{v})\|_2 \leq (1 + \epsilon)\|\mathbf{u} - \mathbf{v}\|_2$$

- This map can be obtained very easily
- Let $\mathbf{M}$ be a $k \times d$ matrix, with $M_{ij} \in \mathcal{N}(0, 1)$, then

$$f(\mathbf{u}) = \frac{1}{\sqrt{k}}\mathbf{M}\mathbf{u}$$