CT Assignment-5

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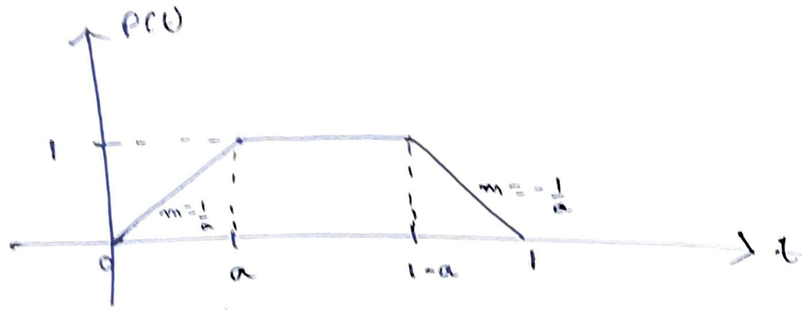
1)

$$p(t) = \begin{cases} \frac{t}{a} & 0 \le t \le a \\ 1 & a \le t \le 1-a \\ \frac{1-t}{a} & 1-a \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$a < \frac{1}{2}$

a)

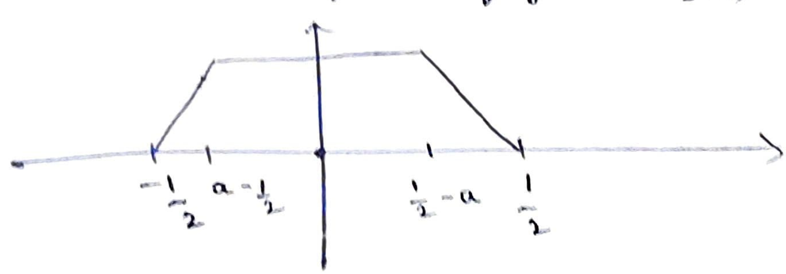


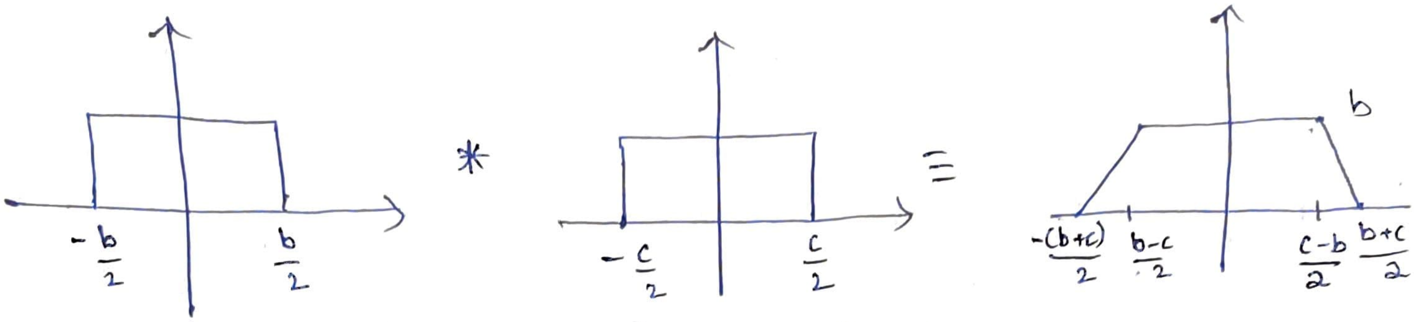
$P(f) = ?$

We can directly calculate P(f) from formula but a better way is to use the convolution property.

We know that we get a $\triangle$le when we convolve two rectangles of same size. Different size rectangle convolution ~~is~~ gives us a trapezium

Consider a shifted figure g(t)





Smaller rectangle   Bigger rectangle

Comparing the shifted figure $\frac{-(b+c)}{2} = -\frac{1}{2}$

$$\frac{b-c}{2} = a - \frac{1}{2}$$

$\Rightarrow$ $b+c=1$ & $b-c = 2a-1$

$\Rightarrow$ $2b = 2a$

$\Rightarrow$ $b = a$

$\Rightarrow$ $c = 1-a$

shifted figure $g(t) = \frac{1}{b}\left(I_{[-\frac{b}{2}, \frac{b}{2}]} * I_{[-\frac{c}{2}, \frac{c}{2}]}(t)\right)$ $\frac{1}{b}$ is for scal the amplitude

We know ~~shifted~~ convolution in time is product in frequency & $I_{[-\frac{b}{2}, \frac{b}{2}]}(t) \Longleftrightarrow b\,\mathrm{sinc}(bf)$

$\Rightarrow$ $G(f) = \frac{1}{b} \cdot \mathrm{sinc}(bf)\,\mathrm{sinc}(cf) \cdot bc$

$= c\,\mathrm{sinc}(bf)\,\mathrm{sinc}(cf)$

$= (1-a)\,\mathrm{sinc}(af)\,\mathrm{sinc}((1-a)f)$

Since $p(t)$ is shifted by $\frac{1}{2}$ towards left to get $g(t)$,

$P(f) = G(f)\, e^{-j2\pi \cdot \frac{1}{2} f}$

$$\Rightarrow \boxed{P(f) = (1-a)\,\mathrm{sinc}(af)\,\mathrm{sinc}((1-a)f)\, e^{-j\pi f}}$$

b)

$u(t) = \Sigma_n b[n] p(t-n)$. where $b[n]$ takes values independently and with equal probability in 4-PAM $\{\pm 1, \pm 3\}$

PSD = ?

We know that $PSD = \frac{|P(f)|^2}{T} \sigma_b^2$ where $\sigma_b^2 = \overline{|b[n]|^2}$

$T = 1$ (given from def^n of $p(t)$)

$$\sigma_b^2 = \frac{1}{4}\cdot(-1)^2 + \frac{1}{4}\cdot(1)^2 + \frac{1}{4}(-3)^2 + \frac{1}{4}(3)^2 = 5$$

$$\therefore PSD = 5\left(\left| (1-a)\, \text{sinc}(af)\, \text{sinc}((1-a)f)\, \underbrace{e^{-j\pi f}}_{\text{Modulus is } 1} \right|\right)^2$$

$$\boxed{PSD = 5(1-a)^2 \,\text{sinc}^2(af)\, \text{sinc}^2((1-a)f)}$$

c) 95% containing bandwidth

Assume bandwidth to be 2B

$$\int_{-B}^{B} PSD\, df = 0.95 \int_{-\infty}^{\infty} PSD\, df$$

$$\Rightarrow \int_{-B}^{B} |P(f)|^2\, df = 0.95 \int_{-\infty}^{\infty} |P(f)|^2\, df$$

From Parsevalls Theorem

$$= 0.95 \int_{-\infty}^{\infty} |p(t)|^2\, dt$$

$$= 0.95\left(\int_0^a \left(\frac{t}{a}\right)^2 dt + \int_a^{1-a} 1\, dt + \int_{1-a}^{1} \left(\frac{1-t}{a}\right)^2 dt \right)$$

$$= 0.95 \left(\frac{1}{a^2} \frac{t^3}{3} \Big|_0^a + t \Big|_a^{1-a} + \left(\frac{t^3}{3a^2} + \frac{t}{a^2} - \frac{t^2}{a^2} \right) \Big|_{1-a}^{1} \right)$$

$$= 0.95 \left(\frac{a}{3} + 1 - 2a + \frac{1}{a^2} \left(\frac{1}{3} - \left(\frac{(1-a)^3}{3} + (1-a) - (1-a)^2 \right) \right) \right)$$

$$= 0.95 \left(1 - \frac{5a}{3} + \frac{1}{a^2} \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} + \cancel{a} - \cancel{a^2} + \frac{a^3}{3} + \cancel{1} - \cancel{a} - \cancel{1} + \cancel{a^2} \right) \right)$$

$$= 0.95 \left(1 - \frac{5a}{3} + \frac{a}{3} \right)$$

$$\int_{-B}^{B} |P(f)|^2 \, df = 0.95 \left(1 - \frac{4a}{3} \right)$$

$$\int_{-B}^{B} (1-a)^2 \, \mathrm{sinc}^2(af) \, \mathrm{sinc}^2((1-a)f) \, df = 0.95 \left(1 - \frac{4a}{3} \right)$$

We can manually solve this when we take $\sin f = f - \frac{f^3}{3!}$ approximation.

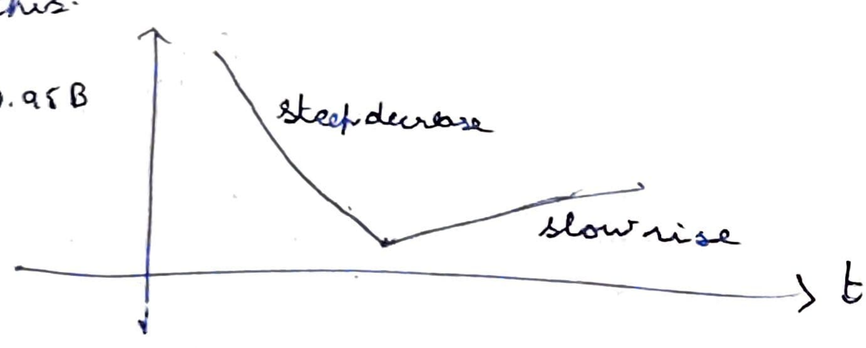
Directly solving this is difficult.

We get an equation in terms of a.

If we fix a, we get B.

Time is 100 ps. $\Rightarrow$ Bandwidth will be in few GHz or tens or hundreds of MHz.

The graph when drawn using approximations using online simulator looks approximately like this.



2)

a) $p(t) = \text{sinc}(at)\,\text{sinc}(bt)$

Say $a < b$

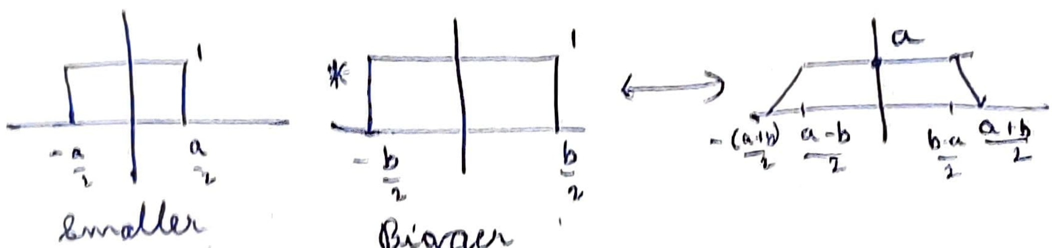
We know $a\,\text{sinc}(at) \longleftrightarrow I_{[-\frac{a}{2}, \frac{a}{2}]}(f)$

$b\,\text{sinc}(bt) \longleftrightarrow I_{[-\frac{b}{2}, \frac{b}{2}]}(f)$

From the convolution property,

$P(f) = \frac{1}{a} I_{[-\frac{a}{2}, \frac{a}{2}]}(f) * \frac{1}{b} I_{[-\frac{b}{2}, \frac{b}{2}]}(f)$

$= \frac{1}{ab} I_{[-\frac{a}{2}, \frac{a}{2}]}(f) * I_{[-\frac{b}{2}, \frac{b}{2}]}(f)$



40 Mbps   M = 16 QAM

50% excess bandwidth

$B_{min} = \frac{R_b}{\eta_B}$

If excessive bandwidth is α, $B = (1+\alpha) B_{min}$

Here, $B_{min} = \frac{40}{\log_2 16} = \frac{40}{4} = 10$

50% $\Rightarrow \alpha = 0.5$

$\Rightarrow B = 1.5 \times B_{min}$

$= 15 = \frac{a+b}{2} - \frac{-(a+b)}{2} = a+b$

$\Rightarrow a+b = 15$ — (1)

We have considered the rectangle pulses to be of width a, b where $b > a$

so $B_{min} = b$

$\Rightarrow b = 10$

Substituting this in (1),

$a = 15 - 10$

$= 5$

$\Rightarrow$ $a = 5\,MHz$; $b = 10\,MHz$; Occupied bandwidth $= 15\,MHz$

b)

16 QAM with 40 Mbps

& 8 PSK with 18 Mbps

$$B_{min_1} = \frac{40}{\log_2 16} = \frac{40}{4} = 10$$

$$B_{min_2} = \frac{18}{\log_2 8} = \frac{18}{3} = 6$$

We need to do Nyquist signalling for both.

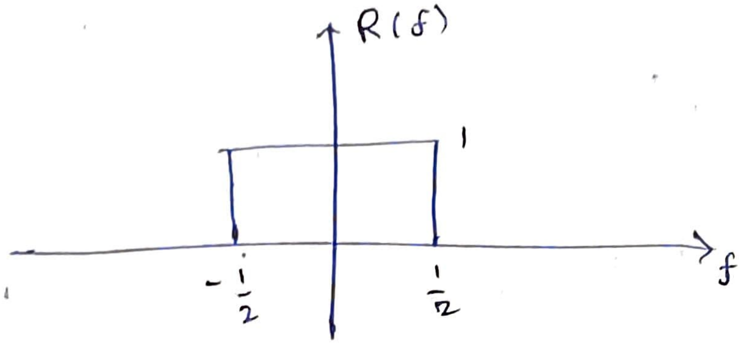
$p(t)$ should have zeros at $\frac{1}{10}$, $\frac{1}{6}$

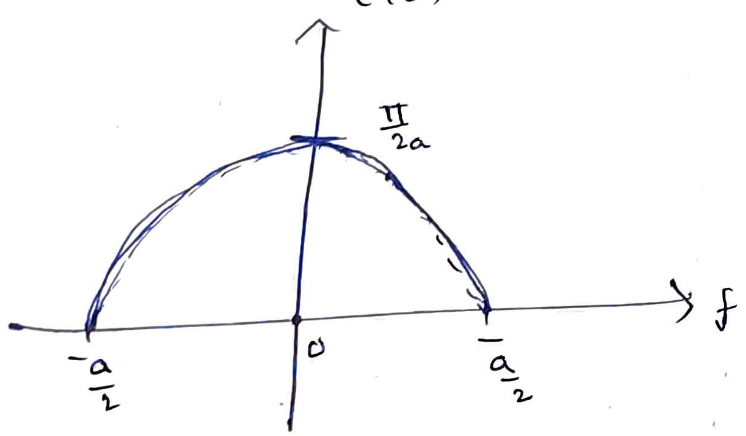
The obvious values that strike to our mind is $a = 6$ & $b = 10$ which make $p(nT)$ zero so for perfect Nyquist, the occupied bandwidth $= 16\,MHz$. Note that we may take multiples of 16 MHz as well but here we tried for the best case

3)

a) $R(f) = I_{[-\frac{1}{2}, \frac{1}{2}]}(f)$ $\quad C(f) = \frac{\pi}{2a} \cos\left(\frac{\pi}{a} f\right) I_{[-\frac{a}{2}, \frac{a}{2}]}$

Assume $0 < a < 1$





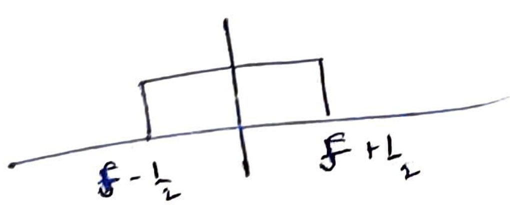
Note that $C(f)$ is symmetric

b) $S(f) = (R * C)(f) = (C * R)(f)$

~~For~~

$$S(f) = \int_{-\infty}^{\infty} C(\tau) R(f - \tau)\, d\tau$$

~~We flip R(f)~~ We flip & shift R(F)



i) For $f+\frac{1}{2} > \frac{a}{2}$ & $f-\frac{1}{2} < -\frac{a}{2}$

$$\Rightarrow f > \frac{a-1}{2} \;\&\; f < \frac{1-a}{2}$$

$$S(f) = \int_{-\frac{a}{2}}^{\frac{a}{2}} 1 \cdot \frac{\pi}{2a} \cos\left(\frac{\pi F}{a}\right) dF$$

$$= \frac{\pi}{2a} \left.\frac{\sin\left(\frac{\pi F}{a}\right)}{\frac{\pi}{a}}\right|_{-\frac{a}{2}}^{\frac{a}{2}}$$

$$= \frac{1}{2}\left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right)$$

$S(f) = 1$ For $f > \frac{a-1}{2}$ & $f < \frac{1-a}{2}$

ii) For $-\frac{a}{2} < f - \frac{1}{2} < \frac{a}{2}$

$$\frac{1-a}{2} < f < \frac{1+a}{2}$$

$$S(f) = \int_{f-\frac{1}{2}}^{\frac{a}{2}} \frac{\pi}{2a} \cos\left(\frac{\pi F}{a}\right) dF$$

$$= \frac{1}{2} \left.\sin\left(\frac{\pi F}{a}\right)\right|_{f-\frac{1}{2}}^{\frac{a}{2}}$$

$$= \frac{1}{2}\left(1 - \sin\left(\frac{\pi\left(f-\frac{1}{2}\right)}{a}\right)\right)$$

For $-\frac{a}{2} < f + \frac{1}{2} < \frac{a}{2}$

$$\frac{-1-a}{2} < f < \frac{a-1}{2}$$

$$S(f) = \int_{-\frac{a}{2}}^{f+\frac{1}{2}} \frac{\pi}{2a} \cos\left(\frac{\pi F}{a}\right) dF$$

$$= \frac{1}{2} \left.\sin\left(\frac{\pi F}{a}\right)\right|_{-\frac{a}{2}}^{f+\frac{1}{2}}$$

$$= \frac{1}{2}\left(\sin\left(\frac{\pi\left(f+\frac{1}{2}\right)}{a}\right) + 1\right)$$

iii) For $f+\frac{1}{2} < -\frac{a}{2}$ & $f-\frac{1}{2} > \frac{a}{2}$

$$f < -\frac{(a+1)}{2} \;\&\; f > \frac{a+1}{2}$$

$$S(f) = 0$$

$$S(f) = \begin{cases} 0 & f < -\frac{(a+1)}{2} \\ \frac{1}{2}\left(1+\sin\left(\frac{\pi}{a}(f+\frac{1}{2})\right)\right) & -\frac{(a+1)}{2} < f < \frac{a-1}{2} \\ 1 & \frac{a-1}{2} < f < \frac{1-a}{2} \\ \frac{1}{2}\left(1-\sin\left(\frac{\pi}{a}(f-\frac{1}{2})\right)\right) & \frac{1-a}{2} < f < \frac{1+a}{2} \\ 0 & f > \frac{1+a}{2} \end{cases}$$

So yes, we can write $S(f) = (R * C)(f)$

c) Find $S(t) = r(t)\, c(t)$

We know $a\,\mathrm{sinc}(at) \leftrightarrow I_{[-\frac{a}{2}, \frac{a}{2}]}(f)$

$$\Rightarrow I_{[-\frac{1}{2}, \frac{1}{2}]}(f) \longleftrightarrow \mathrm{sinc}\, t$$

$$\Rightarrow r(t) = \mathrm{sinc}(t)$$

$$C(f) = \frac{\pi}{2a} \cos\left(\frac{\pi}{a} f\right) I_{[-\frac{a}{2}, \frac{a}{2}]}$$

$$c(t) = \int_{-\infty}^{\infty} C(f)\, e^{j2\pi f t}\, df$$

$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} \frac{\pi}{2a} \cos\left(\frac{\pi}{a} f\right) e^{j2\pi f t}\, df$$

$$= \frac{\pi}{4a} \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(e^{\frac{j\pi}{a} f} + e^{-\frac{j\pi}{a} f} \right) e^{j2\pi f t} \, df$$

$$= \frac{\pi}{4a} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{j\pi\left(\frac{1}{a}+2t\right) f} + e^{j\pi\left(-\frac{1}{a}+2t\right) f} \, df$$

$$= \frac{\pi}{4a} \left(\left. \frac{e^{j\pi\left(\frac{1}{a}+2t\right) f}}{j\pi\left(\frac{1}{a}+2t\right)} \right|_{-\frac{a}{2}}^{\frac{a}{2}} + \left. \frac{e^{j\pi\left(-\frac{1}{a}+2t\right) f}}{j\pi\left(-\frac{1}{a}+2t\right)} \right|_{-\frac{a}{2}}^{\frac{a}{2}} \right)$$

$$= \frac{1}{j4a} \left(\frac{e^{j\pi\left(\frac{1}{2}+at\right)} - e^{j\pi\left(-\frac{1}{2}-at\right)}}{\frac{1}{a}+2t} + \frac{e^{j\pi\left(-\frac{1}{2}+at\right)} - e^{j\pi\left(+\frac{1}{2}-at\right)}}{-\frac{1}{a}+2t} \right)$$

$$\cancel{e^{\frac{j\pi}{2}} = j} \qquad \cancel{e^{-\frac{j\pi}{2}} = -j}$$

$$\cancel{= \frac{1}{4a} \left(\frac{e^{j\pi at} - e^{-j\pi at}}{\frac{1}{a}+2t} - \frac{e^{j\pi at} - e^{-j\pi at}}{-\frac{1}{a}+2t} \right)}$$

$$\cancel{= \frac{1}{2/a}}$$

$$= \frac{1}{2a} \left(\frac{\sin\left(\pi\left(\frac{1}{2}+at\right)\right)}{\frac{1}{a}+2t} + \frac{\sin\left(\pi\left(at-\frac{1}{2}\right)\right)}{2t-\frac{1}{a}} \right)$$

$$= \frac{\cos(\pi at)}{2+4at} - \frac{\cos(\pi at)}{4at-2}$$

$$= \frac{\cos \pi at}{2} \left(\frac{1}{2at+1} - \frac{1}{2at-1} \right)$$

$$c(t) = \frac{\cos \pi at}{-4a^2t^2+1}$$

$\Rightarrow s(t) = \text{sinc}(t) \cdot \dfrac{\cos \pi a t}{1 - 4a^2t^2}$

To find zeros of $s(t)$ $\quad \dfrac{\sin \pi t}{\pi t} = 0$ $\quad$ or $\quad \cos \pi a t = 0$

$\Rightarrow m\pi = \pi t$ $\quad \pi a t = (2m+1)\dfrac{\pi}{2}$ $\quad m \in \mathbb{Z}$

$\Rightarrow t = n$ $\quad t = \dfrac{(2m+1)}{2a}$ $\quad m \in \mathbb{Z}$

where $n \in \mathbb{Z} \setminus \{0\}$ or

S.T. $s(\frac{t}{T})$ ~~has~~ is Nyquist rate $\frac{1}{T}$

If $p(mT) = \delta_{m0} = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$ then, we say $p(t)$ has Nyquist rate $\frac{1}{T}$

As mentioned earlier zero of $s(t)$ is for $t = n \in \mathbb{Z} \setminus \{0\}$

$$s\left(\frac{mT}{T}\right) = s(m) = \text{sinc}(m) \frac{\cos(\pi a m)}{1 - 4a^2m^2}$$

$$\text{sinc}(m) = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$$

$\therefore$ $s(\frac{t}{T})$ has Nyquist rate $\frac{1}{T}$

d) If $s(\frac{t}{T})$ is used for BPSK at rate $\frac{1}{T}$, then magnitude is always finite for the transmitted waveform

$$u(t) = \sum_n b[n]\, s\left(\frac{t - nT}{T}\right) \quad b[n] \text{ corresponds to BPSK symbols}$$

$$= \sum_n b[n]\, s\left(\frac{t}{T} - n\right)$$

$$s\left(\frac{t}{T} - n\right) = \text{sinc}\left(\frac{t}{T} - n\right) \frac{\cos\left(\pi a\left(\frac{t}{T} - n\right)\right)}{1 - 4a^2\left(\frac{t}{T} - n\right)^2}$$

$$= \frac{\sin \pi\left(\frac{t}{T} - n\right) \cos\left(\pi a\left(\frac{t}{T} - n\right)\right)}{\pi\left(\frac{t}{T} - n\right) - 4a^2\pi\left(\frac{t}{T} - n\right)^3}$$

Num: $\sin\theta_1 \cos\theta_2$ is almost 1

Den: Order of $\frac{1}{n^3}$

We know that ~~denominator is~~ $\Sigma \frac{1}{n^3}$ converges to a finite value

$\therefore$ Since we nearly have a constant $\times \Sigma \frac{1}{n^3}$ we can say that $u(t) = \sum_n b[n] s\left(\frac{t}{T} - n\right)$ converges to a finite value and thus has finite magnitude.

4) We have a pulse timelimited to duration $\frac{1}{T}$

Def^n: $g(t)$ is square root Nyquist at rate $\frac{1}{T}$ if $|G(f)|^2$ is Nyquist at rate $\frac{1}{T}$

We know that a pulse $p(t)$ is Nyquist at rate $\frac{1}{T}$ if $p(mT) = \delta_{m0} = \begin{cases} 1 & m = 0 \\ 0 & m \neq 0 \end{cases}$

So we need to find inverse Fourier trasform of $|G(f)|^2$

$$P(f) = |G(f)|^2 \longleftrightarrow g * g_{MF}(t)$$

$$\Rightarrow p(t) = g * g_{MF}(t)$$

$$p(mT) = \int g(t)\, g^*(t - mT)\, dt$$

For $m = 0$,

$$P(0) = \int |g(t)|^2 \, dt$$

Since we can scale (mentioned in question)

$P(0) = 1$

For $m \neq 0$,

we have a product of $g(t)$ and $g^*(t - kT)$

Since we have $g(t)$ to be time limited with T,

they are orthogonal. i.e. one is zero when the other is non zero. This is clearly because $g(t)$ and $g^*(t-T)$... are all nonzero in a particular range only.

$$\therefore \quad g(t)\, g^*(t - mT) = 0$$

$$\Rightarrow \int g(t)\, g^*(t - mT) = 0 \qquad \forall\, m \neq 0$$

$$\therefore \quad P(mT) = \delta_{m0}.$$

$\therefore$ $g(t)$ is square root Nyquist at rate $\frac{1}{T}$ is true statement