Holes in convex drawings*

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Abstract

The Erdős–Szekeres theorem states that, for every positive integer k, every sufficiently large point set in general position contains a subset of k points in convex position – a k-gon. In the same vein, Erdős later asked for the existence of k-holes which are k-gons with no additional points in their convex hulls. Today it is known that every sufficiently large point set contains 6-holes while there exist arbitrarily large point sets without 7-holes. in gen. pos.

(1978) Harborth started the investigation of empty triangles in simple drawings of the complete graph. In a simple drawing, vertices are mapped to points in the plane and edges are drawn as simple curves connecting the corresponding endpoints such that any two edges intersect in at most one point, which is either a common vertex or a proper crossing. For the subclass of convex drawings, which in particular includes point sets, Arroyo et al. showed that quadratically many empty triangles exist. (2022)

In this article, we generalize the concept of k-holes to simple drawings of the complete graph K_n and investigate their existence. We provide arbitrarily large simple drawings without 4-holes, show that convex drawings contain quadratically many 4-holes, and generalize the Empty Hexagon theorem (Gerken 2006; Nicolás 2007) by proving the existence of 6-holes in sufficiently large convex drawings. As a byproduct, we obtain that every convex drawing of K_n yields a pseudolinear drawing of almost logarithmic size; a structural result that might be of independent interest.

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1 Introduction

The study of holes in point sets was motivated by the Erdős–Szekeres theorem [12] and continues to be an active research branch. The theorem states that for every $k \in \mathbb{N}$ every sufficiently large point set in general position (i.e., no three points on a line) contains a subset of k points in convex position – a so called *k-gon*. A variation suggested by Erdős is about the existence of holes. A *k-hole* in a point set S is a k-gon with the property that there are no points of S in the interior of the convex hull. reference?

In this article, we investigate holes in simple drawings of the complete graph K_n. Even though the notation of holes generalizes to simple drawings in a natural manner, we have to introduce some basic notation before we can talk about these structures and our results.

In a *simple drawing*, vertices are mapped to distinct points in the plane (~~resp.~~ or on the sphere) and edges are mapped to simple curves connecting the corresponding points such that two edges have at most one point in common which is either a common endpoint or a proper intersection. Furthermore we assume that no three edges cross in a common point. Simple drawings can be considered as generalization of point sets because a set of n points in general position yields a *geometric drawing* of K_n where the points are the vertices and the edges are ~~drawn as~~ the straight-line segments. connecting the points.

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Further, we investigate the subclass of convex drawings introduced by Arroyo et al. [5]. To define convexity, we consider *triangles* which are subdrawings of K_3 induced by three vertices. Since the three edges of a triangle do not cross, the triangle separates the plane (resp. the sphere) into two connected components. The closure of each of the components is called *side* of the triangle. A side S is *convex* if, for every two vertices ~~from~~ in S, the connecting edge is fully contained in S. A simple drawing is *convex* if every triangle has a convex side. Furthermore, a convex drawing is *f-convex* if there is a marking point f in the plane such that for all triangles the side not containing f is convex. A *pseudolinear drawing* is a simple drawing in the plane such that all edges can be extended to bi-infinite curves such that two curves have at most one point in common. As shown by Arroyo et al. [4], a simple drawing of K_n is pseudolinear if and only if it is f-convex and the marking point f is in the unbounded cell. For more information about the convexity hierarchy, we refer the reader to [4, 5, 10].

Next, we introduce the notions of k-gons in simple drawings of the complete graph. A *k-gon* C_k is a subdrawing isomorphic to the geometric drawing of k points in convex position, see Figure 1(left). Two simple drawings are *isomorphic* if there exists a bijection on the vertex sets such that the same pairs of edges cross. ~~Pause to~~ Note that isomorphism is independent of the choice of the outer cell. Thus, in terms of crossings, a k-gon C_k is a (sub)drawing with vertices $v_1, \ldots, v_k$ such that $\{v_i, v_\ell\}$ crosses $\{v_j, v_m\}$ for $i < j < \ell < m$. In contrast to the geometric setting, where every sufficiently large geometric drawing contains a k-gon, simple drawings of complete graphs do not necessarily contain k-gons [17]. For example, the perfect twisted drawing T_n depicted in Figure 1(right) does not contain any 5-gon. In terms of crossings, T_n can be characterized as the drawing of K_n with vertices $v_1, \ldots, v_n$ such that $\{v_i, v_j\}$ crosses $\{v_\ell, v_m\}$ for $i < j < \ell < m$. However, a theorem by Pach, Solymosi and Tóth [23] states that every sufficiently large drawing of ~~the~~ K_n contains a k-gon or a T_k. The currently best known bound is due to Suk and Zeng [26] who showed that every simple drawing of K_n with $n > 2^{9 \cdot \log_2(a) \log_2(b) a^2 b^2}$ contains a C_a or T_b. Since convex drawings do not contain T_5 as a subdrawing, every convex drawing of ~~the~~ K_n with n sufficiently large contains a k-gon.

To eventually define k-holes for general k, let us first consider the special case of 3-holes, which are also known as empty triangles. A triangle is *empty*, if one of its two sides does not contain any vertices in its interior. For general simple drawings, Harborth [17] proved that there are at least two empty triangles and conjectured that the minimum among all simple drawings on n vertices is $2n - 4$, which is obtained by T_n. García et al. [14] recently showed that the conjecture holds for a larger class containing the perfect twisted drawings, the so called *generalized twisted drawing*. However, the conjecture remains open in general. The best known lower bound is by Aichholzer et al. [3], who proved that there are at least n empty triangles.

In the geometric setting, the number of empty triangles behaves quite differently: every point set has a quadratic number of empty triangles, and this bound is asymptotically optimal [7]. Moreover, determining the minimum number remains a challenging problem [11, Chapter 8.4]. For the current bounds, see [2]. The class of convex drawings behaves similarly as in the geometric setting: the minimum number of empty triangles is asymptotically quadratic as shown by Arroyo et al. [4].

In this article, we go beyond empty triangles and investigate the existence of k-holes in simple drawings for $k \geq 4$. In the subdrawing induced by a k-gon with $k \geq 4$, all triangles have exactly one empty side which is the convex side. We define the *convex side* of a k-gon as the union of all convex sides of its triangles and call a vertex which lies in the interior of its convex side an *interior vertex*. A *k-hole* is a k-gon which has no interior vertices. For a k-gon C_k in a convex drawing, Arroyo et al. [5] showed that the edges from an interior

Given that this is THE central definition of your paper, you ought to give an example that helps to digest the definition! For example, take the 4-hole in T_n.

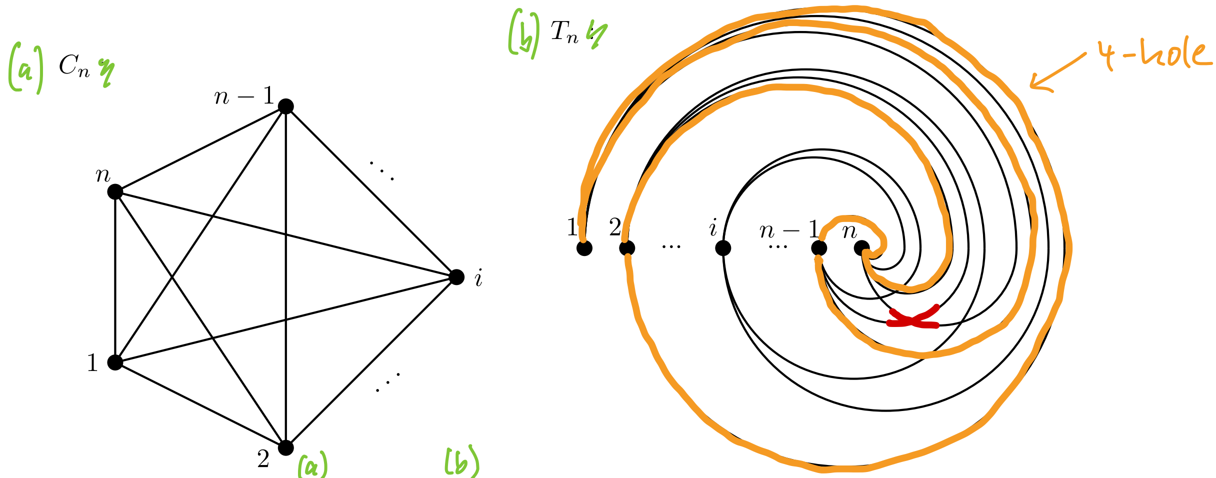

Figure 1 ~~A~~ Drawings of C_n ~~(left)~~ (a) and T_n ~~(right)~~ (b) for $n \geq 4$.

vertex to a vertex of C_k and edges between two interior vertices are fully contained in the convex side of C_k. For more details see Appendix A.

In the geometric setting it is known that for $k \leq 6$ every sufficiently large point set contains a k-hole [16, 15, 21] and that there are arbitrarily large point sets without 7-holes [18]. Since the latter applies to simple drawings, the only remaining questions in simple drawings ~~is~~ are about the existence of 4-, 5- and 6-holes.

Use \paragraph

Our Results: For $n \geq 5$ we present a non-convex simple drawing of K_n without 4-holes (Section 2). Furthermore, we show that – as in the geometric setting – the number of 4-holes in convex drawings of K_n is at least $\Omega(n^2)$ (Theorem 3.1), generalizing a result by Bárány and Füredi [7], and that every sufficiently large convex drawing contains a 5-hole and a 6-hole (Theorem 3.2), generalizing the Empty Hexagon theorem by Gerken [15] and Nicolás [21]. In order to show the latter, we prove that ~~every~~ if a subdrawing of a convex drawing~~, that~~ is induced by a minimal k-gon with $k \geq 5$ together with its interior vertices, then it is f-convex (Theorem 3.3). This result might be of independent interest as it allows to transfer results from the straight-line~~/~~, pseudolinear~~/~~ or f-convex setting to convex drawings.

2 Holes in simple drawings

The perfect twisted drawing T_n depicted in Figure 1(right) has exactly $2n-4$ empty triangles, which are spanned by the vertices $\{1, 2, i\}$ for $3 \leq i \leq n$ and $\{i, n-1, n\}$ for $1 \leq i \leq n-2$ [17]. For $n \geq 4$, T_n has exactly one 4-hole, which is spanned by $\{1, 2, n-1, n\}$.

As illustrated in Figure 2, we extend the drawing by an additional vertex z so that $\{1, 2, n-1, n\}$ is no longer a 4-hole. Moreover, the edge $\{z, i\}$ crosses $\{j, k\}$ for $1 \leq j < k < i \leq n-1$ and the edge $\{z, n\}$ crosses $\{j, k\}$ for $1 \leq j < k < n-1$. ~~We~~ Let T_n^+ denote this non-convex drawing of K_{n+1} ~~by T_n^+~~.

(In English, you cannot denote sth. actively.)

▶ Proposition 2.1. For $n \geq 4$ the drawing T_n^+ does not contain a 4-hole.

Proof. Assume that there is a 4-hole. It has to consist of the new vertex z and three vertices of T_n which ~~already build~~ must form an empty triangle. Hence it is either ~~of the form~~ spanned by $\{1, 2, i, z\}$ or by $\{i, n-1, n, z\}$. The triangle induced by $\{2, i, z\}$ is not empty since 1 and n are separated by it. Further, the subdrawing induced by the ~~4~~ four vertices $\{i, n-1, n, z\}$ has no crossing and hence is not a 4-hole. A contradiction. ◀

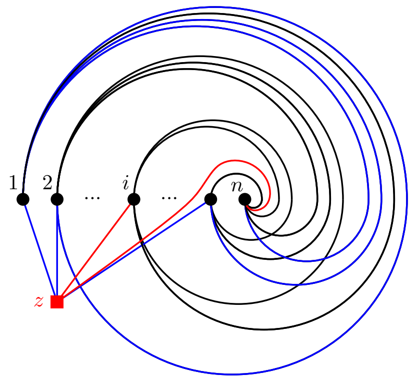

Figure 2 An illustration of the drawing T_n^+ without 4-holes.

3 Holes in convex drawings

In this section, we show that convex drawings of the complete graph behave similarly to geometric point sets when it comes to the existence of holes.

▶ **Theorem 3.1.** *Every convex drawing of ~~the~~ K_n contains at least $\Omega(n^2)$ 4-holes.*

The proof generalizes the idea of B.+F. ~~from~~ [7] and is deferred to Appendix B. The bound is asymptotically best possible as there are point sets (squared Horton sets [8] and random point sets [6]) which only have quadratically many 3-holes, 4-holes, 5-holes, and 6-holes.

Further, we investigate larger holes. We show that every sufficiently large convex drawing contains ~~(a 5-hole and)~~ a 6-hole. (and hence a 5-hole).

▶ **Theorem 3.2.** *Every convex drawing of K_n with n sufficiently large contains a 6-hole.*

For the proof below we use the existence of a k-gon in sufficiently large simple drawings [23, 26]. Even though the existence of 6-holes directly implies the existence of 5-holes, when adapting the proof to 5-holes one can obtain a better bound on the required number of vertices.

An important part of the proof is that the subdrawing induced by a minimal k-gon together with its interior vertices is f-convex, which we can then transform into a pseudolinear drawing. A k-gon is *minimal* if its convex side does not contain the convex side of another k-gon.

▶ **Theorem 3.3.** *Let C_k be a minimal k-gon with $\overbrace{k \geq}^{n \geq}$ 5 in a convex drawing of ~~the~~ K_n. Then the subdrawing induced by the vertices ~~from~~ on the convex side of C_k is f-convex.*

The proof of Theorem 3.3 is deferred to Appendix C.

Since the proof for the existence of 6-holes in point sets [15] also applies to the setting of pseudolinear drawings [25], we can now use Theorem 3.3 to derive Theorem 3.2. Similarly, the text-book proof for the existence of 5-holes in every 6-gon of a point set (see e.g. Section 3.2 in [20]) applies to pseudolinear drawings as it only uses triple orientations. For the sake of completeness, we add this proof in Appendix D. However, proving the existence of 6-holes via 9-gons[1] is far more technical. Hence we refer the interested reader to [25] for a computer-assisted proof and [27] for a simplified proof of the Empty Hexagon theorem with worse bounds.

A sentence should still be readable if you remove the parts in parentheses!

[1] Gerken [15] showed that every 9-gon in a point set yields a 6-hole and Nicolás [21] showed that a 25-gon yields a 6-hole. Both articles involve very long case distinctions.

Proof of Theorem 3.2. By the result of Suk and Zeng [26] every convex drawing of K_n with $n > 2^{225 \log_2 5 \cdot k^2 \log_2 k}$ contains a k-gon. In order to find a 6-hole, we apply this result for $k = 9$. (To find a 5-hole, we can use $k = 6$.) Consider a minimal k-gon. By Theorem 3.3, the subdrawing induced by the vertices from the convex side of the k-gon is f-convex. Since the existence of holes is invariant under the choice of the outer cell, we can choose the cell containing f as the unbounded cell to make the subdrawing pseudolinear. Next we ~~can~~ apply the result~~s~~ concerning ~~of~~ the exist~~ing~~ence of a 6-hole (resp. 5-hole) ~~for~~ in pseudolinear drawings and conclude that the subdrawing induced by the k-gon and the interior vertices contains a 6-hole (resp. 5-hole). This 6-hole (resp. 5-hole) in the subdrawing does not contain vertices of the original drawing of K_n since those vertices would be interior vertices of the k-gon. Therefore it is also a 6-hole (resp. 5-hole) in the original drawing. This completes the argument. ◀

4 Discussion

We have shown that every convex drawing of K_n with $n \geq 5$ contain a quadratic number of 4-holes and that sufficiently large drawings contain 5- and 6-holes, while 7-holes do not exist in general. However, it remains to determine the precise values of $h^{\text{conv}}(5)$ and $h^{\text{conv}}(6)$, where $h^{\text{conv}}(k)$ (resp. $h^{\text{geom}}(k)$) denotes the smallest integer such that every convex (resp. geometric) drawing of size $n \geq h^{\text{conv}}(k)$ contains a k-hole. In the geometric setting it is known that $h^{\text{geom}}(5) = 10$ [16] and $30 \leq h^{\text{geom}}(6) \leq 1717$ [15, 22]. In this article we showed $h^{\text{conv}}(k) \leq 2^{225 \cdot k^2 \cdot \log_2 5 \cdot \log_2 k} + 1$ for $k = 5$ and $k = 6$ (Theorem 3.2). Moreover, we used the SAT framework from [9] to find configurations for $n \leq 10$ and $n = 12$ without 5-holes and to prove that every drawing for $n = 11, 13, 14, 15, 16$ contains a 5-hole. Based on our computational data, we conjecture that $h^{\text{conv}}(5) = 13$. ~~Pause to~~ Note that unlike in the geometric setting, the existence of 5-holes in convex drawings is not monotone as the existence of a 5-hole in all convex drawings of ~~the~~ K_{11} does ~~do~~ not imply ~~apply~~ the ~~existence for $n = 12$.~~ same for K_{12}.

It would be interesting to obtain better bounds on the size of a largest k-gon and on the size of a largest pseudolinear subdrawing in a convex drawing of K_n. The currently best estimate ~~from~~ [26] of Suk & Zeng yields ~~gives~~ $\Omega((\log n)^{1/2-o(1)})$ for both sizes. We conjecture that – as in the geometric setting – every convex drawing of K_n contains a $\Theta(\log n)$-gon.

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A Properties of vertices inside a k-gon

Arroyo, McQuillan, Richter and Salazar [5, Section 3] started the investigations of interior vertices of k-gons. An important part is their Lemma 3.5 which we will use in the following. We introduce the notion of rotation systems: The *rotation* of a vertex u is the cyclic order of outgoing edges labeled with their end-vertex different from u. The collection of rotations for all vertices of the drawing is called the *rotation system*. In the following we will make use of a theorem by Kynčl [19] which asserts that two drawings of the complete graph the are isomorphic if and only if their rotation systems are the same up to relabeling and reversing.

▶ **Lemma A.1** ([5]). *Let C_k be a k-gon in a convex drawing of K_n with vertices $v_1, \ldots, v_k$ and $k \geq 4$. Then for each two vertices u, v contained in the convex side of C_k the edge $\{u, v\}$ is contained in the convex side of C_k. Moreover, for each interior vertex v, the vertices $v_1, \ldots, v_k$ appear in this cyclic order in the rotation around the vertex u.*

In the following we consider *minimal* k-gons, i.e., k-gons whose convex side do not contain the convex side of another k-gon. For the sake of readability, we refer to the vertices $v_1, \ldots, v_k$ of a k-gon with indices modulo k.

▶ **Lemma A.2.** *Let C_k be a minimal k-gon in a convex drawing of the K_n with vertices $v_1, \ldots, v_k$ and $k \geq 4$. Then for all i there are no interior vertices in the convex side of the triangle $\{v_i, v_{i+1}, v_{i+2}\}$. In particular, every minimal 4-gon is a 4-hole.*

Proof. Assume there is an interior vertex v in the convex side of the triangle determined by $\{v_i, v_{i+1}, v_{i+2}\}$. Clearly, the vertices $v_1, \ldots, v_i, v_{i+2}, \ldots, v_k$ span a $(k-1)$-gon and the triangle v_i, v, v_{i+2} is not contained in the convex side of that $(k-1)$-gon. In order to show that $v_1, \ldots, v_i, v, v_{i+2}, \ldots, v_k$ spans a k-gon, we need to show that the rotation system is the same as the one of C_k. By Lemma A.1, the edge $\{v, v_j\}$ does not leave the convex side of C_k and the rotation of v is $v_1, \ldots, v_k$. Further $\{v, v_j\}$ has to enter v_j from the convex side of C_k in the rotation, which is the same as the convex side of the $(k-1)$-gon. Since $\{v, v_j\}$ does not cross boundary edges of the form $\{v_\ell, v_{\ell+1}\}$, the convex side of the $(k-1)$-gon is entered by crossing the edge $\{v_i, v_{i+2}\}$. Moreover, the edge $\{v, v_j\}$ does not cross any of the edges adjacent to v_j. Hence v lies between v_i and v_{i+2} in the rotation around v_j. Figure 3 gives an illustration. Since this is the rotation system of a k-gon, we showed that $v_1, \ldots, v_i, v, v_{i+2}, \ldots, v_k$ span the k-gon C'_k. Furthermore since no edge leaves the convex side of the k-gon, the convex side of the C'_k is contained in the convex side of C_k and hence C_k was not minimal – a contradiction. ◀

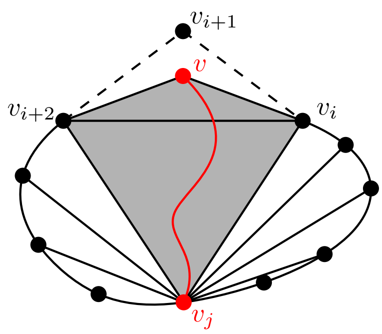

Figure 3 A k-gon with an interior vertex v in the convex side of the triangle v_i, v_{i+1}, v_{i+2}.

B Proof of Theorem 3.1

Let D be a convex drawing of K_n. We show that every edge which is crossed is part of a 4-hole in the sense that it is one of the crossing edges of a 4-hole. Let e be an edge which is crossed by another edge f. The subdrawing induced by the four end vertices of e and f is 4-gon, and we denote it by C_4. We assume the vertices are labeled with v_1, v_2, v_3, v_4 such that $e = \{v_1, v_3\}$ and $f = \{v_2, v_4\}$. If C_4 is minimal, it is a 4-hole.

Hence, we assume that there is an interior vertex x of the 4-gon C_4 as illustrated in Figure 4. By the properties of a 4-gon, x lies in the convex side of exactly two of its triangles. Without loss of generality, we assume that x is in the convex side of the two triangles $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$. By Lemma A.1, the edges $\{x, v_i\}$ are fully contained in the convex side of C_4. Since the edge $\{x, v_4\}$ is fully contained in the convex side of v_2, v_3, v_4, but has to leave the triangle induced by v_1, v_2, v_3 to get to v_4, it crosses the edge $e = \{v_1, v_3\}$. Hence v_1, x, v_3, v_4 spans another 4-gon in which $\{v_1, v_3\}$ is one of the crossing edges. Furthermore, since the edges $\{x, v_1\}, \{x, v_2\}, \{x, v_3\}$ are fully contained in the convex side of C_4, the convex side of the 4-gon $\{v_1, x, v_3, v_4\}$ is fully contained in the convex side of C_4.

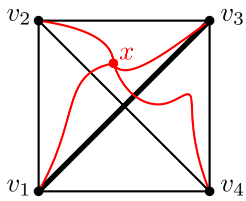

Figure 4 Illustration of the proof of Theorem 3.1.

Now, let f' be an edge crossing e such that the 4-gon determined by the four end vertices of e and f' is minimal. Then, by minimality, e and f' span a 4-hole. This shows that every crossed edge e gives a 4-hole whose diagonal is e.

In total, there are $\binom{n}{2}$ edges in a drawing of the complete graph and at most $3n - 6$ edges are uncrossed since the uncrossed edges form a planar graph. Every 4-hole is counted at most twice since there are two edges involved in the crossing. Hence the total number of 4-holes in D is at least

$$\frac{1}{2}\left(\binom{n}{2} - 3n + 6\right) = \frac{1}{4}n^2 - \frac{7}{4}n + 3.$$

C Proof of Theorem 3.3

In an f-convex drawing D of K_n there is a marking point f such that for every triangle the side which does not contain f is convex. In the subdrawing D' of a k-gon C_k the convex side of all triangles does not contain f. Hence f is not in the convex side of C_k.

Now, suppose towards a contradiction that there exists a triangle determined by vertices t_1, t_2, t_3 from the convex side of C_k, such that the side not containing f is not convex. We denote this side by S_N. Since our drawing is convex, the other side is convex, which we denote by S_C.

If we further assume that S_C is not fully contained in (the closure of) a single cell of the subdrawing D', then some edge $\{v_i, v_j\}$ has a crossing with one of the edges $\{t_i, t_j\}$. This proves that S_C is proves is not convex – a contradiction. Hence, S_C lies in (the closure of) a cell of D'.

By Lemma A.2 and the minimality assumption, there are no interior vertices in the convex side of a triangle $\{v_i, v_{i+1}, v_{i+2}\}$. Since all cells of D' in the convex side of C_k incident to the vertex v_{i+1} are inside this triangle, the vertex v_{i+1} is not part of the triangle spanned by t_1, t_2, t_3. Since the above holds for every index i, the vertices t_1, t_2, t_3 are interior vertices and S_N lies in a common cell of D'.

Since S_N is not convex, there is a vertex z in the interior of S_N such that the subdrawing induced by $\{t_1, t_2, t_3, z\}$ has a crossing [5, Corollary 2.5]. We can assume without loss of generality that the edge $\{t_1, z\}$ crosses $\{t_2, t_3\}$. Moreover, exactly one of the following two conditions holds:

- The triangle $\{t_1, t_3, z\}$ separates t_2 and f.
- The triangle $\{t_1, t_2, z\}$ separates t_3 and f.

We can further assume that the former holds as otherwise we exchange the roles of t_2 and t_3. Figure 5 gives an illustration.

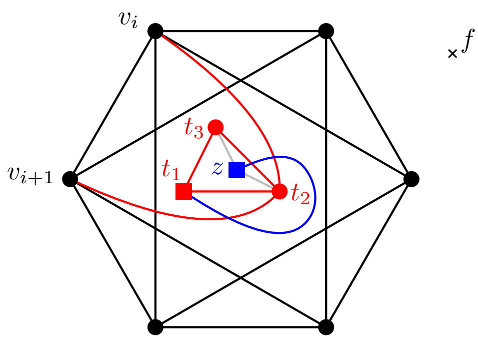

Figure 5 Illustration of the proof of Theorem 3.3. The the vertices t_1, t_2, t_3 are highlighted red and vertex z is highlighted blue. The blue edge violates the convexity.

Now we consider the edges from t_2 to the vertices $v_1, \ldots, v_k$. If the edge $\{t_2, v_i\}$ crosses $\{t_1, t_3\}$, then S_C is not convex – a contradiction.

Consequently, none of the edges $\{t_2, v_i\}$ crosses $\{t_1, t_3\}$ and hence there is an index i such that the three vertices t_1, t_3, z lie in the convex side of the triangle $\{t_2, v_i, v_{i+1}\}$. However, the edge $\{t_1, z\}$ is not fully contained in this side, which is a contradiction to convexity.

This completes the proof of Theorem 3.3.

D 5-holes in Pseudolinear drawings

We here show that, if D is pseudolinear drawing of K_n with a 6-gon, then D contains a 5-hole. The proof is analogous as in the geometric setting (see for example [20]).

We start with a minimal 6-gon C_6 with vertices $v_1, \ldots, v_6$. If C_6 does not have interior points, we clearly have a 6-hole and hence a 5-hole. If C_6 has exactly one interior vertex x, we consider the edges $\{v_i, v_{i+3}\}$ connecting vertices through a diagonal. Those diagonals are fully contained in the convex side of C_6 and hence partition the convex side into two parts. The vertex x is in one part and, together with the four vertices from the other part, it spans a 5-hole.

Let us now assume that C_6 has at least two interior vertices. Since D is a pseudolinear drawing, we can use convex hull edges[2]. Let x, y be two interior vertices such that the edge $\{x, y\}$ is a convex hull edge of all interior vertices. By the definition of pseudolinear drawings, the edge connecting the two vertices can be prolonged to a pseudoline ℓ.

We now consider two different cases, depending on which two edges $\{v_i, v_{i+1}\}$ this pseudoline crosses to leave C_6. If ℓ intersects two opposite edges of C_6, then all other inner points lie on one side of this prolonged pseudoline. In this case there are again three different vertices of C_6 which are on the opposite side as the remaining inner vertices. Those 5 vertices determine a 5-hole; see Figure 6(left). Otherwise, if ℓ intersects two other boundary edges, we find at least four vertices on one of the sides and hence there is a 6-gon with fewer interior vertices as illustrated in Figure 6(right) – a contradiction to the minimality of C_6. This completes the proof.

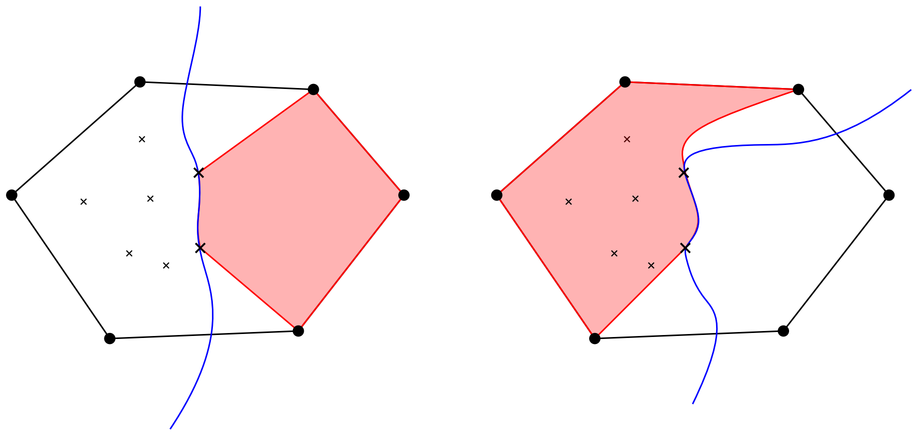

Figure 6 An illustration of the proof for the existence of 5-holes in pseudolinear drawings.

[2] The notion of a convex hull naturally generalizes to pseudopoint configurations, see e.g. [13] or [24]. To be more precise, a convex hull edge is defined by leaving all vertices its pseudoline doesn't contain on the same side of that pseudoline. It follows from above-below preserving duality ([1]) that the convex hull edges are exactly all lines whose dual is an intersection point on the boundary of the top and/or bottom cell of the dual pseudoline arrangement.