Two Watchmen's Routes in Staircase Polygons*

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Abstract

We consider the watchman route problem for multiple watchmen in staircase polygons, which are rectilinear x- and y-monotone polygons. For two watchmen, we propose an optimal algorithm that takes $O(n^2)$ time, improving on the $O(n^3)$ time of the trivial solution. For $m \geq 3$ watchmen, we explain where our approach fails.

use quadratic/cubic, then you don't need to define n.

Lines 172

1 Introduction

The watchman route problem asks for a shortest route inside a polygon, such that every point in the polygon is visible to some point on the route. It was first introduced by Chin and Ntafos [2], who showed that the problem is NP-hard for polygons with holes, but may be solved efficiently for simple polygons. Given a starting point, an optimal route may be computed in $O(n^3)$ time [8], and finding a solution without a fixed starting point takes a linear factor longer [7]. can

name all authors

The Watchman Route Problem has also been considered for multiple watchmen (a problem introduced by Carlsson et al. [1]). For histograms, efficient algorithms have been proposed for ~~the aim of~~ minimizing the total route length (min-sum) [1] and the length of the longest route (min-max) [5]. Here, we are interested in only two watchmen. For this problem, Mitchell and Wynters [3] proved NP-hardness for the min-max objective in simple polygons. Recently, Nilsson and Packer presented a polynomial-time 5.969-approximation algorithm for the same objective in simple polygons [4].

In this paper, we consider a quite restricted class of polygons, staircase polygons, that for two watchmen allows us to assign responsibility for guarding any edge solely to one of the two watchmen (and seeing all of a polygon's boundary is for two watchmen sufficient to see the boundary). Additionally, we show that the two routes can be separated by a diagonal between two reflex vertices. This enables a polynomial-time algorithm to compute the optimal two watchman routes (for both the min-max and the min-sum objective). Despite staircase polygons being so restricted, some of the observations we make do not hold for three or more watchmen. This indicates a discrepancy in the computational complexity between the watchman route problem for one or two watchmen and for multiple watchmen.

polygon?!

* A. B., B. J. N. and C. S. are supported by grant 2021-03810 (Illuminate: provably good algorithms for guarding problems) from the Swedish Research Council (Vetenskapsrådet).

41st European Workshop on Computational Geometry, Liblice, Czech republic, April 9–11, 2025.
This is an extended abstract of a presentation given at EuroCG'25. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

2 Notation and Preliminaries

A polygon is called *rectilinear* if all its edges are parallel to the x- or the y-axis of a given coordinate system, and x*-monotone* (y*-monotone*) if every line that is orthogonal to the x-axis (y-axis) intersects the polygon in exactly one connected interval. A *staircase polygon* is a rectilinear polygon that is both x- and y-monotone. We call the polygonal chains of boundary edges that lie above and below the interior the *ceiling* and the *floor* of P, respectively. We consider the watchman route problem for multiple watchmen in staircase polygons. the polygon

▶ Multiple Watchman Route Problem (m-WRP). Given a polygon P, and a number of watchmen m, find a shortest set of m routes, with respect to the min-sum or min-max criterion, such that every point in P is seen from at least one of the routes.

We denote the length of a route w by $\|w\|$, and refer to a solution of the m-WRP as a set of m watchman routes in P. In the following, we consider the m-WRP for the min-sum and the min-max criterion. Any statement on optimal watchman routes holds for either objective, unless stated otherwise. Due to limited space, the proofs of lemmas marked by ($\star$) are moved to the appendix.

Define! Why don't two guards always suffice?

Let P be a staircase polygon that is not 2-guardable. As P is x- and y-monotone, we make the following observation:

route? Use $x_{\min or \max}$

▶ **Observation 2.1.** *A watchman route w with leftmost point x_{min} and rightmost point x_{max} sees all points $p \in P$ with $x(p) \in [x_{min}, x_{max}]$.*

point or coordinate?

The analogous statement holds for the ~~lower~~ bottommost point $y_{\min}$ and the ~~upper~~ topmost point $y_{\max}$ of watchman w. Watchman w thus sees the contiguous part of the ceiling between $y_{\min}$ and $x_{\max}$, and the contiguous part of the floor between $x_{\min}$ and $y_{\max}$.

We denote the extensions of edges that are incident to reflex vertices as he sees *cuts*, and identify so-called *essential cuts*. ~~For one~~ A single watchman ~~route,~~ sees a simple polygon ~~is seen~~ if all its *essential cuts* are visited. Clearly, visiting all essential cuts is a necessary condition for a set of watchman routes. A staircase polygon has at most four essential cuts: the leftmost vertical extension of the floor v_{left}, the lowe~~rmost~~st horizontal extension of the ceiling h_{bot}, the rightmost vertical extension of the ceiling v_{right}, and the topmost horizontal extension of the floor h_{top}. By "visiting" such an extension, we mean that a watchman route has a point to the left of v_{left}, below h_{bot}, to the right of v_{right}, or above h_{top}. Note that not necessarily all of these four extensions are essential cuts. For the sake of simplicity, we will nevertheless refer to them as such.

Don't lemph twice. Is this the definition of 'essential'?

Show them in Fig. 1!

For one watchman, an optimal solution is given by the shortest route that visits all four essential cuts. An example is shown in Figure 1(a). By the following theorem proven by Chin and Ntafos [2], such a solution may be computed in linear time.

▶ **Theorem 2.2.** *(Theorem 2, Chin & Ntafos [2]) A shortest watchman route in simple rectilinear polygons can be found in $O(n)$ time.* linear

For multiple watchman routes, the watchmen share the responsibility of seeing P. Thus, we aim to find a "good" distribution of responsibilities among the watchmen. For two watchmen, we prove that the polygon may be split into two subpolygons such that an optimal solution to the 2-WRP corresponds to an optimal solution to the WRP in each subpolygon.

3 Computing an Optimal Solution for Two Watchmen

In this section, we investigate the 2-WRP. Let us first state some properties of two optimal watchman routes in staircase polygons.

move to p. 2

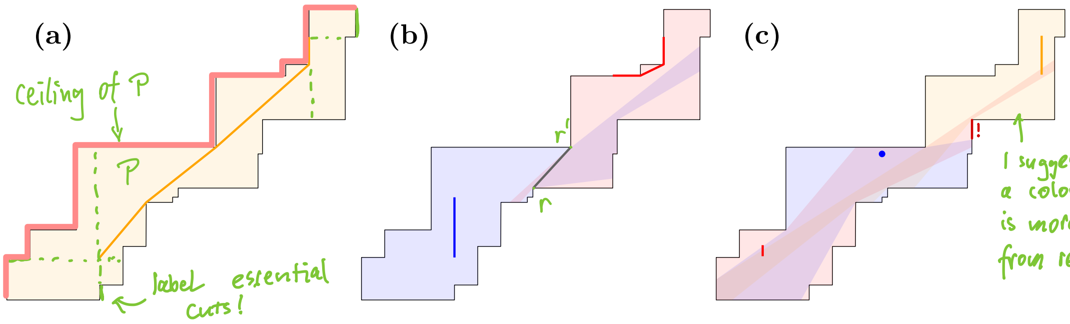


Figure 1 Optimal solutions for (a) 1 watchman, (b) 2 watchmen, (c) 3 watchmen.

one two three

▶ **Lemma 3.1** (⋆). *Let (w_1^*, w_2^*) be an optimal solution to the 2-WRP in a staircase polygon P. Then, the following properties hold:*

1. *w_1^* and w_2^* do not have any common x- and y-coordinate.*
2. *w_1^* visits the essential cuts h_{bot}, v_{left}, and w_2^* visits the essential cuts h_{top}, v_{right}.*
3. *There exists a pair of reflex vertices (r, r') with r on the floor and r' on the ceiling~~,~~ such that $\overline{rr'}$ separates w_1^* and w_2^**; see Fig. 1(b).

In the following, we always assume that an optimal solution (w_1^*, w_2^*) obeys Properties 1-3 of Lemma 3.1. In particular, w_1^* lies below and to the left of w_2^*. 1--3

▶ **Lemma 3.2.** *In an optimal solution to the 2-WRP in a staircase polygon, for every polygon edge there exists a watchman that sees the edge completely.*

Proof. Let (w_1^*, w_2^*) be an optimal solution, and consider w_1^*. As soon as it crosses the extension of a horizontal floor edge e, it sees e completely since nothing blocks the visibility between w_1^* and e along e's extension. Similarly, w_1^* sees a vertical edge on the ceiling completely as soon as it crosses the edge's extension. Before crossing the extension, w_1^* does not see the respective edge at all. Hence, for any horizontal floor edge (vertical ceiling edge) e, if w_1^* sees any point on e, then it sees all points of e. Similarly, for any horizontal ceiling edge (vertical floor edge) e, if w_2^* sees any point on e, then it sees all points of e. Assume w.l.o.g. that there is a horizontal floor edge e such that no point on e is seen by w_1^*. Then, w_2^* sees e completely as otherwise there are points on e that are not seen by any of w_1^* and w_2^*. ◀

With this, we may split two optimal watchman routes in a particular way.

▶ **Lemma 3.3.** *Let (w_1^*, w_2^*) be an optimal solution in a staircase polygon P. There exists a unique diagonal between ~~one~~ a vertex on the floor and ~~one~~ a vertex on the ceiling that cuts P into two subpolygons P_1 and P_2 such that w_1^* sees P_1, and w_2^* sees P_2.*

Proof. By Lemma 3.2, every edge is completely seen by a watchman. For a chain of consecutive edges on the floor or ceiling, there cannot be an alteration in the responsibility of the watchmen: Let e_i, e_{i+1}, e_{i+2} be three consecutive edges (on the floor or ceiling). If one watchman sees e_i, e_{i+2} completely, then it also sees e_{i+1}. Hence, there exist vertices on the floor and the ceiling such that w_1^* sees all edges that lie below and to the left of them completely, and w_2^* sees all edges that lie above and to the right of them completely. We call such vertices *breaking points* and show that there exist two breaking points, one on the floor and one on the ceiling, that see each other. Assume that this is not the case. Let b_f be the lowest-leftmost breaking point on the floor, and b_c be the upper-rightmost breaking point on the ceiling. W.l.o.g., assume that all breaking points on the floor lie to the upper-right of the breaking points on the ceiling (in particular, b_f lies to the upper-right of b_c).

Since b_f and b_c do not see each other, there exist some edges incident to a reflex vertex r that block the visibility. Assume that these edges lie on the ceiling. Then, the horizontal edge incident to r lies above b_c and below b_f, and is seen by w_2^* (by definition of b_c). Hence, w_2^* sees the vertical floor edge v that is hit by the horizontal extension through r (as described in the proof of Lemma 3.2), and thereby also sees the convex vertex on the lower end of v, contradicting the choice of b_f (being the lowest-leftmost breaking point on the floor). ◀

quadratic

We present an algorithm that finds an optimal split, and thus computes an optimal solution for two watchmen in $O(n^2)$ time. Observe that Lemma 3.2 only holds for two watchmen. For three or more watchmen, some edges may only be seen partially by each watchman in an optimal solution. Therefore, an optimal solution for $m \geq 3$ watchmen may induce a split of the polygon's floor and ceiling into more than m parts each, such that every part is seen by ~~one~~ a single watchman. This means that a watchman may be "in charge of" more than one contiguous part of the boundary on the floor and ceiling, respectively. An example is shown in Figure 1(c), where the red watchman is in charge of monitoring a part of a vertical floor edge above the blue watchman's visibility region.

3.1 A Quadratic-Time Algorithm for Two Watchmen

To compute an optimal solution, we consider all diagonals between vertices on the floor and on the ceiling. Any such diagonal splits P into two subpolygons. For each subpolygon, we compute an optimal watchman route using the linear-time algorithm proposed by Chin and Ntafos [2], and then combine the two routes to a solution for the 2-WRP in P.

cubic-time algorithm

As there are at most quadratically many diagonals to consider, this procedure trivially yields a ~~solution in $O(n^3)$ time.~~ However, maintaining a similar structure of the subpolygons by dealing with the diagonals in a certain order allows us to compute many of the watchman routes in amortized constant time.

To this end,

f is not defined. I'd simply use f instead of p_f.

For this, we iterate over the vertices on the floor. For each floor vertex p_f, we compute all its diagonals to points on the ceiling, in clockwise order around p_f. If p_f is a convex vertex, then all diagonals have a negative slope. If p_f is a reflex vertex, some diagonals have positive slope. However, we do not need to consider all diagonals with positive slope, but only those that are followed or preceded by a positive-slope diagonal in the clockwise order. We call those, and the diagonals with negative slopes, *candidate diagonals*; see Figure 2. Every candidate diagonal splits P into two subpolygons; P_1 below and P_2 above the diagonal.

two?

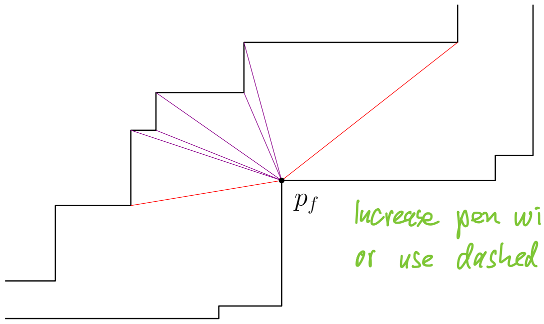
▶ **Lemma 3.4** (⋆). *Any diagonal that is not a candidate diagonal induces a solution that is at least as long as the solution induced by some candidate diagonal.*

Now we compute a solution for each candidate diagonal in the following manner:

- Step 1: Consider ~~the~~ a diagonal~~s~~ with negative slope~~s~~. Cutting along this diagonal only creates convex vertices in each subpolygon, hence all four essential cuts per subpolygon are rectilinear. These define the reflection points of the watchman routes [2]. We compute the optimal solutions for the subpolygons induced by the first diagonal in clockwise order in linear time, Theorem 2.2. For every other diagonal in order, we update the solution in the following way. Moving from one diagonal to the next (i.e., moving from one vertex on the ceiling to the next) alters either the essential cut $v_{\text{right}}(P_1)$ of P_1, or the essential cut $h_{\text{bot}}(P_2)$ of P_2. For P_1, the solution for the previous diagonal is either floating or anchored. If it is floating, the intersection of $h_{\text{bot}}(P_1)$ and $v_{\text{left}}(P_1)$ is visible to the intersection of $h_{\text{top}}(P_1)$ with $v_{\text{right}}(P_1)$. We use the visibility polygon of the intersection

using/by

Define!



Increase pen width (so that colors are visible) or use dashed / dotted line segments.

Figure 2 The candidate diagonals of a reflex vertex p_f: there are two candidate diagonals with positive slope (purple), and several candidate diagonals with negative slope (red).

name it at least!

of $h_{\text{bot}}(P_1)$ and $v_{\text{left}}(P_1)$ and an appropriate data structure [6] to check whether the route is floating. If the route is anchored, then it is anchored on reflex vertices on the floor or the ceiling. When updating the route, we move from one vertical extension $v_{\text{right}}(P_1)$ to the next one $v'_{\text{right}}(P_1)$. During this movement, any reflex vertex on the ceiling that was an anchor point can only be released once per vertex p_f, and any reflex vertex on the floor can only be added as an anchor point once per vertex p_f. Hence, each update takes amortized constant time.

- Step 2: If p_f is a reflex vertex, we also need to consider the two candidate diagonals with positive slope. Here, the subpolygons' essential cuts differ from those of a staircase polygon: There is exactly one non-rectilinear essential cut, namely the extension of the diagonal. We may nevertheless compute an optimal solution, using the algorithm by Chin and Ntafos [2]. Since there are at most five essential cuts, we can try all combinations of subsegments of these essential cuts. The triangulation and reduction to a shortest path problem and computation of its solution all take linear time.

Thus, the computations for each vertex p_f take amortized linear time. As we do this for every vertex on the floor, there are linearly many vertices to consider. With this, we get an optimal solution to the 2-watchman route problem in staircase polygons.

▶ **Theorem 3.5.** *An optimal solution to the 2-WRP in staircase polygons can be computed in $O(n^2)$ time.*

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A Omitted Proofs

▶ **Lemma 3.1** (⋆). *Let (w_1^*, w_2^*) be an optimal solution to the 2-WRP in a staircase polygon P. Then, the following properties hold:*

1. *w_1^* and w_2^* do not have any common x- and y-coordinate.*
2. *w_1^* visits the essential cuts h_{bot}, v_{left}, and w_2^* visits the essential cuts h_{top}, v_{right}.*
3. *There exists a pair of reflex vertices (r, r') with r on the floor and r' on the ceiling, such that $\overline{rr'}$ separates w_1^* and w_2^*.*

Proof. First, we prove Properties 1 and 2. Let w_1 and w_2 be two optimal watchman routes that together see P. Since P is seen, each of the four essential cuts is visited by some watchman. This can be done in four combinatorially different ways.

Case 1: w_1 visits all 4 extensions (Figure 3(a)).
Then, w_1 is a watchman route in P, and w_2 is redundant. Therefore, assume that $\|w_2\| \leq \|w_1\|$. We create a new route w_2' from the point of intersection of h_{top} with v_{right}. Assume w.l.o.g. that v_{right} does not dominate h_{top}. Then, w_2' follows h_{top} until it either hits the ceiling, or the segment has length $\|w_2\|/2$. Moreover, we replace w_1 with a route w_1': let ℓ be the vertical line through the leftmost point of w_2', and replace the parts of w_1 that lie to the right of ℓ with the line segment between points of intersection of w_1 with ℓ. This increases neither the length of w_1 nor of w_2, and the new routes see all of P.

Case 2: w_1 visits three essential cuts, w_2 visits the fourth one (Figure 3(b)).
Assume w.l.o.g. w_1 visits $h_{\text{bot}}, v_{\text{left}}$, and h_{top}. Cut P into two subpolygons along h_{top}, and denote the subpolygon below h_{top} by P_1, the one above h_{top} by P_2. Then, P_1 is seen by w_1. P_2 is star-shaped and may therefore be guarded by a single watchman with route length 0. Such a route w_2' also visits the extension h_{top}. Hence, replacing w_1 with an optimal watchman route w_1' in P_1 yields a solution (w_1', w_2') for P that is shorter than (w_1, w_2), again contradicting its optimality.

Case 3: w_1 visits v_{left} and h_{top}, and w_2 visits h_{bot} and v_{right} (Figure 3(c)).
If there is no x-overlap, observe that w_2 lies to the right of the rightmost point of w_1. We translate w_1 such that its lowermost intersection with v_{left} lies in the point of intersection of h_{bot} with v_{left}. Analogously, we move w_2 such that its uppermost intersection with v_{right} lies in the point of intersection of h_{top} with v_{right}. By construction, the subpolygon below h_{bot} (above h_{top}), and the subpolygon left of v_{left} (right of v_{right}), are star-shaped and w_1 (w_2) visits their kernels. Let P' be the subpolygon between the essential cuts. Watchman w_1 (w_2) sees the ceiling (floor) of P', and moving it vertically does not affect this except possibly losing sight of an interval on h_{top} (h_{bot}), which will be seen by the translated w_2 (w_1). Hence,

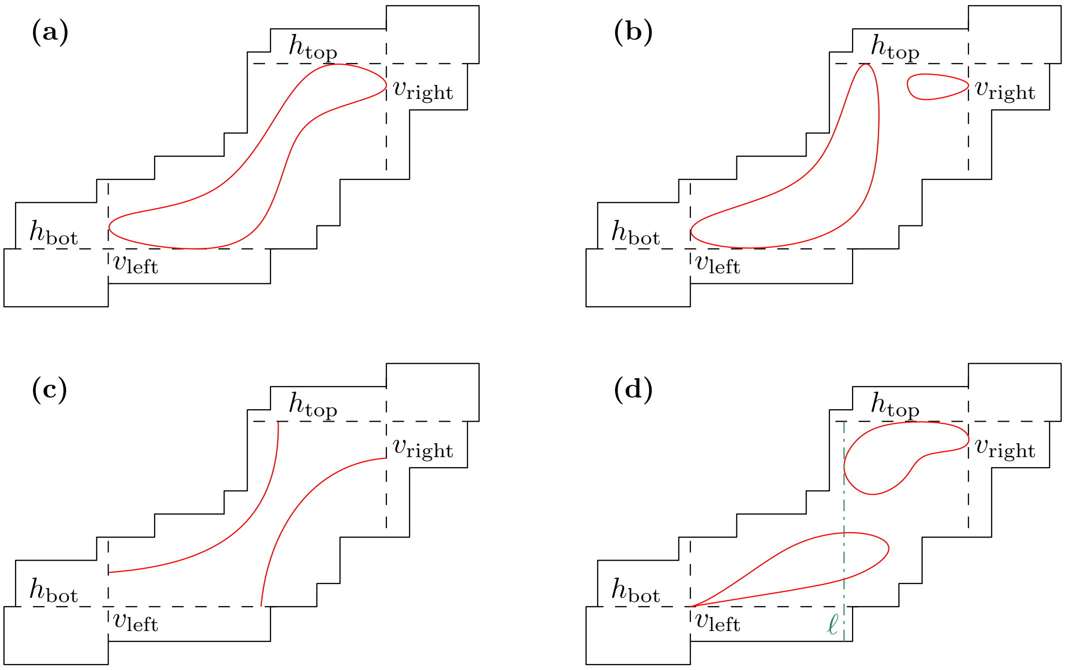

Figure 3 Four possibilities for w_1 and w_2 to visit the up to four essential cuts.

by Lemma 3.1 in [4] (which states that a simple polygon is seen by two watchmen if its boundary is seen), P' is seen as well.

If there is both x- and y-overlap, let q_1 be the point of intersection of v_{left} with h_{bot}, and let q_2 be the point of intersection of v_{right} with h_{top}. Observe that w_1 follows the shortest route between its intersections with v_{left} and h_{top}, which is a geodesic path in P. Consider the route w that follows w_1, h_{top} until q_2, v_{right} until its intersection with w_2, w_2, h_{bot} until q_1, and finally v_{left}. Then, $\|w\| \leq 2(\|w_1\| + \|w_2\|)$. Assume w.l.o.g. $\|w_1\| > \|w_2\|$, and let d be the shortest path inside P between q_1 and q_2. Then, $2\,\|d\| \leq \|w\| \leq 2(\|w_1\| + \|w_2\|)$. For min-sum, we split d into two parts of equal length, and for min-max, we split d such that the ratio of the route lengths equals $\|w_2\| \,/\, \|w_1\|$. We then replace w_1 and w_2 with the routes going back and forth along these paths. This does not increase the route-lengths, and all of P is seen.

Case 4: w_1 visits h_{bot} and v_{left}, and w_2 visits h_{top} and v_{right} (Figure 3(d)). Assume w.l.o.g. that w_1 and w_2 have some x-overlap and let ℓ be the leftmost vertical line that intersects both routes. Then, ℓ cuts P into subpolygons which both are staircase polygons. Denote the subpolygon on the left of ℓ with P_1, and the one to on the right of ℓ with P_2. Let w_1' be the watchman route that consists of the part of w_1 that lies in P_1, together with the straight-line segment between the points of intersection of w_1 with ℓ (this may also be just a single point). By 2.1, P_1 is seen by w_1'. Similarly, P_2 is seen by w_2. Thus, (w_1', w_2) is a shorter solution for P than (w_1, w_2). Furthermore, these routes may still be improved: If both P_1 and P_2 are seen, then P is seen. We may thus replace w_1' with an optimal watchman route in P_1, and w_2 with an optimal watchman route in P_2. None of these optimal tours touches ℓ, and therefore they together yield a solution for P that is shorter than (w_1, w_2) where both routes do not share a common x-coordinate.

Now, assume that w_1 and w_2 are two optimal watchman routes that satisfy Properties 1 and 2. Consider all vertical extensions of reflex vertices ordered from left to right. Assume

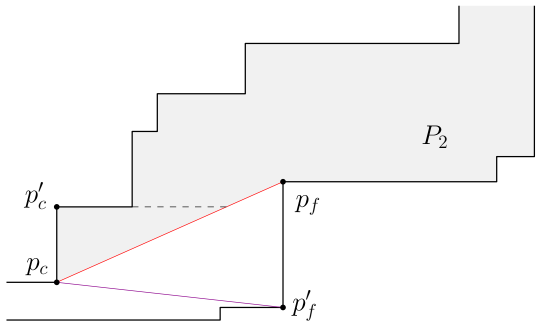

Figure 4 A diagonal with positive slope (red) that is not a candidate: An optimal watchman route in the subpolygon P_2 (marked in gray) needs to visit the same essential cut (dashed line) as an optimal watchman route in the subpolygon induced by $\overline{p_c p_f'}$ (purple).

that the interior of every strip between two consecutive such extensions is entered by w_1 and w_2. Then, we may shorten both w_1 and w_2 similar to the way we shorten in Case 4 (with the two vertical extensions of the entered strip playing the role of ℓ), without losing visibility, since both tours will see the complete rectangle spanned by the two extensions, leading to a contradiction. Hence, there exists such a strip. The diagonal connecting the two opposite reflex vertices of the corresponding extensions then intersects neither w_1 nor w_2. ◀

▶ **Lemma 3.4** ($\star$). *Any diagonal that is not a candidate diagonal induces a solution that is at least as long as the solution induced by some candidate diagonal.*

Proof. First, note that a diagonal of positive slope is spanned between two reflex vertices. Consider w.l.o.g. a non-candidate diagonal $\overline{p_f p_c}$, as seen in Figure 4. Then there is a convex vertex p_c' above p_c that does not yield a diagonal of p_f because $y(p_c) < y(p_f)$. The subpolygon P_2 above $\overline{p_f p_c}$ has the horizontal line through p_c' as an essential cut. Hence, the watchman route in P_2 has points below this cut. There exists a subpolygon induced by a candidate diagonal (incident to p_c and with the other endpoint p_f' below p_f) that also has the horizontal line through p_c' as an essential cut. For this cut, the watchman route in the subpolygon above the diagonal $\overline{p_c p_f'}$ remains the same, and the watchman route in the subpolygon below is not longer than the one induced by $\overline{p_f p_c}$. ◀