COMPUTABILITY THEORY, NONSTANDARD ANALYSIS, AND THEIR CONNECTIONS

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Abstract. In this paper we connect two seemingly unrelated topics, respectively in computability theory and Nonstandard Analysis. In particular, we investigate the following:

(T.1) We introduce the *special fan functional* Θ and establish that it is easy to compute Θ in intuitionistic mathematics but hard to compute in classical mathematics. In particular, we show that the intuitionistic fan functional MUC can compute Θ, but that the Turing jump functional $(\exists^2)$ cannot (and the same for *any* type two functional). We show that the classical *type three* functional $(\mathcal{E}_2)$, which gives rise to full second-order arithmetic, can compute Θ. Thus, *first-order strength* and *computational hardness* diverge significantly for the special fan functional

(T.2) We study the nonstandard counterparts of the 'Big Five' systems WKL_0, ACA_0, and Π^1_1-CA_0 of Reverse Mathematics, resp. the nonstandard compactness of Cantor space STP and the *Transfer* axiom limited to Π^0_1-formulas Π^0_1-TRANS, and limited to Π^1_1-formulas Π^1_1-TRANS. While the Big Five of Reverse Mathematics are linearly ordered, and Π^1_1-$\mathsf{CA}_0 \rightarrow \mathsf{ACA}_0 \rightarrow \mathsf{WKL}_0$ in particular, we show the non-implications Π^0_1-$\mathsf{TRANS} \not\rightarrow \mathsf{STP} \not\leftarrow \Pi^1_1$-$\mathsf{TRANS}$ for the respective nonstandard counterparts.

(T.3) We show that the results (**??**) and (**??**) are intimately connected. In fact, the non-implications in (**??**) are obtained *directly* from the non-computability results in (**??**), and we show that non-computability results also follow from non-implications in Nonstandard Analysis.

Contents

1. Introduction

2. Background: internal set theory and Reverse Mathematics

In this section, we introduce Nelson's syntactic approach to Nonstandard Analysis *internal set theory*, and it fragments based on Peano arithmetic from [?brie]. We also briefly sketch Friedman's foundational program *Reverse Mathematics*.

2.1. Internal set theory and its fragments.

In this section, we discuss Nelson's *internal set theory*, first introduced in [?wownelly], and its fragment P from [?brie]. The latter fragments are essential to our enterprise, especially by Theorem **??** below.

2.1.1. *Internal set theory 101.*

In Nelson's *syntactic* approach to Nonstandard Analysis ([?wownelly]), as opposed to Robinson's semantic one ([?robinson1]), a new predicate 'st(x)', read as 'x is standard' is added to the language of ZFC, the usual foundation of mathematics. The notations $(\forall^{\text{st}}x)$ and $(\exists^{\text{st}}y)$ are short for $(\forall x)(\text{st}(x) \rightarrow \dots)$ and $(\exists y)(\text{st}(y) \wedge \dots)$. A formula is called *internal* if it does not involve 'st', and *external* otherwise. The three external axioms *Idealisation*, *Standard Part*, and *Transfer* govern the new predicate 'st'; They are respectively defined[1] as:

- (I) $(\forall^{\text{st fin}}x)(\exists y)(\forall z \in x)\varphi(z,y) \rightarrow (\exists y)(\forall^{\text{st}}x)\varphi(x,y)$, for internal φ with any (possibly nonstandard) parameters.
- (S) $(\forall^{\text{st}}x)(\exists^{\text{st}}y)(\forall^{\text{st}}z)\big((z \in x \wedge \varphi(z)) \leftrightarrow z \in y\big)$, for any φ.
- (T) $(\forall^{\text{st}}t)\big[(\forall^{\text{st}}x)\varphi(x,t) \rightarrow (\forall x)\varphi(x,t)\big]$, where $\varphi(x,t)$ is internal, and only has free variables t, x.

The system IST is (the internal system) ZFC extended with the aforementioned external axioms; The former is a conservative extension of ZFC for the internal language, as proved in [?wownelly].

In [?brie], the authors study Gödel's system T extended with special cases of the external axioms of IST. In particular, they consider the systems H and P which are conservative extensions of the (internal) logical systems $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$, respectively *Heyting and Peano arithmetic in all finite types and the axiom of extensionality*. We refer to [?kohlenbach3, §3.3] for the exact definitions of the (mainstream in mathematical logic) systems $\mathsf{E\text{-}HA}^\omega$ and $\mathsf{E\text{-}PA}^\omega$. Furthermore, $\mathsf{E\text{-}PA}^{\omega*}$ and $\mathsf{E\text{-}HA}^{\omega*}$ are the definitional extensions of $\mathsf{E\text{-}PA}^\omega$ and $\mathsf{E\text{-}HA}^\omega$ with types for finite sequences, as in [?brie, §2]. For the former systems, we require some notation.

Notation 2.1 (Finite sequences)**.** The systems $\mathsf{E\text{-}PA}^{\omega*}$ and $\mathsf{E\text{-}HA}^{\omega*}$ have a dedicated type for 'finite sequences of objects of type ρ', namely ρ^*. Since the usual coding of pairs of numbers goes through in both, we shall not always distinguish between 0 and 0^*. Similarly, we do not always distinguish between 's^ρ' and '$\langle s^\rho \rangle$', where the former is 'the object s of type ρ', and the latter is 'the sequence of type ρ^* with only element s^ρ'. The empty sequence for the type ρ^* is denoted by '$\langle\rangle_\rho$', usually with the typing omitted. Furthermore, we denote by '$|s| = n$' the length of the finite sequence $s^{\rho^*} = \langle s_0^\rho, s_1^\rho, \dots, s_{n-1}^\rho\rangle$, where $|\langle\rangle| = 0$, i.e. the empty sequence

[1] The superscript 'fin' in (I) means that x is finite, i.e. its number of elements are bounded by a natural number.

has length zero. For sequences s^{ρ^*}, t^{ρ^*}, we denote by '$s * t$' the concatenation of s and t, i.e. $(s*t)(i) = s(i)$ for $i < |s|$ and $(s*t)(j) = t(j - |s|)$ for $|s| \leq j < |s| + |t|$. For a sequence s^{ρ^*}, we define $\overline{s}N := \langle s(0), s(1), \dots, s(N)\rangle$ for $N^0 < |s|$. For a sequence $\alpha^{0\rightarrow\rho}$, we also write $\overline{\alpha}N = \langle \alpha(0), \alpha(1), \dots, \alpha(N)\rangle$ for *any* N^0. By way of shorthand, $q^\rho \in Q^{\rho^*}$ abbreviates $(\exists i < |Q|)(Q(i) =_\rho q)$. Finally, we shall use $\underline{x}, \underline{y}, \underline{t}, \dots$ as short for tuples $x_0^{\sigma_0}, \dots x_k^{\sigma_k}$ of possibly different type σ_i.

2.1.2. *The classical system* P. In this section, we introduce the system P, a conservative extension of $\mathsf{E\text{-}PA}^\omega$ with fragments of Nelson's IST.

To this end, we first introduce the base system $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$. We use the same definition as [?brie, Def. 6.1], where $\mathsf{E\text{-}PA}^{\omega*}$ is the definitional extension of $\mathsf{E\text{-}PA}^\omega$ with types for finite sequences as in [?brie, §2]. The set $\mathcal{T}^*$ is defined as the collection of all the constants in the language of $\mathsf{E\text{-}PA}^{\omega*}$.

Definition 2.2. The system $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ is defined as $\mathsf{E\text{-}PA}^{\omega*} + \mathcal{T}^*_{\mathrm{st}} + \mathsf{IA}^{\mathrm{st}}$, where $\mathcal{T}^*_{\mathrm{st}}$ consists of the following axiom schemas.

(1) The schema[2] $\mathrm{st}(x) \wedge x = y \rightarrow \mathrm{st}(y)$,
(2) The schema providing for each closed term $t \in \mathcal{T}^*$ the axiom $\mathrm{st}(t)$.
(3) The schema $\mathrm{st}(f) \wedge \mathrm{st}(x) \rightarrow \mathrm{st}(f(x))$.

The external induction axiom $\mathsf{IA}^{\mathrm{st}}$ is as follows.

$$\Phi(0) \wedge (\forall^{\mathrm{st}} n^0)(\Phi(n) \rightarrow \Phi(n+1)) \rightarrow (\forall^{\mathrm{st}} n^0)\Phi(n). \qquad (\mathsf{IA}^{\mathrm{st}})$$

Secondly, we introduce some essential fragments of IST studied in [?brie].

Definition 2.3. [External axioms of P]

(1) $\mathsf{HAC_{int}}$: For any internal formula φ, we have
$$(\forall^{\mathrm{st}} x^\rho)(\exists^{\mathrm{st}} y^\tau)\varphi(x,y) \rightarrow \big(\exists^{\mathrm{st}} F^{\rho\rightarrow\tau^*}\big)(\forall^{\mathrm{st}} x^\rho)(\exists y^\tau \in F(x))\varphi(x,y), \qquad (2.1)$$
(2) I: For any internal formula φ, we have
$$(\forall^{\mathrm{st}} x^{\sigma^*})(\exists y^\tau)(\forall z^\sigma \in x)\varphi(z,y) \rightarrow (\exists y^\tau)(\forall^{\mathrm{st}} x^\sigma)\varphi(x,y),$$
(3) The system P is $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}} + \mathsf{I} + \mathsf{HAC_{int}}$.

Note that I and $\mathsf{HAC_{int}}$ are fragments of Nelson's axioms *Idealisation* and *Standard part*. By definition, F in (**??**) only provides a *finite sequence* of witnesses to $(\exists^{\mathrm{st}} y)$, explaining its name *Herbrandized Axiom of Choice*.

The system P is connected to $\mathsf{E\text{-}PA}^\omega$ by the following theorem which expresses that we may obtain effective results as in (**??**) from any theorem of Nonstandard Analysis which has the same form as in (**??**). It is known that the scope of this corollary includes the Big Five systems of Reverse Mathematics ([?sambon]), the Reverse Mathematics zoo ([?samzoo, ?samzooII]), and higher-order computability theory ([?samGH]).

Theorem 2.4 (Term extraction)**.** *If Δ_{int} is a collection of internal formulas and ψ is internal, and*
$$\mathsf{P} + \Delta_{\mathsf{int}} \vdash (\forall^{\mathrm{st}} \underline{x})(\exists^{\mathrm{st}} \underline{y})\psi(\underline{x}, \underline{y}, \underline{a}), \qquad (2.2)$$
then one can extract from the proof a sequence of closed terms t in $\mathcal{T}^*$ such that
$$\mathsf{E\text{-}PA}^{\omega*} + \Delta_{\mathsf{int}} \vdash (\forall \underline{x})(\exists \underline{y} \in t(\underline{x}))\psi(\underline{x}, \underline{y}, \underline{a}). \qquad (2.3)$$

[2]The language of $\mathsf{E\text{-}PA}^{\omega*}_{\mathrm{st}}$ contains a symbol st_σ for each finite type σ, but the subscript is essentially always omitted. Hence $\mathcal{T}^*_{\mathrm{st}}$ is an *axiom schema* and not an axiom.

Proof. See [?samGH, §2] or [?sambon, §2]. □

Curiously, the previous theorem is neither explicitly listed or proved in [?brie]. For the rest of this paper, the notion 'normal form' shall refer to a formula as in (??), i.e. of the form $(\forall^{\text{st}} x)(\exists^{\text{st}} y)\varphi(x, y)$ for φ internal.

Finally, the previous theorems do not really depend on the presence of full Peano arithmetic. We shall study the following subsystems.

Definition 2.5.

(1) Let $\mathsf{E\text{-}PRA}^\omega$ be the system defined in [?kohlenbach2, §2] and let $\mathsf{E\text{-}PRA}^{\omega*}$ be its definitional extension with types for finite sequences as in [?brie, §2].

(2) ($\mathsf{QF\text{-}AC}^{\rho,\tau}$) For every quantifier-free internal formula $\varphi(x, y)$, we have
$$(\forall x^\rho)(\exists y^\tau)\varphi(x, y) \to (\exists F^{\rho\to\tau})(\forall x^\rho)\varphi(x, F(x)) \tag{2.4}$$

(3) The system RCA_0^ω is $\mathsf{E\text{-}PRA}^\omega + \mathsf{QF\text{-}AC}^{1,0}$.

The system RCA_0^ω is Kohlenbach's 'base theory of higher-order Reverse Mathematics' as introduced in [?kohlenbach2, §2]. We permit ourselves a slight abuse of notation by also referring to the system $\mathsf{E\text{-}PRA}^{\omega*} + \mathsf{QF\text{-}AC}^{1,0}$ as RCA_0^ω.

Corollary 2.6. *The previous theorem and corollary go through for* P *and* $\mathsf{E\text{-}PA}^{\omega*}$ *replaced by* $\mathsf{P}_0 \equiv \mathsf{E\text{-}PRA}^{\omega*} + \mathcal{T}^*_{\text{st}} + \mathsf{HAC}_{\mathsf{int}} + \mathsf{I} + \mathsf{QF\text{-}AC}^{1,0}$ *and* RCA_0^ω.

Proof. The proof of [?brie, Theorem 7.7] goes through for any fragment of $\mathsf{E\text{-}PA}^{\omega*}$ which includes EFA, sometimes also called $\mathsf{I}\Delta_0+\mathsf{EXP}$. In particular, the exponential function is (all what is) required to 'easily' manipulate finite sequences. □

Finally, we note that Ferreira and Gaspar present a system similar to P in [?fega], which however is less suitable for our purposes.

2.1.3. *Notations and conventions.* We introduce notations and conventions for P.

First of all, we mostly use the same notations as in [?brie].

Remark 2.7 (Notations)**.** We write $(\forall^{\text{st}} x^\tau)\Phi(x^\tau)$ and $(\exists^{\text{st}} x^\sigma)\Psi(x^\sigma)$ as short for $(\forall x^\tau)\big[\text{st}(x^\tau) \to \Phi(x^\tau)\big]$ and $(\exists^{\text{st}} x^\sigma)\big[\text{st}(x^\sigma) \wedge \Psi(x^\sigma)\big]$. We also write $(\forall x^0 \in \Omega)\Phi(x^0)$ and $(\exists x^0 \in \Omega)\Psi(x^0)$ as short for $(\forall x^0)\big[\neg\text{st}(x^0) \to \Phi(x^0)\big]$ and $(\exists x^0)\big[\neg\text{st}(x^0) \wedge \Psi(x^0)\big]$. Finally, a formula A is 'internal' if it does not involve st, and A^{st} is defined from A by appending 'st' to all quantifiers (except bounded number quantifiers).

Secondly, we use the usual extensional notion of equality.

Remark 2.8 (Equality)**.** The system $\mathsf{E\text{-}PA}^{\omega*}$ includes equality between natural numbers '$=_0$' as a primitive. Equality '$=_\tau$' and inequality $\leq_\tau$ for x^τ, y^τ is:
$$[x =_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k})[xz_1 \dots z_k =_0 yz_1 \dots z_k], \tag{2.5}$$
$$[x \leq_\tau y] \equiv (\forall z_1^{\tau_1} \dots z_k^{\tau_k})[xz_1 \dots z_k \leq_0 yz_1 \dots z_k], \tag{2.6}$$
if the type τ is composed as $\tau \equiv (\tau_1 \to \dots \to \tau_k \to 0)$. In the spirit of Nonstandard Analysis, we define 'approximate equality $\approx_\tau$' as follows:
$$[x \approx_\tau y] \equiv (\forall^{\text{st}} z_1^{\tau_1} \dots z_k^{\tau_k})[xz_1 \dots z_k =_0 yz_1 \dots z_k] \tag{2.7}$$
with the type τ as above. All the above systems include the *axiom of extensionality* for all $\varphi^{\rho\to\tau}$ as follows:
$$(\forall x^\rho, y^\rho)\big[x =_\rho y \to \varphi(x) =_\tau \varphi(y)\big]. \tag{E}$$

However, as noted in [?brie, p. 1973], the so-called axiom of *standard* extensionality $(\mathbf{??})^{\rm st}$ is problematic and cannot be included in P.

Thirdly, the system P proves overspill and underspill, which are quite useful principles.

Theorem 2.9. *The systems* P *and* P_0 *prove* overspill*, i.e.*

$$(\forall^{\rm st}x^\rho)\varphi(x)\to(\exists y^\rho)\big[\neg{\rm st}(y)\wedge\varphi(y)\big], \tag{OS}$$

for any internal formula φ.

Proof. See [?brie, Prop. 3.3]. □

Fourth, we consider the following remark on how $\mathsf{HAC}_{\mathsf{int}}$ and I are used.

Remark 2.10 (Using $\mathsf{HAC}_{\mathsf{int}}$ and I)**.** By definition, $\mathsf{HAC}_{\mathsf{int}}$ produces a functional $F^{\sigma\to\tau^*}$ which outputs a *finite sequence* of witnesses. However, $\mathsf{HAC}_{\mathsf{int}}$ provides an actual *witnessing functional* assuming (i) $\tau=0$ in $\mathsf{HAC}_{\mathsf{int}}$ and (ii) the formula φ from $\mathsf{HAC}_{\mathsf{int}}$ is 'sufficiently monotone' as in: $(\forall^{\rm st}x^\sigma,n^0,m^0)\big([n\leq_0 m\wedge\varphi(x,n)]\to\varphi(x,m)\big)$. Indeed, in this case one simply defines $G^{\sigma+1}$ by $G(x^\sigma):=\max_{i<|F(x)|}F(x)(i)$ which satisfies $(\forall^{\rm st}x^\sigma)\varphi(x,G(x))$. To save space in proofs, we will sometimes skip the (obvious) step involving the maximum of finite sequences, when applying $\mathsf{HAC}_{\mathsf{int}}$. We assume the same convention for terms obtained from Theorem **??**, and applications of the contraposition of idealisation I.

2.2. **Introducing Reverse Mathematics.** Reverse Mathematics (RM) is a program in the foundations of mathematics initiated around 1975 by Friedman ([?fried, ?fried2]) and developed extensively by Simpson ([?simpson2, ?simpson1]) and others. We refer to [?simpson2] for an introduction to RM; we do sketch some of its aspects essential to this paper.

The aim of RM is to find the axioms necessary to prove a statement of *ordinary* mathematics, i.e. dealing with countable or separable objects. The classical[3] base theory RCA_0 of 'computable[4] mathematics' is always assumed. Thus, the aim is:

> *The aim of* RM *is to find the minimal axioms* A *such that* RCA_0 *proves* $[A\to T]$ *for statements* T *of ordinary mathematics.*

Surprisingly, once the minimal axioms A have been found, we almost always also have $\mathsf{RCA}_0\vdash[A\leftrightarrow T]$, i.e. not only can we derive the theorem T from the axioms A (the 'usual' way of doing mathematics), we can also derive the axiom A from the theorem T (the 'reverse' way of doing mathematics). In light of the latter, the field was baptised 'Reverse Mathematics'.

Perhaps even more surprisingly, in the majority[5] of cases for a statement T of ordinary mathematics, either T is provable in RCA_0, or the latter proves $T\leftrightarrow A_i$, where A_i is one of the logical systems $\mathsf{WKL}_0,\mathsf{ACA}_0$, ATR_0 or Π^1_1-CA_0. The latter together with RCA_0 form the 'Big Five' and the aforementioned observation that most mathematical theorems fall into one of the Big Five categories, is called the *Big Five phenomenon* ([?montahue, p. 432]). Furthermore, each of the Big Five has a natural formulation in terms of (Turing) computability (See e.g. [?simpson2, I.3.4,

[3]In *Constructive Reverse Mathematics* ([?ishi1]), the base theory is based on intuitionistic logic.

[4]The system RCA_0 consists of induction $I\Sigma_1$, and the **r**ecursive **c**omprehension **a**xiom Δ^0_1-CA.

[5]Exceptions are classified in the so-called Reverse Mathematics zoo ([?damirzoo]).

I.5.4, I.7.5]). As noted by Simpson in [?simpson2, I.12], each of the Big Five also corresponds (sometimes loosely) to a foundational program in mathematics.

Now, the logical framework for RM is *second-order arithmetic*, i.e. only natural numbers and sets thereof are available. For this reason higher-order objects such as continuous real functions and topologies are not available directly, and are represented by so-called codes (See e.g. [?simpson2, II.6.1] and [?mummy]). Kohlenbach has introduced *higher-order* RM and the associated base theory RCA_0^ω where the language includes all finite types; we refer to [?kohlenbach2, §2] for the definition of the latter system.

Finally, we consider an interesting observation regarding the Big Five systems of Reverse Mathematics, namely that these five systems satisfy the strict implications:

$$\Pi_1^1\text{-}\mathsf{CA}_0 \to \mathsf{ATR}_0 \to \mathsf{ACA}_0 \to \mathsf{WKL}_0 \to \mathsf{RCA}_0. \tag{2.8}$$

By contrast, there are many incomparable logical statements in second-order arithmetic. For instance, a regular plethora of such statements may be found in the *Reverse Mathematics zoo* in [?damirzoo]. The latter is a collection of theorems which fall outside of the Big Five classification of RM.

3. Main results

In this section, we prove the results sketched in the introduction.

3.1. **Computing the special fan functional.**

In this section, we study the relationship between the new *special fan functional* and mainstream functionals like the *Turing jump functional*. As a main result, we show that the latter (and in fact any type two functional) cannot compute (in the sense of Kleene's S1-S9 from [?longmann, §8]) the special fan functional.

As to its provenance, the special fan functional was first introduced in [?samGH, §3] in the study of the Gandy-Hyland functional. The special fan functional is an object of classical mathematics in that it can be defined in a (relatively strong) fragment of set theory by Theorem **??** in Section **??**. Furthermore, the special fan functional may be derived from the *intuitionistic* fan functional, as shown in Section **??**. The latter result shows that the existence of the special fan functional has quite weak first-order strength (in contrast to its computational strength).

3.1.1. *The special and intuitionistic fan functionals.*

In this section, we introduce the functionals from the title and show that the latter computes the former via a term from Gödel's T. In particular, the name 'special fan functional' derives from this relative computability result.

First of all, we define the special fan functional. We reserve the variables S^1, T^1, U^1 for trees and denote by '$T^1 \leq_1 1$' that T is a binary tree. Recall that 1^* is the type of finite sequences of type 1 as in Notation **??**.

Definition 3.1. [Special fan functional] We define $\mathsf{SCF}(\Theta)$ as follows for $\Theta^{(2\to(0\times 1^*))}$:

$$(\forall g^2, T^1 \leq_1 1)\big[(\forall \alpha \in \Theta(g)(2))(\overline{\alpha}g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\exists i \leq \Theta(g)(1))(\overline{\beta}i \not\in T)\big].$$

Any functional Θ satisfying $\mathsf{SCF}(\Theta)$ is referred to as a *special fan functional*.

Note that there is *no unique* special fan functional, i.e. it is in principle incorrect to make statements about 'the' special fan functional.

Secondly, we define the *intuitionistic fan functional* Ω as in [?kohlenbach2, §3] and [?troelstra1, 2.6.6].

Definition 3.2. [Intuitionistic fan functional]

$$(\forall Y^2)(\forall f, g \leq_1 1)(\overline{f}\Omega(Y) = \overline{g}\Omega(Y) \to Y(f) = Y(g)), \qquad (\mathsf{MUC}(\Omega))$$

As to the logical strength of $(\exists \Omega^3)\mathsf{MUC}(\Omega)$, the latter yields a conservative extension of WKL_0 by the following theorem, where 'RCA_0^2' is just the base theory RCA_0 formulated with function variables rather than set variables (See [?kohlenbach2, §2]).

Theorem 3.3. *The system* $\mathsf{RCA}_0^\omega + (\exists \Omega^3)\mathsf{MUC}(\Omega)$ *is a conservative extension of* $\mathsf{RCA}_0^2 + \mathsf{WKL}$ (*for the second-order language of the latter*).

Proof. A very rudimentary sketch of a proof is provided in [?kohlenbach2, §3]. A detailed proof is provided in Theorem **??** of the Appendix. □

Recall that the fan theorem FAN is the classical contraposition of WKL.

$$(\forall T \leq_1 1)\big[(\forall \beta \leq_1 1)(\exists m)(\overline{\beta}m \not\in T) \to (\exists k^0)(\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta}i \not\in T)\big]. \quad (\mathsf{FAN})$$

We also introduce the 'effective version' of the fan theorem as follows.

Definition 3.4. [Effective fan theorem]

$$(\forall T^1 \leq_1 1, g^2)\big[(\forall \alpha \leq_1 1)(\overline{\alpha}g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\overline{\beta}h(g,T) \not\in T)\big]. \quad (\mathsf{FAN}_{\mathsf{ef}}(h))$$

Clearly, the existence of h as in the effective fan theorem implies FAN in RCA_0^ω. Furthermore, with a further minimum of the axiom of choice $\mathsf{QF\text{-}AC}^{2,1}$, the latter also follows form the former. We have the following theorem.

Theorem 3.5. *There are terms s, t such that* $\mathsf{E\text{-}PA}^\omega$ *proves:*

$$(\forall \Omega^3)(\mathsf{MUC}(\Omega) \to \mathsf{SCF}(t(\Omega))) \wedge (\forall \Theta)(\mathsf{SCF}(\Theta) \to \mathsf{FAN}_{\mathsf{ef}}(s(\Theta))). \qquad (3.1)$$

Proof. The second part of (**??**) is immediate. For the first part, let Ω be as in $\mathsf{MUC}(\Omega)$ and note that $\Theta(g)$ as in $\mathsf{SCF}(\Theta)$ has to provide a natural number and a finite sequence of binary sequences. The number $\Theta(g)(1)$ is defined as $\max_{|\sigma|=\Omega(g)\wedge\sigma\leq_{0^*}1} g(\sigma * 00\dots)$ and the finite sequence of binary sequences $\Theta(g)(2)$ consists of all $\tau * 00\dots$ where $|\tau| = \Theta(g)(1) \wedge \tau \leq_{0^*} 1$. We now claim that for all g^2 and $T^1 \leq_1 1$:

$$(\forall \beta \leq_1 1)(\beta \in \Theta(g)(2) \to \overline{\beta}g(\beta) \not\in T) \to (\forall \gamma \leq_1 1)(\exists i \leq \Theta(g)(1))(\overline{\gamma}i \not\in T). \quad (3.2)$$

Indeed, suppose the antecendent of (**??**) holds. Now take $\gamma_0 \leq_1 1$, and note that $\beta_0 = \overline{\gamma_0}\Theta(g)(1) * 00 \dots \in \Theta(g)(2)$, implying $\overline{\beta_0}g(\beta_0) \not\in T$. But $g(\alpha) \leq \Theta(g)(1)$ for all $\alpha \leq_1 1$, by the definition of Ω, implying that $\overline{\gamma_0}g(\beta_0) = \overline{\beta_0}g(\beta_0) \not\in T$ by the definition of β_0, and the consequent of (**??**) follows. □

As it happens, the first part of Theorem **??** was first proved *indirectly* in [?samGH, §3] by applying Theorem **??** to the normal form of $\mathsf{NUC} \to \mathsf{STP}$, where

$$(\forall^{\text{st}} Y^2)(\forall f^1, g^1 \leq_1 1)(f \approx_1 g \to Y(f) =_0 Y(g)), \qquad (\mathsf{NUC})$$

i.e. the statement that every type two functional is nonstandard uniformly continuous on Cantor space in light of Notation **??**.

Furthermore, the 'classical' fan functional is obtained from the intuitionistic one by restricting Y^2 in $\mathsf{MUC}(\Omega)$ to $Y^2 \in C$, i.e. continuous as follows:

$$Y^2 \in C \equiv (\forall f^1)(\exists N^0)(\forall g^1)(\overline{f}N = \overline{g}N \to Y(f) = Y(g)).$$

By combining [?kohlenbach4, Prop. 4.4 and 4.7], the Turing jump functional can compute the classical fan functional (over Kohlenbach's system RCA_0^ω from [?kohlenbach2, §2]).

In light of the previous observations regarding the classical and intuitionistic fan functionals, a special fan functional appears to be a rather weak object. Looks can be deceiving, as we establish in Theorem **??** in the next section that the Turing jump functional (and in fact any type two functional) cannot compute any special fan functional. We finish this section with a remark on the definition of the special fan functional.

Remark 3.6. $\Phi(T, g)$ in INT and CLASS.

3.1.2. *The special fan functional and comprehension functionals.* In this section, we study the relationship between the special fan functional and comprehension functionals. In particular, we show that the former cannot be computed by the *Turing jump functional* (and any type two functional) defined as follows.

Definition 3.7. [Turing jump functional]

$$(\forall f^1)\big[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0\big]. \tag{$\mathsf{TJ}(\varphi)$}$$

We let $(\exists^2)$ stand for $(\exists \varphi^2)\mathsf{TJ}(\varphi)$, and call φ the 'Turing jump functional'.

We make our notion of 'computability' precise as follows.

(1) We adopt ZFC set theory as the official metatheory for all results, unless explicitly stated otherwise.
(2) We adopt Kleene's notion of *higher-order computation* as given by his nine clauses S1-S9 (See [?longmann, §8]) as our official notion of 'computable'.

With these conventions in place, we can prove the following theorem.

Theorem 3.8. *Any functional Θ^3 as in $\mathsf{SCF}(\Theta)$ is not computable in $(\exists^2)$.*

Proof. Assume that Θ as in $\mathsf{SCF}(\Theta)$ is computable in φ as in $\mathsf{TJ}(\varphi)$. Let h^2 be any partial functional computable in φ as in $\mathsf{TJ}(\varphi)$ and total on the class of hyperarithmetical functions, and let g^2 be the total extension of h. Then Θ applied to g will yield a hyperarithmetical finite sequence $\Theta(g)(1)$.

We now define $h_0(\alpha)$, using Gandy selection, using the 'least' number e such that e is an index for α as a hyperarithmetical function in some fixed canonical indexing of the hyperarithmetical sets. By 'least' we mean 'of minimal ordering rank', and then of minimal numerical value among those. In particular, define $h_0(\alpha) = e + 2$ for the aforementioned e, and let g_0 be the associated extension discussed above. Then $\Theta(g_0)(1)$ consists of a finite list $\alpha_1, ..., \alpha_k$ of hyperarithmetical functions, and the neighbourhoods determined by the $\overline{\alpha_i}(g(\alpha_i))$ are not of measure 1, so they do not cover the Cantor space.

However, then there is a non-well founded binary tree T_0 such that $\overline{\alpha_i}(g_0(\alpha_i) \notin T_0$ for all $i = 1, ..., k$, but there is no possible value for $\Theta(g_0)(2)$. □

Corollary 3.9. *Let φ^2 be any type two functional. Any functional Θ^3 as in $\mathsf{SCF}(\Theta)$ is not computable in φ^2.*

Proof. X □

We now list some well-known type two functionals which will also be used below. *Feferman's search operator* (μ^2) (See e.g. [?avi2, §8]) is equivalent to ($\exists^2$) over Kohlenbach's system RCA_0^ω by [?kooltje, §3]:.

$$(\exists \mu^2)\big[(\forall f^1)\big((\exists n^0)(f(n)=0) \to f(\mu(f))=0\big)\big], \tag{μ^2}$$

and is the functional version of ACA_0. The *Suslin functional* and (μ_1) (See [?avi1, §8.4.1], [?kohlenbach2, §1], and [?yamayamaharehare, §3]) are the functional versions of Π^1_1-CA_0, and defined as:

$$(\exists S^2)(\forall f^1)\big[(\exists g^1)(\forall x^0)(f(\overline{g}n)=0) \leftrightarrow S(f)=0\big]. \tag{S^2}$$

$$(\exists \mu_1^{1\to 1})(\forall f^1)\big[(\exists g^1)(\forall x^0)(f(\overline{g}n)=0) \to (\forall x^0)(f(\overline{\mu_1(f)}n)=0)\big]. \tag{μ_1}$$

On the other hand, full second-order arithmetic suffices to compute a special fan functional, as we show now.

Theorem 3.10. *A functional Θ^3 as in $\mathsf{SCF}(\Theta)$ can be computed from ξ as in ($\mathcal{E}_2$):*

$$(\exists \xi^3)(\forall Y^2)\big[(\exists f^1)(Y(f)=0) \leftrightarrow \xi(Y)=0\big]. \tag{$\mathcal{E}_2$}$$

Proof. We first prove the existence of a functional Θ such that $\mathsf{SCF}(\Theta)$ in classical set theory without choice ZF. We then show how the construction can be realised as an algorithm relative to ξ as in ($\mathcal{E}_2$).

Let C be the Cantor space $\{0,1\}^{\mathbb{N}}$ with the lexicographical ordering. If σ is a binary finite sequence, we let C_σ be the set of binary extensions of σ (in C).
We let f, g with indices vary over C and we let α, β etc. vary over the countable ordinals. We let F be a fixed total functional of type 2, and our aim is to define $\Theta(F)$.

By recursion on α we will define an increasing sequence $\{f_\alpha\}_{\alpha<\aleph_1}$ from C. We will let

$$I(\alpha) = \bigcup_{\beta\leq\alpha} C_{\bar{f}_\beta(F(f_\beta))}$$

and we will let

$$I(<\alpha) = \bigcup_{\beta<\alpha} C_{\bar{f}_\beta(F(f_\beta))}.$$

We let $f_0 = \lambda x.0$.
Let $\alpha > 0$.

- If $\lambda x.1 \in I(<\alpha)$, let $f_\alpha = f_\beta$ for the first β such that $\lambda x.1 \in C_{\bar{f}_\beta(F(f_\beta))}$.
- If not, let f_α be the least element not in $I(<\alpha)$.

By construction, the sequence of f_α's will be strictly increasing until we capture $\lambda x.1$, which thus must happen after a countable number of steps.

Clearly, the least α such that $f \in I(\alpha)$ must be a successor ordinal for each f. Let α_0 be this ordinal for $f = \lambda x.1$, and let g_0 be the greatest strict lower bound of $C_{\bar{f}_{\alpha_0}(F(f_{f_{\alpha_0}}))}$.
Let α_1 be this ordinal for $f = g_0$ and let g_1 be the greatest strict lower bound of $C_{\bar{f}_{\alpha_1}(F(f_{f_{\alpha_1}}))}$.
Continue this process, defining a decreasing sequence $\alpha_0, \alpha_1, \cdots$ until $\lambda x.0$ is captured, and we have a finite cover of C of neighborhoods of the form $C_{\bar{f}_{\alpha_i}(F(f_{\alpha_i}))}$ for

$i \leq n$ for some n.
We then let $\Theta(F)$ have the pair of $\{f_{\alpha_i} \mid i \leq n\}$ and $\max\{F(f_{\alpha_i}) \mid i \leq n\}$ as value.

We need far less that ${}^3E$ to capture this construction, but it may be difficult to isolate a simpler functional in which Θ is computable.
Using ${}^3E$ I would proceed as follows:

(1) Let WO be a standard Π^1_1 set of codes for the countable ordinals.
(2) From F and ${}^3E$ we can compute the set of ordinals of order type α_0 of the construction.
(3) For each of these codes, we can compute from F the sequence $\{f_\beta\}_{\beta\leq\alpha_0}$, and for each of these codes, we can compute from F the backtracking to $\alpha_1, \dots, \alpha_n$.
(4) Since the construction only depends on the ordinals, it does not matter which code for an ordinal we use, we end up with the same value for $\Theta(F)$.
(5) Extracting this common value is computable in ${}^3E$.

□

Finally, we show that the special fan functional is not special in the sense that there are a number of functionals with closely related 'computational' properties (although the nonstandard provenance of the former arguably remains 'special'). By way of an example, we consider the following functionals, where we write '$(\forall\alpha^1 \in \gamma)\varphi(\alpha)$' instead of '$(\forall n^0)\varphi(\gamma(n))$' for $\gamma^{0\to 1}$.

Definition 3.11. Define $\mathsf{ALP}(\chi)$ for $\chi^{2\to 1^*}$ as follows:

$$(\forall g^2)(\exists k^0)(\forall T^1 \leq_1 1)\big[(\forall \alpha \in \chi(g))(\overline{\alpha}g(\alpha) \notin T) \to (\forall \alpha^1 \leq_1 1)(\exists i \leq k)(\overline{\alpha}i \notin T)\big].$$

Definition 3.12. Define $\mathsf{BET}(\iota)$ for $\iota^{(2\to(0\times(0\to 1)))}$ as follows:

$$(\forall g^2, T^1 \leq_1 1)\big[(\forall \alpha \in \iota(g)(2))(\overline{\alpha}g(\alpha) \notin T) \to (\forall \alpha^1 \leq_1 1)(\exists i \leq \iota(g)(1))(\overline{\alpha}i \notin T)\big].$$

By the following theorem, the special fan functional is 'sandwiched' between these new functionals and the latter plus Feferman's search operator.

Theorem 3.13. *There are terms s, t, u, v such that $\mathsf{E\text{-}PA}^{\omega*}$ proves that $(\forall\Theta)(\mathsf{ALP}(u(\Theta)) \leftarrow \mathsf{SCF}(\Theta) \to \mathsf{BET}(t(\Theta)))$ and that*

$$(\forall \zeta, \mu)\big((\mathsf{BET}(\zeta) \wedge \mathsf{MU}(\mu)) \to \mathsf{SCF}(s(\zeta,\mu)) \leftarrow (\mathsf{ALP}(\zeta) \wedge \mathsf{MU}(\mu))\big).$$

Proof. For the first part of the proof, define $t(\Theta)(g)(1) := \Theta(g)(1)$ and define $t(\Theta)(g)(2)(k)$ as $\Theta(g)(2)(k)$ for $k < |\Theta(g)(2)|$, and the zero sequence otherwise. For the second part of the proof, we prove that $((\exists^{\text{st}}\iota)\mathsf{BET}(\iota) \wedge \Pi^0_1\text{-}\mathsf{TRANS}) \to \mathsf{STP}$ in P. Applying Theorem **??** to this implication as in Theorem **??** yields the required term s. Now, for standard ι the formula $\mathsf{BET}(\iota)$ implies, since standard inputs yield standard outputs, that

$$(\forall^{\text{st}} g^2, T^1 \leq_1 1)\big[(\forall \alpha \in \iota(g)(2))(\overline{\alpha}g(\alpha) \notin T) \to (\exists^{\text{st}} k^0)(\forall \alpha^1 \leq_1 1)(\exists i \leq k)(\overline{\alpha}i \notin T)\big].$$

Thanks to $\Pi^0_1\text{-}\mathsf{TRANS}$, we may replace the antecedent by $(\forall^{\text{st}}\alpha \in \zeta(g)(2))(\overline{\alpha}g(\alpha) \notin T)$, which is implied by $(\forall^{\text{st}}\alpha \leq_1 1)(\exists^{\text{st}} n^0)(\overline{\alpha}g(\alpha) \notin T)$, and STP follows. The analogous results for χ as in $\mathsf{ALP}(\chi)$ follow from the observation that the consequent of the latter is equivalent to $(\exists k^0)(\forall \alpha^{0^*} \leq_{0^*} 1)(\exists i \leq k)(|\alpha| \leq k \to \overline{\alpha}i \notin T)$, to which (the contraposition of) $\Pi^0_1\text{-}\mathsf{TRANS}$ may be applied. □

Corollary 3.14. *Any functional ι as in $\mathsf{BET}(\iota)$ (resp. χ as in $\mathsf{ALP}(\chi)$) is not computable in $(\exists^2)$.*

Furthermore, the combination of ι and χ as above can compute the special fan functional, but we conjecture that neither functional *alone* suffices. In this way, the special fan functional can easily be 'split' in two similar but independent pieces. Such a 'splitting' is apparently difficult to obtain in Friedman-Simpson Reverse Mathematics (See e.g. [?splitting] for an example).

Remark 3.15 (Further ideas)**.** Normann: We need far less than $(\mathcal{E}_2)$ to capture the construction from the proof, but it may be difficult to isolate a simpler functional in which Θ is computable.

Sam: What about the following one which is the 'Δ^1_1' version of $(\mathcal{E}_2)$

$$(\exists\zeta^3)(\forall Y^2, Z^2)\big[(\forall m^0)[(\exists f^1)(Y(f,m)=0) \leftrightarrow (\forall g^1)(Z(g,m)\neq 0)] \\ \to (\forall n^0)[(\exists f^1)(Y(f,n)=0) \leftrightarrow \zeta(Y,Z,n)=0]\big].$$

The above non-computability results are counterexamples to the heuristic

First-order strength is roughly proportional to computational hardness.

present in the study of the computability of type one objects.

Shall we call the behaviour of the (special) fan functional a 'phase transition' (Andreas Weiermann coined this slogan for his work in incompleteness).

3.2. **A negative result in Nonstandard Analysis.** In section **??**, we observed that the Big Five of RM are linearly ordered as in (**??**). In this section, we show that the *nonstandard* counterparts of Π^1_1-CA_0, ACA_0, and WKL_0 are however *incomparable*. Surprisingly, we make essential use of Theorem **??** to establish this result, rather than taking the 'usual' model-theoretic route. Indeed, the fact that the full axiom *Transfer* does not imply the full axiom *Standard Part* is known (over various systems; see [?blaaskeswijsmaken, ?gordon2]), and is established using model-theoretic techniques.

First of all, Nelson's system IST and the associated fragment P were introduced in Section **??**. The system P includes Nelson's axiom *Idealisation* (formulated in the language of finite types), but to guarantee a conservative extension of Peano arithmetic, Nelson's axiom *Transfer* must be omitted, while *Standard Part* is weakened to $\mathsf{HAC}_{\mathsf{int}}$. Indeed, the fragment of *Transfer* for Π^0_1-formulas as follows

$$(\forall^{\rm st} f^1)\big[(\forall^{\rm st} n)f(n)\neq 0 \to (\forall m)f(m)\neq 0\big] \qquad (\Pi^0_1\text{-}\mathsf{TRANS})$$

is the nonstandard counterpart of arithmetical comprehension (actually the equivalent Π^0_1-comprehension), while the fragment of *Transfer* for Π^1_1-formulas as follows

$$(\forall^{\rm st} f^1)\big[(\exists g^1)(\forall x^0)(f(\overline{g}n)=0) \to (\exists^{\rm st} g^1)(\forall x^0)(f(\overline{g}n)=0)\big] \qquad (\Pi^1_1\text{-}\mathsf{TRANS})$$

is the nonstandard counterpart of Π^1_1-CA_0. The following fragment of *Standard Part* is the nonstandard counterpart of weak König's lemma:

$$(\forall \alpha^1 \leq_1 1)(\exists^{\rm st}\beta^1 \leq_1 1)(\alpha \approx_1 \beta), \qquad (\mathsf{STP})$$

where $\alpha \approx_1 \beta$ is short for $(\forall^{\rm st} n)(\alpha(n) =_0 \beta(n))$. There is no deep philosophical meaning to be found in the words 'nonstandard counterpart': This is just what the principles STP, Π^0_1-TRANS, and Π^1_1-TRANS are called in the literature ([?pimpson, ?sambon]).

Secondly, while Π^1_1-$\mathsf{CA}_0 \to \mathsf{ACA}_0 \to \mathsf{WKL}_0$ by (**??**), we show in Theorem **??** and Corollary **??** that the associated *nonstandard implications* Π^0_1-$\mathsf{TRANS} \to \mathsf{STP}$

and Π^1_1-$\mathsf{TRANS} \to \mathsf{STP}$ do not hold. As noted above, we shall establish this non-implication using Theorem **??**. To establish the aforementioned non-implications, we require the following theorem which provides a normal form for STP and establishes the latter's relationship with the special fan functional.

Theorem 3.16. *In* P_0, STP *is equivalent to the following:*

$$(\forall^{\mathrm{st}} g^2)(\exists^{\mathrm{st}} w^{1^*})\big[(\forall T^1 \leq_1 1)(\exists (\alpha^1 \leq_1 1,\, k^0) \in w)\big((\overline{\alpha} g(\alpha) \not\in T) \\ \to (\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta} i \not\in T)\big)\big]. \tag{3.3}$$

Furthermore, P_0 *proves* $(\exists^{\mathrm{st}}\Theta)\mathsf{SCF}(\Theta) \to \mathsf{STP}$.

Proof. First of all, STP is easily seen to be equivalent to

$$(\forall T^1 \leq_1 1)\big[(\forall^{\mathrm{st}} n)(\exists \beta^0)(|\beta| = n \wedge \beta \in T) \to (\exists^{\mathrm{st}} \alpha^1 \leq_1 1)(\forall^{\mathrm{st}} n^0)(\overline{\alpha} n \in T)\big], \tag{3.4}$$

and this equivalence may also be found in [`?samGH`, Theorem 3.2]. For completeness, we first prove the equivalence $\mathsf{STP} \leftrightarrow$ (**??**). Assume STP and apply overspill to $(\forall^{\mathrm{st}} n)(\exists \beta^0)(|\beta| = n \wedge \beta \in T)$ to obtain $\beta_0^0 \in T$ with nonstandard length $|\beta_0|$. Now apply STP to $\beta^1 := \beta_0 * 00\ldots$ to obtain a *standard* $\alpha^1 \leq_1 1$ such that $\alpha \approx_1 \beta$ and hence $(\forall^{\mathrm{st}} n)(\overline{\alpha} n \in T)$. For the reverse direction, let f^1 be a binary sequence, and define a binary tree T_f which contains all initial segments of f. Now apply (**??**) for $T = T_f$ to obtain STP.

For the implication (**??**)$\to$(**??**), note that (**??**) implies for all standard g^2

$$(\forall T^1 \leq_1 1)(\exists^{\mathrm{st}} (\alpha^1 \leq_1 1,\, k^0)\big[(\overline{\alpha} g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta} i \not\in T)\big], \tag{3.5}$$

which in turn yields, by bringing all standard quantifiers inside again, that:

$$(\forall T \leq_1 1)\big[(\exists^{\mathrm{st}} g^2)(\forall^{\mathrm{st}} \alpha \leq_1 1)(\overline{\alpha} g(\alpha) \not\in T) \to (\exists^{\mathrm{st}} k)(\forall \beta \leq_1 1)(\overline{\beta} k \not\in T)\big], \tag{3.6}$$

To obtain (**??**) from (**??**), apply $\mathsf{HAC}_{\mathsf{int}}$ to $(\forall^{\mathrm{st}} \alpha^1 \leq_1 1)(\exists^{\mathrm{st}} n)(\overline{\alpha} n \not\in T)$ to obtain standard $\Psi^{1 \to 0^*}$ such that $(\forall^{\mathrm{st}} \alpha^1 \leq_1 1)(\exists n \in \Psi(\alpha))(\overline{\alpha} n \not\in T)$, and defining $g(\alpha) := \max_{i < |\Psi|} \Psi(\alpha)(i)$ we obtain g as in the antecedent of (**??**). The previous implies

$$(\forall T^1 \leq_1 1)\big[(\forall^{\mathrm{st}} \alpha^1 \leq_1 1)(\exists^{\mathrm{st}} n)(\overline{\alpha} n \not\in T) \to (\exists^{\mathrm{st}} k)(\forall \beta \leq_1 1)(\overline{\beta} i \not\in T)\big], \tag{3.7}$$

which is the contraposition of (**??**), using classical logic. For the implication (**??**) $\to$ (**??**), consider the contraposition of (**??**), i.e. (**??**), and note that the latter implies (**??**). Now push all standard quantifiers outside as follows:

$$(\forall^{\mathrm{st}} g^2)(\forall T^1 \leq_1 1)(\exists^{\mathrm{st}} (\alpha^1 \leq_1 1,\ k^0)\big[(\overline{\alpha} g(\alpha) \not\in T) \to (\forall \beta \leq_1 1)(\exists i \leq k)(\overline{\beta} i \not\in T)\big],$$

and applying idealisation I yields (**??**). The equivalence involving the latter also immediately establishes the second part of the theorem. $\square$

In light of the previous theorem, the 'nonstandard' provenance of the special fan functional becomes clear. This functional was actually discovered during the study of the Gandy-Hyland functional in Nonstandard Analysis in [`?samGH`, §3-4].

Thirdly, we establish the aforementioned non-implications and related results.

Theorem 3.17. *The system* $\mathsf{P} + \Pi^0_1$-TRANS *does not prove* STP.

Proof. Suppose $\mathsf{P} + \Pi^0_1$-$\mathsf{TRANS} \vdash \mathsf{STP}$ and note that Π^0_1-TRANS is equivalent to

$$(\forall^{\mathrm{st}} f^1)(\exists^{\mathrm{st}} n^0)\big[(\exists m) f(m) = 0 \to (\exists i \leq n) f(i) = 0\big], \tag{3.8}$$

by contraposition. Then the implication 'Π^0_1-TRANS $\rightarrow$ STP' becomes

$$(\forall^{\rm st} f^1)(\exists^{\rm st} n^0)A(f,n) \rightarrow (\forall^{\rm st} g^2)(\exists^{\rm st} w^{1^*})B(g,w) \tag{3.9}$$

where B is the formula in square brackets in (**??**) and where A is the formula in square brackets in (**??**). We may strengthen the antecedent of (**??**) as follows:

$$(\forall^{\rm st} h^2)\big[(\forall^{\rm st} f^1)A(f,h(f)) \rightarrow (\forall^{\rm st} g^2)(\exists^{\rm st} w^{1^*})B(g,w)\big], \tag{3.10}$$

In turn, we may strengthen the antecedent of (**??**) as follows:

$$(\forall^{\rm st} h^2)\big[(\forall f^1)A(f,h(f)) \rightarrow (\forall^{\rm st} g^2)(\exists^{\rm st} w^{1^*})B(g,w)\big], \tag{3.11}$$

Bringing out the standard quantifiers, we obtain

$$(\forall^{\rm st} h^2, g^2)(\exists^{\rm st} w^{1^*})\big[(\forall f^1)A(f,h(f)) \rightarrow B(g,w)\big], \tag{3.12}$$

and applying Corollary **??** to '$\mathsf{P} \vdash$ (**??**)', we obtain a term t such that

$$(\forall h^2, g^2)(\exists w^{1^*} \in t(h,g))\big[(\forall f^1)A(f,h(f)) \rightarrow B(g,w)\big], \tag{3.13}$$

is provable in $\mathsf{E\text{-}PA}^{\omega*}$. Clearly, the antecedent of (**??**) expresses that h is Feferman's search functional (μ^2). Furthermore, it is straightforward to define Θ as in $\mathsf{SCF}(\Theta)$ in terms of $t(h,\cdot)$; However, this implies that the special fan functional is computable in (μ^2) via a term from Gödel's T. This contradicts Corollary **??**, and we are done. □

Corollary 3.18. *The system* $\mathsf{P} + \mathsf{WKL}^{\rm st}$ *does not prove* STP*. The same holds for* $\mathsf{P} + \varphi + \Pi^0_1\text{-}\mathsf{TRANS}$*, where* φ *is any internal sentence such that the latter system is consistent.*

Proof. For the first part, note that $\Pi^0_1\text{-}\mathsf{TRANS} \rightarrow (\mu^2)^{\rm st} \rightarrow \mathsf{WKL}^{\rm st}$ where the second implication follows from the usual proof of $\mathsf{ACA}_0 \rightarrow \mathsf{WKL}_0$ relative to 'st', and where the first implication follows from applying $\mathsf{HAC}_{\mathsf{int}}$ to (**??**) (and taking the maximum of all outputs of the resulting functional).

For the second part, suppose $\mathsf{P} + \varphi + \Pi^0_1\text{-}\mathsf{TRANS} \vdash \mathsf{STP}$ and apply Theorem **??** to $\mathsf{P} + \varphi + \Pi^0_1\text{-}\mathsf{TRANS} \vdash$ (**??**) to obtain a term of Gödel's T which computes the special fan functional in terms of the Turing jump functional. This contradiction immediately yields the second part of the theorem. □

Corollary 3.19. *The system* $\mathsf{P} + \Pi^1_1\text{-}\mathsf{TRANS}$ *does not prove* STP*.*

Proof. Follows from Corolllary **??** in the same way as the theorem. By way of a sketch, suppose that $P \vdash \Pi^1_1\text{-}\mathsf{TRANS} \rightarrow \mathsf{STP}$. Then □

In the same way, Corollary **??** yields that *Transfer* limited to Π^1_k-formulas cannot imply STP. Indeed, the 'comprehension functional' for Π^k_1-formulas has type two, and hence does not compute the special fan functional by Corollary **??**.

Despite the above negative results, we have the following conservation result.

Theorem 3.20. *The systems* $\mathsf{RCA}^\omega_0 + (\exists\Theta)\mathsf{SCF}(\Theta)$ *and* $\mathsf{P}_0 + \mathsf{STP}$ *are conservative over* $\mathsf{RCA}^2_0 + \mathsf{WKL}$ *for sentences in the latter's second-order language.*

Proof. For the first system, by [`?samGH`, Cor. 3.4] and Theorem **??**, the special fan functional can be defined in terms of Φ as in $\mathsf{MUC}(\Phi)$. By Theorem **??**, the first conservation result is now immediate. For the second conservation result, let φ be a (necessarily internal) sentence in the language of RCA^2_0+WKL. If P_0+$\mathsf{STP} \vdash \varphi$, then

$\mathsf{P}_0 \vdash (\exists^{\mathrm{st}}\Theta)\mathsf{SCF}(\Theta) \to \varphi$ by the second part of Theorem **??**. Applying Corollary **??** to $\mathsf{P}_0 \vdash (\forall^{\mathrm{st}}\Theta)(\mathsf{SCF}(\Theta) \to \varphi)$ yields $\mathsf{RCA}_0^\omega \vdash (\forall\Theta)(\mathsf{SCF}(\Theta) \to \varphi)$. □

Finally, we also establish the 'reverse' direction of the fact that Theorem **??** gives rise to Theorem **??**.

Theorem 3.21. *Suppose* $\mathsf{P}+\varphi \nvdash \Pi_1^0\text{-}\mathsf{TRANS} \to \mathsf{STP}$ *for some internal sentence* φ. *Then for every term* t *from Gödel's* T, $\mathsf{E}\text{-}\mathsf{PA}^{\omega*}+\varphi$ *does not prove that* t *expresses a special fan functional in terms of the special fan functional.*

Proof. X □

3.3. **Compactness.** Anti-specker from $(\forall x \in [0,1])(\exists^{\mathrm{st}} y)(x \approx y)$

NSA and INT share notion of compactness!

One of the various definitions of compactness states that 'If a point z is bounded away from a compact space X, z is uniformly bounded away from X'. We consider the following version of this kind of compactness where $\kappa^{2\to(0\times 1^*)}$ provides the uniform bound (namely $\kappa(g)(1)$) for z and the unit interval, and a finite sequence of reals in the unit interval (namely $\kappa(g)(2)=(y_0,y_1,\dots,y_k)$) which have to be bounded away from z as given by g.

$$(\forall g^2, z\in\mathbb{R})\Big[\big[(\forall y\in(\kappa(g)(2)\cap[0,1])\big)(|y-z|>_{\mathbb{R}}\tfrac{1}{g(y)}) \qquad (\mathsf{COMP}([0,1],\kappa))$$
$$\to(\forall x\in[0,1])(|x-z|>_{\mathbb{R}}\tfrac{1}{\kappa(g)(1)})\big]\Big],$$

3.4. **A special case of the special fan functional.** In this section, we consider the principle *weak weak König's lemma*, WWKL for short, first introduced in [?yussie] and defined as in Definition **??**. We shall study the nonstandard counterpart of WWKL as in Definition **??**, and the associated weak version of the special fan functional.

First of all, we have the following definitions.

Definition 3.22. [Weak weak König's lemma]

(1) For a binary tree, define $\mu(T):=\lim_{n\to\infty}\frac{\{\sigma\in T:|\sigma|=n\}}{2^n}$.
(2) For a binary tree T, define '$\mu(T)>_{\mathbb{R}} a^1$' as $(\exists k^0)(\forall n^0)\big(\frac{\{\sigma\in T:|\sigma|=n\}}{2^n}\geq a+\frac{1}{k}\big)$.
(3) We define WWKL as
$$(\forall T\leq_1 1)\big[\mu(T)>_{\mathbb{R}}0\to(\exists\beta\leq_1 1)(\forall m)(\overline{\beta}m\in T)\big],$$
(4) Define WFAN as the classical contraposition of WWKL.

Although WWKL is not part of the 'Big Five' systems of RM, there are *some* equivalences involving the former ([?yussie, ?sayo, ?yuppie, ?simpson2]). The nonstandard counterpart of WWKL was first introduced in [?kei1] as follows:

Definition 3.23. [Nonstandard WWKL] Let T be a binary tree.

(1) Define '$\mu(T)\gg a^1$' as $(\exists^{\rm st}k^0)(\forall^{\rm st}n^0)\big(\frac{\{\sigma\in T:|\sigma|=n\}}{2^n}\geq a+\frac{1}{k}\big)$.
(2) Define '$\mu(T)\approx 0$' as $(\forall^{\rm st}k^0)(\exists^{\rm st}n^0)\big(\frac{\{\sigma\in T:|\sigma|=n\}}{2^n}<\frac{1}{k}\big)$.
(3) Define the 'Loeb measure of T' as $L_N(T):=\frac{\{\sigma\in T:|\sigma|=N\}}{2^N}$.
(4) Define LMP as
$$(\forall T\leq_1 1)\big[(\exists N\in\Omega)(L_N(T)\gg 0)\to(\exists^{\rm st}\beta\leq_1 1)(\forall^{\rm st}n)(\overline{\beta}n\in T)\big]. \tag{3.14}$$

Clearly, WWKL and LMP are weakenings of WKL and STP. Similarly, we introduce the following weak version of the special fan functional.

Definition 3.24. [Weak special fan functional] We define $\mathsf{WCF}(\Lambda)$ for $\Lambda^{(2\to(1\times 1^*))}$:
$$(\forall k^0,g^2,T^1\leq_1 1)\big[(\forall\alpha\in\Lambda(g,k)(2))(\overline{\alpha}g(\alpha)\not\in T)\to(\exists n\leq\Lambda(g,k)(1))(L_n(T)\leq\tfrac{1}{k})\big].$$
Any functional Λ satisfying $\mathsf{WCF}(\Lambda)$ is referred to as a *weak special fan functional*.

We first obtain a normal form for LMP as follows.

Theorem 3.25. *In P_0, the principle LMP is equivalent to:*
$$(\forall^{\rm st}g^2,k^0)(\exists^{\rm st}w^{1^*}) \tag{3.15}$$
$$\big[(\forall T\leq_1 1)(\exists(\alpha\leq_1 1,n)\in w)\big(\overline{\alpha}g(\alpha)\not\in T\to L_n(T)\leq\tfrac{1}{k}\big)\big].$$
Furthermore, P_0 proves $(\exists^{\rm st}\Lambda)\mathsf{WCF}(\Lambda)\to\mathsf{LMP}$.

Proof. Analogous to the proof of Theorem **??**. □

We have the following expected theorem.

Theorem 3.26. *Any functional Λ^3 as in* $\mathsf{WCF}(\Lambda)$ *is not computable in* $(\exists^2)$ (*or any type two functional*).

Proof. Analogous to the proof of Theorem **??**. In fact, the only required modification is that tree T_0 in the proof of the latter just needs to satisfy $\mu(T) >_{\mathbb{R}} 0$. □

Corollary 3.27. *The system* $\mathsf{P} + \Pi_1^0\text{-}\mathsf{TRANS} + \mathsf{WWKL}^{\text{st}}$ *does not prove* LMP.

Proof. Similar to Theorem **??**; note that $\Pi_1^0\text{-}\mathsf{TRANS} \rightarrow (\exists^2)^{\text{st}} \rightarrow \mathsf{WWKL}^{\text{st}}$. □

As to the logical strength of $\mathsf{WCF}(\Lambda)$, we will establish that the latter yields a conservative extension of WWKL, similar to Theorems **??** and **??**. To this end, consider the following uniform and nonstandard principles.

$$(\forall Y^2, k^0)(\forall n \geq \kappa(Y)(k))\big(\tfrac{|\{\tau \leq_{0^*} 1 : |\tau| = n \wedge Y(\hat{\tau}) > n\}|}{2^n} \leq \tfrac{1}{k}\big), \tag{$\mathsf{PUC}(\kappa)$}$$

$$(\forall^{\text{st}} Y^2)(\forall N \in \Omega)\big(\tfrac{|\{\tau \leq_{0^*} 1 : |\tau| = N \wedge Y(\hat{\tau}) > N\}|}{2^N} \approx 0\big), \tag{$\mathsf{PUC}_{\mathsf{ns}}$}$$

where $\hat{\sigma} := \sigma * 00\ldots$ for a finite sequence σ. Intuitively speaking, $\mathsf{PUC}_{\mathsf{ns}}$ expresses that the probability that Y is nonstandard at some sequence is infinitesimal. We refer to κ as in $\mathsf{PUC}(\kappa)$ as the *weak intuitionistic fan functional*.

We have the following results similar to Theorems **??** and **??** .

Theorem 3.28. *The system* $\mathsf{RCA}_0^\omega + (\exists \kappa^3)\mathsf{PUC}(\kappa)$ *is a conservative extension of* $\mathsf{RCA}_0^2 + \mathsf{WWKL}$ (*for the latter's second-order language*).

Proof. A proof may be found in Theorem **??** of the Appendix. □

Theorem 3.29. *The system* P_0 *proves* $\mathsf{PUC}_{\mathsf{ns}} \rightarrow \mathsf{LMP}$. *From this proof, a term t can be extracted such that* RCA_0^ω *proves* $(\forall \kappa)\big(\mathsf{PUC}(\kappa) \rightarrow \mathsf{WCF}(t(\kappa))\big)$.

Proof. For the first part, consider the contraposition of LMP as follows:

$$(\forall T \leq_1 1)\big[(\forall^{\text{st}} \beta \leq_1 1)(\exists^{\text{st}} n)(\overline{\beta} n \notin T) \rightarrow (\forall N \in \Omega)(L_N(T) \approx 0)\big].$$

If the antecedent $(\forall^{\text{st}} \beta \leq_1 1)(\exists^{\text{st}} n)(\overline{\beta} n \notin T)$ holds, apply $\mathsf{HAC}_{\mathsf{int}}$ to obtain standard Y^2 such that $(\forall^{\text{st}} \beta \leq_1 1)(\exists n \leq Y(\beta))(\overline{\beta} n \notin T)$. By $\mathsf{PUC}_{\mathsf{ns}}$, we have that $\frac{1}{2^N}|\{\tau \leq_{0^*} 1 : |\tau| = N \wedge Y(\hat{\tau}) > N\}| \approx 0$ for nonstandard N. By definition, we also have the following for nonstandard N:

$$1 \approx \tfrac{|\{\tau \leq_{0^*} 1 : |\tau| = N \wedge Y(\hat{\tau}) \leq N\}|}{2^N} \leq \tfrac{|\{\tau \leq_{0^*} 1 : |\tau| = N \wedge \tau \notin T\}|}{2^N} = 1 - L_N(T),$$

which yields that $L_N(T) \approx 0$, and LMP follows. For the second part of the proof, note that LMP has an equivalent normal form (**??**), while an (equivalent) normal form for $\mathsf{PUC}_{\mathsf{ns}}$ is as follows:

$$(\forall^{\text{st}} Y^2, k^0)(\exists^{\text{st}} M^0)(\forall N \geq M)\big(\tfrac{|\{\tau \leq_{0^*} 1 : |\tau| = N \wedge Y(\hat{\tau}) > N\}|}{2^N} \leq \tfrac{1}{k}\big), \tag{3.16}$$

which one easily obtains using underspill. Now proceed as in the proof of Theorem **??** to obtain the relative computability result from the theorem. In particular, bring (**??**) $\rightarrow$ (**??**) into normal form as in the aforementioned proof and apply Theorem **??** to obtain the desired term. □

The nonstandard proof in the theorem is rather trivial. We have the following immediate corollary.

Corollary 3.30. *The systems* $\mathsf{RCA}_0^\omega+(\exists\Lambda^3)\mathsf{WCF}(\Lambda)$ *and* $\mathsf{P}_0+\mathsf{LMP}$ *are conservative extensions of* $\mathsf{RCA}_0^2+\mathsf{WWKL}$ *(for the latter's second-order language).*

The following corollary establishes the nonstandard version of the non-implication $\mathsf{WWKL}\not\rightarrow\mathsf{WKL}$, which was first proved in [`?yussie`].

Corollary 3.31. *The system* $\mathsf{P}_0+\mathsf{LMP}$ *does not prove* STP.

Proof. We proceed similar to Theorem **??**. Suppose $\mathsf{P}+\mathsf{LMP}\vdash\mathsf{STP}$; in the same way as for the aforementioned theorem, we obtain some term t such that RCA_0^ω proves $(\forall\Lambda)(\mathsf{WCF}(\Lambda)\rightarrow\mathsf{SCF}(t(\Lambda)))$. In particular $\mathsf{RCA}_0^\omega+(\exists\Lambda)\mathsf{WCF}(\Lambda)$ proves $(\exists\Theta)\mathsf{SCF}(\Theta)$. Since $(\exists\Theta)\mathsf{SCF}(\Theta)\rightarrow\mathsf{WKL}$ over RCA_0^ω, we have that $\mathsf{RCA}_0^\omega+(\exists\Lambda)\mathsf{WCF}(\Lambda)$ proves WKL, contradicting Corollary **??**. $\square$

The following corollary is now straightforward.

Corollary 3.32. *For any term* t *of Gödel's* T, $\mathsf{E\text{-}PA}^{\omega*}\nvdash(\forall\Lambda)(\mathsf{WCF}(\Lambda)\rightarrow\mathsf{SCF}(t(\Lambda)))$.

The following theorem generalises the previous result.

Theorem 3.33. *Any* Θ^3 *as in* $\mathsf{SCF}(\Theta)$ *is not computable in* Λ *such that* $\mathsf{WCF}(\Lambda)$.

Proof. ? $\square$

Finally, we discuss a version of the weak special fan functional more similar to WWKL. To bring out the similarity to the latter, we use write '$(\forall\alpha^1\in\gamma)\varphi(\alpha)$' instead of '$(\forall n^0)\varphi(\gamma(n))$' for $\gamma^{0\rightarrow1}$.

Definition 3.34. [Weak weak special fan functional] Define $\mathsf{WWF}(\zeta)$ for $\zeta^{(2\rightarrow(1\times(0\rightarrow1)))}$:

$$(\forall g^2,T^1\leq_1 1)\big[(\forall\alpha\in\zeta(g)(2))(\overline{\alpha}g(\alpha)\notin T)\rightarrow(\forall k^0)(\exists n\leq\zeta(g)(1)(k))(L_n(T)\leq\tfrac{1}{k})\big].$$

Any functional ζ as in $\mathsf{WWF}(\zeta)$ is referred to as a *weak weak special fan functional*.

Theorem 3.35. *There are terms* s,t *such that* $\mathsf{E\text{-}PA}^{\omega*}$ *proves* $(\forall\Lambda)(\mathsf{WCF}(\Lambda)\rightarrow\mathsf{WWF}(t(\Lambda)))$ *and* $(\forall\zeta,\mu)\big((\mathsf{WWF}(\zeta)\wedge\mathsf{MU}(\mu))\rightarrow\mathsf{WCF}(s(\zeta,\mu))\big)$.

Proof. For the first part of the proof, define $t(\Lambda)(g)(1):=(\lambda k)\Lambda(g,k)(1)$ and $t(\Lambda)(g)(2):=(\lambda k)\Lambda(g,k)(2)$. For the second part of the proof, we prove that $((\exists^{\rm st}\zeta)\mathsf{WWF}(\zeta)\wedge\Pi_1^0\text{-}\mathsf{TRANS})\rightarrow\mathsf{LMP}$ in P. Applying Theorem **??** to this implication as in Theorem **??** yields the required term s. Now, for standard ζ the formula $\mathsf{WWF}(\zeta)$ implies, since standard inputs yield standard outputs, that

$$(\forall^{\rm st}g^2)(\forall T^1\leq_1 1)\big[(\forall\alpha\in\zeta(g)(2))(\overline{\alpha}g(\alpha)\notin T)\rightarrow(\forall^{\rm st}k^0)(\exists^{\rm st}n)(L_n(T)\leq\tfrac{1}{k})\big].$$

Thanks to Π_1^0-TRANS, we may replace the antecedent by $(\forall^{\rm st}\alpha\in\zeta(g)(2))(\overline{\alpha}g(\alpha)\notin T)$, which is implied by $(\forall^{\rm st}\alpha\leq_1 1)(\exists^{\rm st}n^0)(\overline{\alpha}g(\alpha)\notin T)$, and LMP follows. $\square$

TEST2

Appendix A. Conservation results

In this section, we provide proof for the conservation results in Theorems **??** and **??**. To this end, we require some definitions, starting with the notion of *associate* (called 'code' in Reverse Mathematics; see [`?kohlenbach4`, §4]).

Notation A.1. It is customary to define $\alpha(\sigma)$ for α^1 and σ^{0^*} by $\alpha(\pi(\sigma))$ where $\pi^{0^*\rightarrow0}$ is some fixed function coding finite sequences into numbers. Similarly, we define $\tau(\sigma)$ for τ^{0^*},σ^{0^*} as $1+\tau(\pi(\sigma))$ if $\pi(\sigma)<|\sigma|$, and zero otherwise.

Definition A.2. [Associate of a continuous functional] The sequence α^1 is an *associate* for the continuous type two functional Y^2 if

(1) $(\forall\beta^1)(\exists n)\big(\alpha(\overline{\beta}n) > 0\big)$,
(2) $(\forall\beta^1, m^0)(\alpha(\overline{\beta}m) > 0 \to Y(\beta) + 1 = \alpha(\overline{\beta}n))$.

We also define the notion of associate independently, i.e. without referring to the functional it is representing.

Definition A.3. [Associate] The sequence α^1 is an *associate* if

(1) $(\forall\beta^1)(\exists n)\big(\alpha(\overline{\beta}n) > 0\big)$,
(2) $(\forall\sigma_0^{0^*}, \sigma_1^{0^*})(\alpha(\sigma_0) > 0 \wedge \sigma_0 \preceq \sigma_1 \to \alpha(\sigma_0) =_0 \alpha(\sigma_1))$.

Note that $\sigma \preceq \tau$ if $|\sigma| \leq |\tau| \wedge (\forall n < |\sigma|)(\sigma(n) = \tau(n))$, i.e. σ is an initial segment of τ. The second condition is also referred to as α being a 'neighbourhood function'.

Theorem A.4. *The system* $\mathsf{RCA}_0^\omega + (\exists\Omega^3)\mathsf{MUC}(\Omega)$ *is a conservative extension of* $\mathsf{RCA}_0^2 + \mathsf{WKL}$.

Proof. The theorem is listed in [?kohlenbach2, Prop. 3.15] and Kohlenbach states that for its proof, one can adapt the proof of [?troelstra1, Theorem 2.6.6, p. 141]. We discuss the latt'er proo'f and its modification in more detail.

Now, [?troelstra1, Theorem 2.6.6] essentially states that every model $\mathcal{U}$ of the 'full' fan theorem can be extended to a model $\mathsf{ECF}(\mathcal{U})$ which includes a fan functional as in $\mathsf{MUC}(\Omega)$. The 'full' fan theorem is an intuitionistic principle defined as follows (See [?troelstra1, 1.9.24]): For every formula A, we have that

$$(\forall\alpha \leq_1 1)(\exists x^0)A(\alpha, x) \to (\exists z^0)(\forall\alpha^1 \leq_1 1)(\exists y^0)(\forall\beta \leq_1 1)(\overline{\alpha}z =_0 \overline{\beta}z \to A(\beta, y)).$$

As an aside, the 'full' fan theorem implies[6] FAN, i.e. the classical contraposition of WKL, but contradicts classical mathematics[7].

From the proofs of [?troelstra1, Theorems 2.6.4 and 2.6.6], it is clear that the initial model $\mathcal{U}$ only needs to satisfy FAN, the classical contraposition of WKL, and *not the 'full' fan theorem*. In paticular, any model $\mathcal{U}$ satisfying FAN can be extended to a model satisfying the existence of a fan functional; This sketch is the conceptual core of the proof of Theorem **??**.

Secondly, we discuss *how* the fan functional is represented in $\mathsf{ECF}(\mathcal{U})$. For an associate α^1, $(\forall\beta \leq_1 1)(\exists n^0)(\alpha(\overline{\beta}n) > 0)$ implies $(\exists k^0)(\forall\beta \leq_1 1)(\exists n^0 \leq k)(\alpha(\overline{\beta}n) > 0)$ by FAN, and hence a fan functional φ_{uc} can be defined as:

$$\varphi_{\text{uc}}(\alpha) := (\mu k^0)(\forall\beta \leq_1 1)(\exists n^0 \leq k)(\alpha(\overline{\beta}n) > 0).$$

It remains to be shown that φ_{uc} is represented by an object in $\mathsf{ECF}(\mathcal{U})$. This representation is called $[\varphi_{\text{uc}}]'$ in the proof of [?troelstra1, Theorem 2.6.4] and defined as:

$$[\varphi_{\text{uc}}]'(\sigma^{0^*}) := (\mu m \leq z)(\forall\tau^{0^*}, \rho^{0^*} \leq_{0^*} 1)\big((|\tau| = |\rho| = z \wedge \overline{\tau}m = \overline{\rho}m) \to \sigma(\tau) = \sigma(\rho) > 0\big)$$

in case $(\exists k \leq |\sigma|)(\forall\theta^{0^*} \leq_{0^*} 1)(|\theta| = k \to \sigma(\theta) > 0)$ and z is the least such number; otherwise, the number $[\varphi_{\text{uc}}]'(\sigma^{0^*})$ is defined as zero. Finally, it is straightforward to verify that $[\varphi_{\text{uc}}]'$ is extensional in the sense required by $\mathsf{ECF}(\mathcal{U})$. $\square$

[6]Take $A(\alpha, n)$ to be $\overline{\alpha}n \notin T$ for a binary tree T, and FAN follows.
[7]Take $A(\alpha, n)$ to be $(\forall\gamma \leq_1 1)(\overline{\alpha}n = \overline{\beta}n \to Y(\alpha) = Y(\beta))$ for any Y^2, and note that all type two functionals are thus continuous on Cantor space.

Based on the previous proof, we now obtain the following theorem.

Theorem A.5. *The system* $\mathsf{RCA}_0^\omega + (\exists\kappa^3)\mathsf{PUC}(\kappa)$ *is a conservative extension of* $\mathsf{RCA}_0^2 + \mathsf{WWKL}$.

Proof. For an associate α^1, we have $(\forall\beta \leq_1 1)(\exists n^0)(n \geq \alpha(\overline{\beta}n) > 0)$, where the inequality '$\geq$' follows from its status as a neighbourhood function. Now define a binary tree T by $\sigma \in T_0 \leftrightarrow [\alpha(\sigma) = 0 \vee \alpha(\sigma) > |\sigma|]$. By WFAN, we have $(\forall k^0)(\exists n^0)\big(\frac{|\{\sigma\in T_0:|\sigma|=n\}|}{2^n} \leq \frac{1}{k}\big)$, and hence we define the functional φ_{puc} as

$$\varphi_{\text{puc}}(\alpha, k) := (\mu n^0)\big(\tfrac{|\{\sigma:|\sigma|=n\wedge[\alpha(\sigma)=0\vee\alpha(\sigma)>|\sigma|]\}|}{2^n} \leq \tfrac{1}{k}\big),$$

which is as required for the weak intuitionistic fan functional. It remains to be shown that φ_{puc} is represented by an object in $\mathsf{ECF}(\mathcal{U})$. We shall denote this representation by $[\varphi_{\text{puc}}]$, following the proof of Theorem **??**.

$$[\varphi_{\text{puc}}](\tau, k) := (\mu n^0 \leq |\tau|)\big(\tfrac{|\{\sigma:|\sigma|=n\wedge[\alpha(\sigma)=0\vee\alpha(\sigma)>|\sigma|]\}|}{2^n} \leq \tfrac{1}{k}\big).$$

if such n exists, and zero otherwise. □

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