Active earth force in 'cohesionless' unsaturated soils using bound theorems of plasticity

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ABSTRACT: Granular 'cohesionless' soils above the water table are partially saturated but are commonly assumed to be dry in geotechnical practice. The 'dry soil' assumption neglects the effect of suction on shear strength and soil structures are therefore over-designed. This paper presents an approach to the analysis of collapse of soil structures above the water table based on the upper and lower bound theorems of plasticity. As an example, the active earth force on a retaining wall has been calculated and compared to the value obtained from the classical 'dry' approach.

1 INTRODUCTION

Non-clayey 'cohesionless' soils above the water table are generally assumed to be dry in routine engineering calculations. Nonetheless this is rarely the case. Soils above the water table are in fact partially saturated and have shown to exhibit significantly higher shear strength than dry soils.

Practitioners and academicians find it convenient to disregard the contribution of partial saturation to shear strength as this leads to conservative design. This point of view can be questioned. Significant costs might be saved if *new* geostructures are designed to account for the effects of partial saturation. On the other hand, geotechnical engineers are often confronted with *existing* hazardous geostructures, e.g. unstable slopes. In this case, a realistic analysis of the current state of stress, including the characterisation of the partially saturated zone above the water table, is essential to assess the causes of instability and, hence, to design appropriate remedial measures.

To quantify the effects of partial saturation on the stability of geostructures, methods should be developed to analyse collapse conditions in cohesionless partially saturated soils. So far, this problem has received little attention from researchers working on unsaturated soils.

This paper presents an approach based on the upper and lower bound theorems of plasticity. By assuming that the shear strength of partially saturated soils is controlled by the average skeleton stress (as suggested by ample experimental evidence), the classical approach developed for dry and saturated soils can be easily extended to partially saturated soils.

As an example, the problem of the active lateral force on a retaining wall has been assessed. Using a water retention curve modelled by a single-parameter exponential function assuming hydrostatic conditions, analytical lower and upper bound solutions have been obtained for the active earth force.

Finally the active earth forces considering a non-clayey ('cohesionless') soil and different wall heights and water table depths has been calculated and compared with the values obtained using the classical 'dry' approach.

2 THEOREMS OF PLASTIC COLLAPSE APPLIED TO SOILS ABOVE WATER TABLE

The upper and lower bound theorems of plastic collapse set limits to the collapse load of a structure and can be proved for the case of perfectly plastic materials. A perfectly plastic material is one which reaches a state of non-hardening failure where the failure criterion serves as a plastic potential and so the flow rule is associated. At ultimate failure the vector plastic strain increment is normal to the failure envelope and the forces and stresses remain constant for an increment of deformation; thus, since all forces and all stresses remain constant, elastic components of strain are zero and increments of total and plastic strain are identical.

In two-phase soils, the failure envelope under ultimate conditions can be defined by the following equation:

$$\tau = (\sigma - u)\tan\phi' \quad (1)$$

where τ is the shear stress, σ is the normal stress, u is the pore pressure and ϕ' is the effective angle of shearing resistance. Pore pressure equals the pore-water pressure u_w in saturated soils and the pore-air pressure u_a in ideally dry soils. Using the shear strength criterion given by Eq. (1), the ultimate conditions of soil structures such as retaining walls, foundations, vertical cuts, and slopes can be calculated for saturated and dry soil (Atkinson, 1981).

The application of bound theorems of plasticity to soil structures above the water table requires the definition of a suitable failure criterion for partially saturated soils.

For compacted (aggregated) soils, shear strength under partially saturated states can be expressed by the following equation (Tarantino and Tombolato, 2005):

$$\tau = (\sigma - u_w S_{re})\tan\phi' = (\sigma + sS_{re})\tan\phi' \quad (2)$$

where u_w is the pore-water pressure, s is the suction (s=-u_w), and S_{re} is an effective degree of saturation. This is given by:

$$S_{re} = \frac{e_w - e_{wm}}{e - e_{wm}} \quad (3)$$

where e is the void ratio (volume of voids per volume of solids), e_w is the water ratio (volume of water per volume of solids), and e_{wm} is the 'microstructural' water ratio, which separates the region of inter-aggregate porosity from the region of intra-aggregate porosity (Romero & Vaunat, 2000). Eq. (3) characterises the degree of saturation of the macro-pores, i.e. the pores between the aggregates. If the soil is idealised as a granular material where 'grains' are made of aggregates, the degree of saturation of the macro-pores plays the same role as the overall degree of saturation in granular non-materials.

The parameter e_{wm} may be difficult to determine directly, as there might not be a clear-cut distinction between the region of inter-aggregate porosity and the region of intra-aggregate porosity. As a result, the parameter e_{wm} may conveniently be determined by best fitting of shear strength data. The validity of Eq. (2) using Eq. (3) has been proved by Tarantino (2007) for a wide range of compacted soils (clay, sandy clay, silty sand, and gravel) and more recently by Jotisankasa et al. (2009) for another compacted silty clay.

On the other hand, reconstituted and non-clayey soils are generally non-aggregated and the 'microstructural' water ratio e_{wm} may therefore be expected to be zero for these soils. This appears to indeed be the case as shown by Boso (2005) for a reconstituted clayey silt and Papa & Nicotera (in press) for a pyroclastic silty sand. For non-aggregated soils, shear strength can therefore be defined by the following equation:

$$\tau = (\sigma - u_w S_r)\tan\phi' = (\sigma + sS_r)\tan\phi' \quad (4)$$

If Eq. (2) or Eq. (4) are used in place Eq. (1), collapse of geostructures in partially saturated soils can be analysed in a very similar fashion as for saturated and dry soils.

3 RETAINING WALL

Figure 1 schematically shows a gravity wall and a cantilever wall, which are the two most common types of retaining walls. If the water table is located below the wall, pore-water pressure will be negative along the wall interface (indicated by the dashed line in Figure 1). Negative values for the lateral stress σ_x are compatible with the shear strength criterion given by Eq. (2) and, hence, tensile lateral total stress may be established at the soil-wall interface.

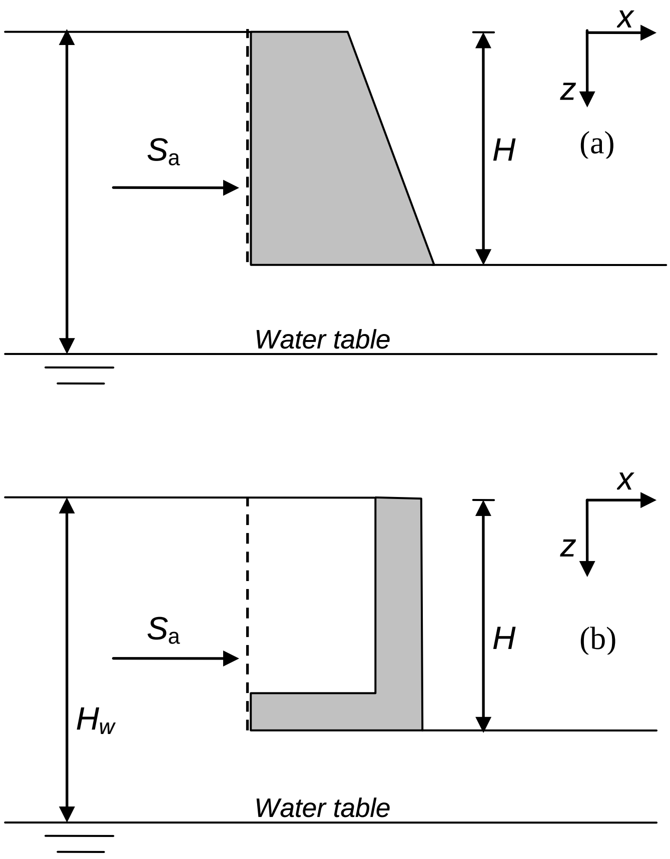

Figure 1. Main types of retaining walls, gravity (a) and cantilever (b).

For the case of the gravity wall shown in Figure 1a, however, negative lateral stresses cannot be transmitted to the backfill soil because the concrete wall is not able to pull the soil (there is no bonding between the soil and the concrete wall). As a result, when evaluating the active lateral force on the wall, it should be assumed $\sigma_x \geq 0$.

On the other hand, negative (tensile) lateral stresses can arise at the soil-soil interface (dotted line) for the cantilever wall shown in Figure 1b. In this case, the suction stresses across the interface allow tensile total stresses to be transmitted from the soil-concrete wall to the backfill soil.

For the sake of simplicity, the paper will only examine the active lateral force S_a for the case shown in Figure 1b. In addition, the case of a non-aggregated soil for which Eq. (4) applies will be considered.

4 ANALYTICAL SOLUTION FOR ACTIVE EARTH FORCE

An analytical solution for the upper and lower bound active thrust can be obtained if an exponential water retention function is considered:

$$S_r = \exp\left(-as\right) \tag{5}$$

where a is a soil parameter. Under hydrostatic conditions suction s is given by

$$s = -u_w = -\gamma_w\left(z - H_w\right) \tag{6}$$

where γ_w is the unit weight of water, z the vertical coordinate positive downward, and H_w is the water table depth (see Figure 1). By combining Eq. (5) and Eq. (6), the degree of saturation under hydrostatic conditions can therefore be expressed as follows:

$$S_r = e^{a\gamma_w\left(z-H_w\right)} = e^{-a\gamma_w H_w} e^{a\gamma_w z} \tag{7}$$

4.1 *Upper bound solution*

To derive the upper bound active thrust, firstly a kinematically admissible mechanism needs to be considered and secondly the external work and the internal energy dissipation need to be equated. The simplest kinematism consists of a single block with planar slip surface as shown in Figure 2.

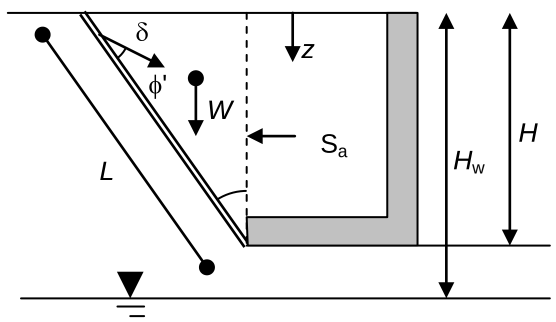

Figure 2. Kinematic mechanism to derive the upper bound solution.

The external work L_e is then given by:

$$L_e = W\delta_v - S_a\delta_h \tag{8}$$

where W is the weight of the block, S_a is the active thrust, and δ_v and δ_h are the vertical and the horizontal components respectively, of the block displacement. Since the unit weight γ of the soil is given by:

$$\gamma = \rho_{dry} + \left(\rho_{sat} - \rho_{dry}\right) S_r \tag{9}$$

where ρ_{dry} and ρ_{sat} are the dry and saturated unit weight respectively, the following relation can be derived for the weight W:

$$W = \tan\alpha \left[\begin{aligned} & \frac{\gamma_{dry} H^2}{2} + \left(\gamma_{sat} - \gamma_{dry}\right) \cdot \\ & \cdot e^{-a\gamma_w H_w} \left(\frac{e^{a\gamma_w H} - a\gamma_w H - 1}{\left(a\gamma_w\right)^2} \right) \end{aligned} \right] \tag{10}$$

By combining Eq. (8) with Eq. (10), the external work L_e can therefore be calculated.

The internal energy dissipation L_i is given by:

$$L_i = \int_V \left(\sigma\varepsilon + \tau\gamma\right) dV \tag{11}$$

where σ and τ are the stresses normal and tangential to the slip surface respectively, and ε and γ are the strains conjugate to the stresses σ and τ. It can be demonstrated that L_i is given by:

$$\begin{aligned} L_i &= \delta \frac{\sin\varphi'}{\cos\alpha} \int_H s S_r dz = \delta\gamma_w \frac{\sin\varphi'}{\cos\alpha} \frac{e^{-a\gamma_w H_w}}{\left(a\gamma_w\right)^2} \cdot \\ & \cdot \left[-a\gamma_w e^{a\gamma_w H} H + \left(a\gamma_w H_w + 1\right)\left(e^{a\gamma_w H} - 1\right) \right] \end{aligned} \tag{12}$$

By equating L_i with L_e, the following equation is obtained for the active thrust:

$$\begin{aligned} S_a &= \left[\begin{aligned} & \frac{\gamma_{dry} H^2}{2} + \\ & + \left(\gamma_{sat} - \gamma_{dry}\right) e^{-a\gamma_w H_w} \left(\frac{e^{a\gamma_w H} - a\gamma_w H - 1}{\left(a\gamma_w\right)^2} \right) \end{aligned} \right] k_a + \\ & + \left(1 - k_a\right) \frac{\gamma_w}{\left(a\gamma_w\right)^2} \cdot \left\{ \begin{aligned} & \left[-1 + a\gamma_w\left(H - H_w\right)\right] e^{a\gamma_w\left(H - H_w\right)} \\ & + \left(1 + a\gamma_w H_w\right) e^{-a\gamma_w H_w} \end{aligned} \right\} \end{aligned} \tag{13}$$

4.2 *Lower bound solution*

To derive the lower bound active thrust, firstly the state of stress in equilibrium with the applied loads needs to be considered and secondly it must be ensured that the stresses do not violate the failure criterion.

If the vertical and horizontal directions are assumed to be principal directions of stress, the equilibrium stress state is given by:

$$\begin{cases} \sigma_z = \gamma z \\ \sigma_x = \sigma_x^0 \end{cases} \tag{14}$$

where σ_z and are the vertical and horizontal directions respectively, and σ_x^0 is a constant. A lower bound value for the horizontal stress σ_x^0 is obtained by imposing the Mohr stress circle in the plane ($\sigma+sS_r$, τ) that is tangent to the failure envelope:

$$\sigma_x^0 + sS_r = k_a \left[\gamma z + sS_r \right] \tag{15}$$

where k_a is the active earth coefficient. The lower bound active lateral force can finally be obtained by combining Eq. (15) with Eqs. (6), (7), (9) and integrating over the height H:

$$S_a = \int_H \sigma_x^0 dz =$$

$$= \left[\begin{array}{l} \dfrac{\gamma_{dry} H^2}{2} + \\ +\left(\gamma_{sat} - \gamma_{dry}\right) e^{-a\gamma_w H_w} \left(\dfrac{e^{a\gamma_w H} - a\gamma_w H - 1}{\left(a\gamma_w\right)^2} \right) \end{array} \right] k_a + \tag{16}$$

$$+\left(1-k_a\right) \frac{\gamma_w}{\left(a\gamma_w\right)^2} \cdot \left\{ \begin{array}{l} \left[-1 + a\gamma_w \left(H - H_w\right)\right] e^{a\gamma_w \left(H - H_w\right)} \\ +\left(1 + a\gamma_w H_w\right) e^{-a\gamma_w H_w} \end{array} \right\}$$

The lower and upper bound solution are found to be coincident when Eq. (13) and Eq. (16) are compared; thus the exact solution for this problem has been obtained.

5 EXAMPLES OF ACTIVE EARTH FORCE IN PYROCLASTIC SILTY SAND

To show the effect of partial saturation on lateral active force in non-clayey soils, we have examined the case of pyroclastic silty sand investigated by Papa & Nicotera (in press) and Nicotera et al. (submitted). The grain distribution shows it to have no clay fraction and silt fraction no greater than 50% (Figure 3).

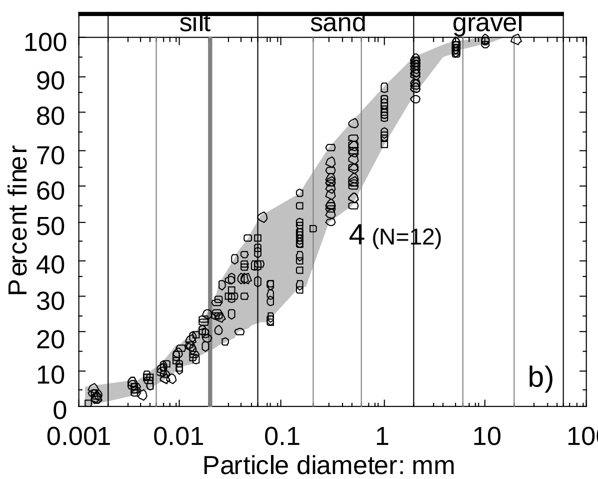

Figure 3. Grain size distribution of the pyroclastic silty sand (after Papa & Nicotera, in press).

The water retention curve was determined in the range from 0 to 20 kPa on several natural samples retrieved from a deposit about 40 km northwest of the volcano Somma-Vesuvius (Naples, Italy). The average water retention curve is shown in Figure 4 together with the exponential water retention function given by Eq. (5) used to fit the water retention curve, by calibrating the single parameter a.

Physical properties are reported in Table 1 together with the critical state friction angle determined on the basis of 48 stress-path controlled triaxial compression tests performed on undisturbed specimens.

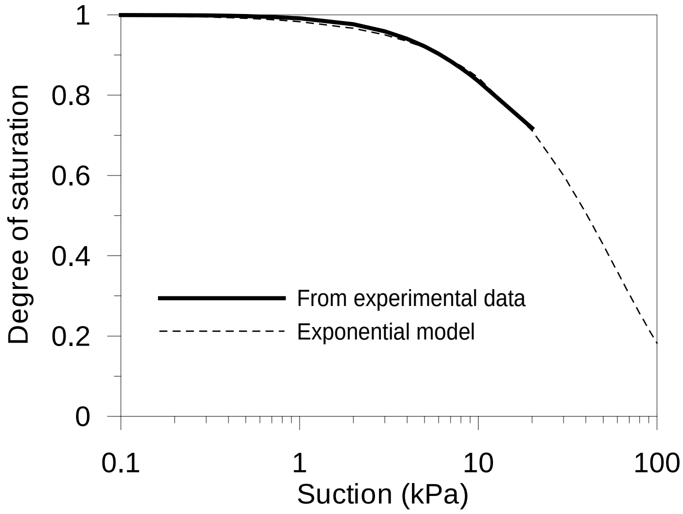

Figure 4. Main drying water retention curve derived from experimental data and fitting using the exponential water retention function.

Table 1. Soil parameters

ρ_s (kN/m^3)	ρ_w (kN/m^3)	γ_{dry} (kN/m^3)	γ_{sat} (kN/m^3)	a kPa^{-1}	ϕ' 36.9
25.7	9.8	8.8	15.2	0.017	36.9

The active thrust on a cantilever wall is shown in Figure 5 as a function of water table depth. Two wall heights were considered, 3 and 6 m respectively. The most striking aspect is negative active thrust for the case of partially saturated soils in contrast to the positive thrust obtained if the soil is assumed to be dry. Under partially saturated conditions, it is the backfill soil which pulls the cantilever wall and not vice versa as usually assumed by the classical solution, where the effect of partial saturation is neglected.

It is worth noticing that negative thrust is associated with negative (tensile) horizontal stresses. It is here assumed that tensile stresses can be transmitted across the interface between the backfill soil and the cantilever wall, i.e. no cracking occurs, provided the shear failure criterion is not violated. This assumption appears to be supported by recent experimental observations (Thusyanthan et al. 2007).

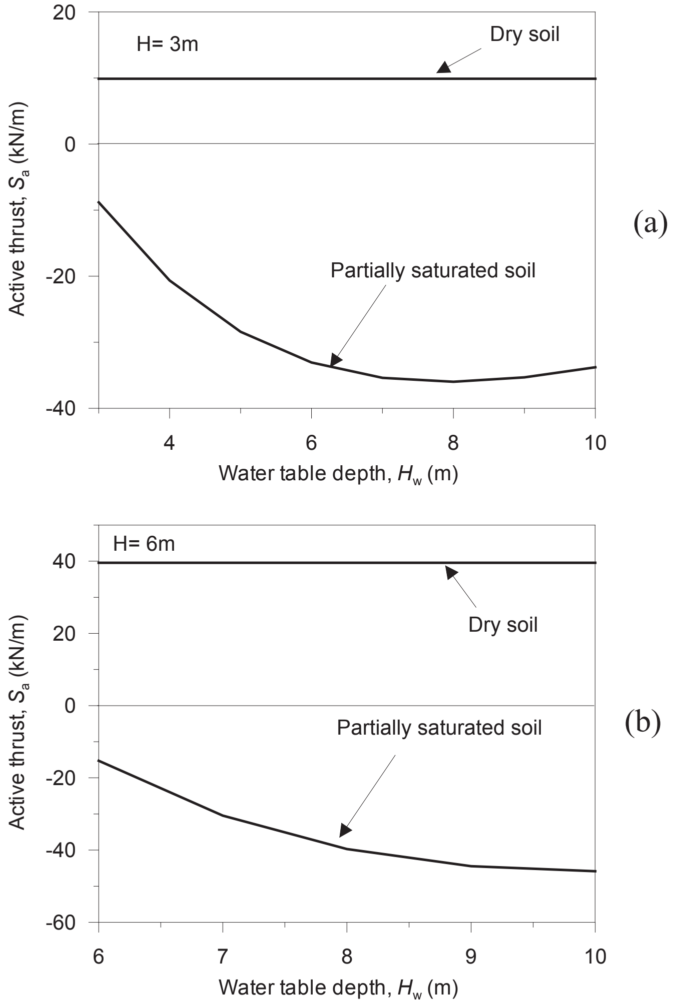

Figure 5. Active thrust at various water table depths for cantilever walls of 3m (a) and 6m (b) height.

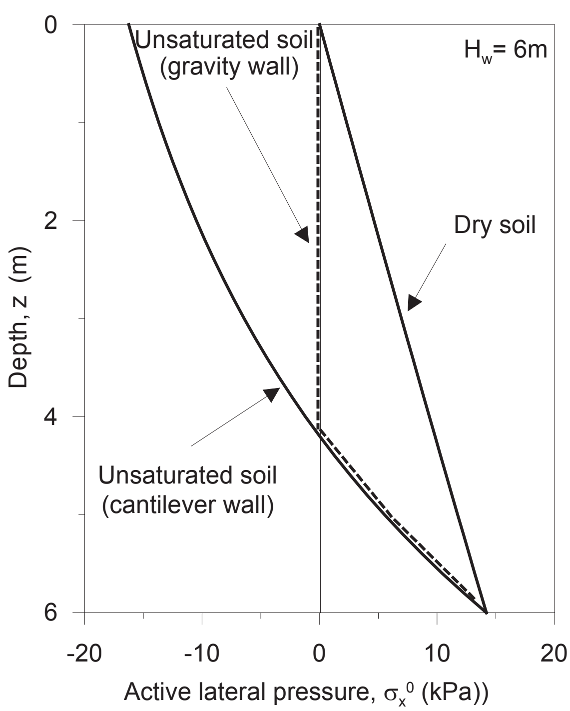

Figure 6. Lateral active pressure for the case of water table at 6m depth.

It is interesting to plot the horizontal stress σ_x^0 derived from the lower bound solution given by Eq. (15) as shown in Figure 6. Lateral stress is always positive in dry soil and no vertical slope can therefore be cut in ideally dry soil. However, if the soil is assumed to be unsaturated, vertical slope can be cut to significant depths amounting to around 2/3 of the water table depth.

It is also interesting to plot lateral pressure for the case of a gravity wall, represented by a dashed line in Figure 6. In this case, boundary conditions at the soil-wall interface require $\sigma_x^0 \geq 0$. If the water table is located at the base of the wall, the active thrust is about one tenth of the thrust calculated under the assumption of dry soil.

6 CONCLUSIONS

The paper has presented a simple approach to assess the active thrust in soils above the water table using the bound theorems of plasticity. By considering an exponential water retention function and under the assumption of hydrostatic conditions, an analytical solution can be derived which can be used for preliminary assessment of the effect of partial saturation on lateral active forces.

Results based on a non-clayey (cohesionless) pyroclastic soil suggests that assuming the soil to be dry above the water leads to significant overestimate of the active lateral force.

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