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Treewidth of Outer k-Planar Graphs

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Abstract

\emph

Treewidth is an important structural graph parameter that quantifies how closely a graph resembles a tree-like structure. It has applications in many algorithmic and combinatorial problems. In this paper, we study the treewidth of *outer k-planar* graphs – graphs admitting a convex drawing (that is, a drawing where all vertices lie on a circle) ~~and~~ s.t. each edge crosses at most k other edges. We also consider ~~a~~ the more general class of *outer min-k-planar* graphs, which are graphs admitting a convex drawing where for every crossing of two edges at least one of these edges is crossed at most k times.

Firman, Gutowski, Kryven, Okada and Wolff [GD 2024] proved that every outer k-planar graph has treewidth at most $1.5k + 2$ and provided a lower bound of $k + 2$ for even k. We establish a lower bound of $1.5k + 0.5$ for every odd k. Additionally, they showed that every outer min-k-planar graph has treewidth at most $3k + 1$. We improve this upper bound to $3 \cdot \lfloor 0.5k \rfloor + 4$. What about $k/2$ instead of $0.5k$?

Our approach also allows us to upper bound the *separation number*, a parameter closely related to treewidth, of outer min-k-planar graphs by $2 \cdot \lfloor 0.5k \rfloor + 4$. This improves upon the previous bound of $2k + 1$ and achieves a bound with an optimal multiplicative constant.

Keywords and phrases treewidth, outer k-planar graphs, outer min-k-planar graphs, separation number

Category Track 1

1 Introduction

In this paper, we study classes of graphs admitting a *convex drawing* with a bounded number of per edge crossings. A convex drawing is a straight-line drawing with all vertices drawn on a common circle. Bannister and Eppstein [1, 2] proved that the treewidth of graphs admitting a convex drawing with at most k crossings in total is bounded by a linear function ~~of~~ in $\sqrt{k}$. For ~~a~~ fixed k, they also provided a linear-time algorithm deciding if a given graph admits such a drawing (using Courcelle's theorem [6]). Another, well-studied ~~before,~~ class of graphs in this area is the class of *outer k-planar* graphs, that is, graphs admitting a convex drawing~~,~~ in which every edge crosses at most k other edges. These graphs have bounded treewidth by a linear function ~~of~~ in k, which was first proven by Wood and Telle [14, Proposition 8.5]. The authors of [5], also using Courcelle's theorem, for a fixed k, ~~showed~~ gave a linear-time algorithm testing whether a given graph is maximal outer k-planar. Recently, Kobayashi, Okada and Wolff [11], for a fixed k, provided a polynomial-time algorithm for testing whether a given graph is outer k-planar and proved that recognising outer k-planar graphs is XNLP-hard.

For disambiguation, we recall the definition of *k-outerplanar* graphs. A graph is *outerplanar* if it has a planar drawing with all vertices lying on the outer face. A graph is *1-outerplanar* when it is outerplanar. A graph is *k-outerplanar* for $k > 1$ when it has a planar drawing such that after removing the vertices of the outer face, each of the remaining components is $(k - 1)$-outerplanar.

We mainly study the treewidth of outer k-planar graphs and *outer min-k-planar* graphs. A graph is outer min-k-planar if it admits a convex drawing~~,~~ in which, for every crossing of two edges, at least one of these edges is crossed at most k times. The authors of [8] proved that outer k-planar graphs have treewidth at most $1.5k + 2$ and outer min-k-planar graphs have treewidth at most $3k + 1$. To obtain these results, they showed that every outer k-planar graph admits a *triangulation* of the outer cycle such that every edge of the triangulation is

list them once (if space permits)

before, you just used "cycle".

crossed at most k times by the edges of the graph. A similar property was proven for outer min-k-planar graphs.

Another property closely related to treewidth is the separation number of a graph. A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ such that $A \cup B = V(G)$ and there are~~n't any~~ no edges between the sets $A \setminus B$ and $B \setminus A$. The *order* of a separation is $|A \cap B|$. A separation is *balanced* if $|A \setminus B| \leq \frac{2}{3}|V(G)|$ and $|B \setminus A| \leq \frac{2}{3}|V(G)|$. The *separation number* of a graph G, denoted $\mathrm{sn}(G)$, is the minimum integer a such that every subgraph of G has a balanced separation of order at most a. Robertson and Seymour [12] proved that $\mathrm{sn}(G) \leq \mathrm{tw}(G) + 1$ for every graph G. From the other side, Dvořák and Norin [7] showed that $\mathrm{tw}(G) \leq 15\,\mathrm{sn}(G)$. Recently, Houdrouge, Miraftab and Morin [9] provided a more constructive proof of an analogous inequality, but with a worse multiplicative constant.

Our contribution. The authors of [8] proved that ~~for~~ every outer k-planar graph ~~its~~ has treewidth ~~is~~ at most $1.5k + 2$. They also presented a lower bound of $k + 2$ for every even k. We present an infinite family of outer k-planar graphs with treewidth at least $1.5k + 0.5$, showing that the multiplicative constant 1.5 in the upper bound cannot be improved; see Section 3.

We also improve the upper bounds for the treewidth and separation number of outer min-k-planar graphs. It was previously known that the treewidth of such graphs is at most $3k + 1$ and the separation number is at most $2k + 1$ [8]. We give an upper bound of $3 \cdot \lfloor 0.5k \rfloor + 4$ for treewidth; see Section 4, and an upper bound of $2 \cdot \lfloor 0.5k \rfloor + 4$ for the separation number; see Section 5. Both multiplicative constants are optimal, as the lower bounds for outer k-planar graphs also hold for outer min-k-planar graphs – namely, our lower bound of $1.5k + 0.5$ for the treewidth and ~~a~~ the lower bound of $k + 2$ for the separation number presented in [8].

2 Preliminaries

you use this already in line 43.

Let G be a graph. By $V(G)$ and $E(G)$ we denote the set of vertices and edges of G, respectively. For an edge connecting vertices u and v, we use the compact notation uv, instead of $\{u, v\}$. For a directed edge, we stick to the standard notation (u, v). Let $\deg(v)$ denote the degree of a vertex v, and let $\Delta(G)$ ~~be~~ denote the maximum degree of a vertex of G.

For a graph G, a subgraph *induced* by a set $U \subseteq V(G)$, denoted $G[U]$, is a subgraph with vertex set U and all edges of G between the vertices of U. A *spanning tree* of a graph G is a subgraph of G containing all the vertices of G that is a tree. By $\mathrm{dist}_G(v, w)$ we denote the distance (i.e. the length of the shortest path) between v and w in a graph G. For any tree T rooted at vertex r, we define the depth of a vertex v as $\mathrm{depth}_T(v) = \mathrm{dist}_T(r, v)$. We may omit subscripts if they are clear from the context.

A *tree decomposition* $\mathcal{T} = (T, B)$ of a graph G is a collection ~~of nodes, called~~ *bags*, $\{B_x : x \in V(T)\}$, indexed by the vertices of a tree T. Every bag is a subset of $V(G)$ with following properties ~~satisfied~~:

1. for every vertex $v \in V(G)$, the set $\{x : v \in B_x\}$ induces a non-empty subtree in T;
2. for every edge $uv \in E(G)$, there exists a bag containing both u and v.

The *width* of a given tree decomposition is the size of the largest bag minus one. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of any tree decomposition of G.

A set $\mathcal{B}$ of non-empty subsets of $V(G)$ is a *bramble* if:

1. for every $X \in \mathcal{B}$, the induced subgraph $G[X]$ is connected;

2. for every $X_1, X_2 \in \mathcal{B}$, the induced subgraph $G[X_1 \cup X_2]$ is connected. In other words, Th~~at~~ is means, each $X_1, X_2 \in \mathcal{B}$ share a common vertex or there exists an edge of G incident to both X_1 and X_2.

A *hitting set* of a bramble is a set of vertices with non-empty intersection with every element of $\mathcal{B}$. The *order* of a bramble is the size of its smallest hitting set. The *bramble number* of a graph G, denoted by $\mathrm{bn}(G)$, is the maximum order of any bramble of G.

The following result by Seymour and Thomas shows the relation between bramble number and treewidth.

▶ **Theorem 1** (Seymour and Thomas, [13]). *For every graph G, $\mathrm{tw}(G) = \mathrm{bn}(G) - 1$.*

We say that a graph G is a \emph minor of a graph H~~,~~ if G can be obtained from H by a sequence of vertex deletions, edge deletions or edge contractions. Edge contraction of an edge uv is an operation that replaces vertices u and v with a new vertex adjacent to every vertex other than u and v that was adjacent to u or v. It is a well-known fact that if G is a minor of H, then $\mathrm{tw}(G) \leq \mathrm{tw}(H)$. The proof of this fact can be found in [3].

In the remainder of this section, we introduce some notation and simple observations regarding drawings. A *convex drawing* of a graph G is a straight-line drawing where the of V(G) vertices of G are placed on different points of a circle. ~~in the~~ Given a cyclic order $(v_1, \ldots, v_n)$, We say that an edge $v_i v_j$ with $i < j$ *crosses* an edge $v_{i'} v_{j'}$ with $i' < j'$ if either $1 \leq i < i' < j < j' \leq n$ or $1 \leq i' < i < j' < j \leq n$. We consider only ~~such~~ convex drawings ~~that~~ where no three edges pass through the same point. An *outer k-planar drawing* of a graph is a convex drawing such that every edge crosses at most k other edges. An *outer min-k-planar drawing* of a graph is a convex drawing such that, for every crossing of two edges, at least one of these edges crosses at most k other edges.

An outer k-planar graph G is *maximal outer k-planar* if, for every pair $e \in V^2(G) \setminus E(G)$, the graph $G + e$ is not outer k-planar.

▶ **Observation 2.** *Let G be a maximal outer k-planar graph with at least three vertices. Then, in every outer k-planar drawing of G, the outer face is bounded by a simple cycle.*

Proof. Consider an outer k-planar drawing Γ of G. Let u and v be consecutive vertices in the cyclic order defined by Γ. Suppose that $uv \notin E(G)$. Notice that the graph $G + uv$ has an outer k-planar drawing defined by the same cyclic order as Γ, which contradicts the maximality of G. ◀

A graph G is *expanded outer k-planar* if G is an outer k-planar graph with $\Delta(G) \leq 3$ and its outer face is bounded by a simple cycle in some outer k-planar drawing of G.

▶ **Observation 3.** *Every outer k-planar graph G is a minor of an expanded outer k-planar graph G'.*

Proof. Let us assume that G is maximal outer k-planar. Now, in order to obtain G' from G, we perform the following transformation to every vertex v of G with $\deg(v) \geq 4$. The transformation is depicted in Figure 1. Let $w_0, w_1, \ldots, w_s, w_{s+1}$ be all neighbors of v in clockwise order, with edges vw_0 and vw_{s+1} adjacent to the outer face of G. We replace v with a path $v_1, \ldots, v_s$, put it on the outer face of G in counter clockwise order, in the place of v. We connect this path to vertices w_0 and w_{s+1} by adding edges $v_1 w_0$ and $v_s w_{s+1}$. Finally, for every $1 \leq i \leq s$, we add an edge $v_i w_i$ that corresponds to an edge vw_i in the original graph. It is easy to see that G is a minor of G' and the ordering of corresponding edges in G' matches the one in G. The crossings in the resulting graph naturally correspond to the crossings in the original graph. ◀

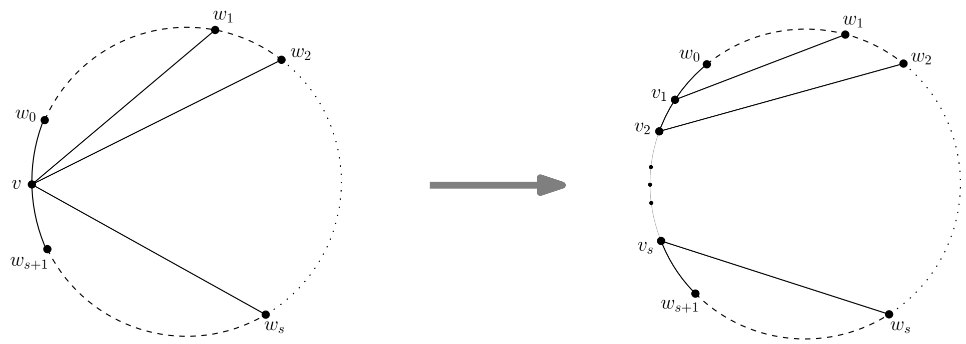


Figure 1 The transformation described in Observation 3.

The vertices which we write as $v_1, \ldots, v_s$ defined in the proof are called *images* of v, and v is the *origin* of these vertices, ~~denoted~~ $\operatorname{org}(v_i) = v$. If the transformation was not performed for some vertex v of G, i.e. $\deg(v) \leq 3$, then v is an image and origin of itself.

We remark that the analogous definitions and Observations 2, 3 hold for outer min-k-planar graphs. Since adding edges increases neither the treewidth nor the separation number, we are interested in the properties of maximal graphs. Also, taking a minor does not increase the treewidth, so we work with expanded graphs when bounding treewidth.

removing?

3 Lower bound for treewidth of outer k-planar graphs

In this section, we construct an infinite family of outer k-planar graphs with treewidth at least $1.5k + 0.5$. This improves the previous lower bound of $k + 2$ that was presented in [8]. We begin by defining the necessary graphs.

For positive integers m and n, let $X_{m,n}$ denote the grid of m rows and n columns, i.e. a graph with

$$V(X_{m,n}) = \{x_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\} \text{ and } E(X_{m,n}) = \{x_{i,j}x_{k,l} : |i-k| + |j-l| = 1\}.$$

For a positive integer k, let ~~consider~~ Q_k = ~~– a copy of grid~~ $X_{2k,2k}$, and let R_k = ~~– a copy of~~ $X_{2k(k+1),k}$. Let $v_{i,j}$, for $1 \leq i, j \leq 2k$, be ~~a~~ the vertex in the i-th row and the j-th column of Q_k, and let $u_{i,j}$, for $1 \leq i \leq 2k(k+1)$, $1 \leq j \leq k$, be ~~a~~ the vertex in the i-th row and the j-th column of R_k. Let G_k be a graph such that $V(G_k) = V(Q_k) \cup V(R_k)$ and

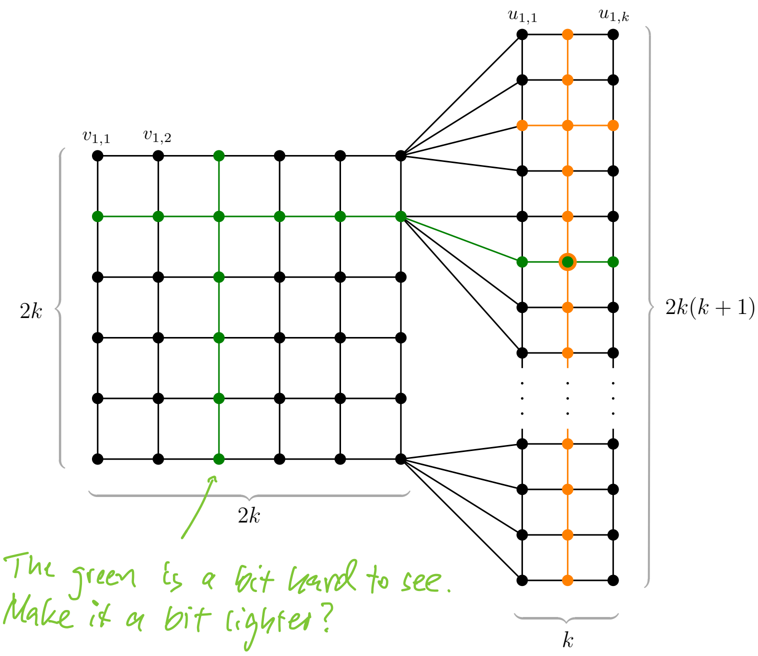
$$E(G_k) = E(Q_k) \cup E(R_k) \cup \left\{v_{i,2k}u_{(i-1)(k+1)+j,1} : 1 \leq i \leq 2k, 1 \leq j \leq k+1\right\};$$

see Figure 2. For $1 \leq i \leq 2k(k+1)$, let the i-th *extended row* of G_k be the union of the i-th row of R_k and the $\lceil \frac{i}{k+1} \rceil$-th row of Q_k. Not~~ic~~e that each row of Q_k is contained in $k+1$ extended rows and the graph induced by each extended row is a path.

The graph G_k was previously defined by Kammer and Tholey ~~in~~ [10] as an example of tightness of the upper bound for the treewidth of k-outerplanar graphs. They used the *cops and robber game* to establish ~~the~~ a lower bound for the treewidth of G_k. Below, we present a proof using brambles.

▶ **Theorem 4** (Kammer and Tholey, [10]). *For every $k \geq 1$, $\operatorname{tw}(G_k) = 3k - 1$.*

Proof. Notice that the drawing of G_k in Figure 2 is k-outerplanar. By the fact that k-outerplanar graphs have treewidth at most $3k - 1$ ~~(Bodlaender,~~ [4]~~)~~, we get $\operatorname{tw}(G_k) \leq 3k - 1$.



The green is a bit hard to see. Make it a bit lighter?

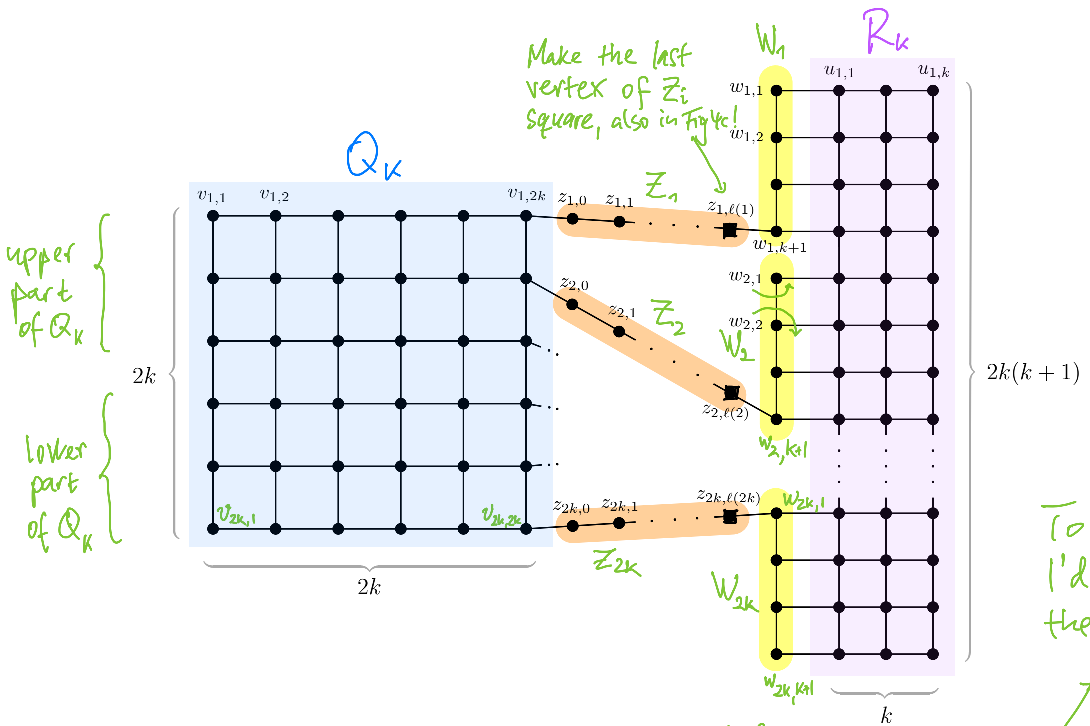
Figure 2 The graph G_k, for $k = 3$, with a subgraph of $\mathcal{B}_1$ colored green and a subgraph of $\mathcal{B}_2$ colored orange.

To prove that $\text{tw}(G_k) \geq 3k - 1$, we ~~will~~ construct a bramble of order $3k$. Then, using Theorem 1, we ~~will~~ get $\text{tw}(G_k) \geq 3k - 1$. Let $\mathcal{B}_1$ be a family consisting of every ~~subgraph of G_k~~ subset of $V(G_k)$ that is a union of an extended row of G_k and a column of Q_k. Let $\mathcal{B}_2$ be a family consisting of every ~~subgraph~~ set of $V(G_k)$ that is a union of a row of R_k and a column of R_k. The set $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ forms a bramble of G_k, as each subgraph ~~in~~ induced by an element of $\mathcal{B}$ is connected and every two such subgraphs have at least one common vertex.

Consider any hitting set S of $\mathcal{B}$. Let q and r be the number of vertices of S in $V(Q_k)$ and in $V(R_k)$, respectively. ~~Now, we~~ We would like to show that $|S| = q + r \geq 3k$. Notice that $r \geq k$, as otherwise there is a row and a column of R_k not containing any element of S, and thus there is an element of $\mathcal{B}_2$ not hit by S.

If $q \geq 2k$ then $q + r \geq 3k$. Otherwise, let $q = 2k - l$ for some positive integer l. Now, we can find at least l columns and at least l rows of Q_k not intersecting S. These l rows are contained in $l(k+1)$ extended rows. ~~Every one~~ Each of them has to intersect S at some vertex of R_k, because otherwise we can find a column of Q_k and an extended row not intersecting S that form an element of $\mathcal{B}_1$. The extended rows restricted to R_k are pairwise disjoint, so we have $r \geq l(k+1)$. Summing up, we get $q + r \geq 2k - l + l(k+1) = 2k + lk \geq 2k + k = 3k$, which concludes the proof. ◀

Let F_k be the following modification of G_k depicted in Figure 3. We set $\ell(i) = (k-i)(k+1)$ for $1 \leq i \leq k$, and $\ell(i) = (i-k-1)(k+1)$ for $k+1 \leq i \leq 2k$. For every $1 \leq i \leq 2k$, we remove every edge between $v_{i,2k}$ and any vertex of the grid R_k. We add a path Z_i of length $\ell(i)$ on vertices $z_{i,0}z_{i,1}\ldots z_{i,\ell(i)}$. We add a path W_i of length k on vertices $w_{i,1}w_{i,2}\ldots w_{i,k+1}$. We connect $v_{i,2k}$ with Z_i by adding ~~an~~ the edge $v_iz_{i,0}$; and we connect Z_i with W_i by adding ~~an~~ the edge $z_{i,\ell(i)}w_{i,k+1}$ for $1 \leq i \leq k$ or ~~an~~ the edge $z_{i,\ell(i)}w_{i,1}$ for $k+1 \leq i \leq 2k$. Finally, we connect W_i with R_k by adding ~~an~~ the edge $w_{i,j}u_{(i-1)(k+1)+j,1}$ for every $1 \leq j \leq k+1$.



Make the last vertex of Z_i square, also in Fig 4c!

To clarify this, I'd probably show the complete example for $k=3$!

Figure 3 The graph F_k is a modification of the graph G_k, here for $k = 3$. Note that $k^2-1 = |V(Z_1)| > |V(Z_2)| > \dots > |V(Z_k)| = 1$.

To see that G_k is a minor of F_k, it is enough to contract, for every $1 \leq i \leq 2k$, vertex $v_{i,2k}$ with all vertices of the paths Z_i and W_i. Since taking a minor does not increase the treewidth, we obtain the following corollary.

▶ **Corollary 5.** *For every $k \geq 1$, $\mathrm{tw}(F_k) \geq 3k - 1$.*

▶ **Theorem 6.** *The graph F_k has an outer $(2k-1)$-planar drawing for every $k \geq 1$.*

Proof. We describe an outer $(2k-1)$-planar drawing of F_k as depicted in Figure 4. We call the set of vertices $\{v_{i,j} : 1 \leq i \leq k, 1 \leq j \leq 2k\}$ the \emph upper part of Q_k. The other vertices of Q_k are called the lower part of Q_k. We define a cyclic order of the vertices of F_k by arranging them in ~~a~~ clockwise ~~direction~~ order from some selected starting point on a circle.

First, we put the vertices ~~from~~ of the upper part of Q_k in ~~the~~ column-by-column order (see Figure 4b):

$v_{k,1}, \dots, v_{2,1}, v_{1,1},$

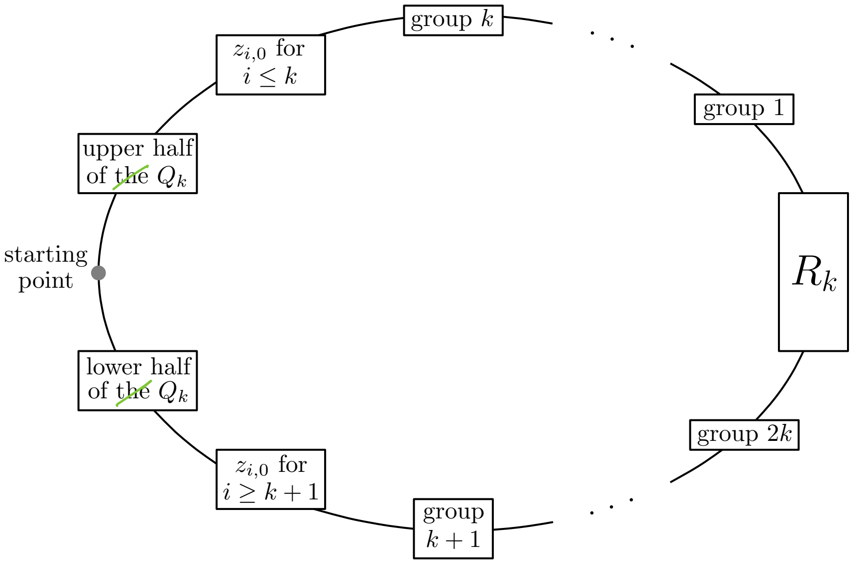
$v_{k,2}, \dots, v_{2,2}, v_{1,2},$

$\vdots$

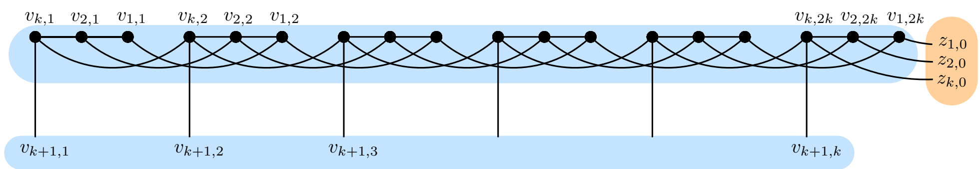
$v_{k,2k}, \dots, v_{2,2k}, v_{1,2k}.$

After that, we put vertices $z_{k,0}, \dots, z_{2,0}, z_{1,0}$, in this order.

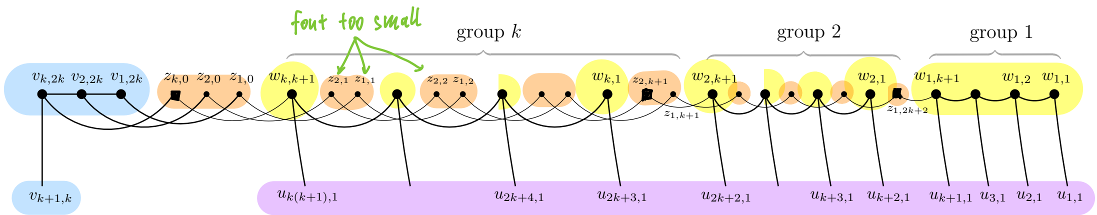
We divide the remaining vertices of the paths Z_i and W_i, for $1 \leq i \leq k$, into k groups, as follows. The i-th group contains the vertices of W_i, and if $i \geq 2$, it also includes the vertices $z_{a,b}$ for $1 \leq a < i$ and $(k-i)(k+1) < b \leq (k-i+1)(k+1)$. ~~Next, on~~ in the drawing, we ~~respectively put~~ place the groups of indices $k, k-1, \dots, 1$ in this order. We arrange the vertices in the i-th group, for



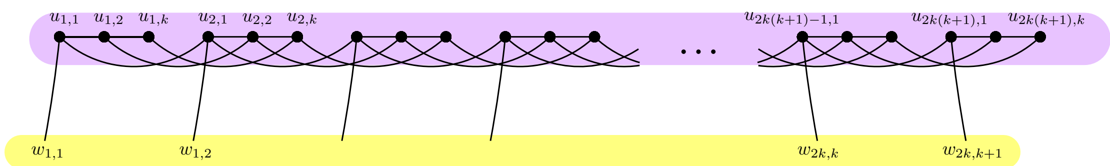
(a) Overview of the outer $(2k-1)$-planar drawing of F_k.



(b) Drawing of the upper part of the grid Q_k.



(c) Drawing of the groups $k, k-1, \ldots, 1$ connected to the upper parts of Q_k. and R_k.



(d) Drawing of the grid R_k.

Figure 4 Fragments of the outer $(2k-1)$-planar drawing of F_k, where $k=3$.

$2 \le i \le k$, in the order (see Figure 4c):

$$w_{i,k+1}, z_{i-1,(k-i)(k+1)+1}, z_{i-2,(k-i)(k+1)+1}, \dots, z_{1,(k-i)(k+1)+1},$$
$$w_{i,k}, z_{i-1,(k-i)(k+1)+2}, z_{i-2,(k-i)(k+1)+2}, \dots, z_{1,(k-i)(k+1)+2},$$
$$\vdots$$
$$w_{i,1}, z_{i-1,(k-i+1)(k+1)}, z_{i-2,(k-i+1)(k+1)}, \dots, z_{1,(k-i+1)(k+1)}.$$

The group of index 1 has vertices arranged in the order: $w_{1,k+1}, w_{1,k}, \dots, w_{1,1}$.

Next, we put the vertices of R_k in ~~the~~ row-by-row order (see Figure 4d):

$$u_{1,1}, u_{1,2}, \dots, u_{1,k},$$
$$u_{2,1}, u_{2,2}, \dots, u_{2,k},$$
$$\vdots$$
$$u_{2k(k+1),1}, u_{2k(k+1),2}, \dots, u_{2k(k+1),k}.$$

The vertices of F_k that are not placed yet are in the lower part of Q_k or in the paths Z_i, W_i, ~~for~~ with $k+1 \le i \le 2k$. We arrange them in ~~a~~ counter clockwise direction from the starting point and place them between the starting point and the vertices of R_k. The order is symmetric, with respect to the starting point, to the one used to arrange the upper part of Q_k and the paths Z_i, W_i ~~for~~ with $1 \le i \le k$. Every vertex $v_{i,j}$, where $k+1 \le i \le 2k$ and $1 \le j \le 2k$, is placed symmetrically to $v_{2k-i+1,j}$. Vert~~ices~~ex $z_{i,j}$, where $k+1 \le i \le 2k$ and $0 \le j \le \ell(i)$, ~~are~~ is placed symmetrically to $z_{2k-i+1,j}$, and vert~~ices~~ex $w_{i,j}$, where $k+1 \le i \le 2k$ and $1 \le j \le k+1$, is placed symmetrically to $w_{2k-i+1,k-j+2}$. The symmetrical drawing of the i-th group, for every $1 \le i \le k$, forms the group of index $2k-i+1$.

Now, we ~~will~~ show that every edge crosses at most $2k-1$ other edges, by partitioning ~~them~~ edges into several types.

1. The "column" edges in the upper or lower part of Q_k ~~–~~, that is, the edges $v_{i,j}v_{i+1,j}$, for $1 \le i \le 2k-1, i \ne k$ and $1 \le j \le 2k$. These edges cross no other edges.
2. The "column" edges between the upper and the lower part of Q_k – edges $v_{k,j}v_{k+1,j}$, for $1 \le j \le 2k$. Each of these edges crosses $k-1$ edges of type 3 from the upper part of Q_k, and $k-1$ edges from the lower part. The edge $v_{k,1}v_{k+1,1}$ crosses no other edges.
3. The "row" edges of Q_k – edges $v_{i,j}v_{i,j+1}$, for $1 \le i \le 2k$ and $1 \le j \le 2k-1$. Each of these edges crosses $2(k-1)$ edges of types 3, 4 and additionally at most one edge of type 2.
4. Each edge $v_{i,2k}z_{i,0}$, for $1 \le i \le 2k$, crosses at most $2(k-1)$ edges either of type 3 or edges incident to vertices $z_{j,0}$. for $j<i$.
5. Each edge $z_{i,\ell(i)}w_{i,k+1}$, for $1 \le i \le k$, crosses exactly $2(i-1)$ edges incident to vertices $z_{i-1,0}, \dots, z_{1,0}$. Symmetrically, each edge $z_{i,\ell(i)}w_{i,1}$, for $k+1 \le i \le 2k$, also crosses at most $2(k-1)$ edges.
6. Each edge $z_{i,0}z_{i,1}$, for $1 \le i \le 2k$, $i \notin \{k, k+1\}$, crosses exactly $2(k-2)$ other edges from the paths Z_j or incident to vertices $z_{j,0}$; and exactly ~~3~~ three edges incident to $w_{k,k+1}$ or $w_{k+1,1}$.
7. Every other edge from paths Z_i crosses at most $2(k-2)$ edges from other paths Z_j and at most 3 edges incident to some vertex $w_{a,b}$.
8. Each edge ~~from~~ of a path~~s~~ W_i crosses at most $2(k-1)$ edges ~~from~~ of any path~~s~~ Z_j.
9. Each edge $w_{i,j}u_{(i-1)(k+1)+j,1}$, for $1 \le i \le 2k$ and $1 \le j \le k+1$, crosses at most $k-1$ edges ~~from~~ of some path~~s~~ Z_a and at most $k-1$ edges from R_k.
10. The "row" edges of R_k ~~–~~, that is, the edges $u_{i,j}u_{i,j+1}$, for $1 \le 2k(k+1)$ and $1 \le j \le k-1$, ~~They~~ cross no other edges.

These "place holders" should be quantified for clarity.

I suggest to place Figs 4b-d as separate figures next to the items in the list where the types of edges in the figure are discussed.

11. The “column” edges of R_k that is, the edges $u_{i,j}u_{i+1,j}$, for $1 \leq 2k(k+1)-1$ and $1 \leq j \leq k$, ~~Each of these edges~~ cross~~es~~ at most $2(k-1)$ other edges of this type and at most one edge of type 9. ◀

▶ **Theorem 7.** *For every odd positive integer k, there exists an outer k-planar graph G with* $\mathrm{tw}(G) \geq 1.5k + 0.5$.

Proof. By Theorem 6, the graph $F_{\frac{k+1}{2}}$ is outer k-planar, and by Corollary 5 it has treewidth at least $3\frac{k+1}{2} - 1 = 1.5k + 0.5$. ◀

4 Upper bound ~~for~~ on the treewidth of outer min-k-planar graphs

In this section, we upper bound the treewidth of outer min-k-planar graphs. We improve the previous bound of $3k+1$ presented in [8] to $3 \cdot \lfloor 0.5k \rfloor + 4$. We begin by introducing required notation.

This is commonly called "planarization" of G w.r.t. Γ.

For an outer min-k-planar graph G with a given drawing Γ, we define ~~a~~ the *crossing graph* G_C as a graph~~, in which the~~ whose vertex set is ~~a~~ the union of $V(G)$ and all crossing points of the edges of G. We say that a vertex $w \in V(G_C)$ *lies* on an edge $uv \in E(G)$ if w is an endpoint of uv or the crossing point corresponding to w belongs to the segment that is a drawing of the edge uv in Γ. Graph G_C contains an edge between two vertices if and only if they are consecutive vertices lying on the drawing of some edge of G. Observe that G_C is a planar graph. We say that an edge $xy \in E(G_C)$ *lies* on an edge $uv \in E(G)$ if both x and y lie on uv in Γ. Furthermore, we say that a vertex $v \in V(G_C)$ is *outer* if it is adjacent to the outer face of G_C. Otherwise, v is an *inner* vertex. As we consider only maximal graphs G, the outer vertices of G_C are exactly the vertices of G. By f_o we ~~will~~ denote the outer face of G_C.

w.r.t. what?

Use f_o

For a planar graph G, denote Let G^* ~~as~~ the graph dual to G. ~~By~~ Let $f^* \in V(G^*)$ ~~we~~ denote the vertex dual to the face f of G, and ~~by~~ let $e^* \in E(G^*)$ ~~we~~ denote the edge dual to ~~an~~ the edge $e \in E(G)$. We remark that G^* can be drawn on the drawing of G in a way that f^* is on the face f and the drawing of e^* is a curve that passes through crosses the edge e and the faces corresponding to the endpoints of e^*.

The following lemma shows a bijection between a spanning tree T of a planar graph G and a spanning tree T^* of G^*, ~~that we denote by~~ where $T^* = \mathrm{dual}(T)$. We also use the notation $\mathrm{dual}(T^*)$ for T.

▶ **Lemma 8** (Folklore)**.** *Let T be a spanning tree of a planar graph G. Then T^* with $V(T^*) = V(G^*)$ and $E(T^*) = \{e^* : e \in E(G) \setminus E(T)\}$ is a spanning tree of G^*.*

The next lemma proves that there exists a spanning tree preserving shortest paths from a given vertex. Such tree can be found via ~~a BFS algorithm.~~ breadth-first search.

▶ **Lemma 9** (Folklore)**.** *Let G be a graph and let r be a vertex of G. Then there exists a spanning tree T of G rooted at r such that* $\mathrm{depth}_T(v) = \mathrm{dist}_G(r, v)$ *for every vertex v of G.*

▶ **Lemma 10.** *Let G be an expanded outer min-k-planar graph with ~~its~~ crossing graph G_C. Then* $\mathrm{dist}(f^*, f_o^*) \leq \lfloor 0.5k \rfloor + 1$ *for every vertex $f^* \in V(G_C^*)$.*

Proof. Let f be a ~~non-outer~~ inner face of G_C. If f is adjacent to f_o, then $\mathrm{dist}(f^*, f_o^*) = 1$. Otherwise, let v be a vertex of G_C adjacent to f. As G is expanded, the vertex v is inner, so it lies on an edge e of G that crosses at most k other edges. Let $v_0, v_1, \ldots, v_s, v_{s+1}, \ldots, v_{s+t+1}$ be all vertices lying on e, listed in the ~~consecutive~~ order along e, where $v_s = v$ and v_{s+1} is a neighbor of v that is adjacent to f. We may assume that $s \le t$, i.e. v_s is closer to an endpoint of the edge e than v_{s+1} to the other endpoint of e. Notice that at most $k+2$ vertices lie on e (two endpoints and at most k crossing points), so $s + t + 2 \le k + 2$. Together with the previous inequality, this implies $s \le 0.5k$. The number s is an integer, so $s \le \lfloor 0.5k \rfloor$.

$k/2$

We inductively define a sequence $w_s, w_{s-1}, \dots, w_0$ of vertices. Vertex w_s is the neighbor of v_s that is adjacent to f and does not ~~lying~~ lie on e. For every $i \in \{s-1, \dots, 0\}$ the vertex w_i is one of the two neighbors of v_i not lying on e such that w_i, v_i, v_{i-1} and w_{i-1} are adjacent to the same face of G_C. Let e_i denote the edge $v_i w_i$. Now, notice that the path formed by the edges $e_s^*, \dots, e_0^*$ connects f^* with f_o^* (e_0 is adjacent to f_o). So, $\mathrm{dist}(f^*, f_o^*) \le s + 1 \le \lfloor 0.5k \rfloor + 1$. ◀

Let v be an inner vertex of G_C with neighbors w_1, w_2, w_3, w_4 in clockwise order. As we forbid common crossing points of three edges of G, every such vertex has degree 4. We *subdivide* the vertex v by replacing it with two vertices v_1 and v_2 connected with an edge, and adding edges v_1w_1, v_1w_2 and v_2w_3, v_2w_4. We fix a planar embedding of the new graph by drawing v_1, v_2 very close to where v was drawn. We say that vertices v_1 and v_2 *lie* on the same edges as the vertex v. Moreover, we say that the edge v_1v_2 in an *auxiliary* edge. After subdividing a vertex, every edge of G_C has an edge corresponding to it and every face of G_C corresponds to a new one in a natural way. Also, the dual graph has one additional edge dual to v_1v_2.

Let ~~By~~ G_S ~~we~~ denote the *subdivided crossing graph*, that is, the graph G_C with all inner vertices subdivided. See Figure 5 for an example. Observe that there is a one-to-one correspondence between $V(G_C^*)$ and $V(G_S^*)$. Further, every edge of G_C^* has a corresponding edge of G_S^*. The following lemma shows how we can preserve the properties of a spanning tree T_C of G_C and a spanning tree $\mathrm{dual}(T_C)$ of the dual graph after subdividing vertices of G_C.

▶ **Lemma 11.** *Let G be an expanded outer min-k-planar graph with its subdivided crossing graph G_S. Then there exists a spanning tree T_S of G_S and a spanning tree $T_S^* = \mathrm{dual}(T_S)$ of G_S^* rooted at f_o^*, such that $\mathrm{depth}(f^*) \le \lfloor 0.5k \rfloor + 1$ for every vertex $f^* \in V(G_S^*)$ and $E(T_S)$ contains all auxiliary edges of G_S.*

Proof. Let G_C be a crossing graph of G. By Lemmas 9 and 10 there exists a spanning tree T_C^* of G_C^* that is rooted at f_o^* and whose vertices have depth ~~with maximal depth of a vertex being~~ at most $\lfloor 0.5k \rfloor + 1$. Let $T_C = \mathrm{dual}(T_C^*)$ be a spanning tree of G_C.

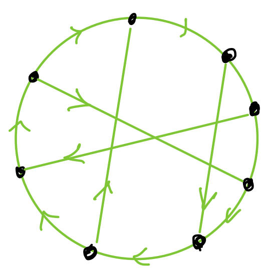
Let G_S denote the graph obtained from G_C by subdividing every inner vertex. After this transformation, let T_S^* be a tree constructed of edges corresponding to the edges of T_C^*. Clearly, T_S^* is a spanning tree of the graph G_S^*. The spanning tree $T_S = \mathrm{dual}(T_S^*)$ of G_S contains all auxiliary edges of G_S, because none of the duals of auxiliary edges are in $E(T_S^*)$, as they were not in $E(T_C^*)$. ◀

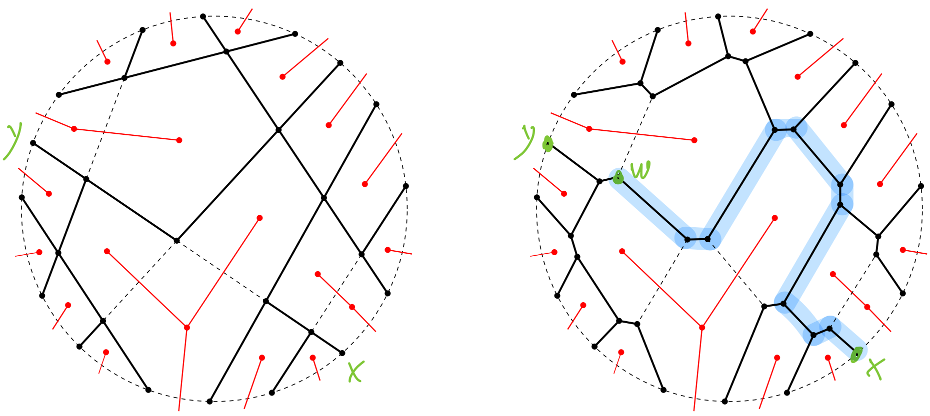
Now, we are ready to prove the main result of this section.

▶ **Theorem 12.** *Let G be an outer min-k-planar graph. Then $\mathrm{tw}(G) \le 3 \cdot \lfloor 0.5k \rfloor + 4$.*

Proof. By Observation 3 we may assume that G is an expanded outer min-k-planar graph. Let G_S be the subdivided crossing graph of G. By Lemma 11 there exists a spanning tree T_S of G_S and a spanning tree $T_S^* = \mathrm{dual}(T_S)$ of G_S^* rooted at f_o^*, such that $\mathrm{depth}(f^*) \le \lfloor 0.5k \rfloor + 1$ for every vertex $f^* \in V(G_S^*)$ and $E(T_S)$ contains all auxiliary edges of G_S.

We ~~Lets~~ orient the ~~every~~ edges of G. Edges adjacent to the outer face are oriented clockwise, the other edges are oriented arbitrarily. Observe that every vertex of G has at most two incoming edges.





(a) The graph G_C with its spanning tree T_C colored black and a spanning tree T_C^* of G_C^* in red. **(b)** The graph G_S with its spanning tree T_S colored black and a spanning tree T_S^* of G_S^* in red.

Figure 5 Drawings of an example graphs with their spanning trees and spanning trees of the dual graphs, The vertex f_o^* is missing ~~on~~ in both ~~figures~~ drawings.

Now, we construct the tree decomposition $\mathcal{T} = (T_S, B)$ of the graph G. The bags of $\mathcal{T}$ are indexed by the vertices of T_S. We put vertices of G into bags using the following rules.

1. For every outer vertex $v \in V(G_S)$, we put v in the bag B_v.
2. For every oriented edge (x, y) of G, we put x in the bag B_y.
3. For every inner vertex $v \in V(G_S)$, lying on the edge (x, y) of G, we put x in the bag B_v.
4. For every vertex $v \in V(G_S)$, every non-outer face f adjacent to v and every edge e^* belonging to the path from f^* to f_o^* in T_S^* that is dual to the edge e of G_S lying on the edge (x, y) of G, we put x in the bag B_v. Try to simplify!

Notice that, in rule 4, the edge e is not in $E(T_S)$, so it is not an auxiliary edge, which implies that it is lying only on a single edge of G.

For every edge (x, y) of G, by rules 1 and 2, the bag B_y contains both x and y. Also, every vertex of G is present in some bag. So to prove that $\mathcal{T}$ is a proper tree decomposition of G it is enough to prove that, for every vertex $x \in V(G)$, the set $\{w : x \in B_w\}$ induces a connected subtree in T_S.

~~Lets~~ Let's fix a vertex x and an edge (x, y) of G. Let $e = uv$ be an edge lying on (x, y) such that $e \notin E(T_S)$. We assume that $e^* = f_1^* f_2^*$ and $\mathrm{depth}_{T_S^*}(f_2^*) = \mathrm{depth}_{T_S^*}(f_1^*) + 1$. Let T_e^* be a subtree of T_S^* induced ~~on all~~ by the set of descendants of f_2^*, including f_2^*. Define $F_e = \{f : f^* \in V(T_e^*)\}$, forbid line break and let boundary(F_e) be the set of vertices adjacent to some face in F_e. Notice that, by rule ~4, 4, x is put into all bags indexed by boundary(F_e). The set boundary(F_e) induces a connected subgraph of T_S, i.e. $T_S[\text{boundary}(F_e)]$ is connected, containing both u and v. Observe that the bags of $\mathcal{T}$, into which we put x by rule 4, are exactly the bags of vertices of T_S that are in boundary(F_e) for some edge $e \notin E(T_S)$ lying on some edge (x, y) of G.

or: $V(\partial F_e)$

By rules 1, 2 and 3, x is contained in all bags B_w such that w lies on (x, y) for some edge (x, y) of G. We claim that the vertices indexing these bags, together with the vertices indexing bags we put x into by rule 4, form a connected subgraph of T_S. To see that, we show that, for every vertex w ~~such that~~ with $x \in B_w$, w is connected to x by a walk in T_S such that bags indexed by the vertices of this walk contain x.

If w lies on an edge (x, y) of G then, in order to construct this walk, we start at vertex w. We iterate over consecutive edges lying on (x, y) between w and x, starting at the edge incident to w. If ~~given~~ the current edge e of our walk is in $E(T_S)$, th~~a~~en we extend the walk by e. Otherwise, $e \notin E(T_S)$.

Add examples x, y, w and the w-x walk to Fig. 5(b).

As $T_S[\text{boundary}(F_e)]$ is connected and every bag of a vertex in boundary(F_e) contains x, we can extend the walk by some path in $T_S[\text{boundary}(F_e)]$ connecting the endpoints of e.

If w is in the set boundary(F_e) for some edge e lying on (x, y), th~~a~~en we begin the walk with a path contained in boundary(F_e) between w and an endpoint v of e. We extend this walk by a walk between v and x, ~~which~~ whose existence we have already proven.

Next, we bound the size of the bags in $\mathcal{T}$. Consider an inner vertex v of T_S. It lies on exactly two edges of G, so by rule 3 we put two vertices into B_v. Also, v is adjacent to three non-outer faces of G_S. For every such face f and every edge e^* belonging to the path from f^* to f_o^* in T_S^*, by rule 4, we put one vertex into B_v. Every such path has at most $\lfloor 0.5k \rfloor + 1$ edges. So $|B_v| \le 2 + 3 \cdot (\lfloor 0.5k \rfloor + 1)$. Now, ~~let v be~~ consider an outer vertex v of T_S. By rules 1 and 2, the bag B_v contains v and at most two other endpoints of the edges incoming to v in G. Also, v is adjacent to two non-outer faces of G_S. Thus, we derive a bound $|B_v| \le 3 + 2 \cdot (\lfloor 0.5k \rfloor + 1)$. The width of the constructed tree decomposition is at most

$$\max\{2 + 3 \cdot (\lfloor 0.5k \rfloor + 1), 3 + 2 \cdot (\lfloor 0.5k \rfloor + 1)\} - 1 = 2 + 3 \cdot (\lfloor 0.5k \rfloor + 1) - 1 = 3 \cdot \lfloor 0.5k \rfloor + 4.$$

◀

5 The Separation number of outer min-k-planar graphs

The inequality $\text{sn}(G) \le \text{tw}(G) + 1$ bounding the separation number holds for every graph G. We remark that Theorem 12 directly implies that $\text{sn}(G) \le 3 \cdot \lfloor 0.5k \rfloor + 5$ for every outer min-k-planar graph G. By carefully choosing some bag B_x of the tree decomposition, we can construct ~~such~~ a balanced separation (C, D) satisfying $C \cap D = B_x$. To establish a better upper bound, we first ~~we~~ prove a general lemma showing how ~~from a tree decomposition~~ satisfying some additional properties, we can obtain a balanced separation (C, D) such that $C \cap D = B_x \cap B_y$ for some two neighboring vertices x, y of ~~the~~ a tree decomposition.

▶ **Lemma 13.** *Let $\mathcal{T} = (T, B)$ be a tree decomposition of a graph G. Assume that $\Delta(T) \le 3$ and that every vertex $v \in V(G)$ is in at least two bags of $\mathcal{T}$. Let a be an integer such that $|B_x \cap B_y| \le a$ for any edge $xy \in E(T)$. Then G has a balanced separation of order at most a.*

Proof. For every edge $xy \in E(T)$, after removing it from T, we obtain two connected components C_x and C_y of T such that $x \in V(C_x)$ and $y \in V(C_y)$. We define $S_{x,y} = \bigcup_{v \in V(C_x)} B_v$ and $S_{y,x} = \bigcup_{v \in V(C_y)} B_v$. It is a well known fact that the pair $(S_{x,y}, S_{y,x})$ is a separation of G of order $|S_{x,y} \cap S_{y,x}| = |B_x \cap B_y| \le a$.

We claim that there exists an edge $xy \in E(T)$ such that $(S_{x,y}, S_{y,x})$ is a balanced separation of G. Suppose the contrary. Now, we orient every edge of T. ~~For~~ Then for every $xy \in E(T)$ it holds that $|S_{x,y} \setminus S_{y,x}| > \frac{2}{3}n$ or $|S_{y,x} \setminus S_{x,y}| > \frac{2}{3}n$, where $n = |V(G)|$. If the first inequality holds, then, we orient xy as (y, x), in the other case as (x, y). Also, notice that $|S_{x,y} \setminus S_{y,x}| > \frac{2}{3}n$ is equivalent to (implies ?) $|S_{y,x}| < \frac{1}{3}n$.

The tree T with oriented edges ~~in~~ is an acyclic graph, so there exists a sink, that is, a vertex in T such that all edges incident to x are oriented towards x. Let $\{y_1, \dots, y_d\}$, where $d \le 3$, be ~~all~~ the set of neighbors of x in T. We have $|S_{y_i,x}| < \frac{1}{3}n$. Also $\bigcup_{1 \le i \le d} S_{y_i,x} = \bigcup_{v \in V(T) \setminus \{x\}} B_v = V(G)$, as every vertex of G is in at least two bags of $\mathcal{T}$. We obtain the following inequalities

$$|V(G)| = \left| \bigcup_{1 \le i \le d} S_{y_i,x} \right| \le \sum_{1 \le i \le d} |S_{y_i,x}| < d \cdot \frac{1}{3}n \le n,$$

which ~~gives~~ is a contradiction. ◀

Now, we are ready to upper bound the separation number of outer min-k-planar graphs.

I find it a bit strange to see first $\mathcal{T}'$ and B', and then $\mathcal{T}$ and B. Swap?

▶ **Theorem 14.** *Let G be an outer min-k-planar graph. Then* $\mathrm{sn}(G) \leq 2 \cdot \lfloor 0.5k \rfloor + 4$.

Proof. The class of outer min-k-planar graphs is closed under taking subgraphs. So it is enough to find a balanced separation of order at most $2 \cdot \lfloor 0.5k \rfloor + 4$ for every maximal outer min-k-planar graph G. Let H be an expanded outer min-k-planar graph obtained from G by Observation 3. By Theorem 12, there exists a tree decomposition $\mathcal{T}' = (T_S, B')$ of H, where T_S is a spanning tree of the ? subdivided graph H. From the proof of Theorem 12, it follows that $\Delta(T_S) \leq 3$ and that every vertex $v \in V(H)$ is in at least two bags of $\mathcal{T}'$ (there is an oriented edge (v, w) in H, so $v \in B'_v$ and $v \in B'_w$).

We construct a tree decomposition $\mathcal{T} = (T_S, B)$ of G with $B_x = \{\mathrm{org}(v) : v \in B'_x\}$. remind reader Every vertex $v \in V(G)$ is in at least two bags of $\mathcal{T}$ as every image of v is in at least two bags of $\mathcal{T}'$. Every edge $vw \in E(G)$ is realised in some bag of $\mathcal{T}$, because in H there is an edge corresponding to vw between an image of v and an image of w. To prove that, for every vertex v of G, the bags of $\mathcal{T}$ containing v are spanning a connected subtree of T_S, ~~lets~~ we denote ~~all~~ the images of v by $v_1, \ldots, v_s$, in the ~~consecutive order~~ order along the outer face. As H is maximal, for every $1 \leq i < s$, ← better: $i \in \{1, .., s-1\}$ there is an edge $v_i v_{i+1}$ in $E(H)$. So the two subtrees of T_S induced by bags of $\mathcal{T}'$ containing v_i, and by bags of $\mathcal{T}'$ containing v_{i+1} share a common vertex. Bags containing v in $\mathcal{T}$ are spanning a connected subtree of T_S, because this subtree is a union of subtrees spanned by the images of v. So $\mathcal{T}$ is a proper tree decomposition of G. you mean in the sense of Lemma 13?

Rephrase & simplify

We say that a vertex v was put into a bag B'_x of $\mathcal{T}'$ due to rule 4 of constructing the tree decomposition being applicable to a vertex v and a face f adjacent to x, if there exists an edge e lying on an edge (v, w) of H such that e^* belongs to the path in T^*_S between f^*_o and f^*. Now, we want to show that, for every edge $xy \in E(T_S)$, we have $|B_x \cap B_y| \leq 2 \cdot \lfloor 0.5k \rfloor + 4$. Let f_1 and f_2 be the faces of H_S adjacent to xy.

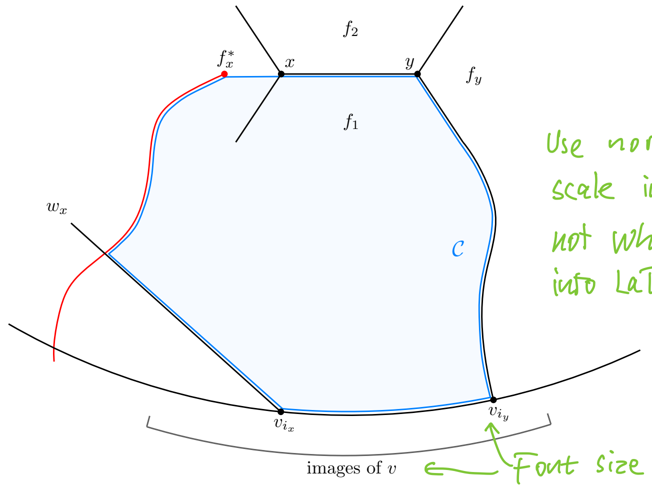
▷ Claim. If $v \in B_x \cap B_y$ then there is an image v_t of v such that (i) xy lies on an edge (v_t, w) defined? of H or (ii) v_t was put into both B'_x and B'_y due to rule 4 of constructing $\mathcal{T}'$ being applicable to v_t and face f_1 or face f_2.

Proof. If ~~$\deg(x)$~~ x has degree $= 3$ in H_S, then let $f_x \notin \{f_1, f_2\}$ be the face of H_S~~, different from f_1 and f_2~~ such that f_x is adjacent to the vertex x. Similarly, if $\deg(y) = 3$ in H_S, then let $f_y \notin \{f_1, f_2\}$ be the face of H_S~~, different from f_1 and f_2~~ such that f_y is adjacent to the vertex y. Assume that $v \in B_x \cap B_y$, but that no images of v were put into B'_x and B'_y due to the reasons in the claim statement. So there exist v_{i_x} and v_{i_y} that are, not necessarily distinct, images of v such that $v_{i_x} \in B'_x$ and $v_{i_y} \in B'_y$. For $t \in \{x, y\}$, vertex v_{i_t} was put in B'_t because

1. t lies on an edge (v_{i_t}, w_t) of H such that xy is not lying on $v_{i_t} w_t$; or
2. when $\deg(t) = 3$ in H_S, rule 4 of constructing $\mathcal{T}'$ is applicable to vertex v_{i_t} and face f_t, i.e. there exists an edge (v_{i_t}, w_t) of H and an edge e_t of H_S lying on the edge $v_{i_t} w_t$ of H such that e^*_t belongs to the path between f^*_t and f^*_o in T^*_S.

Now, we draw a curve $\mathcal{C}$ on the drawing of H_S. The curve $\mathcal{C}$ consists of the drawing of the edge xy and the drawing of an arc of the outer face between v_{i_x} and v_{i_y} that contains only images of v (the images of v are spanning a single arc of the outer face). Next, we add to $\mathcal{C}$ curves connecting v_{i_x} with x and v_{i_y} with y. For $t \in \{x, y\}$, to determine how to draw these curves, we case over the reason v_{i_t} was put in B'_t, in the order as listed above.

1. We draw along the drawing of (v_{i_t}, w_t), starting at vertex v_{i_t} and ending at vertex t.
2. Let p_t be a path between f^*_t and f^*_o in T^*_S. We draw along the drawing of (v_{i_t}, w_t), staring at vertex v_{i_t} and ending at the crossing point with the drawing of p_t. We continue along p_t till vertex f^*_t. Finally, we connect vertices f^*_t and t with a segment.



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Font size too small.

Figure 6 Drawing of an example curve $\mathcal{C}$.

Note that if $x = v_{i_x}$ and $y = v_{i_y}$, then $\mathcal{C}$ is degenerated to the arc between v_{i_x} and v_{i_y}, implying that xy connects two consecutive images of v – contradiction. Otherwise, we claim that one of the closed regions induced by $\mathcal{C}$ contains f_1 or f_2. Indeed, $\mathcal{C}$ follows edges of H_S and edges of T_S^*, but cannot contain f_1^* nor f_2^*, because then rule 4 of constructing $\mathcal{T}'$ would be applicable to vertex v_{i_x} or v_{i_y} and face f_1 or f_2. The segments between f_t^* and t does not intersect f_1 nor f_2. We may assume that f_1 is contained inside a closed region induced by $\mathcal{C}$. Consider a path p_1 between f_1^* and f_o^* in T_S^*. As f_1^* is inside $\mathcal{C}$ and f_o^* is outside $\mathcal{C}$, drawing of p_1 has to intersect $\mathcal{C}$. We consider where the first intersection point is located.

- Path p_1 cannot intersect e nor the segments between f_t^* and t.
- If p_1 intersects an edge (v_{i_t}, w_t) then rule 4 of constructing $\mathcal{T}'$ is applicable to v_{i_t} and f_1.
- If p_1 intersects p_t then p_1 follows along p_t up to the intersection point with (v_{i_t}, w_t), so the previous case applies.
- If p_1 intersects the arc of the outer face between v_{i_x} and v_{i_y} then it has to intersect an edge (v_r, v_{r+1}), where v_r and v_{r+1} are consecutive images of v on the outer face. So rule 4 of constructing $\mathcal{T}'$ is applicable to v_r and f_1.

In each case, we have obtained a contradiction. ◁

We proved that if $v \in B_x \cap B_y$, then there is an image v_t of v such that (i) xy lies on an edge (v_t, w) of H or (ii) v_t was put into both B'_x and B'_y due to rule 4 of constructing the tree decomposition $\mathcal{T}'$ being applicable to vertex v_t and face f_1 or f_2. Notice that xy lies on at most two edges of H (two if xy is an auxiliary edge, one otherwise) and each of the paths from f_o^* to f_1^* or f_2^* in T_S^* has at most $\lfloor 0.5k \rfloor + 1$ edges. So $|B_x \cap B_y| \le 2 + 2 \cdot (\lfloor 0.5k \rfloor + 1) = 2 \cdot \lfloor 0.5k \rfloor + 4$. By Lemma 13 ~~applied~~ applying to $\mathcal{T}$, we get that G has a balanced separation of order at most $2 \cdot \lfloor 0.5k \rfloor + 4$. ◀

To give a lower bound, we define a graph called *stacked prism*. A stacked prism $Y_{m,n}$ is an $m \times n$ grid with additional edges connecting the vertices of the first and the last row that are in the same column. The $Y_{m,n}$ has an outer $(2n-2)$-planar drawing, thus also an outer min-$(2n-2)$-planar drawing. In the cyclic order of the drawing, we place rows consecutively, one after another. The edges from rows cross no other edges and the edges from columns cross exactly $2n-2$ other edges. The Authors of [8] showed that for every number n and for every sufficiently large even number m, $\mathrm{sn}(Y_{m,n}) = 2n$. This leads to the following theorem.

▶ **Theorem 15.** *For every even number k, there exists an outer min-k-planar graph G such that* $\mathrm{sn}(G) = k + 2$.

We remark that the multiplicative constant of 1 in the upper bound given in Theorem 14 is tight, as it matches that of the lower bound in Theorem 15.

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