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Recognition Complexity of Subgraphs of k-Connected Planar Cubic Graphs

Anonymous author(s)
Anonymous affiliation(s)

Abstract

We study the recognition complexity of subgraphs of k-connected planar cubic graphs for $k \in \{1, 2, 3\}$. Recently, Goetze, Jungeblut and Ueckerdt presented [ESA 2022] a quadratic-time algorithm to recognize subgraphs of planar cubic bridgeless (but not necessarily connected) graphs, both in the variable and fixed embedding setting (the latter only for 2-connected inputs). Here, we extend their results in two directions: First, we present polynomial-time algorithms to recognize subgraphs of 1- and 2-connected planar cubic graphs in the fixed embedding setting, even for disconnected inputs. Second, we prove NP-hardness of recognizing subgraphs of 3-connected planar cubic graphs in the variable embedding setting.

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Keywords and phrases planar cubic graphs, k-connectedness, generalized factors, recognition problem, NP-hardness

1 Introduction

Given a planar graph G of maximum degree at most 3 (i.e., a *subcubic* planar graph), we want to augment G by adding further vertices and edges to obtain a 3-regular (i.e., *cubic*) planar graph H, which is then called a 3-*augmentation* of G. An embedding[1] of H induces an embedding $\mathcal{E}$ of G, and each face f of $\mathcal{E}$ may contain edges of $E(H) - E(G)$, called *new edges*, or even vertices of $V(H) - V(G)$, called *new vertices*, or none of these. It is easy to see that a 3-augmentation always exists, as ~~already~~ observed ~~in~~ [4]: by G., J. and U.

▶ **Observation 1.1** ([4]). *Every subcubic planar graph G has a 3-augmentation H extending its embedding. If G is connected, then so is H.*

However, this becomes non-trivial if we require the 3-augmentation H to be k-connected for some $k \in \{1, 2, 3\}$. See Figure 1 for some problematic cases. Here, we study whether a subcubic planar graph G admits some k-connected 3-augmentation H, or equivalently, the recognition of subgraphs of k-connected cubic planar graphs. We consider several variants where the input graph G is given with a fixed embedding $\mathcal{E}$ and the desired 3-augmentation H must extend $\mathcal{E}$, and/or where the input graph G is already k'-connected for some $k' \in \{0, 1, 2\}$. (If G is 3-connected, then $H = G$ is the only connected 3-augmentation.)

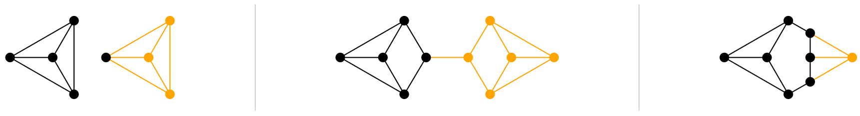
Previous Results. Motivated by the problem of deciding whether a given planar graph G is 3-edge-colorable (whose complexity is still open), Goetze, Jungeblut and Ueckerdt recently considered the sufficient (although not necessary) condition of whether G admits a bridgeless 3-augmentation. In fact, this can be tested in quadratic-time in the variable embedding setting, and also in the fixed embedding setting, provided that G is 2-connected [4].

Hartmann, Rollin and Rutter [5] studied, for each $k, r \in \mathbb{N}$, whether a planar graph G can be augmented by adding edges (but no vertices!), to a k-connected r-regular planar

[1] We consider combinatorial (crossing-free) embeddings with no specific choice of an outer face.

Planar 2-Vertex Connectivity Augmentation (in the variable embedding setting, without vertex addition and without degree restriction) was shown NP-hard by Kant and Bodlaender (WADS 1991).

Planar 2-Edge Connectivity Augmentation (as above) was shown NP-hard by Rutter and Wolff (JGAA 2012). They also investigated a geometric version (i.e., a specific fixed embedding setting) of both problems.



$G_1 \subset H_1$ $G_2 \subset H_2$ $G_3 \subset H_3$

Figure 1 For $k = 1, 2, 3$, the planar subcubic graph G_k (in black) admits a $(k-1)$-connected 3-augmentation H_k (new vertices and edges in orange), but no k-connected 3-augmentation.

graph H. In particular, for $r = 3$, they show that the problem is NP-complete in the variable embedding setting for all $k \in \{0, 1, 2, 3\}$, as well as in the fixed embedding setting when $k = 3$. For the remaining cases of fixed embedding and $k \in \{0, 1, 2\}$, they present a polynomial-time algorithm.

of Hartmann et al.

▶ Remark 1.2. In fact, several concepts and techniques ~~in~~ [5] are very similar to ours. In case of G having a fixed embedding $\mathcal{E}$, any 3-augmentation H (with or without new vertices) extending $\mathcal{E}$ induces an assignment of each new edge e that is incident to an "old" vertex of G to the face of $\mathcal{E}$ that contains e. These are called *free valencies* in [5]. what?
Hartmann et al. ~~The authors of~~ [5] present conditions of this assignment that are necessary and sufficient for a connected or 2-connected 3-augmentation *without new vertices*. (~~and which~~ also allow for a polynomial-time algorithm to find such an assignment). Their *matching condition* and *planarity condition* become obsolete in our setting. However, their *connectivity condition* and *biconnectivity condition* demand roughly twice as many free valencies to be assigned to a face with several connected components. (Intuitively, these components must be stringed together in [5], while we can do a star-like connection.) Most crucially, their *parity condition*, which requires the number of free valencies assigned to each face to be even, is no longer necessary nor sufficient in our setting. It is for example violated in every example in Figure 1. Their conditions strung?

One might be tempted to find a connected or biconnected 3-augmentation in ~~our setting,~~ the ~~with~~ fixed embedding setting, by preprocessing the input by inserting extra vertices, so as to always fulfil the parity condition (and then handle the connectivity or biconnectivity condition somehow). A reasonable attempt would be to subdivide some edges in the input graph. However, this might turn a No-instance into a Yes-instance, as shown for example in Figure 2.

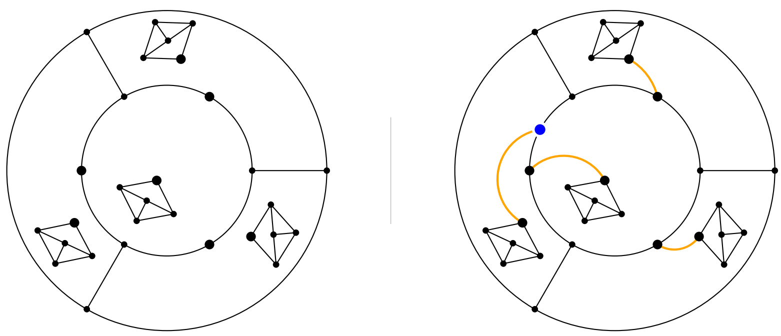


Figure 2 Left: A graph G with no connected 3-augmentation extending its embedding. Right: After adding an extra degree-2 vertex (blue) to G, there is a connected 3-augmentation.

Finally, for $k \leq 2$, finding a k-connected 3-augmentation in the variable embedding setting is in P [4], while the version without new vertices is NP-complete [5]. So to summarize, there is probably no direct reduction between these problems. ◀ which?

We refer to [4] for more related work and other augmentation problems.

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Our Results. We resolve the complexity of finding a k-connected 3-augmentation for a given subcubic planar graph G (with or without a given embedding), except when $k = 3$ and the embedding of G is given. See also Figure 3. For a graph G, let $\Delta(G)$ be the max. deg. of G.

▶ **Theorem 1.3.** *Let G be a ~~an n-vertex~~ planar graph with ~~maximum degree~~ $\Delta(G) \leq 3$. ~~and embedding $\mathcal{E}$~~.* Let n be the number of vertices of G, and let $\mathcal{E}$ be the embedding of G.

1. *We can compute, in time $\mathcal{O}(n^2)$, a connected 3-augmentation H extending $\mathcal{E}$, or conclude that none exists.*
2. *We can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H extending $\mathcal{E}$, or conclude that none exists. If G is connected, $\mathcal{O}(n^2)$ time suffices.*
3. *It is* NP*-complete to decide whether G admits a 3-connected 3-augmentation.*

Note that Statements 1 and 2 concern the fixed embedding setting, while Statement 3 concerns the variable embedding setting.

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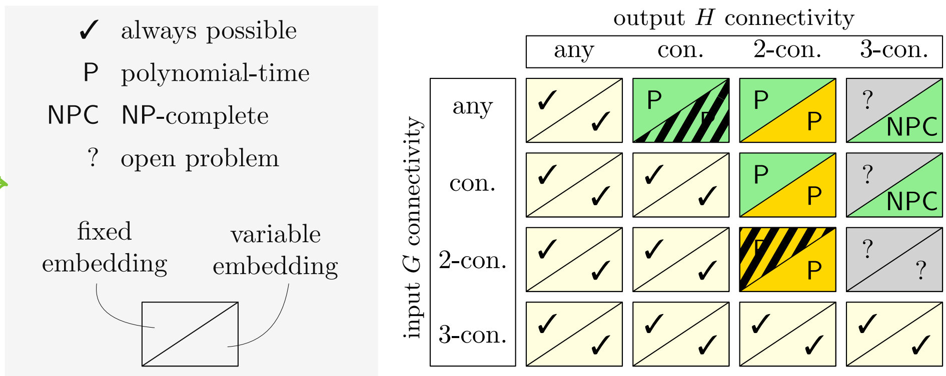

Figure 3 Complexity of finding 3-augmentations. Colors: In this paper and in [4].

▶ Remark 1.4. For our polynomial-time algorithms in the fixed embedding setting, we shall reduce the problem to a particular version of the GENERALIZED FACTOR problem (definitions in Section 2). This approach is similar to the treatment of 2-connected input graphs in [4]. But here, for graphs containing bridges or consisting of several connected components, additional tools and a more refined analysis are needed, which is also reflected in the increased runtime.

2 Preliminaries

A graph G ~~$= (V, E)$~~ is *k-connected* if $G - S$ is connected for every set $S \subseteq V(G)$ of at most $k-1$ vertices in G. Similarly, G is *k-edge-connected* if $G - S$ is connected for every set $S \subseteq E(G)$ of at most $k-1$ edges in G. We denote by $\theta(G)$ the largest k for which G is k-edge-connected. If G has maximum degree at most 3, then G is k-connected if and only if $\theta(G) \geq k$.

For an integer $\ell \geq 3$, let W_ℓ be the graph obtained from $C_\ell \square P_2$ by subdividing each edge in one cycle C_ℓ exactly once. See Figure 4 (left) for an illustration. Consider a planar graph G with an embedding $\mathcal{E}$, and a vertex $v \in V(G)$ with $\deg_G(v) = \ell \geq 3$. A *wheel-extension at v* is the graph and the embedding obtained by replacing v with W_ℓ, and by attaching v's incident edges to the subdivision vertices of W_ℓ in a one-to-one non-crossing way. See Figure 4 (right).

▶ **Observation 2.1.** *Let G be a graph (possibly with multi-edges, but no loops), let $v \in V(G)$ be a vertex with $\deg_G(v) \geq 3$, and let G' be obtained from G by a wheel-extension at v. Then $\theta(G') \geq \min\{\theta(G), 3\}$.*

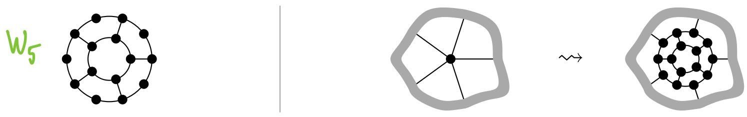

Figure 4 Left: $C_5 \square P_2$ with subdivision vertices. Right: Wheel-extension.

Proof. If $\theta(G') \leq 2$ (otherwise there is nothing to show), let S be an edge-cut of size $\theta(G') \leq 2$ in G'. As S is minimal, S does not consist of the two edges at a subdivision vertex of W_ℓ. Thus, as $C_\ell \square P_2$ is 3-connected, it follows that $S \cap E(W_\ell) = \emptyset$. But then, S is also an edge-cut in G and hence $\theta(G) \leq |S| = \theta(G')$, as desired. ◀

Generalized Factors. Let H be a graph with a set $B(v) \subseteq \{0, \dots, \deg_H(v)\}$ assigned to each vertex $v \in V(H)$. Following Lovász, a spanning subgraph $G \subseteq H$ is called a *B-factor* of H if and only if $\deg_G(v) \in B(v)$ for every vertex $v \in V(H)$ [6]. Deciding whether a graph H admits a B-factor is known as the Generalized Factor problem. In general, the Generalized Factor problem is NP-complete [6]. Still, for certain well-behaved sets $B(\cdot)$, the problem becomes polynomial-time solvable. A set $B(v)$ is said to have a *gap of length* $\ell \geq 1$ if there is an integer $i \in B(v)$ such that $i+1, \dots, i+\ell \notin B(v)$, and $i+\ell+1 \in B(v)$. If all gaps of each $B(v)$ have length 1, then an algorithm by Cornuéjols can compute a B-factor in time $\mathcal{O}(|V(H)|^4)$ [2]. Moreover, if there are no two consecutive forbidden degrees $\{i, i+1\} \subseteq \{0, \dots, \deg_H(v)\}$ for any v, i.e., $\{i, i+1\} \not\subseteq B(v)$, then a B-factor can be computed in time $\mathcal{O}(|V(H)| \cdot |E(H)|)$ by a result of Sebő [7]. (The latter condition is slightly stronger than requiring gaps of length at ~~most~~ 1, explain~~ing~~s the better runtime.)

, Section 3 which

3 2-Connected 3-Augmentations for a Fixed Embedding

plural singular

We consider the 3-augmentation problem for arbitrary input graphs G and 2-connected output graphs H, corresponding to the third column of the table in Figure 3. For the variable embedding setting, a quadratic-time algorithm is given ~~in~~ [4, Theorem 2]. For the fixed embedding setting here, we present a quartic-time algorithm. We start with a reduction to graphs G with $\delta(G) \geq 2$.

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▶ **Lemma 3.1.** *Let G be a planar graph with embedding $\mathcal{E}$. There is a planar supergraph $G' \supseteq G$ with $\delta(G') \geq 2$ whose embedding $\mathcal{E}'$ extends $\mathcal{E}$, such that G has a 2-connected 3-augmentation extending $\mathcal{E}$ if and only if G' has one extending $\mathcal{E}'$.*

Proof. Consider the following two replacement rules, also shown in Figure 5 (left/middle): Each isolated vertex is replaced by a copy of K_3, and each vertex v of degree 1 is replaced by a copy of K_3 with one vertex connected to the other neighbor of v. Let G' be the obtained graph such that its planar embedding $\mathcal{E}'$ extends $\mathcal{E}$.

Let H be a 2-connected 3-augmentation of G. We obtain a 2-connected 3-augmentation of G' as follows: For each vertex v of degree 0 (or 1) in G, let $N(v)$ be its three (two) new neighbors in H. In H, replace v by its corresponding copy of K_3. Connect its three (two) degree-2-vertices with one vertex of $N(v)$ such that the embedding remains planar.

The other direction works similar: In a 2-connected 3-augmentation of G', contract each copy of K_3 that was introduced for a vertex v of G into a single vertex. If this creates multi-edges, replace each duplicated edge by the gadget shown in Figure 5 (right) to obtain a simple graph. ◀

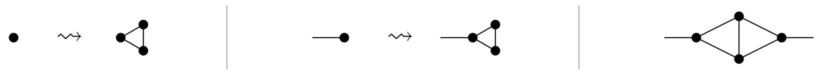


Figure 5 Left/Middle: Replacement rules. Right: Gadget to avoid parallel edges.

▶ **Lemma 3.2.** *Let G be a planar n-vertex graph with an embedding $\mathcal{E}$, $\delta(G) \geq 2$, and $\Delta(G) \leq 3$. Then we can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H of G extending $\mathcal{E}$, or conclude that none exists. If G is connected, then time $\mathcal{O}(n^2)$ suffices.*

Proof. The proof is by a linear-time reduction to an equivalent instance A of the Generalized Factor problem, such that A fulfills the necessary condition to apply the ~~an~~ $\mathcal{O}(n^4)$-time algorithm by Cornuéjols [2, ~~Section 3~~], or even the ~~an~~ $\mathcal{O}(n^2)$-time algorithm by Sebő [7, ~~Section 3~~]; see Section 2.

We construct the 2-connected 3-augmentation H of G by adding new edges and vertices into the faces of $\mathcal{E}$. Therefore, the obtained embedding of H extends $\mathcal{E}$.

Some faces of $\mathcal{E}$ stand out, as these *must* contain new edges (and possibly vertices) to reach 2-connectedness. We call the set of these ~~the~~ *connecting faces* F_c. Obviously, all faces incident to at least two connected components are connecting faces. Further, for each bridge e of G, the unique face f incident to both sides of e is a connecting face because the only way to add new connections between the components separated by e is through f. Recall that a 3-regular graph is 2-connected if and only if it is connected and bridgeless, so these are the only two types of connecting faces. All other faces are considered to be *normal* faces, denoted by F_n.

A variable c does not exist. I suggest to write F_conn and ... {norm}

Let the set of normal faces be

Can you give some intuition here (as in the last paragraph)?

For a connecting face the set of $f \in F_c$, let G_f be the subgraph of G on the vertices and edges incident to f, let B_f be its blocks (i.e., maximal 2-connected components or bridges), and let T_f be its block-cut-forest. We partition B_f into $S_f \cup I_f \cup L_f$, where we call S_f the set of *singleton blocks*, I_f the set of *inner blocks*, and L_f the set of *leaf blocks*:

$$S_f := \{b \in B_f \mid b \text{ forms a trivial (i.e., single-vertex) tree in } T_f\}$$
$$I_f := \{b \in B_f \mid b \text{ is an inner vertex of a non-trivial tree in } T_r\}$$
$$L_f := \{b \in B_f \mid b \text{ is a leaf in a non-trivial tree in } T_f\}$$

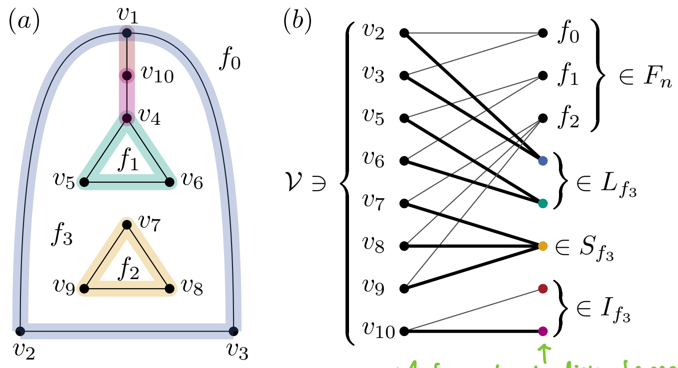
The Generalized Factor instance A is a bipartite graph with bipartition classes $\mathcal{V}$ and $\mathcal{F}$. Here, $\mathcal{V} := \{v \in V(G) \mid \deg_G(v) = 2\}$ contains all vertices of G not yet having degree 3. Similarly, vertices in $\mathcal{F}$ represent the faces of $\mathcal{E}$. Edges of a B-factor of A will determine the faces of $\mathcal{E}$ containing the new edges. In particular, $\mathcal{F}$ contains one vertex corresponding to each normal face in F_n of $\mathcal{E}$. Additional vertices in $\mathcal{F}$ are needed to handle the connecting faces. For each connecting face $f \in F_c$, we add all blocks in B_f as vertices to $\mathcal{F}$. (If there are two faces f, g in $\mathcal{E}$ such that B_f and B_g contain blocks corresponding to the same subgraph of G, then $\mathcal{F}$ contains two such vertices: one corresponding to the block in B_f, and another to the block in B_g.)

You are saying the same thing twice.

In A, each $x \in \mathcal{F}$ is incident to exactly the following $v \in \mathcal{V}$: If x ~~is~~ represents a normal face f~~$_n \in F_n$~~, then x is connected to ~~all~~ every $v \in \mathcal{V}$ that ~~are~~ is incident to f~~$_n$~~ in $\mathcal{E}$. Otherwise, if x ~~is~~ represents a block $b \in B_f$ for some connecting face ~~$f \in F_c$~~ f, then x is connected to ~~all~~ every $v \in \mathcal{V}$ that ~~are~~ is contained in b. See Figure 6 for an example.

Lastly, we need to assign a set $B(x) \subseteq \{0, 1, \ldots, \deg_A(x)\}$ of possible degrees to each

Very helpful figure!



Make colored disks larger and use same transparency as in (a).

Figure 6 (a) A planar subcubic graph G. (b) Its corresponding Generalized Factor instance. Thick edges denote a possible solution. (c) A 2-connected 3-augmentation of G (with an indicated wheel-extension). the unique?

vertex $x \in (\mathcal{V} \cup \mathcal{F})$:

What about using the notation $[k]_0$ for $\{0,1,\dots,k\}$? Then you can write, e.g., $[\deg_A(x)]_0 \setminus \{1\}$

$$B(x) := \begin{cases} \{1\}, & x \in \mathcal{V} \\ \{0,2,3,\dots,\deg_A(x)\}, & x \in F_n \\ \{0,1,2,\dots,\deg_A(x)\}, & x \in I_f \text{ for some connecting face } f \in F_c \\ \{1,2,3,\dots,\deg_A(x)\}, & x \in L_f \text{ for some connecting face } f \in F_c \\ \{2,3,4,\dots,\deg_A(x)\}, & x \in S_f \text{ for some connecting face } f \in F_c \end{cases}$$

By the following claim, the above reduction is linear.

▷ Claim 3.3. The order and size of A is linear in n. Moreover, A can be computed in linear time.

Proof. If $v \in V(G)$ is a vertex incident to a face f of $\mathcal{E}$, then it lies in at most three blocks of G_f, since its degree in G_f is at most 3. Further, every vertex is incident to at most three different faces of $\mathcal{E}$. Thus, there are at most nine blocks in B_f containing v, which shows that $|B_f|$ is linear in n. As the number of faces of a planar embedding is linear in n, so is $|A|$. A vertex $v \in \mathcal{V}$ is incident to (at most) two faces of $\mathcal{E}$, and therefore it is contained in at most six distinct blocks in B_f for some faces f. Hence, we see that each vertex $x \in \mathcal{V}$ is adjacent to at most six vertices in $B_f \subseteq \mathcal{F}$ and at most two vertices in $F_n \subseteq \mathcal{F}$. Thus, the bipartite graph A contains at most $8n$ edges. Note that, in particular, A can be computed in linear time. ◁

The next two claims establish that A admits a B-factor if and only if G admits a 2-connected 3-augmentation H extending $\mathcal{E}$.

▷ Claim 3.4. If A admits a B-factor, then G has a 2-connected 3-augmentation H extending $\mathcal{E}$.

Proof. Let A' be a B-factor of A, i.e., a subgraph A' such that $\deg_{A'}(x) \in B(x)$ for every ~~all~~ $x \in V(A)$. We construct a connected and bridgeless supergraph H' of G as follows: For each edge $vx \in E(A')$ with $v \in \mathcal{V}$ and $x \in \mathcal{F}$, we add a new half-edge from v into a face f of $\mathcal{E}$. If x is a face, then $f = x$. Otherwise, let f be the face such that x is a block in B_f. represents

Now, for each face f of $\mathcal{E}$, all half-edges ending inside f are connected to a new vertex v_f. Obviously, H' is planar. To see that H' is connected, consider a connecting face f of $\mathcal{E}$. We have $\deg_{A'}(b) \geq 1$ for ~~all~~ every $b \in \cancel{(}S_f \cup L_f\cancel{)}$, so each such b is connected to v_f by at least one edge. Lastly, to prove that H' is bridgeless, we consider three cases:

- A non-bridge of G is a non-bridge in H' as $H' \supseteq G$.
- A bridge e of G has a unique face f of $\mathcal{E}$ incident to both its sides. The leaf blocks in L_f are subgraphs of the blocks separated by e. As we have $\deg_{A'}(b) \geq 1$ for all $b \in L_f$, there is at least one edge from each leaf block to v_f in H'. Thus, e is a non-bridge in H'.
- No edge incident to a new vertex v_f (for some face f of $\mathcal{E}$) is a bridge, because v_f has at least two edges to every incident component of H': If f is a normal face, then $\deg_{H'}(v_f) \geq 2$ because $\deg_{A'}(f) \neq 1$. Now assume that f is a connecting face, and consider a component C of G_f. If C consists of a single block b (which would be in S_f), then v_f is connected to at least two vertices of b, because we have $\deg_{A'}(b) \geq 2$. Otherwise, if C consists of multiple blocks, then its block-cut-tree has at least two leaves. In this case, v_f is connected to at least one vertex per leaf block $b \in L_f$ because $\deg_{A'}(b) \geq 1$.

Since $\deg_{A'}(v) = 1$ for each $v \in \mathcal{V}$, we see that all vertices in $V(G)$ have degree 3 in H'. We apply a wheel-extension (Observation 2.1) at each new vertex v_f of degree larger than 3, and replace each vertex v_f of degree 2 by the gadget represented on the right of Figure 5 (this simulates replacing it with a single edge connecting its neighbors, but without the risk of creating a multi-edge). We obtain a 3-regular graph H that is planar, connected, and bridgeless. Further, H is 2-connected, because a connected graph with maximum degree 3 is 2-connected if and only if it is bridgeless. $\lhd$

$\rhd$ Claim 3.5. If G has a 2-connected 3-augmentation H extending $\mathcal{E}$, then A has a B-factor.

Proof. Since H extends $\mathcal{E}$, its new vertices and edges must have been added solely into the faces of $\mathcal{E}$.

We have to construct a B-factor A' of A. To this end, we consider the vertices $v \in \mathcal{V}$, i.e., vertices of G with $\deg_G(v) = 2$. Each such v has $\deg_H(v) = 3$, so there is exactly one new edge e incident to v. Let f be the face of $\mathcal{E}$ that e is inside. If f is a normal face, we add the edge vf to A'. Now assume that f is a connecting face. As $\deg_G(v) = 2$, vertex v can be in at most two blocks of G_f. If v is in a singleton block, then this is the only block in B_f containing v. If v lies in exactly two leaf blocks $b_1, b_2 \in B_f$, then both must be bridges, whose other endpoints have degree 1; a contradiction to $\delta(G) \geq 2$. Thus, ~~we may note~~ there is at most one block in $L_f \cup S_f$ containing v. If there exists a block in $L_f \cup S_f$ containing v, let b be this (unique) block. Otherwise, choose an arbitrary $b \in B_f$ containing v. We add the edge vb to A'.

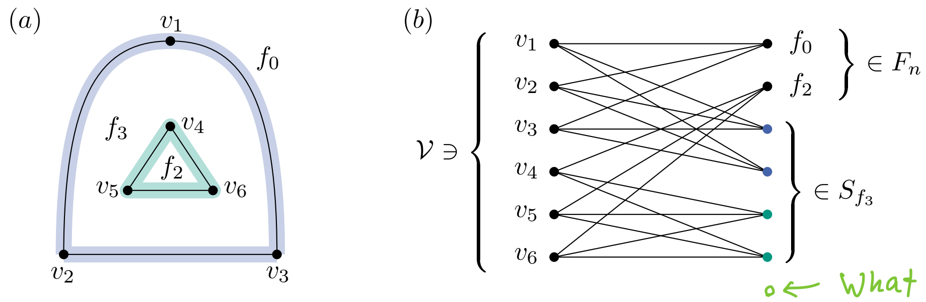
We prove that $\deg_{A'}(x) \in B(x)$ for all $x \in \cancel{(}\mathcal{V} \cup \mathcal{F}\cancel{)}$. For each vertex $v \in \mathcal{V}$, we added exactly one edge to A', therefore we have $\deg_{A'}(v) = 1$ as required.

For a normal face f ($\in F_n$), it holds that $\deg_{a'}(f)$ is either 0 or at least 2 because H is 2-connected and therefore there are either no or at least two new edges inside f.

Now, consider a connecting face f ($\in F_c$). Each $b \in S_f$ is a singleton block of G_f. Since H is 2-connected, there are (at least) two paths leaving different vertices $v_1, v_2 \in b$ via new edges through f. Therefore, A' contains the edges $v_1 b$ and $v_2 b$, i.e., $\deg_{A'}(b) \geq 2$ as required. Similarly, each $b \in L_f$ is a leaf-block. Since H is 2-connected, there is (at least) one path leaving $v \in b$ via a new edge through f. Therefore, we have $vb \in E(A')$ and thus $\deg_{A'}(b) \geq 1$ as required. $\lhd$

each

It remains to argue that we can compute a B-factor of A efficiently. By inspecting the set~~s~~ $B(x)$ for ~~all~~ $x \in \cancel{(}\mathcal{V} \cup \mathcal{F}\cancel{)}$, we can see that none of them contains a gap of size 2 or greater. Therefore, we are in a special case of the Generalized Factor problem that can be solved, in $\mathcal{O}(n^4)$ time, by Cornuéjols' algorithm [2].



o ← What about placing f_2 here?

Figure 7 (a) A planar subcubic graph G. (b) Its modified Generalized Factor instance where each singleton block appears twice.

A closer inspection yields that only for $x \in S_f$ the sets $B(x)$ contain two forbidden degrees. (Note that $\deg_A(v) \leq 2$ for all $v \in \mathcal{V}$: If there is a face f such that v is contained in two blocks of G_f, then both edges incident to v are bridges; thus v is incident to no other face. Otherwise, this follows from $\deg_G(v) \leq 2$, i.e., v being incident to at most two faces.) Therefore, if $S_f = \emptyset$ for ~~all~~ every connecting face~~s~~ $f \in F_C$, then we can even apply the algorithm by Sebő, ~~taking~~ which takes only $\mathcal{O}(n^2)$ time [7]. In particular, this is the case if G is connected. ◀

▶ Remark 3.6. The attentive reader might be tempted to think that we can modify the Generalized Factor instance in the proof above so that it satisfies the conditions of Sebő, even when $S_f \neq \emptyset$. One such attempt resides consists (?) in splitting each vertex x representing a singleton block into two vertices x_1, x_2 with associated sets $B(x_1), B(x_2)$ which only exclude the value 0. Indeed, the sets $B(x_1), B(x_2)$ fulfill the condition of Sebő, but now some of the vertices in $\mathcal{V}$ might not; see Figure 7. In the obtained Generalized Factor instance, all vertices in $\mathcal{V}$ have degree 3, yet $B(x) = \{1\}$ for ~~all~~ every $x \in \mathcal{V}$. (Why is this a problem? See line 175.)

4 Connected 3-Augmentations for a Fixed Embedding

In this section, we ~~consider~~ for arbitrary (and possibly disconnected) input graphs~~, G, finding~~. find connected 3-augmentations ~~H~~. Refer to the second column of the table in Figure 3. For the variable embedding setting, the problem can be solved in linear time (using the same argument as ~~in~~ Goetze et al. [4, Proposition 4]):

▶ **Observation 4.1.** *A planar subcubic graph G admits a connected 3-augmentation, unless G is disconnected and has a 3-regular component.*

~~In this paper,~~ We now present a quadratic-time algorithm for the fixed embedding setting. By Observation 1.1, this is equivalent to deciding whether a given embedding can be extended to a subcubic planar connected graph.

▶ **Lemma 4.2.** *Let G be a planar subcubic ~~n-vertex~~ graph with an embedding $\mathcal{E}$. Then we can compute, in* quadratic *time ~~$\mathcal{O}(n^2)$~~, a connected subcubic planar supergraph H of G extending $\mathcal{E}$, or conclude that none exists.*

Essentially, the proof of Lemma 4.2 is a simpler version of the proof of Lemma 3.2.

Proof. Using the same ideas as in the proof of Lemma 3.2, we reduce the problem of finding a connected supergraph which extends $\mathcal{E}$, to an instance of the Generalized Factor problem A which fulfills the necessary condition to apply ~~an $\mathcal{O}(n^2)$~~ the quadratic-time algorithm by Sebő [7, Section 3].

We construct the supergraph H of G by adding vertices and edges to some faces of $\mathcal{E}$. Thus, the obtained embedding of H extends $\mathcal{E}$.

In order to obtain a connected supergraph, faces of $\mathcal{E}$ which are incident to at least two connected components *must* contain new edges (and possibly vertices). We call ~~these~~ these the set of *component-connecting faces* F_{cc}.

For a component-connecting face f ($\in F_{cc}$), we denote by G_f the subgraph of G on the vertices and edges incident to f (using the same notation as in the proof of Lemma 3.2).

The Generalized Factor instance A is a bipartite graph with bipartition classes $\mathcal{V}$ and $\mathcal{F}$. Here, $\mathcal{V} := \{v \in V \mid \deg_G(v) \leq 2\}$ contains all vertices of G not yet having degree 3. Similarly, vertices in $\mathcal{F}$ represent component-connecting faces of $\mathcal{E}$. For each component-connecting face f ($\in F_{cc}$), we add all components of G_f as vertices to $\mathcal{F}$. (If there are two faces f, g in $\mathcal{E}$ such that G_f and G_g contain two components corresponding to the same subgraph of G, then $\mathcal{F}$ contains two such vertices: one corresponding to the component of G_f, and another to the component of G_g.)

to what?

Each $c \in \mathcal{F}$ corresponds to a component of G_f for some face of $\mathcal{E}$. In the graph A, c is incident to all $v \in \mathcal{V}$ which are incident to the component c in $\mathcal{E}$. It remains to assign a set $B(x) \subseteq \{0, 1, \ldots, \deg_A(x)\}$ of possible degrees to each vertex $x \in (\mathcal{V} \cup \mathcal{F})$:

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$$B(x) := \begin{cases} \{k \in \mathbb{N}_0 \mid k \leq \min(3 - \deg_G(x), \deg_A(x))\}, & \text{if } x \in \mathcal{V} \\ \{1, 2, 3, \ldots, \deg_A(x)\}, & \text{if } x \in \mathcal{F}, \end{cases}$$

i.e., each vertex $x \in \mathcal{V}$ can be incident to up to $3 - \deg_G(x)$ edges, and each $c \in \mathcal{F}$ is incident to at least one edge in any B-factor.

We now observe that the order and size of A are linear in n: Every vertex $u \in \mathcal{V}$ is incident to at most two faces of $\mathcal{E}$ as $\deg_G(u) \leq 2$. Since u belongs to at most one component of G_f for each component-connecting face f, we obtain $|\mathcal{F}| \leq 2n$ and $\deg_A(v) \leq 2$ for ~~all~~ every $v \in \mathcal{V}$. In particular, it follows that $|E(A)| \leq 2n$. Note that A can be computed in linear time.

Using a similar argument as in Lemma 3.2, we observe that A admits a B-factor if and only if G has a connected subcubic planar supergraph H extending $\mathcal{E}$.

Recall that $\deg_A(v) \leq 2$ for all $v \in \mathcal{V}$. Thus, for each $v \in \mathcal{V}$, the set $B(v)$ excludes at most one possible degree. As the same holds for all vertices in $\mathcal{F}$, we can apply the algorithm by Sebő [7]. We can therefore compute a B-factor of A in time $\mathcal{O}(n^2)$. ◀

▶ Remark 4.3. The Generalized Factor instance we constructed in the proof above can be reduced to an instance of Maximum Flow, achieving a better, namely almost-linear [1], runtime. The graph of the flow instance is obtained by adding two vertices s and t (namely the source and sink) and connecting s to all vertices in $\mathcal{V}$ and t to all vertices in $\mathcal{F}$. Edges incident to s are outgoing, edges incident to t are incoming. Edges between $\mathcal{V}$ and $\mathcal{F}$ are oriented from $\mathcal{V}$ to $\mathcal{F}$. The edge capacities of edges sx with $x \in \mathcal{V}$ encode the sets $B(x) = \{0, 1, 2, \ldots, 3 - \deg_G(x)\}$. This is achieved by setting the edge capacity to $3 - \deg_G(x)$. All other edges have a capacity of 1. It can be easily verified that the obtained graph admits an s-t flow of value at least $|\mathcal{F}|$ if and only if A admits a B-factor.

Then I would make a theorem with this runtime. The proof of this theorem would simply combine the proofs of Lemma 4.2 and Remark 4.3.

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5 NP-Hardness for 3-Connected 3-Augmentations

In this section, we ~~shall~~ prove that deciding whether a given planar graph G admits a 3-connected 3-augmentation is NP-complete. In particular, we show that the problem remains NP-complete when restricted to connected graphs G. This implies the NPC-results represented in the fourth column of the table in Figure 3.

Recall that an embedding of any 3-connected 3-augmentation H induces an embedding $\mathcal{E}$ of G. ~~and~~ for convenience, ~~let us~~ we call the pair $(H, \mathcal{E})$ a *solution* for G. Let us also define a (≤ 2)-*subdivision* of a graph R to be the result of subdividing each edge in R with up to two vertices. Note that, if R is 2-connected, then so is every (≤ 2)-subdivision of R.

▶ **Lemma 5.1.** *Let G be a graph obtained from a (≤ 2)-subdivision R_2 of a 3-connected planar graph R by attaching a degree-1 vertex to each subdivision vertex. Then G admits a solution $(H, \mathcal{E})$ if and only if no face of $\mathcal{E}$ has exactly one or two incident degree-1 vertices.*

Proof. First, assume that $(H, \mathcal{E})$ is a solution for G. Assume for the sake of contradiction that f is a face of $\mathcal{E}$ incident to a set S of exactly one or two degree-1 vertices of G. As H is 3-regular, each vertex in S is incident to two new edges in f. But then, S forms a vertex-cut of cardinality at most 2 in H; a contradiction to H being 3-connected.

For the other direction, let $\mathcal{E}$ be an embedding of G in which no face has exactly one or two incident degree-1 vertices. Our task is to find a solution $(H, \mathcal{E})$ for G, i.e., to insert new vertices and new edges into the faces of $\mathcal{E}$ to obtain a 3-connected 3-regular planar graph H.

To this end, consider any face f of $\mathcal{E}$. If f has no incident degree-1 vertices of G, we insert nothing in f. Otherwise, f has at least $\ell \geq 3$ incident degree-1 vertices, and we identify all these vertices into one vertex v_f of degree ℓ. Let H_1 be the planar graph we obtain by doing this for all faces of $\mathcal{E}$. Clearly, H_1 is planar, $\delta(H_1) \geq 3$, and $R_2 \subset H_1$.

We claim that H_1 is 3-edge-connected, i.e., $\theta(H_1) \geq 3$. First, H_1 is connected, as R_2 is connected. It remains to show that the plane dual H_1^* of H_1 has no loops (i.e., H_1 has no bridges) and no pairs of parallel edges (i.e., H_1 has no 2-edge-cuts). For this, consider any edge e^* of H_1^* and its primal edge e of H_1. If $e \notin E(R_2)$, then e is incident to a vertex v_f of degree $\ell \geq 3$ in a face f of $\mathcal{E}$. In this case e^* is neither a loop nor has a parallel edge in H_1^*.

If $e \in E(R_2)$, then e^* is not a loop, since R_2 is 2-connected. It remains to rule out that two edges $e_1, e_2 \in E(R_2)$ form a 2-edge-cut, i.e., their dual edges e_1^*, e_2^* in H_1^* are parallel. Let f, f' be the two faces of $\mathcal{E}$ incident to e_1 and e_2. As R is 3-connected, e_1 and e_2 both originate from the subdivision(s) of the same edge e_R of R. Consider a subdivision vertex of R_2 between e_1 and e_2. Let v be its new neighbor in H_1; say $v = v_f$ for face f. Then v_f has at least two further neighbors, at least one of which is not a subdivision vertex of e_R, because e_R is subdivided at most twice. But then in the dual H_1^*, the edges e_1^* and e_2^* are incident to different vertices inside f; hence are not parallel.

Finally, we apply a wheel-extension to every vertex v_f, resulting in a planar 3-regular graph H. Further, H contains G as a subgraph and Observation 2.1 yields $\theta(H) \geq \min(\theta(H_1), 3) = 3$. In other words, H is the desired 3-connected 3-augmentation of G. ◀

By Lemma 5.1, any graph G as described in the lemma admits a 3-connected 3-augmentation if and only if it admits an embedding $\mathcal{E}$ with no face incident to exactly one or two degree-1 vertices. Testing such graphs for such embeddings, however, turns out to be NP-complete.

▶ **Theorem 5.2.** *Deciding whether a given graph is a subgraph of a 3-regular 3-connected planar graph is* NP*-complete.*

Proof. First, we show that the problem is in NP. Let G be a graph that admits a 3-connected 3-augmentation H. We need to show that G also admits a 3-connected 3-augmentation whose size is polynomial in the size of G. To this end, consider the subgraph N of H induced by all new vertices. In H, contract each connected component of N into a single vertex, keeping

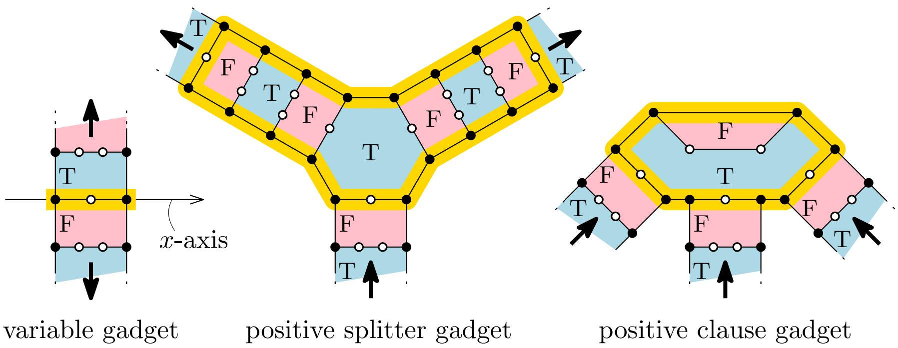

Figure 8 Gadgets used in the NP-hardness reduction.

parallel edges but removing loops. The resulting graph H' is planar and 3-edge-connected, since so was H. Note in particular that the vertices we obtained by contraction have degree at least 3. Next, we apply a wheel-extension to each vertex obtained from the contractions that has degree larger than 3. By Observation 2.1, the resulting graph H'' is 3-regular and 3-edge-connected, hence also 3-connected; in particular, a 3-connected 3-augmentation of G. Moreover, each vertex in H'' has distance at most three to some vertex of G and the maximum degree is bounded by 3, and thus H'' has only $O(|V(G)|)$ many vertices. Thus, our decision problem is in NP.

To show NP-hardness, we reduce from PLANAR-MONOTONE-3SAT. An instance of PLANAR-MONOTONE-3SAT is a monotone 3SAT-formula Ψ together with its bipartite variable-clause incidence graph I_Ψ and a planar embedding $\mathcal{E}_\Psi$ of I_Ψ. Each clause in Ψ contains either only positive literals (called a *positive clause*) or only negative literals (called a *negative clause*). Moreover, Ψ and the given embedding $\mathcal{E}_\Psi$ satisfy the following:

(then the clause is called positive)

- Each variable lies on the x-axis and no edge crosses the x-axis.
- Positive clauses lie above the x-axis, negative clauses lie below the x-axis.
- Each clause has at most three literals.

It is known that PLANAR-MONOTONE-3SAT is NP-complete [3]. Note that the problem remains NP-hard if we assume that each clause contains exactly three literals and that each variable appears in at least one positive and at least one negative clause. The latter can be achieved by the following reduction rule: Any variable that only appears in positive (negative) form can be set to TRUE (FALSE).

Next, we ~~shall~~ construct a planar graph G_Ψ that admits a 3-connected 3-augmentation if and only if there exists a truth assignment of the variables in Ψ satisfying all clauses. The graph G_Ψ is obtained from the embedding $\mathcal{E}_\Psi$ of I_Ψ using the gadgets illustrated in Figure 8. In particular, we replace each variable by a copy of the variable gadget in Figure 8 (left). As illustrated by the arrows in Figure 8, these gadgets form the starting point of one *upper* and one *lower corridor* for each variable. Each corridor consists of alternating red (standing for false) and blue (standing for true) faces. Above the x-axis, for each variable with k occurrences in positive clauses, we use $k-1$ positive splitter gadgets, see Figure 8 (middle), to split its upper corridor into k upper corridors. Then we replace each positive clause by a copy of the positive clause gadget in Figure 8 (right), which forms the end of one corridor for each variable appearing in the clause. These corridors are routed without overlap and entirely above the x-axis by following the given embedding $\mathcal{E}_\Psi$ of I_Ψ. We proceed symmetrically below the x-axis, with red and blue swapped, using otherwise isomorphic negative splitter gadgets and negative clause gadgets. See Figure 9 for a full example.

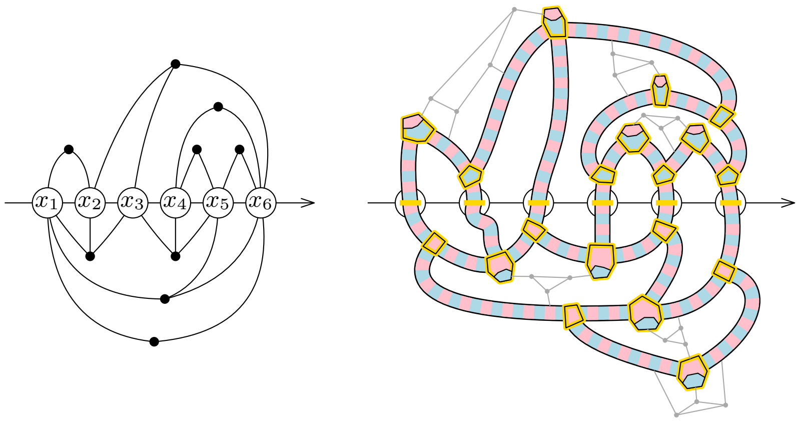

Figure 9 Illustration of a PLANAR-MONOTONE-3SAT embedding $\mathcal{E}_\Psi$ and a corresponding graph G_Ψ. Extra vertices and edges added for 3-connectivity of R are shown in gray.

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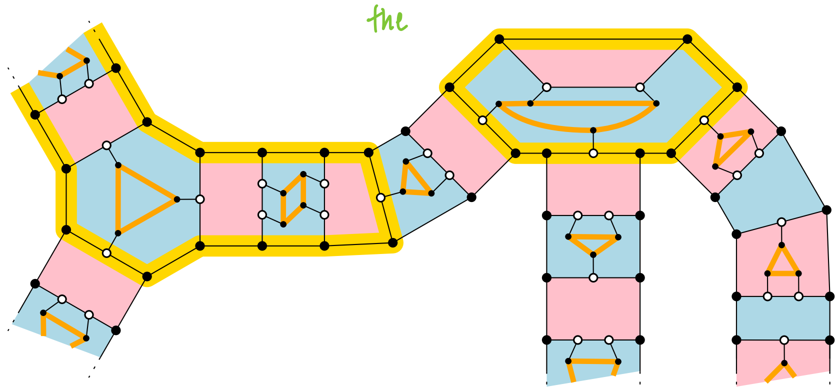

Figure 10 Part of the graph G_Ψ together with a 3-connected 3-augmentation in orange. This corresponds to a clause with two true variables (left with a splitter gadget, and middle) and one false variable (right).

The resulting graph is a (≤ 2)-subdivision of a 3-regular planar graph R. We refer to the vertices of R as black vertices, and the subdivision vertices as white vertices. Each of the white (subdivision) vertices is incident to one red and one blue face, while each black vertex (of R) is incident to an *uncolored* (neither red nor blue) face. By adding additional vertices into uncolored faces and connecting these to incident edges, we modify R to obtain a 3-connected 3-regular planar graph R' which still has a (≤ 2)-subdivision R_2 including all gadgets and corridors. As R' is 3-connected, the plane embedding of R_2 is unique (up to the choice of the outer face). Finally, we attach a degree-1 vertex to each subdivision vertex, which completes the construction of G_Ψ. See Figure 10.

By Lemma 5.1, G_Ψ admits a 3-connected 3-augmentation if and only if G_Ψ admits an embedding $\mathcal{E}$ in which no face has exactly one or two incident degree-1 vertices. Since the embedding of the subgraph R_2 of G_Ψ is fixed, such embedding $\mathcal{E}$ exists if and only if for each subdivision vertex we can choose either the incident red or the incident blue face in such a way that no face is chosen exactly once or twice. Except for the two highlighted faces in the clause gadgets, any pair of neighboring red and blue faces has in total at most five subdivision vertices. Thus, for each variable either all blue faces in all corridors are chosen, corresponding to the variable being set to true, or all red faces in all corridors are chosen, corresponding to the variable being set to false. In each positive clause gadget, the highlighted red face has only two incident subdivision vertices and hence cannot be chosen at all. Thus, the blue face of each positive clause gadget is chosen by two subdivision vertices, and thus must be chosen by at least one further subdivision vertex at the end of a variable corridor. This means that in each positive clause, at least one variable must be set to true. See again Figure 10 for an illustration. Symmetrically, at least one variable in each negative clause must be set to false, i.e., we have a satisfying truth assignment for Ψ. In the same way, we obtain from a satisfying truth assignment for Ψ, a valid choice for each subdivision vertex, i.e., an embedding of G_Ψ as required by Lemma 5.1.

To summarize, we obtain a planar graph G_Ψ that admits a 3-connected 3-augmentation if and only if the Planar-Monotone-3SAT-formula Ψ is satisfiable. The size of G_Ψ is polynomial in the size of Ψ. ◀

▶ Remark 5.3. The (≤ 2)-subdivision in the above reduction behaves quite similar to G_Ψ. The only problem is that a face f may have been chosen by exactly *two* incident degree-2 vertices to contain their third (new) edge without creating a 2-edge-cut; namely, with a direct edge. Thus, the above reduction also yields NP-completeness of recognizing *induced* subgraphs of 3-connected 3-regular planar graphs, even for 2-connected inputs with a unique embedding.

6 Conclusions

Consulting the table in Figure 3, our results together with [4] show that for $k \leq 2$ finding k-connected 3-augmentations is possible in polynomial time, both in the variable and the fixed embedding setting. On the other hand, Theorem 5.2 shows that finding 3-connected 3-augmentations is NP-complete in the variable embedding setting, even if the input graph is connected. The case of a fixed embedding and/or a 2-connected input graph remains open. We suspect these cases for 3-connected 3-augmentations to be NP-complete as well. For one thing, the graphs in our reduction in Section 5 are “almost 2-connected” and have “almost a unique embedding”, as discussed in Remark 5.3.

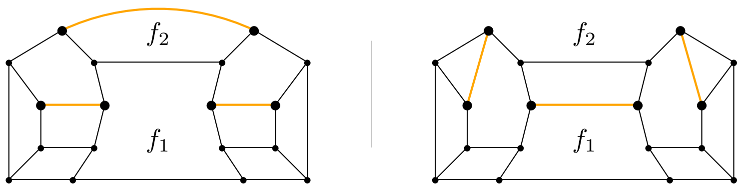

Figure 11 Two 3-connected 3-augmentations of the same 2-connected graph with fixed embedding.

Additionally, finding 3-connected 3-augmentations for fixed embedding seems to crucially require a coordination among the new edges, which cannot be modeled as a Generalized Factor problem with gaps of length 1. For example, if the input graph G is 2-connected (but not already 3-connected) with a fixed embedding, then there is an edge-cut of size 2. See Figure 11 for an illustration. To establish 3-connectivity in a 3-augmentation $H \supset G$, we must connect both sides of the cut through one of the two incident faces f_1, f_2, requiring both sides to coordinate and agree on which of f_1, f_2 to choose.

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