Artificial Intelligence using Python

Vectors

Tejinder Singh Bawa

Overview

1.Physical Quantities.

2.Vector and Scalars

3.Representation of vectors.

4.Where are vectors used in Machine Learning

5.Types of Vectors

6.Operations on Vectors

What are Physical Quantities

Any quantity that can be measured in terms of numbers is called a physical quantity.

Measurements in terms of size, mass etc.

Let us understand it better with an example, your mom has asked you to buy 2 Kilograms of tomatoes, you convey the amount you want to buy to the vendor. He "measures" the mass and then gives you the desired quantity of tomatoes. Here since the mass can be measured in terms of numbers, it will be called a physical quantity.

Physical quantities can be categorized into 2 categories: Scalars and Vectors.

Scalars and Vectors

Concisely, Scalars can be understood as the quantities which do not vary with direction example: mass, time, length etc. whereas vector quantities are the ones which do vary with direction example: force, velocity, momentum etc.

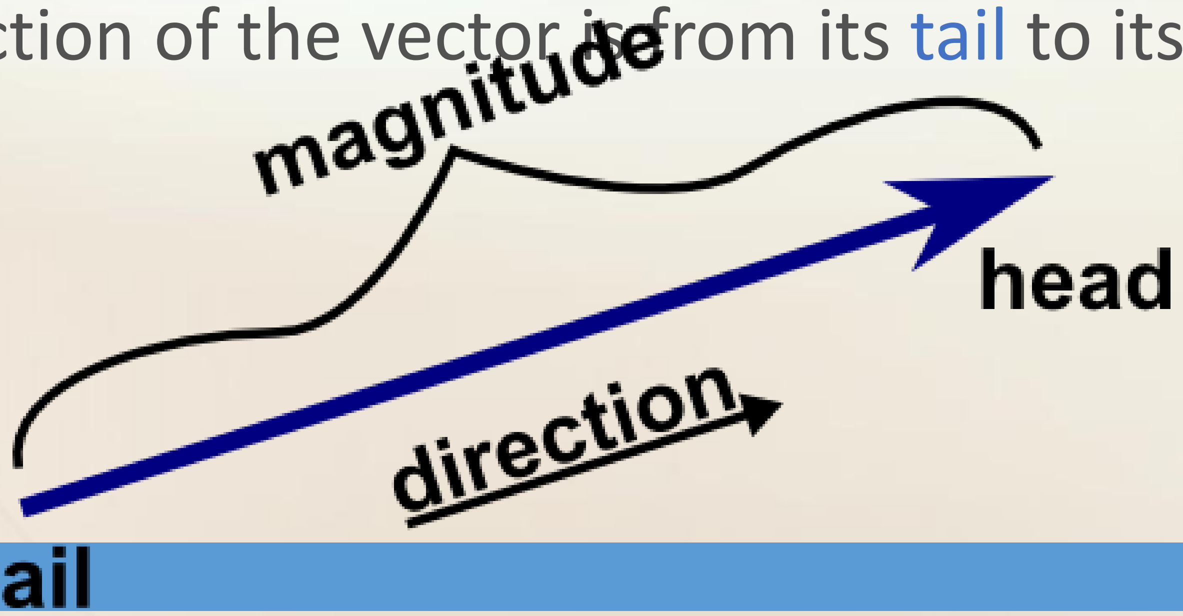
So if we talk about 1 meter of cloth, it will stay the same no matter how to see and from where you see it. It is independent of direction, place it is seen from.

But this is not the case when it comes to the term "force", we see the value of force changes when we vary direction where it is exerted from. The same is for velocity, acceleration etc. And this is exactly why they are called vector quantities.

What are Vectors

Definition of a vector

- A vector is an object that has both a magnitude and a direction.
- Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction.
- The direction of the vector is from its tail to its head.



What are Vectors

- Two examples of vectors are those that represent force and velocity.
- Both force and velocity are in a particular direction.
- The magnitude of the vector would indicate the strength of the force or the speed associated with the velocity.
- Two vectors are the same if they have the same magnitude and direction.
- This means that if we take a vector and translate it to a new position (without rotating it), then the vector we obtain at the end of this process is the same vector we had in the beginning.

Denoting Vectors

- We can denote vectors using boldface as in **a** or **b**.
- Especially when writing by hand where one cannot easily write in boldface, people will sometimes denote vectors using arrows as in $\vec{a}$ or $\vec{b}$, or they use other markings.
- We denote the magnitude of the vector **a** by ||**a**||.
- When we want to refer to a number and stress that it is not a vector, we can call the number a scalar.
- We can denote scalars with italics, as in *a* or *b*.
- There is one important exception to vectors having a direction.
- The zero vector, denoted by boldface **0**, is vector of zero length.

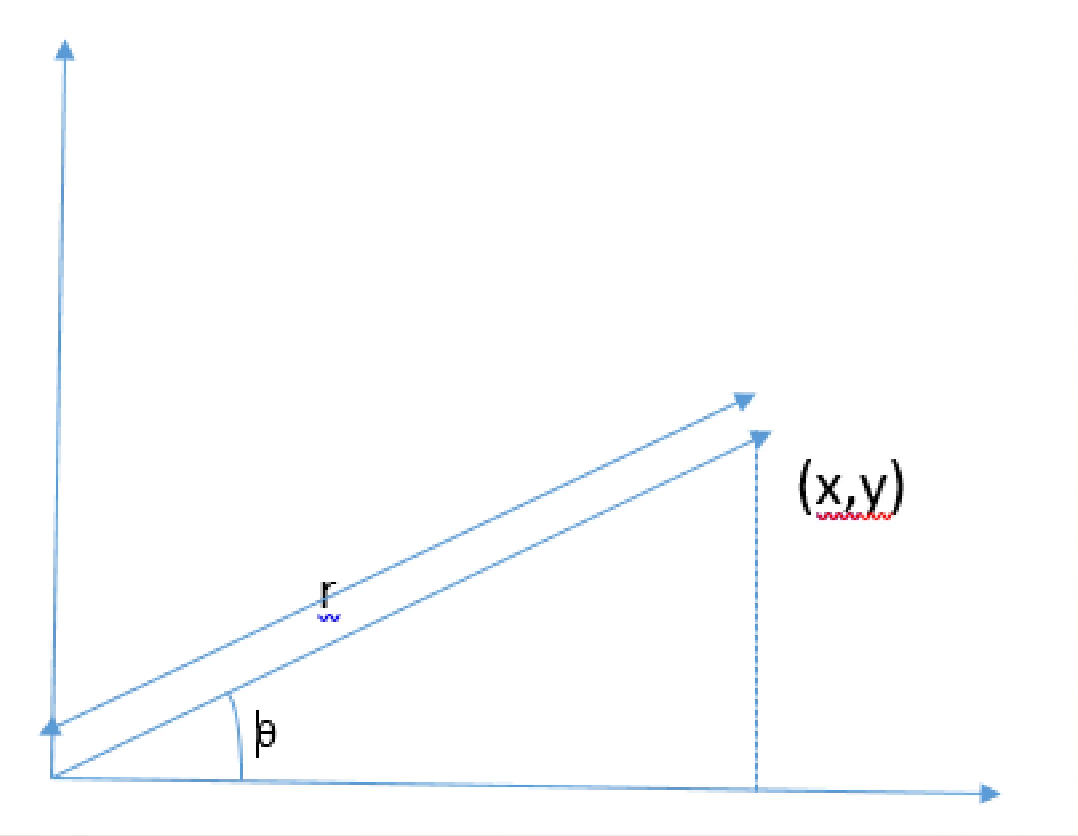
Let's say the magnitude of the vector is denoted by r and the angle at which it is inclined is theta.
Then, there is a correlation between all the values,

$X = r\cos\theta$

$Y = r\sin\theta$

$r = \sqrt{(x^2 + y^2)}$

$\theta = \tan^{-1}(y / x)$



Array creation

First, we must import the module "numpy"

```
import numpy as np
```

np is the alias used for accessing to the routines of the package 'numpy '.

Converting Python array_like objects (e.g. list)

```
a = np.array([1.2, 2.5, 3.2, 1.8])
```

[ ] is a list of values (float)

```
#object type
print(type(a))  #<class 'numpy.ndarray'>
#data type
Print (a.dtype) #float64
```

Information about the structure

```
#number of dimensions
print(a.ndim)  #1 (we have 2 if it is a matrix, etc.)
#number of rows and columns
print(a.shape)  #(4,) (tuple! 4 elements for the 1st  dim
#total number of elements
print(a.size)  #4, nb rows x nb columns
```

Finding Magnitude in Numpy

```
import numpy as np

x = np.array([1,2,3,4,5])

print("Original array:")
print(x)

print("Magnitude of the vector:")
print(np.linalg.norm(x))
```

Note: The NumPy linear algebra functions called by using linalg provide efficient low level implementations of standard linear algebra algorithms.

Finding Angle Between Vectors

USE numpy.arccos() TO GET THE ANGLE BETWEEN TWO VECTORS

Steps

- Use lists to represent vectors.
- Use vector / np.linalg.norm(vector) to get the unit vector of vector.
- Do this for both vectors.
- Call np.dot(u1, u2) to get the dot product of the previous results u1 and u2.
- Call np.arccos(x) with the previous result as x to get the angle between the two original vectors.

$$\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$

```
v1 = [0, 1]
v2 = [1, 0]

u1 = v1 / np.linalg.norm(v1)
u2 = v2 / np.linalg.norm(v2)
dot_product = np.dot(u1, u2)
angle = np.arccos(dot_product)
print(angle)
```

Why Learn Vectors ?

Why Data is represented as a ‘Vector’ in Data Science Problems?

- Let's say you are collecting some data about a group of students in a class.
- You are measuring the height and weight of each student and the data collected for 5 students is as follows:

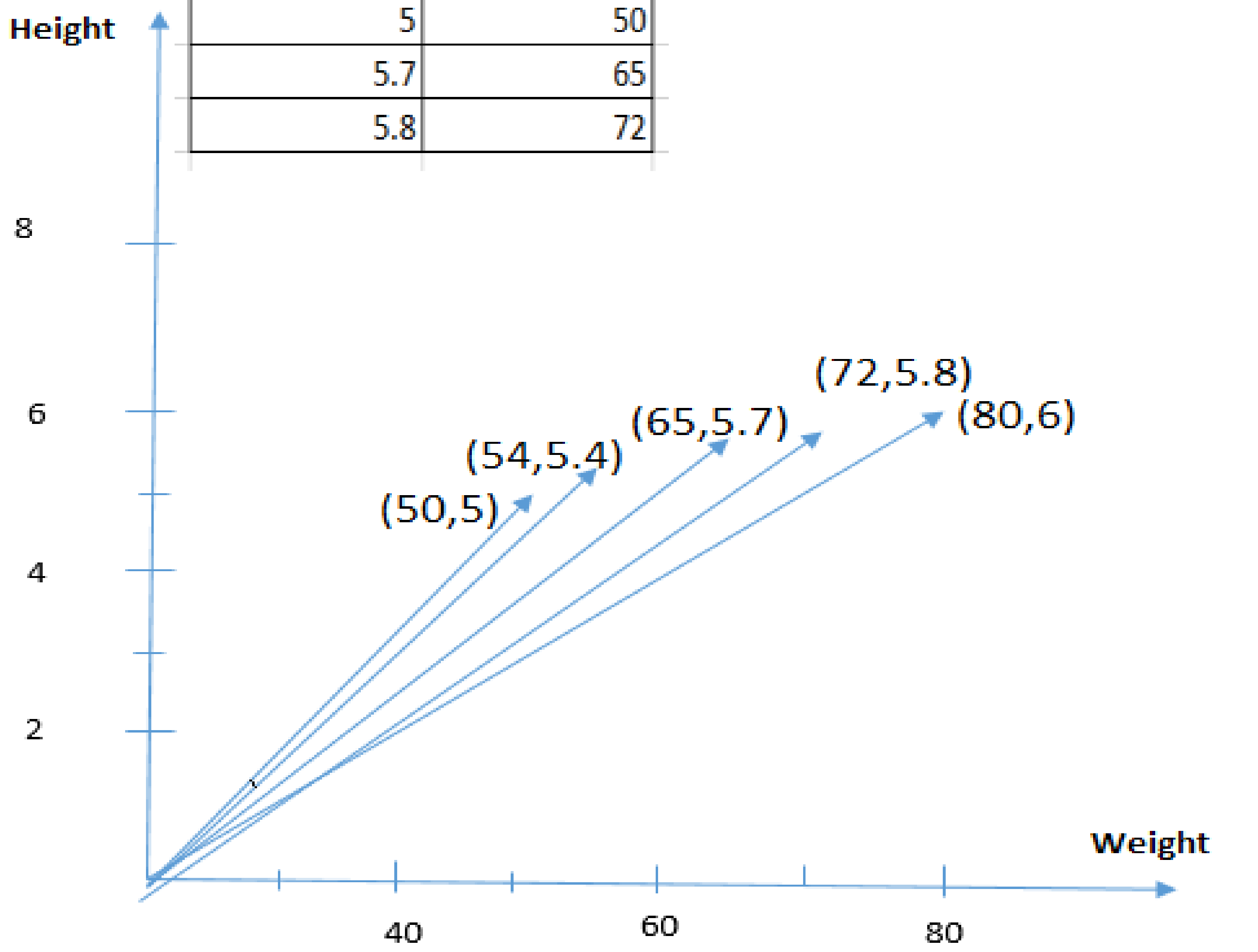
Heigt (in feets)	Weight (in Kgs)
6	80
5.4	54
5	50
5.7	65
5.8	72

- When you look at the observation about each student as a whole i.e height and weight together for every student, you can think of it as a vector.

Why Learn Vectors ?

Why Data is represented as a 'Vector' in Data Science Problems?

Heigt (in feets)	Weight (in Kgs)
6	80
5.4	54
5	50
5.7	65
5.8	72



Why Learn Vectors ?

- Every observation in a given data set can be thought of as a vector.
- All possible observations of a data set constitute a "Vector Space".
- Its a fancy way of saying that there is a space out there and every vector has its location within that space.
- The benefit of representing data as vectors is :— we can leverage vector algebra to find patterns or relationships within our data.

How can vectors help?

- Looking at vectors in vector space, you can quickly compare them to check if there is a relationship.
- E.g Similar vectors will have smaller angle between them i.e their orientation will be close to each other.

- In sample data, students (5.4,54) and (5, 50) are quite similar.
- Angle between vectors indicates "similarity" between them.

- The vectors in the same direction (close to 0 degrees angle) are similar while vectors in the opposite direction (close to 180 degrees angle) are dissimilar.
- Theoretically, if the vectors are at 90 degrees to each other (orthogonal), then there is no relationship between them. .

How can vectors help?

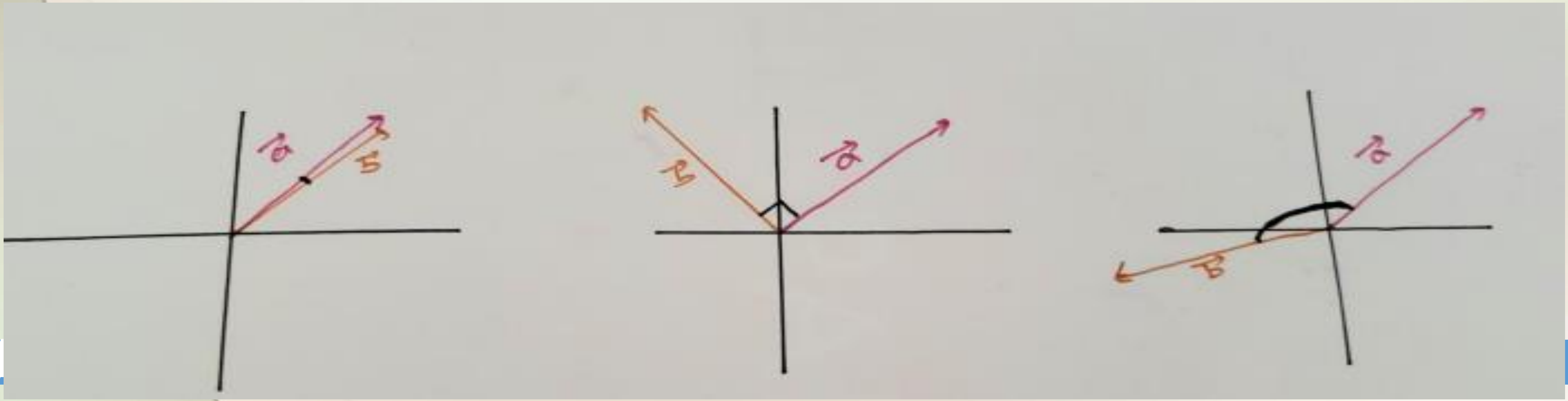
Cosine Similarity

- Cosine Similarity is a metric that gives the cosine of the angle between vectors.
- It can be calculated using the "dot product" of 2 vectors. Mathematically,

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

Degrees	0^0	90^0	180^0
Cosine	1	0	-1



How can vectors help in Text?

- "Word Embedding" is the process of representing words or text as Vectors
- There are a number of techniques of converting/representing text as Vectors.
- One of the simplest methods is Count Vectorizer.
- Below are few lines of text :
- *It was the best of times,*
- *it was the worst of times,*
- *it was the age of wisdom,*
- *it was the age of foolishness*
- Unique words from collection of your text are:

  'age', 'best', 'foolishness', 'it', 'of', 'the', 'times', 'was', 'wisdom', 'worst'

How to Use Vectors in Text Documents?

- **Step 1**: Get unique words from collection of your text.
- The total text you have is called “corpus”
- The unique words (ignoring case and punctuation) here are:
- ['age', 'best', 'foolishness', 'it', 'of', 'the', 'times', 'was', 'wisdom', 'worst']
- This is a vocabulary of 10 words from a corpus containing 24 words.
- **Step 2**: For each sentence, create a list of 10 zeroes
- **Step 3**: For each sentence, start reading word one by one.
- For each word, count total occurrence in the sentence.
- Now identify position of word in vocabulary list above and replace zero with this count at that position.

How can vectors help in Text?

- For our corpus, vectors we got are:
- ['age', 'best', 'foolishness', 'it', 'of', 'the', 'times', 'was', 'wisdom', 'worst']
- “It was the best of times“= [0 1 0 1 1 1 1 1 0 0]
- “it was the worst of times” = [0 0 0 1 1 1 1 1 0 1]
- “it was the age of wisdom” = [1 0 0 1 1 1 0 1 1 0]
- “it was the age of foolishness” = [1 0 1 1 1 1 0 1 0 0]
- The cosine similarity between ’it was the age of wisdom’ and ’it was the age of foolishness’ is

```
Cosine Similarity : 0.8333333333333335
```

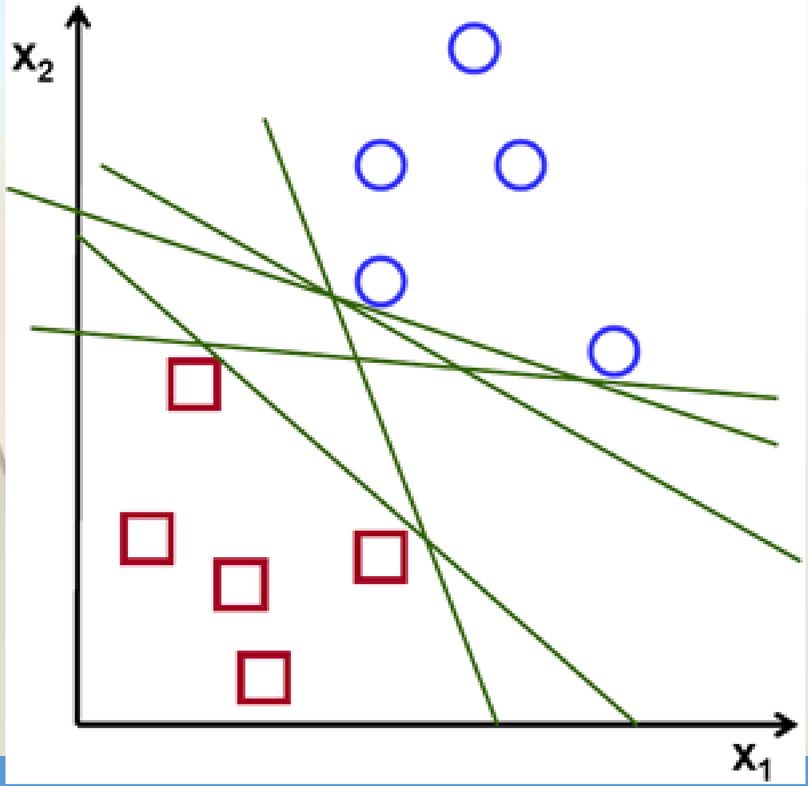
- The cosine similarity will be a number between 0 and 1. The more the result is close to 1, the greater the similarity.
- 0.83 value suggests that the two sentences in this example are pretty similar.

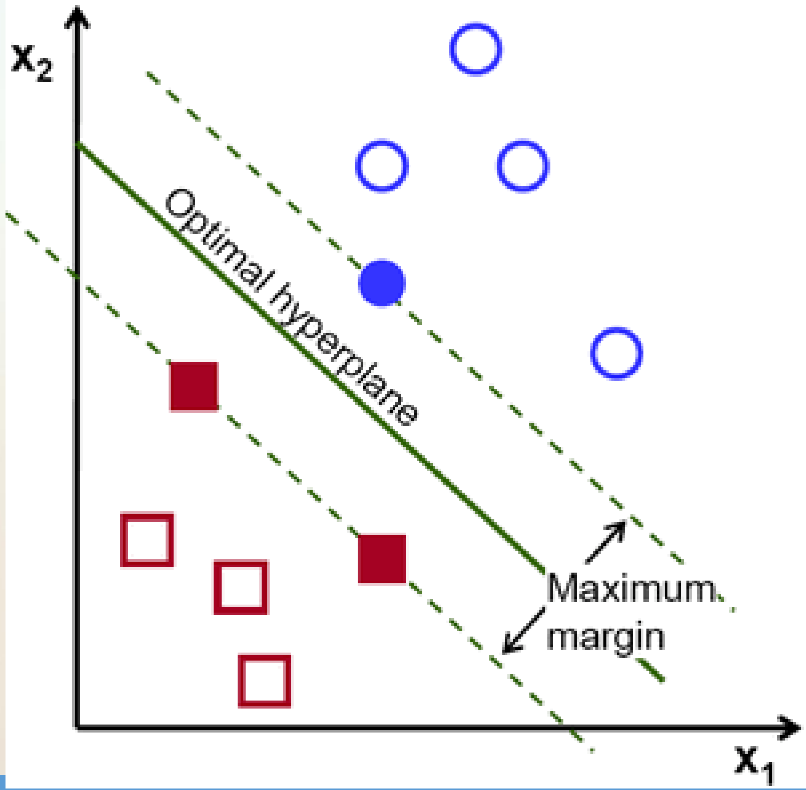
Vectors and Machine Learning

- Vectors are commonly used in machine learning as they lend a convenient way to organize data.
- Many times one of very first steps in making a machine learning model is vectorizing data.
- They are also relied upon heavily to make up the basis for some machine learning techniques as well.
- One example in particular is support vector machines.
- It analyzes vectors across n-dimensional space to find optimal hyperplane for given data set.
- It will attempt to find a line that have maximum distance between data sets of both classes.
- This allows for future data points to be classified with confidence

Vectors and Machine Learning

- Support vector machines will attempt to find a line that have maximum distance between data sets of both classes.
- This allows for future data points to be classified with confidence



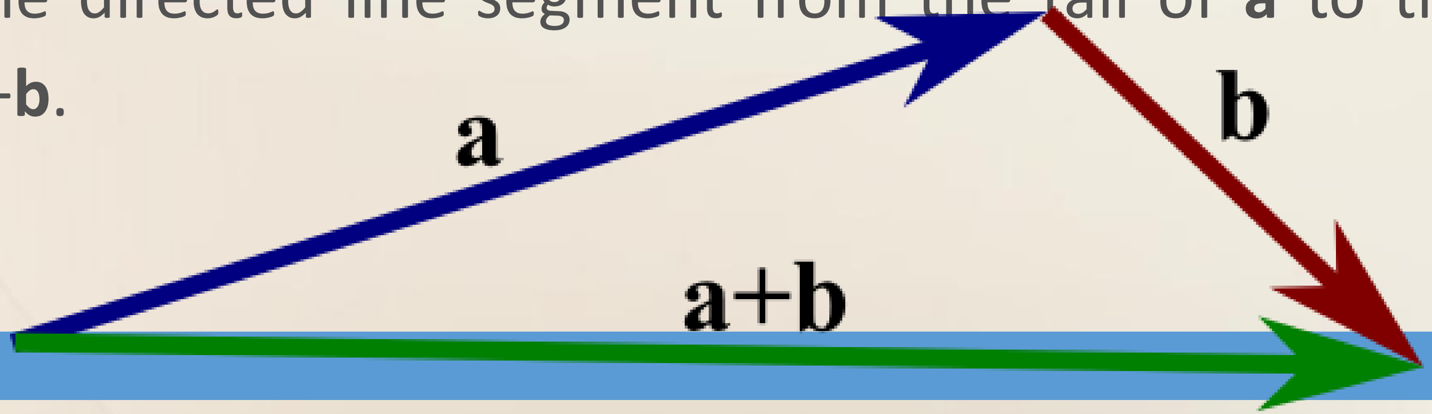


Operations on vectors

- We can define a number of operations on vectors geometrically without reference to any coordinate system.
- Here we define addition, subtraction, and multiplication by a scalar.

Addition of vectors

- Given two vectors **a** and **b**, we form their sum **a+b**, as follows:-
- We translate the vector **b** until its tail coincides with the head of **a**. (Recall such translation does not change a vector.)
- Then, the directed line segment from the tail of **a** to the head of **b** is the vector **a+b**.

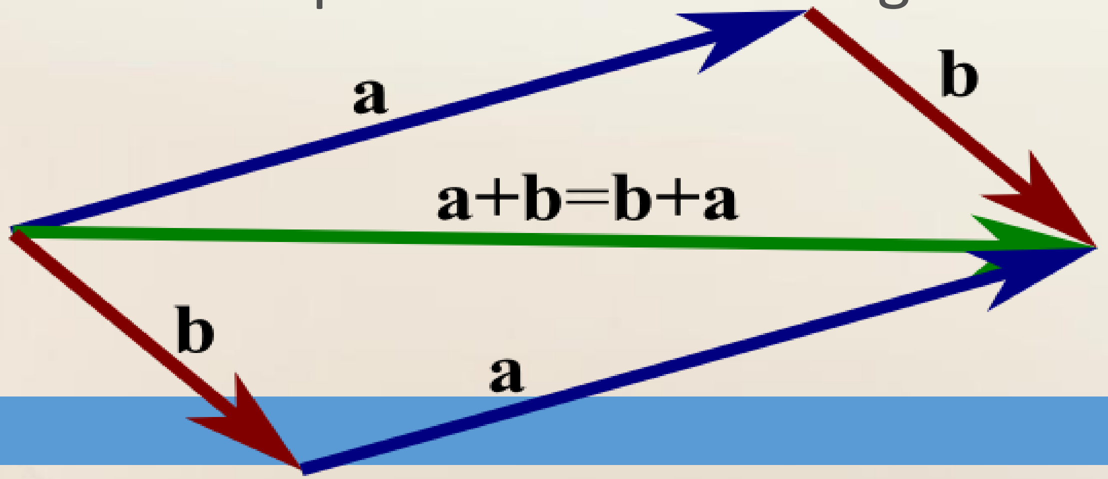


What are Vectors

- Addition of vectors satisfies two important properties.
- The commutative law, which states the order of addition doesn't matter:

  a+**b**=**b**+**a**.

- This law is also called the parallelogram law, as illustrated in the below image.
- Two of the edges of the parallelogram define **a**+**b**, and the other pair of edges define **b**+**a**.
- But, both sums are equal to the same diagonal of the parallelogram.



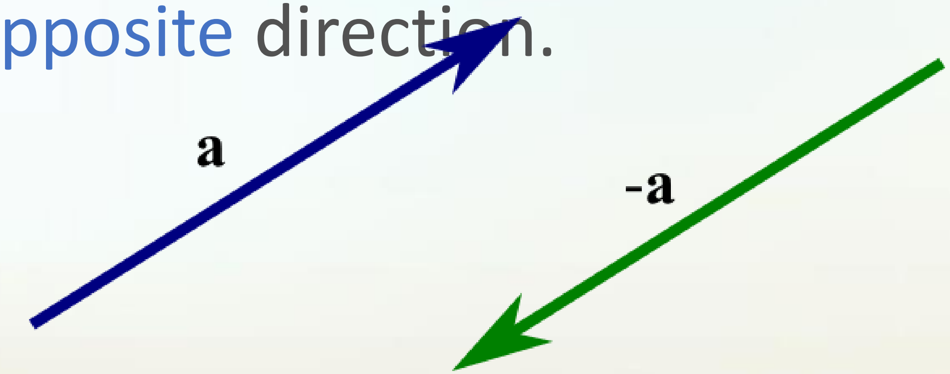
Laws of Vectors

- The associative law, which states that the sum of three vectors does not depend on which pair of vectors is added first:

$$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}).$$

Vector subtraction

- Before we define subtraction, we define the vector −**a**, which is the opposite of **a**.
- The vector −**a** is the vector with the same magnitude as **a** but that is pointed in the opposite direction.



- We define subtraction as addition with the opposite of a vector:
- b−a=b+ (−a).

Vector subtraction

- The vector **x** in the above figure is equal to **b**−**a**?
- Notice how this is the same as stating that **a**+**x**=**b**, just like with subtraction of scalar numbers.

Scalar multiplication

- Given vector **a** & real number(scalar) λ, we form vector λ**a** as follows:
- If λ is positive, then λ**a** is vector whose direction is same as direction of **a** & whose length is λ times length of **a**.
- In this case, multiplication by λ simply stretches (if λ>1) or compresses (if 0<λ<1) vector a.
- If λ is negative, then we have to take the opposite of **a** before stretching or compressing it.
- In other words, the vector λ**a** points in the opposite direction of **a**, and the length of λ**a** is |λ| times the length of **a**.
- No matter the sign of λ, magnitude of λa is |λ| times magnitude of a: $\|\lambda a\| = |\lambda| \, \|a\|$

Scalar multiplication

- Scalar multiplications satisfies many of the same properties as the usual multiplication.
  - $s(\mathbf{a}+\mathbf{b})=s\mathbf{a}+s\mathbf{b}$ (distributive law, form 1)
  - $(s+t)\mathbf{a}=s\mathbf{a}+t\mathbf{a}$ (distributive law, form 2)
  - $1\mathbf{a}=\mathbf{a}$
  - $(-1)\mathbf{a}=-\mathbf{a}$
  - $0\mathbf{a}=\mathbf{0}$
- In the last formula, the zero on the left is the number 0, while the zero on the right is the vector **0**, which is the unique vector whose length is zero.

Types of vectors

1. **Zero Vector or NULL Vector***: A vector with zero magnitudes is called a NULL vector. The tail and the head of this vector are the same. Denoted by vector* ***O.***

2. **Unit Vector***: A unit vector is the one with the value of its magnitude equaling to one. Denoted by .*

u=r/|r|

3. Collinear Vector*: Two vectors are collinear if they are either parallel to each other or lie on same line irrespective of their direction.*

4. Coplanar vector: *A set of vectors is called coplanar if they all lie on the same plane.*

Operations on Vectors

1. **Vector addition CODE**

```
import numpy as np #pip install numpy

a=np.array([1,2,3])

b=np.array([4,5,3])

print(a+b)

Output:  [5,7,6]
```

Let us say

A = 1i+2j+3k and B = 4i+5j+3k

So A+B would be (1+4)i+(2+5)j+(3+3)k = 5i+7j+6k

2. Vector Subtraction CODE

```
import numpy as np

a=np.array([3,5])

b=np.array([4,-2])

print(a-b)

Output: [-1,7]
```

Let us take A = 3i+5j and B = 4i-2j

So A-B= (3-4)i+(5-(-2))j = -i+7j

3. Multiplication of scalars to vectors:

CODE

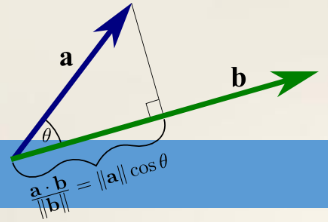
```
import numpy as np

a=np.array([3,5])

 print(3*a)

Output: [9, 15]
```

1. 3*a=a*3
2. x(a+b)=xa+xb
3. x(a+b)=(a+b)x

A=4i-3j,
So, 4A=4(4)i-4(3)j = 16i - 12j

The dot product

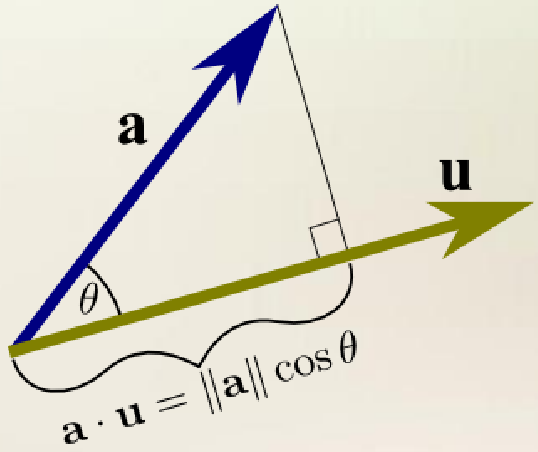
- The dot product between two vectors is based on the projection of one vector onto another.
- Let's imagine we have two vectors **a** and **b**, and we want to calculate how much of **a** is pointing in the same direction as the vector **b**.
- We want a quantity that would be positive if the two vectors are pointing in similar directions, zero if they are perpendicular, and negative if the two vectors are pointing in nearly opposite directions.
- **We will define the dot product between vectors to capture these quantities.**



The dot product

- But first, notice that question “how much of **a** is pointing in the same direction as vector **b**” does not have anything to do with the magnitude (or length) of **b**; it is based only on its direction.
- Let's scale the vector so that it has length one.
- In other words, let's replace b with the unit vector that points in the same direction as b.
- We'll call this vector u, which is defined by

  u=**b** / ‖**b**‖.



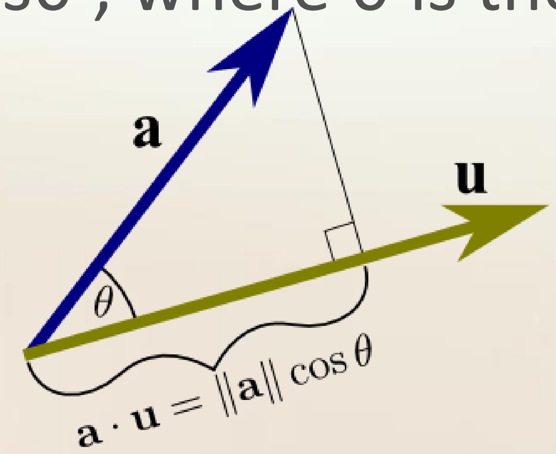
The dot product

- The dot product of a with unit vector **u**, denoted **a·u**, is defined to be the projection of **a** in the direction of **u**, or the amount that **a** is pointing in the same direction as unit vector **u**.

- By forming a right triangle with **a** and this shadow, you can use geometry to calculate that

$$\frac{x}{||a||} = \cos\theta$$

- **a**·**u**=||**a**||cosθ , where θ is the angle between **a** and **u**.



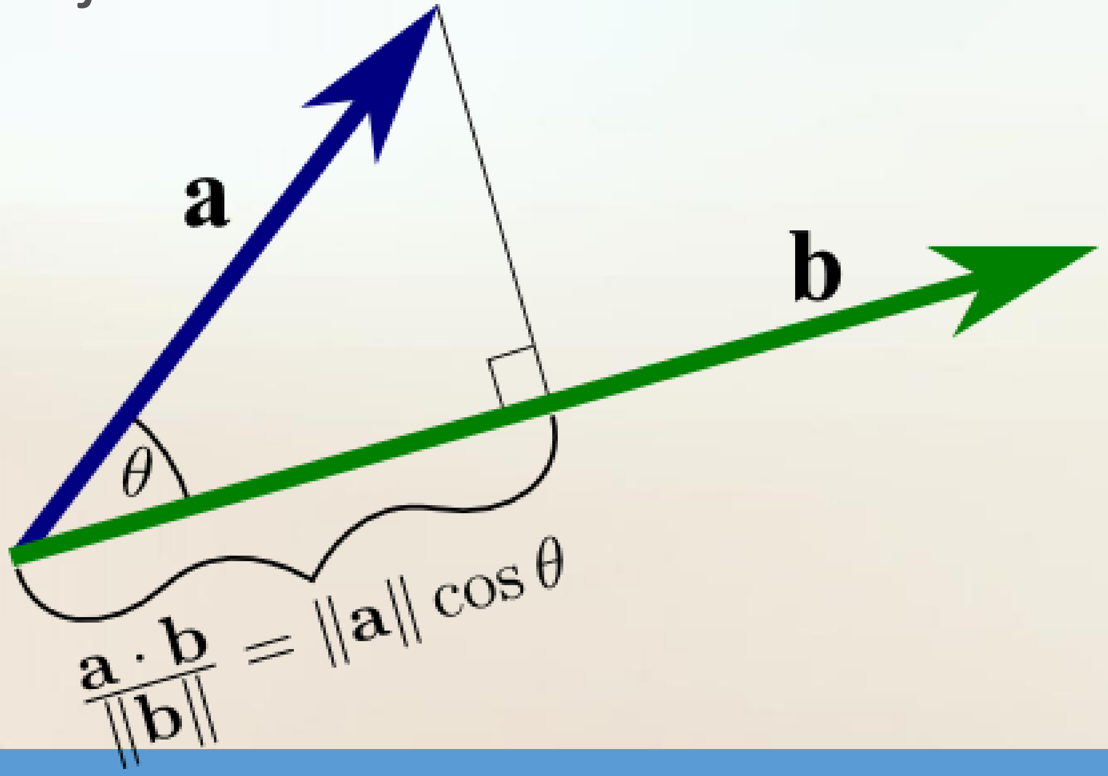
The dot product

- If **a** and **u** were perpendicular, there would be no shadow.
- That corresponds to the case when $\cos\theta=\cos\pi/2=0$ and $\mathbf{a}\cdot\mathbf{u}=0$.
- If the angle θ between **a** and **u** were larger than $\pi/2$, then the shadow wouldn't hit **u**.
- Since in this case $\cos\theta<0$, the dot product $\mathbf{a}\cdot\mathbf{u}$ is also negative.
- You could think of $-\mathbf{a}\cdot\mathbf{u}$ (which is positive in this case) as being the length of the shadow of **a** on the vector $-\mathbf{u}$, which points in the opposite direction of **u**.
- Replace **u** by $\mathbf{b} / \|\mathbf{b}\|$ in above formula & get $\mathbf{a}\cdot\mathbf{b}/\|\mathbf{b}\|=\|\mathbf{a}\|\cos\theta$.

Finally we get :- $\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta$

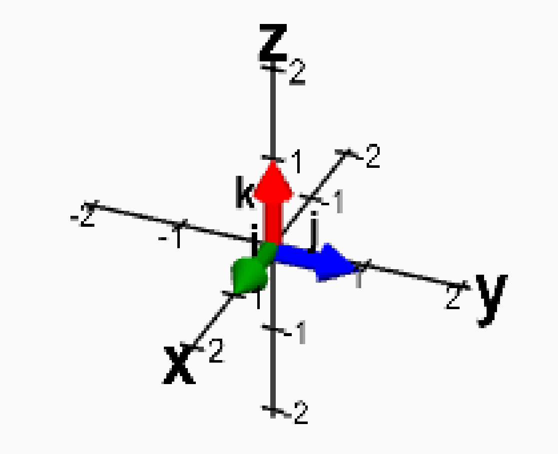
The dot product

- The picture of the geometric interpretation of **a**·**b** is almost identical to the above picture for **a**·**u**.
- We just have to remember that we have to divide through by ||**b**|| to get the projection of **a** onto **b**.



The dot product

- For any three vectors **a**, **b**, and **c**, and any scalar λ,
  - **a.b=b.a**
  - (λ**a**)·**b** = λ(**a**·**b**) = **a**·(λ**b**)
  - (**a**+**b**)·**c**=**a**·**c**+**b**·**c**.
- For a 3-Dimentional space, the standard unit vectors in three dimensions, **i** (green), **j** (blue), and **k** (red) are length one vectors that point parallel to the x-axis, y-axis, and z-axis respectively.



- Consider two three-dimensional vectors:-
  - **a**=(a1,a2,a3)=a1**i**+a2**j**+a3**k**
  - **b**=(b1,b2,b3)=b1**i**+b2**j**+b3**k**

The dot product

- Since the standard unit vectors are orthogonal, we immediately conclude that the dot product between a pair of distinct standard unit vectors is zero: $\mathbf{i}\cdot\mathbf{j}=\mathbf{i}\cdot\mathbf{k}=\mathbf{j}\cdot\mathbf{k}=0$.
- The dot product between a unit vector and itself is also simple to compute. In this case, the angle is zero and $\cos\theta=1$.
- Given that the vectors are all of length one, the dot products are

  $\mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=\mathbf{k}\cdot\mathbf{k}=1$
- we can expand the dot product $\mathbf{a}\cdot\mathbf{b}$ in terms of components,

  $\mathbf{a}\cdot\mathbf{b}=(a1\mathbf{i}+a2\mathbf{j}+a3\mathbf{k})\cdot(b1\mathbf{i}+b2\mathbf{j}+b3\mathbf{k})=a1b1\mathbf{i}\cdot\mathbf{i}+a2b2\mathbf{j}\cdot\mathbf{j}+a3b3\mathbf{k}\cdot\mathbf{k}+(a1b2+a2b1)\mathbf{i}\cdot\mathbf{j}+(a1b3+a3b1)\mathbf{i}\cdot\mathbf{k}+(a2b3+a3b2)\mathbf{j}\cdot\mathbf{k}$

Dot product in matrix notation

- If we multiply x^T (a 1×n matrix) with any n-dimensional vector y (viewed as an n×1 matrix), we end up with a matrix multiplication equivalent to the familiar dot product of x·y:

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3 + \ldots + x_ny_n = \mathbf{x} \cdot \mathbf{y}$$

Vector Spaces

- We can think of a vector space in general, as a collection of objects that behave as vectors do in R^n
- The objects of such a set are called vectors.
- Definition: A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition (**+**) and multiplication by scalars (real numbers), subject to the ten axioms below:-
- The axioms must hold for all u, v and w in V and for all scalars c and d.
- 1. **u + v** is in V.
- 2. **u + v** = **v + u**.
- 3. **(u + v) + w = u + (v + w)**

Vector Spaces

- 4.There is a vector (called the zero vector) **0** in V such that **u + 0** = **u**.
- 5. For each **u** in V, there is vector **–u** in V satisfying **u + (–u)** = **0**.
- 6. c**u** is in V.
- 7. c (**u + v**) =c**u**+c**v**.
- 8. (c + d)**u** = c**u** + d**u**.
- 9. (cd)**u** = c(d**u**).
- 10. 1**u** = **u**.

- The above is true for all u, v and w in V and for all scalars c and d.

Vector Spaces

Example:

Let $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,\ b,\ c,\ d \text{ are real} \right\}$

In this context, note that the $\mathbf{0}$ vector is $\begin{bmatrix} \quad & \quad \end{bmatrix}$.

Example:

- Let $n \geq 0$ be an integer and let P_n = the set of all polynomials of degree at most $n \geq 0$.
- Members of P_n have the form $p(t) = a0 + a1t + a2t^2 + \cdots + an\ t^n$ where a0, a1, . . . , an are real numbers and t is a real variable.
- The set P_n is a vector space.
- We will just verify 3 out of the 10 axioms here.

Vector Spaces

- Let $p(t) = a0 + a1t + \cdots + an\, t^n$ and
  $q(t) = b0 + b1t + \cdots + bn\, t^n$.
  Let c be a scalar.

- Axiom 1: The polynomial p + q is defined as follows:

$(p + q)(t) = p(t)+q(t)$
$= (a0+b0) + (a1+b1)t + \cdots + (an+bn\,)t^n$

which is also a polynomial of degree at most n .

So p + q is in P_n.

Vector Spaces

- Axiom 4:
- $\mathbf{0} = 0 + 0t + \cdots + 0t^n$ (zero vector in P_n)
- $(p + \mathbf{0})(t) = p(t) + \mathbf{0} = (a0 + 0) + (a1 + 0)t + \cdots + (an + 0)t^n$
- $= a0 + a1t + \cdots + an\, t^n = p$
- Axiom 6:
- Let c be scalar e.g. a real number
- $(cp)(t) = cp(t) = (ca0) + (ca1)t + \cdots + (can)t^n$ which is in P^n.
- The other 7 axioms also hold, so P^n is a vector space

Sub Spaces

- Vector spaces may be formed from subsets of other vectors spaces. These are called subspaces.
- Defintion: A subspace of a vector space V is a subset H of V that has three properties:
- a. The zero vector of V is in H.
- b. For each **u** and **v** are in H, **u + v** is in H. (In this case we say H is closed under vector addition.)
- c. For each **u** in H and each scalar c, c**u** is in H. (In this case we say H is closed under scalar multiplication.)
- If the subset H satisfies these three properties, then H itself is a vector space.

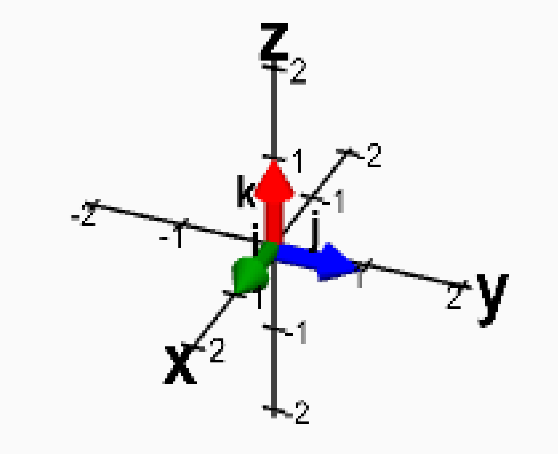
Sub Spaces

- Example:

Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.

- Note that $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} c \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ 0 \\ b+d \end{bmatrix}$ is in H

- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is the zero vector which belongs to H



- $c\begin{bmatrix} a \\ 0 \\ \ldots \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ \ldots \end{bmatrix}$ *also belongs to H.*

Vector Spans

Defn: Span of Vectors: Let v1,v2………vn be vectors in a Vector Space V, the Span{v1,v2………vn} is set of all vectors of the form c1v1+c2v2+…….+cnvn in the vector space V i.e. all the linear combinations of vectors v1,v2………vn

Note: If v1,v2………vn are vectors in a Vector Space V then Span{v1,v2………vn} is a subspace of V.

$\mathbb{R}^2$

Example: Is V = {(a + 2b, 2a – 3b) : a and b are real} a subspace of R^2 ?

(a+2b, 2a – 3b) = (a ,2a) + (2b, -3b) = a(1,2)+b(2,-3)

So V is span of vectors (1,2) and (2,-3) i.e. Span{(1,2), (2,-3)}

Sub Spaces

Example: Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times 2}$? Explain.

- $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix} = a\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$

- $H = \text{Span}\left\{\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\right\}$
- So H is a subspace of M_{2x2}

The cross product

- The cross product is defined only for three-dimensional vectors.
- If **a** and **b** are two three-dimensional vectors, then their cross product, written as **a×b** and pronounced "a cross b," is another three-dimensional vector.
- We define cross product vector **a×b** by following three requirements:
  - **a×b** is a vector that is perpendicular to both **a** and **b**.
  - The magnitude (or length) of the vector **a×b**, written as ||**a×b**||, is the area of the parallelogram spanned by **a** and **b** (i.e. parallelogram whose adjacent sides are vectors **a** and **b**, as shown in below figure).

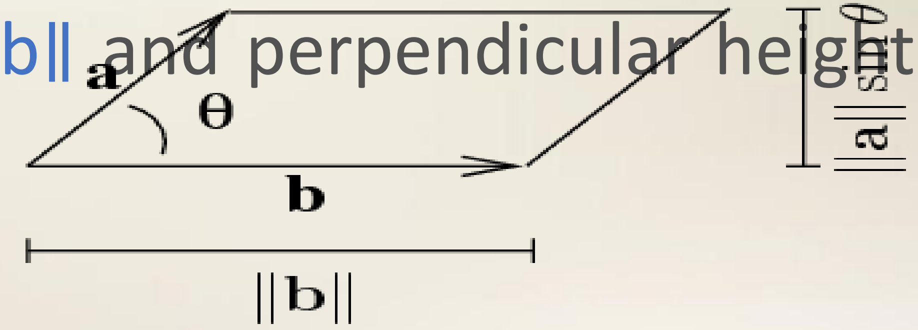
The cross product

- The direction of **a×b** is determined by the right-hand rule. (This means that if we curl the fingers of the right hand from **a** to **b**, then the thumb points in the direction of **a×b**.)

- Below figure illustrates how, using trigonometry, we can calculate that area of the parallelogram spanned by a and b is

$\|a\| \, \|b\| \sin\theta$,

- where θ is the angle between a and b. The figure shows the parallelogram as having a base of length $\|b\|$ and perpendicular height $\|a\| \sin\theta$.



- Area of parallelogram: Base × Height

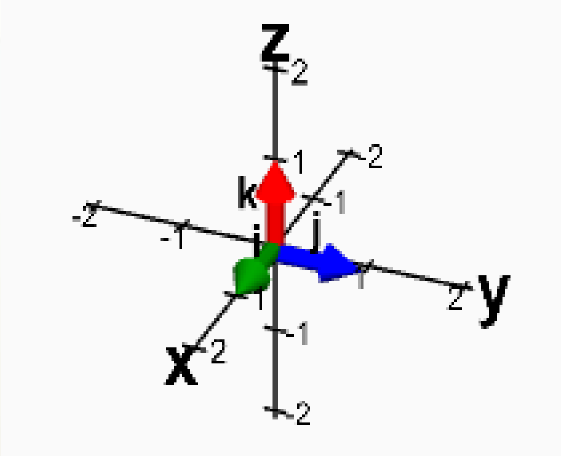
The cross product

- $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\ \|\mathbf{b}\|\sin\theta$
- This formula shows that the magnitude of the cross product is largest when **a** and **b** are perpendicular.
- On the other hand, if **a** and **b** are parallel or if either vector is the zero vector, then the cross product is the zero vector.

The formula for the cross product

Cross product of unit vectors

- Let **i, j** and **k** be the standard unit vectors in R3. (We define cross product only in three dimensions. Note that we are assuming a right-handed coordinate system.)

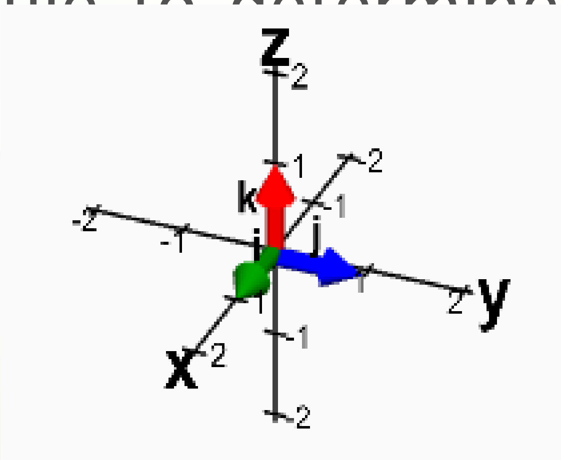


- The parallelogram spanned by any two of these standard unit vectors is a unit square, which has area one.
- Hence, by geometric definition, cross product must be a unit vector.
- Since cross product must be perpendicular to two unit vectors, it must be equal to other unit vector or opposite of that unit vector.

The formula for the cross product

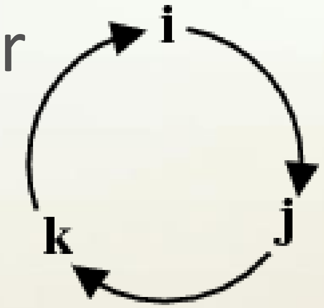
Cross product of unit vectors Contd….

- Looking at this graph, you can use the right-hand rule to determine the following results.



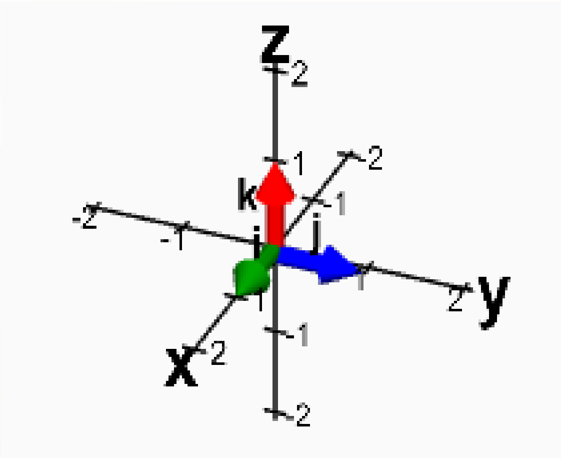
$i \times j$ = k, $j \times k = i$, $k \times i = j$

- This little cycle diagram can help you remember these results



The formula for the cross product

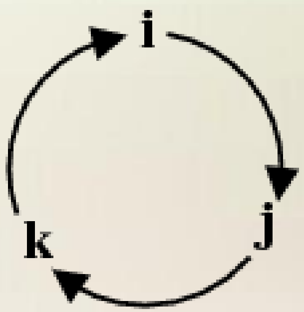
- What about **i×k**?
- By right-hand rule, it must be −**j**.



- By remembering that b×a=−a×b, you can infer that

  j×i=-k, **k×j=-I**, **i×k=−j**

- Finally, cross product of any vector with itself is the zero vector (**a**×**a**=**0**).



- In particular, cross product of any standard unit vector with itself is zero vector.

  i×i=0, **j×j=0**, **k×k=0**

Laws of Cross Products

- With exception of two special properties viz $\mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b}$ & $\mathbf{a}\times\mathbf{a} = 0$), cross product behaves like regular multiplication.

- It obeys the following properties:

$$(y\mathbf{a})\times\mathbf{b} = y(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(y\mathbf{b}),$$
$$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c},$$
$$(\mathbf{b}+\mathbf{c})\times\mathbf{a} = \mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}$$

where **a**, **b**, and **c** are vectors in R^3 and y is a scalar.

- We can use these properties, along with cross product of standard unit vectors, to write formula for cross product in terms of components.

Laws of Cross Products

- We write components of **a** and **b** as:

  $\mathbf{a}=(a1,a2,a3)=a1\mathbf{i}+a2\mathbf{j}+a3\mathbf{k}$

  $\mathbf{b}=(b1,b2,b3)=b1\mathbf{i}+b2\mathbf{j}+b3\mathbf{k}$
- First, we'll assume that $\mathbf{a3=b3=0}$.
- We calculate:
- $\mathbf{a}\times\mathbf{b} = (a1\mathbf{i}+a2\mathbf{j})\times(b1\mathbf{i}+b2\mathbf{j})$
- $= a1b1(\mathbf{i}\times\mathbf{i})+a1b2(\mathbf{i}\times\mathbf{j})+a2b1(\mathbf{j}\times\mathbf{i})+a2b2(\mathbf{j}\times\mathbf{j})$

- Since we know that $\mathbf{i}\times\mathbf{i}=0=\mathbf{j}\times\mathbf{j}$ and that $\mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}$, this quickly simplifies to
- $\mathbf{a}\times\mathbf{b}=(a1b2-a2b1)\mathbf{k}$

$$= \det \begin{vmatrix} a1 & a2 \\ b1 & b2 \end{vmatrix} . \mathbf{k}$$

Laws of Cross Products

- We start with by expanding out the product
- $\mathbf{a}\times\mathbf{b} = (a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\times(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$
- $= a_1b_1(\mathbf{i}\times\mathbf{i}) + a_1b_2(\mathbf{i}\times\mathbf{j}) + a_1b_3(\mathbf{i}\times\mathbf{k}) + a_2b_1(\mathbf{j}\times\mathbf{i}) + a_2b_2(\mathbf{j}\times\mathbf{j})$
  $+ a_2b_3(\mathbf{j}\times\mathbf{k}) + a_3b_1(\mathbf{k}\times\mathbf{i}) + a_3b_2(\mathbf{k}\times\mathbf{j}) + a_3b_3(\mathbf{k}\times\mathbf{k})$
  and then calculate all cross products of unit vectors

- $\mathbf{a}\times\mathbf{b} = a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i}$
  $= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$
- Using determinants, we can write result as
- $\mathbf{a}\times\mathbf{b} =$

$$\det\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} . \mathbf{i} + \det\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} . \mathbf{j} + \det\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} . \mathbf{k}$$

Cross Products Of Vectors

- We see that formula for a cross product looks a lot like formula for 3×3 determinant.
- If we allow a matrix to have vectors **i, j,** and **k** as entries, 3×3 determinant gives a handy mnemonic to remember the cross product:

$$\mathbf{a \times b} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \mathbf{a1} & \mathbf{a2} & \mathbf{a3} \\ \mathbf{b1} & \mathbf{b2} & \mathbf{b3} \end{vmatrix}$$

- This is a compact way to remember how to compute the cross product.