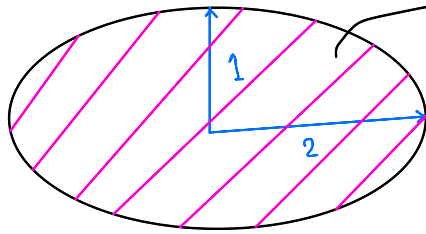
1.



PDF of the R.V. inside the ellipse should be uniformly distributed.

a) Let major axis be $a=2$
and minor axis be $b=1$

Let (X,Y) be a point, let us consider an extended polar transformation

$$\Omega : \{ (x,y) : (X,Y) \text{ inside ellipse} \}$$

$$(X,Y) \to (ar\cos\theta,\ br\sin\theta)$$

$$r \in [0,1] \text{ and } \theta \in [0,2\pi)$$

Let $f_{R,\theta}(r,\theta)$ be the multi-variate density function of R. Vector.

Since $X = ar\cos\theta$, $Y = br\sin\theta$,
We can find Jacabian and its determinant

$$\Rightarrow \quad J = \begin{bmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{bmatrix}$$

$$|\det J| = \left| \begin{vmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{vmatrix} \right| = abr$$

- Since the distribution is uniform over ellipse in x,y and area of ellipse is πab

$$f_{X,Y}(x,y) = \frac{1}{\pi ab} \quad \left[\text{from } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) \cdot dx \cdot dy = 1 \right]$$

And now from the jacobian and $f_{X,Y}(x,y)$ we get

$$f_{R,\Theta}(r,\theta) = \frac{1}{\pi ab} \cdot abr = \frac{r}{\pi}$$

$$\Rightarrow \quad \boxed{f_{R,\Theta}(r,\theta) = \frac{r}{\pi}}$$

$$f_R(r) = 2r \quad , \quad f_\Theta(\theta) = \frac{1}{2\pi}$$

- If we assume r, θ are picked independant of each other then,

$$f_{R,\Theta}[r,\theta] = f_R(r)\cdot f_\Theta(\theta) = r/\pi$$

- Hence we got an implementable algorithm such that we pick r, θ from $[0,1]$ and $[0,2\pi)$ independent with PDF's

$$f_R(r) = 2r \quad \text{and} \quad f_\Theta(\theta) = \frac{1}{2\pi} \text{ respectively.}$$

From this we can generate uniform distribution inside ellipse $x^2/a^2 + y^2/b^2 = 1$ by making inverse Transform,

$$(x, y) \leftarrow (ar\cos\theta, br\sin\theta)$$

- Now we need method to generate $f_R(r)$ and $f_\Theta(\theta)$ from $U(0,1)$

  - $f_\Theta(\theta)$ can be trivially taken as $2\pi \times U(0,1)$.
  - $f_R(r) = 2r$. This can be generated by using a transformation on the R.V. with PDF $U(0,1)$

    i.e,

$g(x) = \sqrt{x}$

$r = g(u) \quad , u \to U(0,1)$

$$\Rightarrow \boxed{\begin{array}{ll} r = \sqrt{u} & , u \to U(0,1) \\ \theta = 2\pi \cdot u & , u \to U(0,1) \end{array}}$$

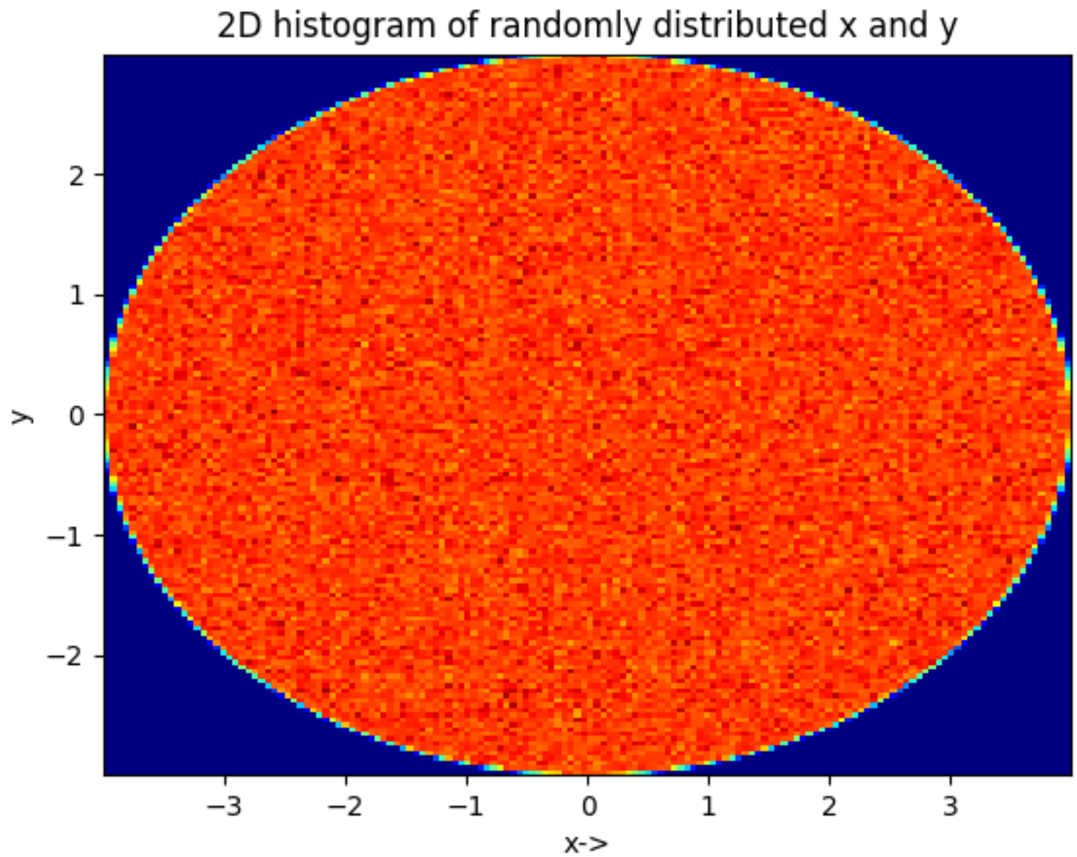
b. Implementing the Algorithm :

- Created two arrays of size 10^7 with values distributed uniformly from $[0,1]$ one for r_values and one for θ_values.
- Applied $\sqrt{u}$ transformation on r and $2\pi.u$ transformation on θ.
- Applied inverse transformation from $(r,\theta) \to (x,y)$ and plotted the

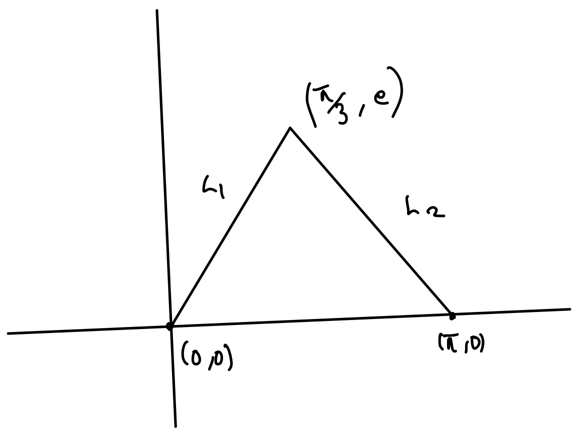
Histogram.

IMG NAME $\Rightarrow$ hist_1b.png

CODE NAME $\Rightarrow$ q_1b.py


2D histogram of randomly distributed x and y
y
2
1
0
−1
−2
−3
−2
−1
0
1
2
3
x->

C.



$$X > 0 \,,\, Y > 0$$

$$\frac{Y}{X} < \frac{e}{\pi/3} = \frac{3e}{\pi}$$

Let $\frac{Y}{X} = \frac{3e}{\pi}\alpha_1$, then $0 < \alpha_1 < 1$ → ①

Equation of L_2 is given by,

$$Y + \frac{3eX}{2\pi} = \frac{3e}{2} \rightarrow L_2$$

Now, $0 < Y + \frac{3e}{2\pi}X < \frac{3e}{2}$

Let $Y + \frac{3e}{2\pi}X = \frac{3e}{2}\beta_1$, then $0 < \beta_1 < 1$ → ②

- Solving ① & ② both we get,

$$\boxed{X = \frac{\pi \beta_1}{2\alpha_1+1} \quad , \quad Y = \frac{3e\,\alpha_1 \beta_1}{2\alpha_1+1}}$$

Let $f_{\alpha_1,\beta_1}(U,V)$ be density function at

random vectol U,V,

Now finding Jacobian fol the toansfolmation,

$$dx\,dy = \begin{vmatrix} -\frac{2\pi\beta_1}{(2\alpha_1+1)^2} & \frac{\pi}{2\alpha_1+1} \\ \frac{3e\beta_1}{(2\alpha_1+1)^2} & \frac{3\alpha_1 e}{2\alpha_1+1} \end{vmatrix} d\alpha_1\, d\beta_1$$

$$\Rightarrow \underset{\substack{\Downarrow \\ \text{det of Jacobian.}}}{J} = \frac{6\pi e\alpha_1\beta_1}{(2\alpha_1+1)^3} + \frac{3e\beta_1\pi}{(2\alpha_1+1)^3}$$

$$= \frac{3e\pi\beta_1(2\alpha_1+1)}{(2\alpha_1+1)^3} = \frac{3e\pi\beta_1}{(2\alpha_1+1)^2}$$

$$f_{(X,Y)}(x,y) = \frac{1}{\text{Area of triangle}} = \frac{1}{\frac{e\pi}{2}} = \frac{2}{e\pi}$$

$$\left[\text{because } \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1\right]$$

Now,

$$f_{\alpha_1,\beta_1}(u,v) = f_{X,Y}(x,y)\cdot J = \frac{1}{\left(\frac{e\pi}{2}\right)}\cdot\frac{3e\pi\beta_1}{(2\alpha_1+1)^2}$$

$$= \frac{6\beta_1}{(2\alpha_1+1)^2}$$

Let us consider α_1, β_1 are independent.

Now we can take

$$f_\alpha(\alpha) = \frac{3}{(2\alpha_1+1)^2}$$

$$f_\beta(\beta) = 2\beta_1$$

$$F_\alpha(X) = \frac{-3}{2}\cdot\left[\frac{1}{2\alpha_1+1}\right]_0^X = -\frac{3}{2}\left[\frac{1}{2X+1}-1\right]$$

CDF

$$F_\alpha(X) = \frac{3}{2}\left[\frac{2X}{2X+1}\right] = \frac{3X}{2X+1}$$

$$2YX + X = 3Y$$

$$\Rightarrow \boxed{Y = \frac{X}{3-2X}}$$

(To find inverse function.)

Now $\alpha_1 = \dfrac{x_0}{3-2x_0}$ $\quad x_0 \to U(0,1)$

$\beta_1 = \sqrt{y_0}$ $\quad y_0 \to U(0,1)$

$\therefore$ The above is the algorithm similar to the question 1a we have implemented.

and $X = \dfrac{\pi \beta_1}{2\alpha_1+1}$, $Y = \dfrac{3 e^{\alpha_1} \beta_1}{2\alpha_1+1}$

d.

Implementation of Algorithm:

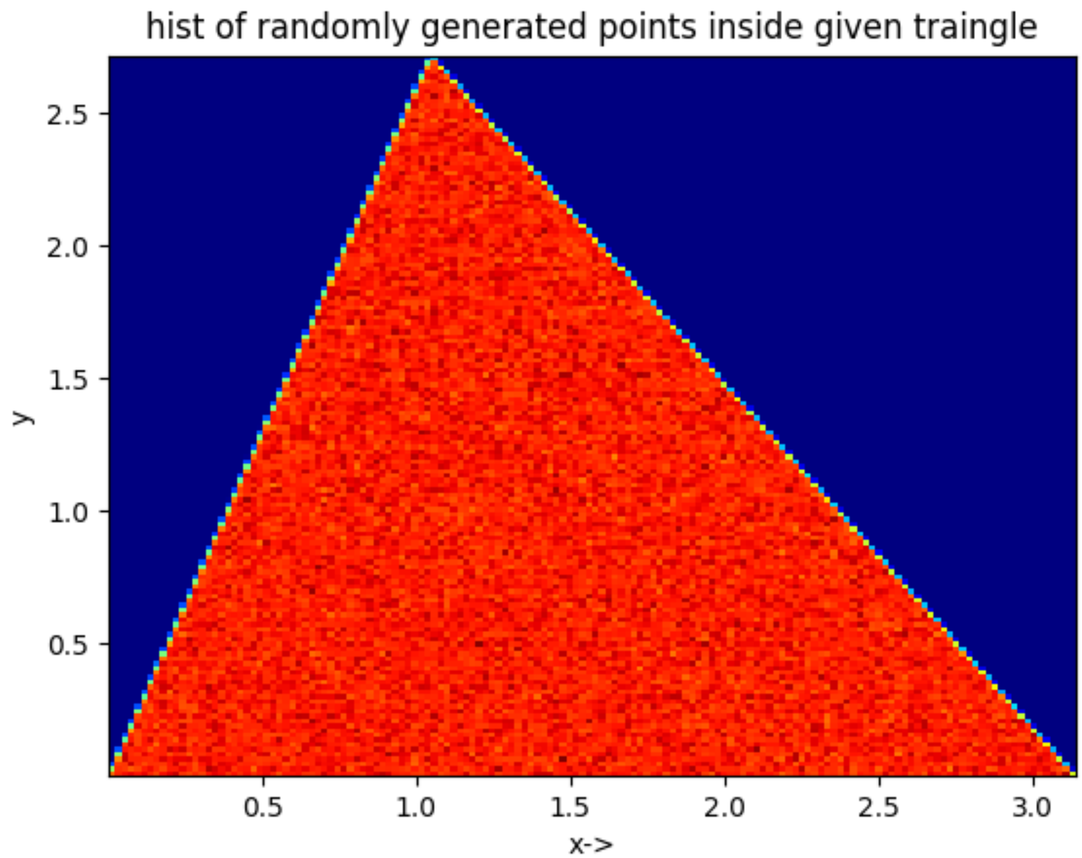
- Created 2 arrays of 10^7 from (0, 1) uniform distribution each for α_1, β_1.
- Got α and β sets from arrays using

$$\alpha = \frac{X}{3-2X} \text{ and } \beta = \sqrt{Y}$$

- Done the inverse transformation from $\alpha, \beta \to x, y$ and plotted the histogram in x, y coordinates.

CODE ⇒ q1_d.py

HISTOGRAM ⇒ hist_1C.png.


hist of randomly generated points inside given traingle
2.5
2.0
1.5
1.0
0.5
y
0.5
1.0
1.5
2.0
2.5
3.0
x->

Histogram