Approximation Algorithms for Lawn Mowing with Obstacles

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Abstract

Put this behind the problem definition.

We consider a geometric optimization problem that generalizes both the Lawn Mowing Problem of covering all of a given region with a unit-sized cutter and the Milling Problem of not leaving the covered area during coverage: For a given polygonal region P and a set of obstacles $\mathcal{O}$, the *Lawn Mowing Problem with Obstacles*, asks for a shortest tour that has Euclidean distance 1 to each point in $P \setminus \mathcal{O}$ and distance at least 1 to every point in $\mathcal{O}$. We present constant factor approximations. For the case where the obstacles are strictly contained in P, we present a 21.5-approximation algorithm and a 6.5-approximation for large obstacles. If the obstacles are additionally well-separated, i.e., at least distance $2 + \pi$ apart, we provide a polynomial time 4.96-approximation algorithm.

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1 Introduction

(the lawn) (the cutter)

The *Lawn Mowing Problem* (LMP) is a well-studied problem in geometric optimization that occurs in a wide range of appplications, such as sensing, surveillance and manufacturing: For a given region P and a unit-radius disk cutter D, find a closest roundtrip of shortest Euclidean length that moves the center of D within distance 1 from every point in P. If in addition, the disk is not allowed to cover any point outside of P, we are dealing with the *Milling Problem* (MP), a natural variant motivated by applications such as cutting a desired shape from a block of material. As generalizations of the *Traveling Salesman Problem* (TSP), both problems are NP-hard, with previous work [2] providing approximation algorithms. Namely?

In this paper, we consider a generalization of both problems: In the *Lawn Mowing Problem with Obstacles* (LMPO), we seek a shortest tour of D that covers a given region P without intersecting the interior of a designated set $\mathcal{O}$ of obstacles. We focus on the *enclosed LMPO* (e-LMPO) with convex polygonal obstacles of positive area strictly contained in P and separated by at least a distance of 2 from each other to ensure the existence of a feasible tour. Figure 1 illustrates the~~se~~ three problems. LMP, MP, and (e-)LMPO.

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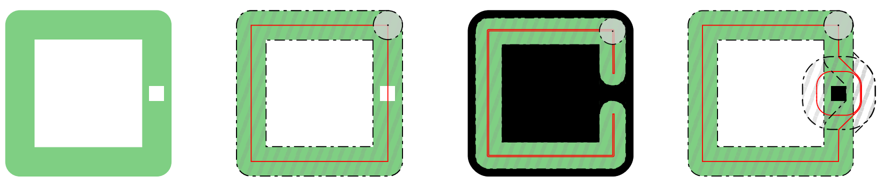

Figure 1 Example of a feasible tour in LMP, MP, and LMPO with a circular cutter.

41st European Workshop on Computational Geometry, Liblice, Czech republic, April 9–11, 2025.
This is an extended abstract of a presentation given at EuroCG'25. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

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1.1 Our contribution

with

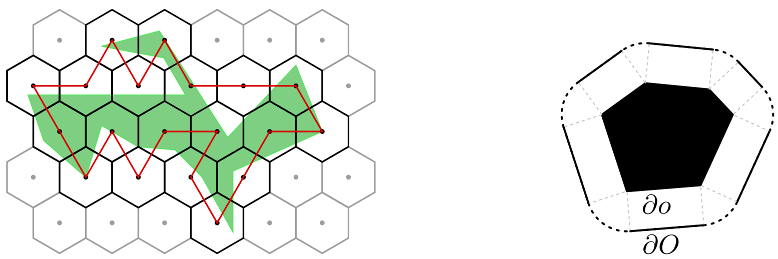
We provide a $\gamma = (4\pi + 4\sqrt{3} + 2) < 21.5$ γ-approximation algorithm for the e-LMPO that can be improved to 6.46 for instances with large obstacles. For the de-LMPO, in which obstacles are *well-separated*, i.e., at least $2 + \pi$ apart, we provide a $(2\sqrt{3}\alpha + 1.5) < 5$-approximation algorithm, with α being the performance guarantee for a TSP approximation algorithm.

1.2 Related work

There is a wide range of practical applications for lawn mowing variants, including manufacturing [3, 14, 15], cleaning [7], robotic coverage [8, 9, 13, 16], inspection [12], CAD [11], farming [5, 10, 18] and pest control [6]. The LMP was first introduced by Arkin et al. [1], expand! who later gave the currently best approximation algorithm with a performance guarantee of $2\sqrt{3}\alpha < 3.5\alpha$ [2], where α can be $(1 + \varepsilon)$ based on the methods of Arora [4] or Mitchell [17]. The algorithm computes a TSP tour on the dual graph of a hexagonal tiling of the lawn; see Figure 2 for an example.

(left) set to $1+\varepsilon$ for any $\varepsilon > 0$

▶ **Theorem 1.1.** *(Theorem 3 in [2]) The lawn mowing problem has a $2\sqrt{3}\alpha$-approximation algorithm.*



■ **Figure 2** (Left) A hexagonal tiling of the lawn. (Right) The offset boundary ∂O consists of segments and circular arcs. Its total lentgth is given by $|\partial O| = |\partial o| + 2\pi$.

2 e-LMPO approximation

In this section, we present an approximation algorithm for the e-LMPO. For our analysis, we make use of the following simple fact on the offset boundaries of the obstacles; the offset boundary ∂O of an obstacle $o \in \mathcal{O}$ consists of all points at distance 1 of the boundary ∂o of o. For convex obstacles, we have $|\partial O| = |\partial o| + 2\pi$, and we define $\partial\mathcal{O} := \sum_{o \in \mathcal{O}} \partial O$; see Figure 2 (right).

▶ **Lemma 2.1.** *For the e-LMPO, any feasible tour contains the segments of $\partial\mathcal{O}$.*

Proof. For an obstacle o, its offset boundary ∂O consists of segments and circular arcs, see Figure 2. For each inner point p of a segment of ∂o, there exists a unique point in ∂O at distance 1. Hence, all segments of $\partial\mathcal{O}$ belong to any feasible tour, see Figure 2. ◀

We now adapt the $2\sqrt{3}\alpha$-approximation algorithm by Arkin et al. [2] to handle obstacles.

▶ **Theorem 2.2.** *The e-LMPO ~~has an~~ admits γ $\cancel{(4\pi + 4\sqrt{3} + 2) < 21.5}$-approximation algorithm.* with $\gamma =$

Proof. For an instance $(P, \mathcal{O})$, the idea is to first cover the boundary of the obstacles and then cover the rest of P using a tiling of the plane with regular hexagons of sidelength 1; see

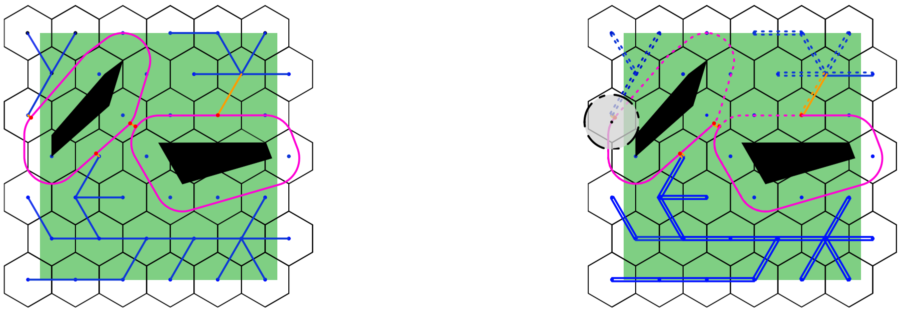


Figure 3 Illustration for the proof of Theorem 2.2. (Left) Spanning trees of the components of $G[V_p - V_o]$ are depicted in blue, offset boundaries in pink and connectors in orange. (Right) The partially traversed tour T is obtained by *walking around H and the offset boundary once*.

2

also Figure 3. Let $G \overset{*}{=} (V, E)$ denote the plane graph ~~which~~ that has a vertex for each hexagon center and an edge (of length $\sqrt{3}$) between any two hexagons sharing a side. Let $V_p \subset V$ and $V_o \subset V$ denote the sets of vertices whose hexagons ~~intersects~~ intersect $P \setminus \mathcal{O}$ and an obstacle boundary, respectively. We compute a (minimal) um? spanning tree for each connected component of $G[V_p - V_o]$. We enhance the union of all spanning trees and the offset boundaries to an (abstract) tree H by inserting so-called *connector* edges in a Kruskal-fashion; the length of an edge between $v \in V_p$ (or an ∂O_j) to some ∂O_i is the minimum Euclidean distance between any point of ∂O_i and v (or any point of ∂O_j). Note that each connector has length at most $\sqrt{3}$. Moreover, each obstacle of positive area intersects some hexagon in an interior point. Such a hexagon is not intersected by any other obstacle as they have pairwise distance 2. Consequently, $|\mathcal{O}| \leq |V_o|$. Therefore, we insert at most $|V_p| - |V_o| + |\mathcal{O}| \leq |V_p|$ connectors, and H has at most $(2|V_p| - |V_o| - 1)$ edges of length $\sqrt{3}$.

p and o are not defined. Use V_p and V_o ?!

By doubling all edges of H and inserting the offset boundaries as curves, we obtain an Eulerian graph. ~~which yields~~ It contains a tour T of length at most $2(2|V_p| - |V_o|)\sqrt{3} + |\partial\mathcal{O}|$ that visits all vertices $V_p \setminus V_o$ and traverses all offset boundaries of the obstacles; see Figure 3.

By Lemma 2.1, the segments of the offset boundary of an obstacle are contained in any feasible tour. The total length of all segments is $|\partial\mathcal{O}| - |\mathcal{O}|2\pi$. Because any point p in the interior of a segment belongs to at most two offset boundaries, we have $\frac{1}{2}(|\partial\mathcal{O}| - |\mathcal{O}|2\pi) \leq$ OPT; here we use the fact that the obstacles are convex. Together with the fact $|\mathcal{O}| \leq |V_o|$, it follows that

$$|\partial\mathcal{O}| \leq 2\text{OPT} + |\mathcal{O}|2\pi \leq 2\text{OPT} + |V_o|2\pi \tag{1}$$

and hence

$$|T| \leq (4|V_p| - 2|V_o|)\sqrt{3} + |\partial\mathcal{O}| \leq (4|V_p| + 2(\pi/\sqrt{3} - 1)|V_o|)\sqrt{3} + 2\text{OPT}.$$

Note that by disregarding the obstacles, a lawn mowing tour of $P \setminus \mathcal{O}$ is a natural lower bound for an optimal tour in our instance. The tour computed in Theorem 1.1 has length at least $\sqrt{3}|V_p|$ and is a $2\sqrt{3}\alpha$-approximation where α can be arbitrarily close to 1 [4, 17]. This ~~gives~~ yields an approximation ratio of

Hence, $\text{OPT} \geq \frac{|V_p|}{2\alpha}$.

$$\frac{|T|}{\text{OPT}} \leq \left(\frac{4\sqrt{3} + 2(\pi/\sqrt{3} - 1)\sqrt{3}}{\sqrt{3}} \cdot 2\sqrt{3} + 2\right) \cancel{\text{OPT}} = (4\pi + 4\sqrt{3} + 2)\cancel{\text{OPT}}. \qquad ◀$$

A better approximation factor can be achieved by restricting the e-LMPO to well-separated obstacles allowing for better lower bounds.

* G is defined properly only once you've fixed the position/orient. of the tiling w.r.t. to the coordinate system of P and O.

3 A better approximation for well-separated obstacles

In contrast to the LMP, the presence of obstacles imposes specific structures on the optimal (and any feasible) tour, which can be utilized to establish lower bounds; cf. Lemma 2.1.

3.1 Traversing the boundary of obstacles

Not completely clear what you are arguing here.

Lemma 2.1 motivates the use of the length $|\partial\mathcal{O}|$ as a lower bound for the length of an optimal tour. However, when the obstacles are close to each other, this bound may not hold; see the example in Figure 4. The (black dotted) circular arcs are longer than the connecting (orange) segments. Decreasing the height of the triangular obstacles ~~gives~~ yields a detour of $\pi - \epsilon$ for any $\epsilon > 0$ around the obstacle, and a minimum distance of $\geq \pi$. In the case of e-LMPO, at least we can show that $|\partial\mathcal{O}|$ is a lower bound to the length of an optimal tour if and only if obstacles are *well-separated*, i.e., each pair of obstacles has distance $\geq 2 + \pi$. We call this variant de-LMPO.

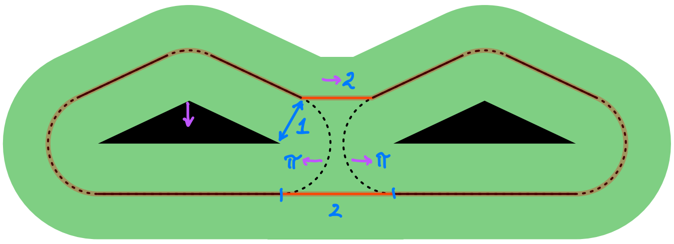


Figure 4 When obstacles are close, then $|\partial\mathcal{O}|$ may not be a lower bound for OPT (in red).

▶ **Theorem 3.1.** *For an instance of de-LMPO with a set of well-separated obstacles $\mathcal{O}$, the optimal solution has length at least $|\partial\mathcal{O}|$. Moreover, the distance bound is best possible, i.e., for each $\varepsilon > 0$, there exists an example where the obstacles have distance at least $2 + \pi - \varepsilon$ and the length of the optimal solution is $< |\partial\mathcal{O}|$.*

less than

Proof. As each obstacle $o \in \mathcal{O}$ is enclosed, its entire boundary ∂o must be visited. ∂O ← Don't start a sentence with a variable. consists of all points of the cutter center that visit ∂o. Because o_i is a convex polygon, ∂O_i consists of segments and circular arcs where each circular arc has length at most π and the total length of the circular arcs sums to 2π; see Figure 2. By Lemma 2.1, all segments of ∂O belong to any feasible tour, which ~~have~~ a total length of $|\partial O| - 2\pi|\mathcal{O}|$. hence has

Before T was the tour of the alg. Use T*!

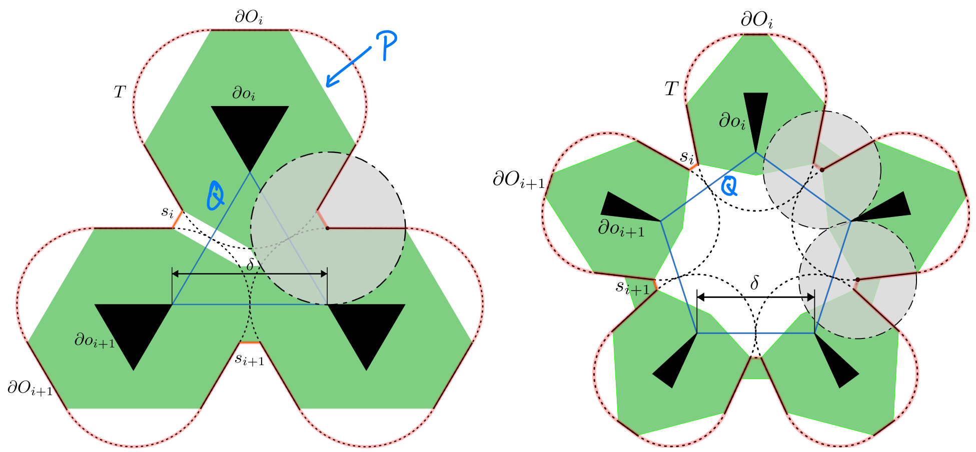
Let T be an optimal tour. We call a (maximal) subcurve γ of T a *part visiting* $o \in \mathcal{O}$ if its endpoints belong to segments of ∂O and γ contains no point of another $\partial O'$. ~~and~~ We call a subcurve connecting a part visiting o with a part visiting o' a *connector*. Note that each connector has length at least π. When traversing T in some direction, we associate each part visiting o with its proceeding connector. We aim to show that the parts visiting o and their connectors contribute 2π besides the contained segments ~~of~~ in ∂O.

obstacle at least ?!

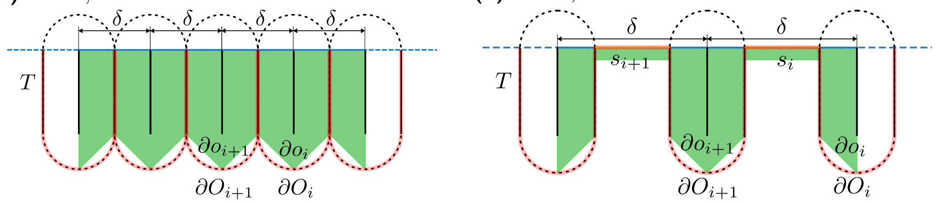
If each o has at least two parts visiting it, then its associated connectors sum to 2π. If an o is visited by just one part, then this part is shortest if it consists of ∂O minus one arc and hence the contribution is at least π (as each arc has a length of at most π). Together, with the associated connector, this yields a total contribution of at least 2π. !

Now, we show that the bound is best possible. Let $\delta := 2 + \pi - \varepsilon$, and consider an n-gon P → Q with side length δ. Each corner of P Q is incident to a triangular obstacle, and the lawn consists of the neighborhood of the obstacle as illustrated in Figure 5; see Appendix A for details.

ε>0, let P



(a) $n = 3$, $\delta = 2$ **(b)** $n = 5$, $\delta = 2$



(c) For very large n, $\delta = 2$ **(d)** For very large n, $\delta = 4$

Figure 5 Red tour T is feasible and has length $|T| < |\partial O|$ iff $\delta < \pi$. −2 ?!
In each example, the polygon P is shaded green, and the

Except for the inner circular arcs, the optimal tour T traverses $\partial \mathcal{O}$ and the small connecting segments. The lawn is defined such that T covers it. When increasing n and decreasing the width of the obstacles, the unused arc of each offset boundary converges to a length of π, and the length of each connecting segment coverges to $\delta - 2 = \pi - \varepsilon$. Consequently, in the limit, the tour has length $|\partial \mathcal{O}| - n\varepsilon < |\partial \mathcal{O}|$. Thus the bound is best-possible. ◀

3.2 Approximation algorithm for the de-LMPO

In the de-LMPO variant, all obstacles have distance at least $2+\pi$ to all other obstacles which ~~enables the~~ allows us to use ~~of~~ Theorem 3.1 to obtain a better approximation factor than that of Theorem 2.2.

▶ **Theorem 3.2.** *The de-LMPO has an* $(2\sqrt{3}\alpha + 1.5) \leq 5$*-approximation algorithm.*

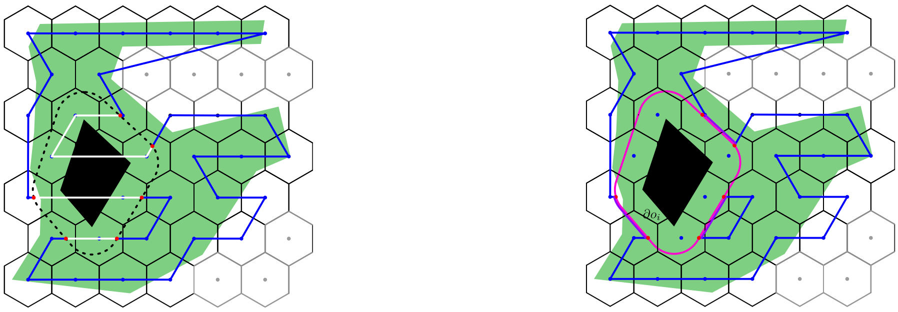
Proof. For a well-seperated instance $(P, \mathcal{O})$, the idea is to cover P using the approximation algorithm of Arkin et al. from [2] that uses a tiling of the plane with regular hexagons of sidelength 1 and then introduce detours following $\partial \mathcal{O}$ to cover the lawn around the obstacles. Let $G = (V, E)$ be the plane graph ~~which~~ that corresponds to the tiling that has a vertex for each hexagon center and an edge (of length $\sqrt{3}$) between any two hexagons sharing a side. Let $V_p \subset V$ and $V_o \subset V$ denote the sets of vertices whose hexagons intersects $P \setminus \mathcal{O}$ and an obstacle boundary, respectively. We compute an α-approximate TSP tour T' that visits all hexagon centers in V_p, where α can be arbitrarily close to 1 [4, 17].

We proceed by removing parts of T' that lie in the offset region of the obstacles $\mathcal{O}$ and obtain a set of disconnected paths $\{\pi_1, \pi_2, \dots\}$; see Figure 6a. Each path $\pi_i = (v_1, v_2, \dots, v_{n_i-1}, v_{n_i})$ contains points $v_2, \dots, v_{n_i-1} \in V_p \setminus V_o$ and intersects $\partial \mathcal{O}$ in its endpoints v_1, v_{n_i}. Let $k_i \geq 0$ be the number of endpoints that lie on the offset boundary of an

Avoid this single line by top-aligning Fig. 6.

Variable c is not defined.

obstacle o_i. We call the union of all endpoints the *connection points* V_c with $|V_c| = \sum_{o_i \in \mathcal{O}} k_i$.



(a) TSP approximation on V_p. **(b)** Eulerian graph H'.

Figure 6 de-LMPO approximation with blue TSP tour, pink graph H and Eulerian graph H'.

Consider the graph H with vertices $(V_p \setminus V_o) \cup V_c$ and edges according to the paths $\pi_1, \ldots, \pi_k$ that is further enhanced by adding edges between the connection points on the offset boundaries of the obstacles. We order the k_i connection points on each offset boundary ∂O_i in counterclockwise order and connect them via edges that follow ∂O_i. By Theorem 3.1, the total length of the newly added edges is $|\partial\mathcal{O}| \leq \text{OPT}$. Adding a second copy of every second edge around each offset boundary ensures that every connection point has an even degree, see Figure 6b. The last step can be done by inserting edge of total length at most $\frac{1}{2}|\partial\mathcal{O}| \leq \frac{1}{2}\text{OPT}$. The resulting Eulerian graph H' contains a feasible tour T ~~which~~ that traverses all offset boundaries $\partial\mathcal{O}$ and visits all vertices in $V_p \setminus V_o$ as well as all connection points ~~V_c~~ (or: all pts in V_c).

By Theorems 1.1 and 3.1, the edges in H cost at most $2\sqrt{3}\alpha\text{OPT}$ and the additional edges in H' cost at most 1.5OPT. Thus, ~~APX has a~~ in the worst case, ~~factor of~~ $|T'|$ $\leq (2\sqrt{3}\alpha + 1.5)\text{OPT}$. ◀

4 Approximation for large obstacles

In some practical applications, the perimeter of the obstacles is large compared to the cutter. This motivates e-LMPO[ρ] where each obstacle has perimeter at least ρ. For e-LMPO[ρ], we can bound $|\partial\mathcal{O}|$ by inserting ρ into Equation (1), which ~~gives us~~ yields $|\partial\mathcal{O}| \leq 2\left(1 + \frac{\pi}{\rho}\right)\text{OPT}$. Using this bound, we modify the analysis of the algorithm from Theorem 3.2 from $1.5|\partial\mathcal{O}| \leq 1.5\text{OPT}$ to $1.5|\partial\mathcal{O}| \leq 3\left(1 + \frac{\pi}{\rho}\right)\text{OPT}$. For large ρ, the approximation factor converges to $2\sqrt{3}\alpha + 3 < 6.5$.

▶ **Corollary 4.1.** *The e-LMPO[ρ] has an* $\left(2\sqrt{3}\alpha + 3\left(1 + \frac{\pi}{\rho}\right)\right)$*-approximation algorithm.*

5 Conclusion

We introduced the e-LMPO and provided a (< 21.5)-approximation algorithm. For the de-LMPO with obstacles at least $2 + \pi$ apart, we achieved a (< 5)-approximation algorithm. A new analysis of the second algorithm also leads to a 6.46-approximation for large obstacles. Several open questions remain, such as algorithms for LMPO with arbitrary obstacles (not necessarily convex or inside P) or the existence of a PTAS. Better lower bounds for any variant could lead to improved approximations and exact algorithms.

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A Construction for Theorem 3.1

Our construction is based on a regular n-gon with side length δ. At every corner of the n-gon, we add triangular obstacles with a height of 1, see Figure 7.

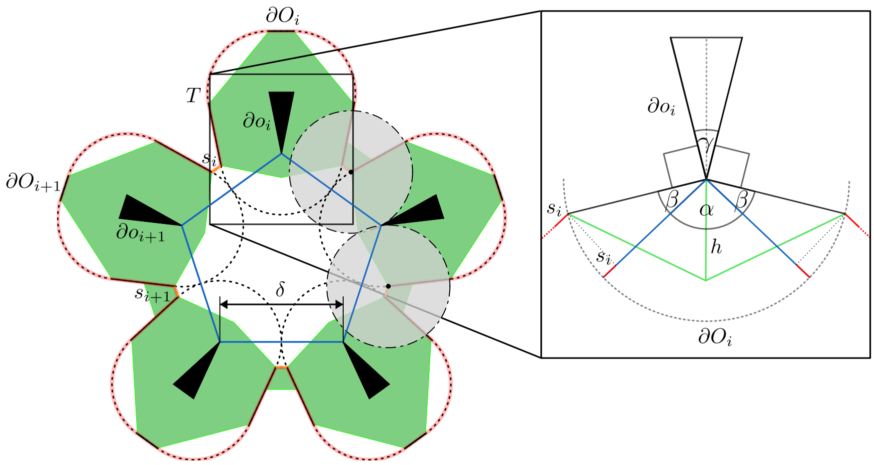

Figure 7 The boundary length $|\partial O|$ is not an upper bound since a shorter red feasible covering tour covers the full lawn.

The tour T is constructed by connecting the offset boundaries ∂O_i with segments s_i, short cutting the circular arcs in the center. To ensure that the tour does not self-intersect all obstacles ∂o have to have distance > 2 (excluding the tips of the obstacles). We can guarantee this by choosing a $\gamma < \frac{\pi-\alpha}{2}$, with $\alpha = \pi - \frac{2\pi}{n}$ being the interior angle of the regular n-gon.

The length of segment $s_i = 1 - \cos(\beta) + (\delta - 2)$ is depended on $\beta = \pi - \frac{\gamma+\alpha}{2}$. It is easy to see that for very small γ and very large n, s_i converges against $\delta - 2$.

The lawn is constructed based on the straight segments of the offset boundary ∂O_i, see Figure 7. To ensure that every obstacle is enclosed, we place some lawn above the corner, so that the lawn above the segment can be covered by the tour T, see Figure 7. Since when increasing n the area above the obstacles covered by the Tour T decreases, we need to adjust the height h accordingly. Therefore, we set

$$h = \frac{1}{\sin(\pi - \frac{\alpha}{2} - \beta)} \sin\left(\left(\pi - \frac{\alpha}{2} - \beta\right) - \arcsin\left(\sin\left(\pi - \frac{\alpha}{2} - \beta\right) 2\right)\right).$$

This concludes our construction.