CT Assignment - 1

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1) $x(t) = e^{-at} u(t)$

We first need 95% of energy

$$\text{Energy} = \int_{-\infty}^{\infty} |X(f)|^2 df$$

But from Parseval's theorem,

$$= \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |e^{-at} u(t)|^2 dt$$

$$= \int_{0}^{\infty} e^{-2at} dt$$

$$= \frac{e^{-2at}}{-2a} \Big|_0^{\infty}$$

$$= \frac{1}{2a}$$

We know that

$$e^{-at} u(t) \longleftrightarrow \frac{1}{a+2\pi f}$$

95% of energy is now equal to the energy spectral density of the fourier transform

Let W be the one sided bandwidth

$$\Rightarrow \int_{-W}^{W} \frac{1}{|a+2\pi f|^2} df = \frac{0.95}{2a}$$

$$\Rightarrow \int_{-W}^{W} \frac{1}{a^2+(2\pi f)^2} df = \frac{0.95}{2a}$$

$$\Rightarrow \int_{-W}^{W} \frac{1}{(2\pi)^2}\,\frac{1}{f^2+\left(\frac{a}{2\pi}\right)^2}\,df = \frac{0.95}{2a}$$

$$\Rightarrow \frac{1}{(2\pi)^2}\cdot\frac{2\pi}{a}\cdot\operatorname{Tan}^{-1}\left(\frac{f}{\frac{a}{2\pi}}\right)\Bigg|_{-W}^{W} = \frac{0.95}{2a}$$

$$\Rightarrow \frac{1}{2\pi a}\left(\operatorname{Tan}^{-1}\left(\frac{2\pi W}{a}\right)-\left(-\operatorname{Tan}^{-1}\left(\frac{2\pi W}{a}\right)\right)\right) = \frac{0.95}{2a}$$

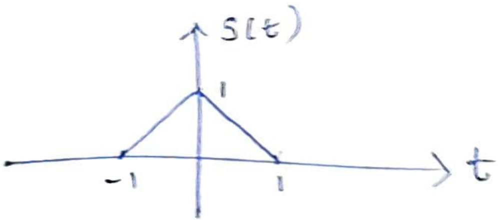
$$\Rightarrow \frac{1}{\pi}\operatorname{Tan}^{-1}\left(\frac{2\pi W}{a}\right) = \frac{0.95}{2}$$

$$\Rightarrow W = \frac{a}{2\pi}\tan\left(\frac{0.95\pi}{2}\right)$$

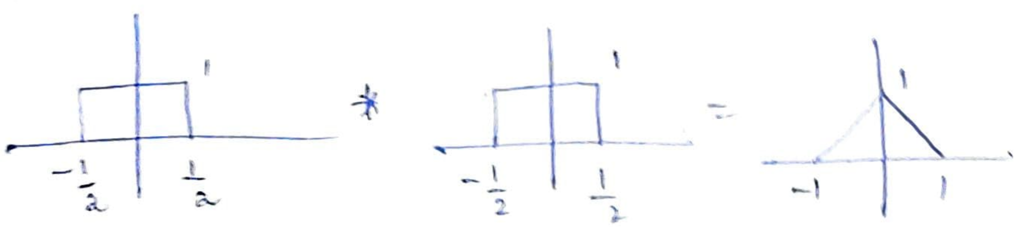
$\therefore$ Bandwidth (Double sided) $= \frac{a}{\pi}\tan\left(\frac{0.95\pi}{2}\right)$

$= 4.044a$ Hz

2) a) $s(t) = (1-|t|)\,I_{[-1,1]}(t)$



We know that convolution of two rectangular pulses gives △lar pulse.



$\Rightarrow s(t) = I_{[-\frac{1}{2},\frac{1}{2}]}(t) * I_{[-\frac{1}{2},\frac{1}{2}]}(t).$

We have the convolution property i.e convolution in time is multiplication in frequency domain.

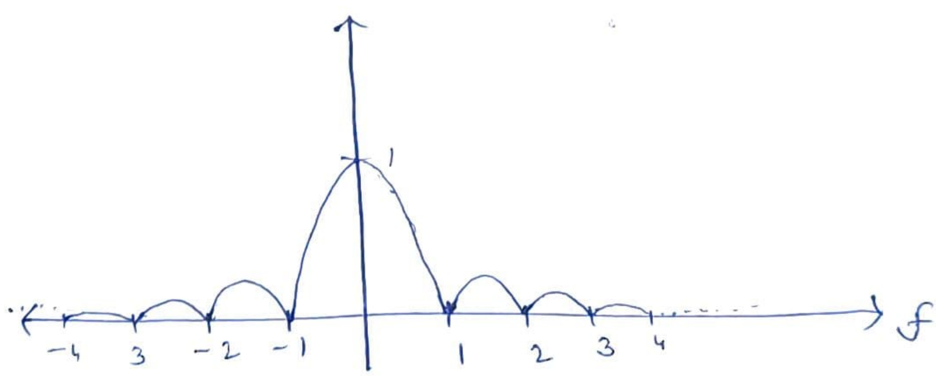
- We know $I_{[-\frac{T}{2},\frac{T}{2}]}(t) \longleftrightarrow T\text{sinc}(fT)$

$$\Rightarrow I_{[-\frac{1}{2},\frac{1}{2}]}(t) \longleftrightarrow \text{sinc}(f)$$

$$\Rightarrow S(f) = \text{sinc}(f) \times \text{sinc}(f)$$

$$= (\text{sinc}(f))^2$$

$$\boxed{S(f) = \left(\frac{\sin(\pi f)}{\pi f}\right)^2}$$



b) This follows a similar approach to Question (1)

$$\text{Energy} = \int_{-\infty}^{\infty} |S(f)|^2 \, df$$

$$= \int_{-\infty}^{\infty} |s(t)|^2 \, dt \quad \text{(From Parsevall's theorem}$$

$$= \int_{-1}^{1} |s(t)|^2 \, dt$$

$$= 2\int_{0}^{1} (1-t)^2 \, dt$$

$$= 2\left(0 - \left(-\frac{1}{3}\right)\right)$$

$$= \frac{2}{3}$$

Now, If W is the bandwith (one sided), then

$$\int_{-W}^{W} |S(f)|^2 \, df = 0.99 \times \text{Energy}$$

$$\Rightarrow 2\int_{0}^{W} \text{sinc}^4 f \, df = 0.99 \times \frac{2}{3}$$

$$\Rightarrow \int_{0}^{W} \text{sinc}^4 f \, df = 0.33$$

$$\Rightarrow \int_{0}^{W} \left(\frac{\sin(\pi f)}{\pi f}\right)^4 df = 0.33$$

3)

We have $x(t)$ and $y(t)$ which are periodic with period T_0. x_n and y_n are fourier series coefficient

a) S.T $\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)y^*(t)dt = \sum_{n=-\infty}^{\infty} x_n y_n^*$

We know $x(t) = \sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}$

$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{jn\omega_0 t}$

Let us substitute these values in LHS

$$\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\, y^*(t)\, dt = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \left(\sum_{n=-\infty}^{\infty} x_n e^{jn\omega_0 t}\right)\left(\sum_{n=-\infty}^{\infty} y_n^* e^{-jn\omega_0 t}\right) dt$$

We notice that in the expansion if the n is same, then we get $x_n y_n^*$ otherwise we get e powers

$$= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \left(\sum_{n=-\infty}^{\infty} x_n y_n^*\right) + \sum_{\substack{l=-\infty \\ l\neq m}}^{\infty}\sum_{m=-\infty}^{\infty} x_l e^{j(l+m)t} y_m^* \, dt$$

Note that we manipulated $jn\omega_0 t$ to look like lt and mt

$$= \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} \left(\sum_{n=-\infty}^{\infty} x_n y_n^*\right) + \left(\sum\sum k_i e^{jpt}\right) dt$$

where $k_i = x_l y_m^*$; $p = l+m$;

Here $\int_{\alpha}^{\alpha+T_0} \sum\sum k_i e^{jpt} dt = \sum\sum k_i \underbrace{\int_{\alpha}^{\alpha+T_0} e^{jpt} dt}$

Periodic with period T_0

Since it contains sine & cos with period T_0, integration is 0

$$\Rightarrow \sum\sum k_i \int_{\alpha}^{\alpha+T_0} \cos pt + j\sin pt = 0 + j0 = \underline{0}$$

$$\therefore \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\, y^*(t)\, dt = \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0}\left(\sum_{n=-\infty}^{\infty} x_n y_n^*\right) dt$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} x_n y_n^* \,(\alpha + T_0 - \alpha)$$

$$= \boxed{\sum_{n=-\infty}^{\infty} x_n y_n^*}$$

For Rayligh's relation, take $y^*(t) = x^*(t)$

$$\Rightarrow \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} x(t)\, x^*(t)\, dt = \sum_{n=-\infty}^{\infty} x_n x_n^*$$

$$\Rightarrow \frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2\, dt = \boxed{\sum_{n=-\infty}^{\infty} |x_n|^2}$$

b)

$x(t)$ is Finite Power signal,

$\Rightarrow$ Power $< \infty$ $\qquad \Rightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt < \infty$

$\cancel{\frac{1}{T_0}\int_{\alpha}^{\alpha+T_0} |x(t)|^2 dt < \infty}$ $\qquad \Rightarrow \lim_{T\to\infty}\frac{1}{T}\int_{\alpha}^{\alpha+T} |x(t)|^2 dt < \infty$

~~But from~~

$\Rightarrow \cancel{\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2}$ $\qquad \Rightarrow \boxed{\sum_{n=-\infty}^{\infty} |x_n|^2 < \infty}$

~~Take~~ $\cancel{\alpha = -\frac{T}{2}}$

$\cancel{\lim_{\alpha\to-\infty}\frac{1}{T}\int_{\alpha}^{\alpha+T} |x(t)|^2}$

Clearly, sum of squares converges.

We know that if $\sum_{n=-\infty}^{\infty} |x_n|^2$ converges to a finite number, this means that

$$\lim_{n \to \infty} |x_n|^2 = 0$$

$$\Rightarrow \lim_{n \to \infty} |x_n| = 0$$

$$\Rightarrow \lim_{n \to \infty} x_n = 0$$

c) R.T.P

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots + \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$$

Let, $f(x) = x^2$. $x \in [-\pi, \pi]$

$$\therefore a_k = \frac{1}{T_0} \int_{T_0} f(t) e^{-jk\omega_0 t} dt$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 e^{-jk\omega_0 t} dt \qquad \omega_0 = \frac{2\pi}{T_0} = 1$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} t^2 e^{-jkt} dt$$

$$= \frac{1}{2\pi} \left(\frac{t^2 e^{-jkt}}{-jk} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{2t e^{-jkt}}{+jk} dt \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{t e^{-jkt}}{jk} dt$$

$$= \frac{1}{\pi jk} \left(\frac{t e^{-jkt}}{-jk} \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{e^{-jkt}}{jk} dt \right)$$

$$= \frac{1}{\pi k^2} (2\pi)(-1)^k + \frac{1}{\pi k^2} \int_{-\pi}^{\pi} e^{-jkt} dt$$

$$\Rightarrow a_k = \frac{2(-1)^k}{k^2}$$

But we have Rayleigh's relation from which we can write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t^4\,dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\Rightarrow \frac{1}{10\pi}(\pi^5+\pi^5) = \left(2\sum_{k=1}^{\infty}|a_k|^2\right) + |a_0|^2$$

$$\Rightarrow \frac{\pi^4}{5} = 2\times 4\times \sum_{k=1}^{\infty}\frac{1}{(k^2)^2} + \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\,dt\right|^2$$

$$\Rightarrow \frac{\pi^4}{5} = 8\sum_{k=1}^{\infty}\frac{1}{k^4} + \left(\frac{\pi^2}{3}\right)^2$$

$$\Rightarrow \frac{\pi^4}{5} - \frac{\pi^4}{9} = 8\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$$

Required is $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}$ which are odd terms

Add and subtract even terms i.e $\sum_{k=1}^{\infty}\frac{1}{(2k)^4}$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} = \underbrace{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4} + \sum_{k=1}^{\infty}\frac{1}{(2k)^4}}_{\text{whole series}} - \sum_{k=1}^{\infty}\frac{1}{(2k)^4}$$

$$= \left(\sum_{k=1}^{\infty}\frac{1}{k^4}\right) - \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$= \sum_{k=1}^{\infty}\frac{15}{16}\cdot\frac{1}{k^4}$$

$$= \frac{15}{16}\cdot\frac{\pi^4}{90} = \boxed{\frac{\pi^4}{96}}$$

4) a) $x(t) = \text{sinc}^3(t)$

We know that $W\,\text{sinc}(Wt) \xleftrightarrow{FT} I_{[-\frac{W}{2},\frac{W}{2}]}(f)$

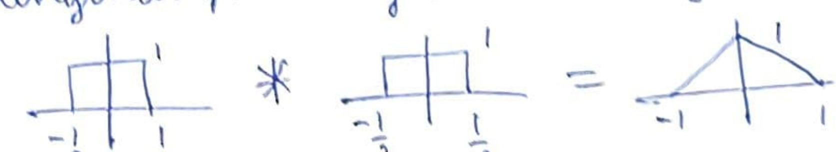
$\Rightarrow \text{sinc}(t) \xleftrightarrow{FT} I_{[-\frac{1}{2},\frac{1}{2}]}(f)$

Also Multiplication in time domain is convolution in frequency domain

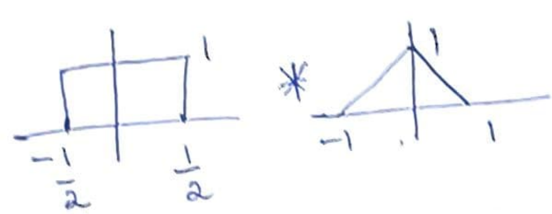
$\Rightarrow X(f) = I_{[-\frac{1}{2},-\frac{1}{2}]}(f) * I_{[-\frac{1}{2},\frac{1}{2}]}(f) * I_{[-\frac{1}{2},\frac{1}{2}]}(f)$

We know convolution ~~in time~~ of two rectangular pulses gives a triangle

i.e



$\Rightarrow X(f) =$



flip and shift

i) for $f+\frac{1}{2} < -1 \Rightarrow f < -\frac{3}{2}$; $X(f) = 0$

ii) for $f-\frac{1}{2} < -1 < f+\frac{1}{2} \Rightarrow -\frac{3}{2} < t < -\frac{1}{2}$

$X(f) = \int_{-1}^{f+\frac{1}{2}} (1+F)\,dF$

$= \frac{(1+F)^2}{2}\Big|_{-1}^{f+\frac{1}{2}}$

$= \frac{(f+\frac{3}{2})^2}{2}$

iii) $f - \frac{1}{2} < 0 < f + \frac{1}{2}$

$\Rightarrow -\frac{1}{2} < f < \frac{1}{2}$

$$X(f) = \int_{f-\frac{1}{2}}^{0} (1+F)\,dF + \int_{0}^{f+\frac{1}{2}} (1-F)\,dF$$

$$= F + \frac{F^2}{2}\Big|_{f-\frac{1}{2}}^{0} + F - \frac{F^2}{2}\Big|_{0}^{f+\frac{1}{2}}$$

$$= -\left(f-\tfrac{1}{2}\right) - \frac{\left(f-\frac{1}{2}\right)^2}{2} + \left(f+\tfrac{1}{2}\right) - \frac{\left(f+\frac{1}{2}\right)^2}{2}$$

$$= 1 - \frac{1}{2}\left(2\left(f^2 + \tfrac{1}{4}\right)\right)$$

$$= \frac{3}{4} - f^2$$

iv) $f - \frac{1}{2} < 1 < f + \frac{1}{2}$

$\Rightarrow \frac{1}{2} < f < \frac{3}{2}$

$$X(f) = \int_{f-\frac{1}{2}}^{1} (1-F)\,dF$$

$$= -\frac{(1-F)^2}{2}\Big|_{f-\frac{1}{2}}^{1}$$

$$= \frac{\left(f-\frac{3}{2}\right)^2}{2}$$

v) $f > \frac{3}{2}$; $X(f) = 0$

$$\therefore X(f) = \begin{cases} 0 & f < -\frac{3}{2} \\ \frac{\left(f+\frac{3}{2}\right)^2}{2} & -\frac{3}{2} < f < -\frac{1}{2} \\ \frac{3}{4} - f^2 & -\frac{1}{2} < f < \frac{1}{2} \\ \left(f - \frac{3}{2}\right)^2 & \frac{1}{2} < f < \frac{3}{2} \\ 0 & f > \frac{3}{2} \end{cases}$$

4 b)

$$x(t) = t\,\mathrm{sinc}(t)$$
$$= t\,\frac{\sin(\pi t)}{\pi t}$$
$$= \frac{\sin(\pi t)}{\pi}$$
$$= \frac{\sin(2\pi \frac{1}{2} t)}{\pi}$$

We know $\sin(2\pi f_0 t) \longleftrightarrow \frac{1}{2j}\left(\delta(f-f_0) - \delta(f+f_0)\right)$

$$\Rightarrow X(f) = \frac{1}{2\pi j}\left(\delta(f-\tfrac{1}{2}) - \delta(f+\tfrac{1}{2})\right) \quad //$$

4 c)

$$x(t) = t e^{-\alpha|t|}\cos(\beta t)$$

We know that

$$\cos(\beta t) \longleftrightarrow \frac{1}{2}\left[\delta(f-\tfrac{\beta}{2\pi}) + \delta(f+\tfrac{\beta}{2\pi})\right]$$

Idea is to see to it that multiplic

Let, $x_1(t) = t e^{-\alpha|t|}$

$$X_1(f) = \int_{-\infty}^{\infty} t e^{-\alpha|t|} e^{-j2\pi f t}\,dt$$

$$= \int_{-\infty}^{0} t e^{\alpha t} e^{-j2\pi f t}\,dt + \int_{0}^{\infty} t e^{-\alpha t} e^{-j2\pi f t}\,dt$$

$$= \left.\frac{t e^{t(\alpha - j2\pi f)}}{\alpha - j2\pi f}\right|_{-\infty}^{0} - \int_{-\infty}^{0}\frac{e^{t(\alpha - j2\pi f)}}{\alpha - j2\pi f}$$

$$+ \left.\frac{t e^{t(-\alpha - j2\pi f)}}{(-\alpha - j2\pi f)}\right|_{0}^{\infty} - \int_{0}^{\infty}\frac{e^{t(-\alpha - j2\pi f)}}{-\alpha - j2\pi f}$$

$$= 0 - \left.\frac{e^{t(\alpha - j2\pi f)}}{(\alpha - j2\pi f)^2}\right|_{-\infty}^{0} - \left.\frac{e^{t(-\alpha - j2\pi f)}}{(\alpha + j2\pi f)^2}\right|_{0}^{\infty}$$

$$= -\frac{1}{(\alpha - j2\pi f)^2} - \left[-\frac{1}{(\alpha + j2\pi f)^2} \right]$$

$$= \frac{2 \cdot j \cdot 4\pi f \alpha}{(\alpha + j2\pi f)^2 (\alpha - j2\pi f)^2}$$

$$X_1(f) = \frac{j \cdot 8\pi f \alpha}{(\alpha^2 - 4\pi^2 f^2 j^2}$$

$$X(F) = X_1(f) * \frac{1}{2}\left[\delta\left(f - \frac{\beta}{2\pi}\right) + \delta\left(f + \frac{\beta}{2\pi}\right) \right]$$

5 a) $y(t) = \int_0^{\infty} e^{-\alpha t} \operatorname{sinc}^2 t \, dt$

Let $x_1(t) = e^{-\alpha t} u(t)$

$x_2(t) = \operatorname{sinc}^2(t)$

$$x_1(t) \longleftrightarrow \frac{1}{\alpha + j\omega}$$

$$x_2(t) \longleftrightarrow \begin{cases} 1-|f| & \text{for } -1 \le f \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, $X(\omega) = \int_{-\infty}^{\infty} e^{-\alpha t} u(t) \operatorname{sinc}^2(t) e^{-j\omega t} dt$

$= \int_0^{\infty} e^{-\alpha t} \operatorname{sinc}^2(t) e^{-j\omega t} dt$

take $\omega = 0$

$X(0) = \int_0^{\infty} e^{-\alpha t} \operatorname{sinc}^2(t) \, dt$

$\therefore X(0) = \int_{-\infty}^{\infty} X_1(\omega) X_2(\omega) d\omega$

From Parseval's theorem

Take $\int_0^{\infty} e^{-\alpha t} \operatorname{sinc}^2 dt = \int_{-\infty}^{\infty} X_1(f) X_2(f) \, df$

$$= \int_{-1}^{0} \frac{f+1}{\alpha + j2\pi f} + \int_0^1 \frac{1-f}{\alpha + j2\pi f} df$$

Solving i.e integrating, we get

$$y(t) = \frac{1}{j\alpha\pi} \log_e \left(\frac{\alpha + j2\pi}{\alpha - j2\pi} \right) + \frac{-\alpha}{4\pi^2} \log_e \left(\frac{a^2 + 4\pi^2}{a^2} \right)$$

5 b)

$$y(t) = \int_0^{\infty} e^{-\alpha t} \cos\beta t \, dt$$

Here take $x_1(t) = e^{-\alpha t} u(t)$

$$x_2(t) = \cos\beta t$$

$$x_1(t) \longleftrightarrow \frac{1}{\alpha + j 2\pi f}$$

$$x_2(t) \longleftrightarrow \frac{1}{2}\left[\delta\left(f - \frac{\beta}{2\pi}\right) + \delta\left(f + \frac{\beta}{2\pi}\right)\right]$$

$$\Rightarrow Y(f) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, e^{-j2\pi f t}\, dt$$

$$\cancel{Y(0) = \int_{-\infty}^{\infty} x_1(t)\, x_2(t)\, dt}$$

$$\cancel{= \int_{-\infty}^{\infty} X_1(f) * X_2(f)\, df}$$

In frequency domain

$$Y(f) = X_1(f) * X_2(f)$$

This is because multiplication in time domain is ~~multi~~ convolution in frequency

$$= \frac{1}{\alpha + j2\pi f} * \frac{1}{2}\left[\delta\left(f - \frac{\beta}{2\pi}\right) + \delta\left(f + \frac{\beta}{2\pi}\right)\right]$$

$$Y(f) = \frac{1}{2}\left[\frac{1}{\alpha + j2\pi\left(f - \frac{\beta}{2\pi}\right)} + \frac{1}{\alpha + j2\pi\left(f + \frac{\beta}{2\pi}\right)}\right]$$

$$Y(0) = \frac{1}{2}\left(\frac{1}{\alpha - j\beta} + \frac{1}{\alpha + j\beta}\right)$$

$$\boxed{y(t) = \frac{\alpha}{\alpha^2 + \beta^2}}$$

6) a) Given,

$u_p(t) = \text{sinc}(2t)\cos(100\pi t)$

$v_p(t) = \text{sinc}(t)\sin(101\pi t + \frac{\pi}{4})$

i) $u_p(t) = u_c(t)\cos(2\pi f_c t) - u_s(t)\sin(2\pi f_c t)$

$u_c(t) = \text{sinc}(2t) \quad u_s(t) = 0 \;\&\; f_c = 50$

$u(t) = u_c(t) + j u_s(t)$

$= \text{sinc}(2t)$

// ly

ii) $v_p(t) = V_c(t)\cos(2\pi f_c t) - V_s(t)\sin(2\pi f_c t)$

But $V_p(t) = \text{sinc}(t)\left(\sin(100\pi t)\cos(\pi t + \frac{\pi}{4}) + \cos(100\pi t)\sin(\pi t + \frac{\pi}{4})\right)$

$\Rightarrow V_c(t) = \text{sinc}(t)\sin(\pi t + \frac{\pi}{4})$

$\Rightarrow V_s(t) = -\text{sinc}(t)\cos(\pi t + \frac{\pi}{4})$

$\Rightarrow V(t) = \text{sinc}(t)\sin(\pi t + \frac{\pi}{4}) - j\,\text{sinc}(t)\cos(\pi t + \frac{\pi}{4})$

$= \text{sinc}(t)\left(\sin(\pi t + \frac{\pi}{4}) - j\cos(\pi t + \frac{\pi}{4})\right)$

$= -j\,\text{sinc}(t)\left(\cos(\pi t + \frac{\pi}{4}) + j\sin(\pi t + \frac{\pi}{4})\right)$

$= -j\,\text{sinc}(t)\, e^{j(\pi t + \frac{\pi}{4})}$

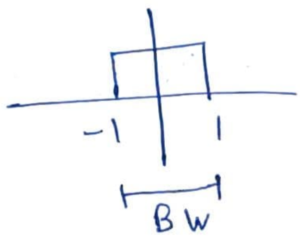
$= \text{sinc}(t)\, e^{j(\pi t - \frac{\pi}{4})}$

b) i) $u_p(t) = \text{sinc}(2t)\cos(100\pi t)$

$u(t) = \text{sinc}(2t)$

$2\,\text{sinc}(2t) \xleftrightarrow{FT} I_{[-1,1]}(f)$

$\text{sinc}(2t) \xleftrightarrow{FT} \frac{1}{2} I_{[-1,1]}(f)$



Bandwidth of baseband = $\underline{\underline{2}}$

$\Rightarrow$ For passband signal

One sided Bandwidth = 2

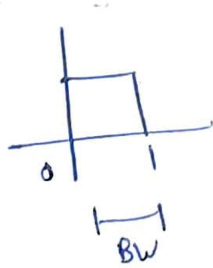
ii) $v(t) = \text{sinc}(t)\, e^{j(\pi t - \frac{\pi}{4})}$

$\text{sinc}(t) \xleftrightarrow{FT} I_{[-\frac{1}{2}, \frac{1}{2}]}(f)$

$e^{j2\pi \cdot \frac{1}{2} t} \xleftrightarrow{FT} \delta(f + \frac{1}{2})$

$\Rightarrow V(f) = I_{[-\frac{1}{2}, \frac{1}{2}]}(f) * \delta(f + \frac{1}{2})\, e^{-\frac{j\pi}{4}}$

$= I_{[0,1]}(f)\, e^{-\frac{j\pi}{4}}$



Bandwidth = $\underline{\underline{1}}$

$\Rightarrow$ For passband also we get similar answer on one side.

c) $\langle u_p, v_p \rangle$

We know that $\langle u_p, v_p \rangle = \frac{1}{2} \mathrm{Re}\{\langle u, v \rangle\}$

Using Parseval's $\langle u, v \rangle = \langle U(f), V(f) \rangle$

$$\langle u_p, v_p \rangle = \frac{1}{2} \mathrm{Re}\{\langle U(f), V(f) \rangle\}$$

$$= \frac{1}{2} \mathrm{Re}\left\{ \int_{-\infty}^{\infty} \frac{1}{2} I_{[-1,1]}(f)\, I_{[0,1]}(f)\, e^{-j\frac{\pi}{4}}\, df \right\}$$

$$= \frac{1}{4} \mathrm{Re}\left\{ \int_0^1 e^{-j\frac{\pi}{4}}\, df \right\}$$

$$= \frac{1}{4} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}} \;//$$

d) We got $y_p(t) = u_p * v_p(t)$

$$Y_p(f) = U_p(f) V_p(f) = \frac{1}{2} U(f) V(f)$$

$$= \frac{1}{4} I_{[-1,1]}(f)\, I_{[0,1]}(f)\, e^{-j\frac{\pi}{4}}$$

$$= \frac{1}{4} I_{[0,1]}(f) e^{-j\frac{\pi}{4}}$$

$$\Big\updownarrow \text{ FT}$$

$$v_p(t)$$

$$\Rightarrow y_p(t) = \frac{1}{4} v_p(t)$$

$$= \frac{1}{4} \mathrm{sinc}(t) \sin(10\pi t + \tfrac{\pi}{4})$$

7) $u_p(t) = I_{[-1,1]}(t) \cos 100\pi t$

$h_p(t) = I_{[0,3]}(t) \sin 100\pi t$

$y(t) = \frac{1}{2} u(t) * h(t)$

$u_p(t) = u_c(t) \cos(2\pi f_c t) - u_s(t) \sin(2\pi f_c t)$

Assume $f_c = 50$ Hz for the filter

$u_c(t) = I_{[-1,1]}(t)$; $u_s(t) = 0$

$\Rightarrow u(t) = I_{[-1,1]}(t)$

||ly $h_c(t) = 0$, $h_s(t) = -I_{[0,3]}(t)$

$\Rightarrow h(t) = -j I_{[0,3]}(t)$

$\therefore y_p(t) = -\frac{j}{2} \left(I_{[-1,1]} * I_{[0,3]} \right)$

i) For $t < -1$ $x(t) = 0$

ii) For $t-1 < 0 < t+1$ $\quad -1 < t < 1$ ~~$x(t) =$~~

$$x(t) = \int_0^{t+1} dt = t+1$$

iii) For $t-1 > 0$ & $t+1 < 3$ $\quad$ i.e $1 < t < 2$

$$x(t) = \int_{t-1}^{t+1} dt$$

$= 2$

iv) For $t-1<3<t+1$ i.e. $2<t<4$

$$x(t) = \int_{t-1}^{3} dt = 4-t$$

v) For $t>4$ 0

$$x(t) = \begin{cases} 0 & t<-1 \\ t+1 & -1<t\leq 1 \\ 2 & 1<t<2 \\ 4-t & 2<t<4 \\ 0 & t>4 \end{cases}$$

$$y_p(t) = -j\frac{x(t)}{2}$$

$\Rightarrow$ $y_c(t) = 0$; $y_s(t) = -\frac{x(t)}{2}$

$\Rightarrow$ $\boxed{y_p(t) = +\frac{x(t)}{2}\cos(100\pi t)}$

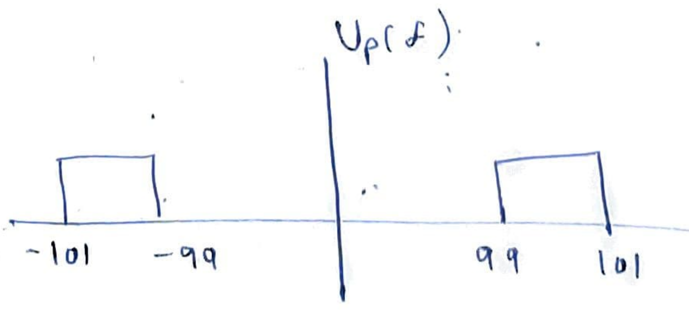
8) a) $u_p(t) = a(t)\cos(200\pi t)$

$$U_p(f) = A(f) * \frac{(\delta(f-100)+\delta(f+100))}{2}$$

$$= \frac{1}{2}(A(f-100)+A(f+100))$$

$$A(f) = \frac{1}{2} I_{[-1,1]}(f)$$

∴ graph



$\Rightarrow$ Band 99-101 MHz

b)

$u_p(t)\cos 199\pi t$ is passed through LPF.

$\Rightarrow$ We get the output to be similar but $\frac{1}{2}u_c(t)$

Initially, $u_p(t) = \text{sinc}(2t)\cos(200\pi t)$

$$= \text{sinc}(2t)\cos(199\pi t + \pi t)$$

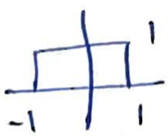
$$= \text{sinc}(2t)\left(\cos(199\pi t)\cos(\pi t) - \sin(199\pi t)\sin(\pi t)\right)$$

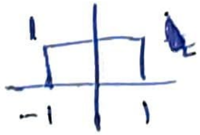
$$= \text{sinc}(2t)\cos(\pi t)\cos(199\pi t) - \text{sinc}(2t)\sin(\pi t)\sin(199\pi t)$$

$u_c(t) = \text{sinc}(2t)\cos(\pi t)$ & $u_s(t) = \text{sinc}(2t)\sin(\pi t)$

$\Rightarrow \frac{1}{2}u_c(t) = \frac{1}{2}\text{sinc}(2t)\cos(\pi t)$

Now $U_c'(f) = \frac{1}{2}FT\{\text{sinc}(2t)\} * FT\{\cos(\pi t)\}$

$$= \frac{1}{2}\Bigg(\frac{1}{2}\cdot$$  $$* \ \frac{1}{2}\left(\delta\left(f-\tfrac{1}{2}\right) + \delta\left(f+\tfrac{1}{2}\right)\right)\Bigg)$$

$$= \frac{1}{8}\Bigg($$  $$* \left(\delta\left(f-\tfrac{1}{2}\right) + \delta\left(f+\tfrac{1}{2}\right)\right)\Bigg)$$

$$= \frac{1}{8}\left(I_{[-\frac{1}{2},\frac{3}{2}]}(f) + I_{[-\frac{3}{2},\frac{1}{2}]}(f) \right)$$