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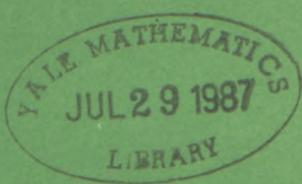
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A. *Algebraic approach.* Let us look at the differential operator Δ_{ω} from the point of view of the theory of Bernstein algebras [1].
A *Bernstein algebra* is a Lie algebra with a nondegenerate bilinear form (\cdot, \cdot) and a left-invariant metric tensor g . The object Δ_{ω} is called a *basic algebra* (longleftrightarrow ω is a weight) [1].

Example. Let \mathcal{F} be a linear operator in a Euclidean space over \mathbb{R} , i.e., in linear form ($x \mapsto \langle \mathcal{F}x, x \rangle$). We define the multiplication

$$xy = \langle \mathcal{F}x, \mathcal{F}y \rangle - \det(\mathcal{F})y.$$

Then $\Delta_{\omega}(x) = \omega \sin(x)$, and we obtain a *Bernstein algebra* Δ_{ω} , generated by the *harmonics* contained in \mathcal{F} (see [1]).

¹For now, [3] is a reformulation of [5]. In [4], Bernstein's idea [4]

²and [9] contains a number of results previously obtained in [8].

³in [10] the author's book [12].

we obtain

$$l_k(z) = \Phi \left[\int_0^u E_k(v) f(u+z-v) dv \right] = \Phi \left[\int_0^u E_k(u-v) f(v+z) dv \right].$$

Consequently $l_k(z) \in T_j$, and Dickson's theorem follows.

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Generalized Conformal Structures on Manifolds

A. B. Goncharov*

Introduction. This paper consists of several parts which may be of interest separately, unified by a common idea.

In Sections 1 and 2 we introduce the notion of an F -structure, which generalizes both the notions of a distribution and of the \mathcal{P} -structure of Bernstein and Gindikin [12]. Obstructions to the integrability of F -structures are constructed. Sections 1 and 2 are auxiliary.

In Sections 3–5 we introduce, on a manifold, a notion of a generalized conformal structure related to a compact Hermitian symmetric space (CHSS) X of rank greater than 1. The differential geometry of this structure is studied and relations between F -structures and G -structures are established. If X is a quadric in CP^{n+1} , i.e. a CHSS of type IV in E. Cartan's classification, then the generalized conformal structure of type X on the complex manifold \mathcal{X} is the usual conformal structure, i.e. the conformal class of a complex metric on \mathcal{X} .

In Section 6 we describe a construction of representations of some simple Lie algebras in spaces of the lowest functional dimension. This construction is an analogue of the Weil representation for $Sp(2n)$.

In section 7 we study manifolds with quaternionic structure.

Let us summarize the results of the paper.

0.1. Frobenius structures. Suppose that at each point x of an n -dimensional manifold \mathcal{X} there is defined a family $\mathcal{F}(x)$ of subspaces of dimension k in the tangent space $T_x\mathcal{X}$ that depend analytically (smoothly) on $x \in \mathcal{X}$, i.e. the subset $\mathcal{F}(x) \subset \text{Gr}_k(T_x\mathcal{X})^\dagger$ is given. Then we will say that a *Frobenius structure* \mathcal{F} is defined on \mathcal{X} .

A submanifold $\mathcal{Z} \subset \mathcal{X}$ such that $T_z\mathcal{Z} \subset \mathcal{F}(z)$ for every $z \in \mathcal{Z}$ is called *integral*.

A Frobenius structure is *completely integrable* if for any $x \in \mathcal{X}$ and for any subspace $V(x) \subset \mathcal{F}(x)$ there is an integral manifold tangent to $V(x)$.

* Previously unpublished. Received in translation, March, 1985.

† We have denoted by $\text{Gr}_k(V)$ the complex Grassmann manifold of k -dimensional subspaces in V . If $\dim V = n$, then instead of $\text{Gr}_k(V)$ we write Gr_k^n .

Examples (of completely integrable Frobenius structures).

- (1) A distribution on a manifold.
- (2) Let $B \xleftarrow{\pi_1} A \xrightarrow{\pi_2} \Gamma$ be a double fibration (see [9]),

$$B_\xi = \pi_1 \pi_2^{-1}(\xi), \quad \Gamma_x = \pi_2 \pi_1^{-1}(x).$$

Then $\{T_\xi \Gamma_x | x \in B_\xi\}$ and $\{T_x B_\xi | \xi \in \Gamma_x\}$ define Frobenius structures on Γ and B , respectively. In this situation Γ_x and B_ξ are integral manifolds.

Let us say that a double fibration is *admissible* if there is a set $\{\varphi_\xi\}$ of densities on B_ξ such that for the integral transformation $f \mapsto \int_{B_\xi} f \varphi$ there is a local inversion formula. J. Bernstein and S. Gindikin have shown [12] that the admissibility condition imposes the following restriction on the Frobenius structure in Γ :

for any $\xi \in \Gamma$ there exists $A(\xi): \mathbf{C}^2 \otimes \mathbf{C}^k \rightarrow T_\xi \Gamma_x$
such that $\{ \operatorname{Im} A(\xi)(v \otimes \mathbf{C}^k) = \{T_\xi \Gamma_x\} \}$.

This is precisely the definition of the Bernstein-Gindikin \mathcal{P} -structure.

It turns out that if $\dim \Gamma > \dim B > 2$ then the following uniqueness theorem holds:

Exactly 1 integral manifold passes through each subspace.

In this case B is recovered from a (completely integrable) \mathcal{P} -structure in Γ as the family of integrable manifolds and the obtained double fibration is admissible.

Two problems arise:

- (1) under what conditions is the Frobenius structure completely integrable?
- (2) for a completely integrable Frobenius structure, how many integral manifolds are tangent to the given subspace?

0.2. F-structures. Suppose that $\pi: \mathcal{F}^1 \rightarrow \mathcal{X}$ is a bundle with fiber $\mathcal{F}(x)$ over x . In what follows, V_f stands for the subspace in $T_x \mathcal{X}$ corresponding to $f \in \mathcal{F}(x)$.

In order to solve problem (1) of section 0.1, we construct a sequence of obstructions to the integrability of the Frobenius structure. The k th obstruction is the obstruction to the existence of the $(k+1)$ -jet of the integral submanifold through V_f .

Definition. Let $F \subset \operatorname{Gr}_k(W)$ be a manifold with the transitive action of a subgroup of $\operatorname{GL}(W)$. We say that an *F-structure* is given on the manifold \mathcal{X} if, for every point $x \in \mathcal{X}$, there is a linear isomorphism $B(x): W \rightarrow T_x \mathcal{X}$ such that $B(x)(F) = \mathcal{F}(x)$.

It happens that for *F*-structures the process of construction of the full set of obstructions to the existence of integral manifolds becomes much simpler.

The reason is that a lot of information on the nature of integrability may be obtained from the study of $F \subset \operatorname{Gr}_k(W)$ itself, regardless the concrete type of the *F*-structure. In particular, the answer to problem (2) depends only on *F* (see Theorem A, sec. 2.4).

Examples (of F -structures). (1) F is a point. Then an F -structure is a k -dimensional distribution on \mathcal{X} and a completely integrable F -structure is a foliation on \mathcal{X} .

(2) $F = \mathbf{CP}^1$ and there are a linear space L and linear mappings $A_1, A_2: L \rightarrow W$ such that $F = \{\text{Im}(\lambda_1 A_1 + \lambda_2 A_2), \text{ where } (\lambda_1, \lambda_2) \in \mathbf{C}^2 \setminus (0, 0)\}$ and $\bigcap_{\lambda \neq 0} \text{Im}(\lambda_1 A_1 + \lambda_2 A_2) = 0$.

Let E be a bundle on \mathbf{CP}^1 whose fiber over $(\lambda_1 : \lambda_2)$ is $W/\text{Im}(\lambda_1 A_1 + \lambda_2 A_2)$. Then ([12]) $\Gamma(\mathbf{CP}^1, E) \cong W$. In particular, $\text{GL}(2)$ acting on \mathbf{CP}^1 acts transitively on F . The corresponding F -structure is the Bernstein-Gindikin \mathcal{P} -structure that plays a key role in the problem of the description of admissible families of curves in integral geometry [12].

F -structures arise in integral geometry in the following situation. Suppose that on the manifold \mathcal{Y} there is given a family of 1-dimensional submanifolds \mathcal{L}_x parameterized by points of the manifold \mathcal{X} . Let $\mathcal{V}(y) \subset \mathcal{X}$ be a set of points x such that \mathcal{L}_x contains y . Then if $y \in \mathcal{L}_{x_0}$, the subspace $V_y(x_0)$ is induced in $T_{x_0}\mathcal{X}$ by $\mathcal{V}(y)$. This, in $T_{x_0}\mathcal{X}$ we have defined a family of subspaces, parameterized by points of \mathcal{L}_{x_0} . Clearly, $\mathcal{V}(y)$ is an integral manifold.

For finite-dimensional admissible families of curves on \mathcal{X} this is the way that there arises a completely integrable F -structure ([12]) that describes almost everywhere the admissible family of curves. Let us clarify this important remark. In Chapter I the full set of obstructions to the existence of an integral manifold is constructed according to the method of §0.2 so that the k -obstruction is defined when the $(k - 1)$ -th vanishes, and it is the section of the vector bundle $\mathcal{H}^{k-1,2}(F)$ over \mathcal{F}^1 ; see §1. The k -th obstruction vanishes at the point $f \in \mathcal{F}^1$ if and only if the $(k + 1)$ -jet of the integral manifold is tangent to V_f . In this situation $\dim \mathcal{H}^{k-1,2}_{f(x)}$ depends only on $F \subset \text{Gr}_k(W)$ regardless of the specific type of the F -structure.

Important remark. In fact, in the construction of the full set of obstructions and in the proof of Theorem A (§2.4) we make use not of the homogeneity of F but of a much weaker condition.

It suffices that for any $f_1, f_2 \in F$ there exists $A \in \text{GL}(W)$ such that

$$A(T_{f_1}F) = T_{f_2}F.$$

For example, any one-dimensional submanifold in Gr_2^3 satisfies this condition, since $\text{GL}(W_f)$ is transitive on $\text{Hom}(W_f, \mathbf{C}^3/W_f) \setminus 0$, where W_f is a subspace of W of dimension 1.

Let $G_F \subset \text{GL}(W)$ be the group of all transformations that preserve F . On a manifold with an F -structure, a G_F -structure may be introduced. (Definition of G -structures and their main properties may be found in ([16], [20], [26]).

Our results on the relations of the structure functions of the G_F -structure with obstructions to the integrability of the F -structure generalize Penrose's theorem that states that the antiselfdual part of the Weyl tensor on a 4-

dimensional manifold with a conformal structure vanishes if and only if there exist α -surfaces [21].

0.3. The geometry of generalized conformal manifolds. In $T_x X$, the cone K_x is canonically constructed, so that the cones K_x for different x 's are linearly equivalent. That is, let S be a simple complex Lie group, and P a parabolic subgroup with the Levi decomposition $P = GN$, so that the radical N is Abelian. As we know [17], N is Abelian if and only if $X = S/P$ is CHSS and in that case $G = G_0 \mathbf{C}^*$, where G_0 is semisimple. Let N_- be a subgroup opposite to N and let $\mathfrak{s}, \mathfrak{p}, \mathfrak{g}, g_0, \mathfrak{n}, \mathfrak{n}_-$ be corresponding Lie algebras.

Let $P_x = G_x N_x$ be the Levi decomposition of the stabilizer of $x \in X$ in S . Put K_x for the cone of highest weight vectors in the G -module $T_x X$, i.e. each element in K_x is highest with respect to a Borel subgroup in G_x .

Clearly, $s \in S$ transforms K_x in K_{sx} ; therefore with X there is associated the cone $K(X) \subset \mathfrak{n}_-$, which is linearly equivalent to all cones K_x . It will be convenient to identify \mathfrak{n}_- with $T_e X$ and $K(X)$ with $K_{\tilde{e}}$, where \tilde{e} is the image of the unit $e \in G$ in X .

Now let $\text{rk } X > 1$, i.e. $X \neq \mathbf{C}P^n$. We will say that a *generalized conformal structure of type X* is given on the manifold \mathcal{X} , if the cone $\mathcal{K}_{\mathcal{X}}$ analytically depending on x and \mathbf{C} -linearly equivalent to the cone $K(X)$ (i.e. there is a \mathbf{C} -linear isomorphism $A_x: \mathfrak{n}_- \rightarrow T_x \mathcal{X}$ such that $A_x(K(X)) = \mathcal{K}_{\mathcal{X}}$) is defined at each point $x \in \mathcal{X}$.

Remark. By Corollary 3.7 $\text{rk } X = 1$ if and only if $\mathcal{K}_x = T_x \mathcal{X} \setminus 0$, so that the extra infinitesimal structure on \mathcal{X} is defined only for $\text{rk } X > 1$.

Generalized conformal structures on manifolds \mathcal{X} and \mathcal{Y} are *equivalent* if there is a diffeomorphism $f: \mathcal{X} \rightarrow \mathcal{Y}$ such that $f(\mathcal{K}_x) = \mathcal{K}_{f(x)}$.

Example. Let X be a non-degenerate quadric in $\mathbf{C}P^{n+1}$. Then K_x is the non-degenerate quadratic cone consisting of lines in $\mathbf{C}P^{n+1}$ passing through x and belonging to X . The family of cones $\mathcal{K}_{\mathcal{X}}$ defines a conformal structure on \mathcal{X} and $\mathcal{K}_{\mathcal{X}}$ is the zero cone.

Among all manifolds \mathcal{X} the manifold X is distinguished as having the *flat* structure so that in any domain on X local diffeomorphisms that preserve the family of cones $\mathcal{K}_{\mathcal{X}}$ are extended to holomorphic automorphisms of X . We obtain an infinitesimal characterization of CHSS or rank > 1 . In fact, we may recover a CHSS of rank > 1 from any simply connected domain with a flat family of cones. Perhaps this fact may be useful in the theory of analytic functions on bounded complex symmetric domains of rank > 1 .

We will show that a generalized conformal manifold is a manifold with G -structure, and compute all structure functions of this G -structure. For the conformal structure these structure functions constitute the Weyl tensor.

For a generalized conformal structure the group G is reductive and its centre is one-dimensional. The reduction of the structure group to its

semisimple part, G_0 , is an analogue of distinguishing a metric on a conformal manifold.

Structure functions of the G_0 -structure form an analogue of the Riemann tensor for the metric. They include the structure functions of the G -structure and two more irreducible components that are analogues of the traceless Ricci tensor and of the scalar curvature (Theorem 4.6).

More precisely, the structure functions of the G -structure are defined as the part of the structure functions of the G_0 -structure obtained by a reduction of the G -structure that does not depend on the choice of reduction. In other words, this is (generally) the conformally invariant part of the structure functions of the G_0 -structure.

There is a striking similarity between generalized conformal structures and the classical geometry of conformal manifolds. The algebraic background of this similarity is that generalized conformal structures are G -structures of order 2. In the flat case this means that an automorphism of CHSS X is not defined by its differential at a point. More precisely, in an appropriate coordinate system infinitesimal automorphisms of X are vector fields whose coefficients are polynomials of degree ≤ 2 . The geometric background is that our G -structure is defined by a family of cones that play an important role in various constructions.

For example, there are often many isotropic linear subspaces, i.e. subspaces that belong to $K_x \cup 0$ (see Proposition 3.10). Therefore, an F -structure may be introduced on a generalized conformal manifold. This F -structure is completely integrable if and only if the Weil tensor (or its corresponding part, when $X = \text{Gr}_k^n$) vanishes (see Theorem 5.2). For $X = \text{Gr}_2^4$ this result is due to R. Penrose. Therefore, we have obtained a (non-standard) geometrical interpretation of structure functions of generalized conformal manifolds.

On generalized conformal manifolds there are no canonical linear connections. However, the infinitesimal transport of an isotropy plane along vectors that belong to this plane may be defined (Theorem 4.9).

In particular, curves $\gamma(s)$ are defined with the property that the tangent at each point is an isotropy line and its infinitesimal transport from $\gamma(t)$ to $\gamma(t + dt)$ is the tangent to $\gamma(s)$ at the point $t + dt$. Such curves will be called 0-geodesics.

Instead of the canonical linear connection on a generalized conformal manifold \mathcal{X} of type X , a *Cartan connection* may be introduced. Roughly speaking, this means that with each point $x \in \mathcal{X}$ a CHSS X_x isomorphic to X is associated so that a parallel transport $\tau_{\gamma(0), \gamma(1)}: X_{\gamma(0)} \rightarrow X_{\gamma(1)}$ along the curve $\gamma(t)$, where $0 \leq t \leq 1$, is defined and $\tau_{\gamma(0), \gamma(1)}$ is a diffeomorphism.

It turns out that K_x is defined in $T_x \mathcal{X}$ by a system of quadratic equations. (Therefore the G_0 -structure may be considered as a “metric” with values in the bundle E^* , where the fiber of E over x is the linear space of quadratic equations that singles \mathcal{K}_x out of $T_x \mathcal{X}$.)

Remark. In a paper to appear we will show that the G_0 -structure on \mathcal{X} defines a second-order differential operator acting from functions into E . In case of a metric, E is the trivial 1-dimensional bundle and the operator is the classical Laplace operator. The characteristic manifold of the Laplace operator is $\cup_x \mathcal{K}_x^*$, where $\mathcal{K}_x^* \subset T_x^*\mathcal{X}$ is the cone linearly equivalent to \mathcal{K}_x and in a sense dual to \mathcal{K}_x . On a generalized conformal manifold a conformal analogue of the Laplace operator may be defined.

Until now the ground field has been \mathbf{C} . Let us consider real forms of CHSS, i.e. \mathbf{R} -symmetric Nagano spaces [20]. For example, forms of Gr_{2k}^{2n} are $\mathbf{R}\mathrm{Gr}_{4k}^{4n}$ and $\mathbf{H}\mathrm{Gr}_k^n$ and forms of the quadric in $\mathbf{C}P^n$ are quadrics in $\mathbf{R}P^n$.

For us it is important only that each \mathbf{R} -symmetric Nagano space is presentable in the form $S_{\mathbf{R}}/P_{\mathbf{R}}$, where the Lie algebras of $S_{\mathbf{R}}$ and $P_{\mathbf{R}}$ are $\mathfrak{s} = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1$ and $\mathfrak{p} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$, respectively.

To each of these \mathbf{R} -symmetric spaces we will assign a differential-geometric structure which is a real form of the corresponding conformal structure. For a quadric in $\mathbf{R}P^{n+1}$ this is a way of obtaining a conformal class of a (pseudo) Riemannian metric on a real manifold of dimension n .

All the results obtained for generalized conformal structures can be easily extended to their real forms. Moreover, several real forms obtain some extra interesting properties. The typical example is $\mathbf{H}P^n$ and the corresponding theory of quaternionic manifolds.

The complexification of $\mathbf{H}P^n$ is Gr_2^{2n+2} . Therefore, if \mathcal{X} is a quaternionic manifold then in $T_x\mathcal{X} \otimes \mathbf{C}$ the cone \mathcal{K}_x linearly equivalent to the cone $K(\mathrm{Gr}_2^{2n+2})$ is canonically constructed.

Let $\pi: \mathcal{P} \rightarrow \mathcal{X}$ be the bundle with the fiber over x being the set of all α -subspaces in \mathcal{K}_x , i.e. the set of isotropic subspaces of \mathbf{C} -dimension $2n$ in \mathcal{K}_x .

We say that a unitary connection in a bundle on \mathcal{X} is *self-dual* if its curvature form vanishes on α -subspaces. Assigning to a bundle on \mathcal{X} with a self-dual connection a certain holomorphic bundle on \mathcal{P} we establish an equivalence of the category of such bundles on \mathcal{X} with a certain category of holomorphic bundles on \mathcal{P} .

By Theorem 4.9 each α -subspace is uniquely lifted by the corresponding point of \mathcal{P} . Since $\pi^{-1}(x) \cong \mathbf{C}P^1$, we obtain an almost complex structure on \mathcal{P} . This structure is integrable if and only if the antiselfdual part of structure functions vanishes. To a bundle on such an \mathcal{X} with a self-dual connection there corresponds a holomorphic bundle on \mathcal{P} . This fact is a multidimensional generalization of the Atiyah-Ward-Belavin-Zakharov construction (see [2-5]).

After the section on quaternionic manifolds was written, Yu. I. Manin informed me about a preprint by S. Salamon "Quaternionic manifolds," that contains results similar to ours.*

* This paper was written in Autumn 1981, but as the result of the pressure of various matters it was submitted for publication only now. During this time Salamon's preprint has been published, see [23].

0.5. The Weil-type representations. Put $K^*(K_x^*)$ for the cone of highest vectors of the G -module $\mathfrak{n}(T_x X)$.

We will construct the embedding of \mathfrak{s} in the Lie algebra $\mathcal{D}(K^*)$ of regular differential operators on K^* , yielding the representation of \mathfrak{s} in the space of regular functions on K^* . In the coadjoint representation of S there is the unique orbit \mathcal{O}_g of minimal dimension. It passes through the highest weight vector in \mathfrak{g}^* . The constructed representation corresponds to \mathcal{O}_g meaning that $2 \dim K(X) = \dim \mathcal{O}_g$. But for $S \neq SL(n)$ there is no polarization of \mathcal{O}_g and the usual methods of recovering a representation from an orbit fail. The problem of construction of "minimal" representations was considered by different authors but their construction was the restriction of the Weil representation of $sp(2n)$ onto $\mathfrak{s} \rightarrow sp(2n)$. Our construction is more universal, and for $sp(2n)$ we get a new Weil-type construction of representations. It is easily extended to a wide class of Lie algebras, in particular, to split algebras over k , where $\text{char } k = 0$, that have a parabolic subalgebra with Abelian radical.

Roughly speaking, the construction is as follows. S acts on sections of a linear S -bundle E_λ , associated with the character $\chi_\lambda: P \rightarrow \mathbb{C}^*$ so that $d\chi_\lambda(t) = \lambda t$ for $t \in \text{Center } G \cong \mathbb{C}$. Let $\exp(\mathfrak{n}_-)P$ be a neighborhood of the unit in S . Then let us identify points of $P \cdot e$ in X with \mathfrak{n}_- and consider an N_- -invariant trivialization of E_λ over \mathfrak{n}_- . For $s \in S$ the operator \mathcal{L}_s that will be constructed below is just a coordinate expression of the corresponding S -action in canonical coordinates on \mathfrak{n}_- . If λ is as in Theorem 6.4, then $F(\mathcal{L}_s)$, the image of \mathcal{L}_s under the Fourier transform (see 6.1) belongs to $\mathcal{D}(K^*)$. The reason for this is that $s \in S$ transforms K_{sx}^* into K_x^* .

Note that $F(\mathcal{L}_n)$ is of order 2 for $n \in \mathfrak{n}$ and cannot be expressed in terms of first-order operators of $\mathcal{D}(K^*)$. It is possible to prove that $\mathcal{D}(K^*)$ is generated by operators $F(\mathcal{L}_s)$, where $s \in \mathfrak{s}$. Therefore we have obtained new interesting examples of rings of regular differential operators on a singular algebraic variety.

Remark. The results of sections 3–5 and section 6 are partially announced in [3] and [14], respectively.

In the remarkable paper by M. A. Akivis [1], the generalized conformal structure related to a Grassmannian (an "almost Grassmann structure" in terms of [1]) was considered in connection with the geometry of webs. In [1] the results of §6 are obtained for CHSS of type 1.

I learned about [1] only after [13] was sent to press, and that is why there is no reference to [1] in [12].

Note that several results of this paper may be interpreted as a multidimensional generalization of results of R. Penrose from twistor theory.

The paper originated from the S. G. Gindikin's question on infinitesimal structures on CHSS. I am indebted to S. G. Gindikin and J. N. Bernstein for valuable discussions, advice and attention. I thank D. A. Leites for editing and translation.

Chapter I

The Frobenius Theorem for F -structures

§1. The first obstruction to integrability of the F -structure

1. Preliminaries. An m -jet \mathcal{K}^m of the germ of a k -dimensional submanifold $\mathcal{K} \subset \mathcal{X}$ at $x \in \mathcal{X}$ is the class of m -tangent k -dimensional submanifolds at x . Denote by $\mathcal{T}_k^m : \mathcal{F}_k^m \rightarrow \mathcal{X}$ the bundle of m -jets of germs of k -dimensional submanifolds (shortly, k -submanifolds). For example, $(\mathcal{T}_k^s)^{-1}(x) = \text{Gr}_k(T_x \mathcal{X})$. (In what follows, the subscript k will be omitted whenever misunderstandings are unlikely).

Select a local coordinate system x^1, \dots, x^n on \mathcal{X} . An m -germ at $(0, \dots, 0)$ of a k -submanifold close to the submanifold $x^{k+1} = \dots = x^n = 0$ may be defined by the formulas $x^{k+i} = y^i(x^1, \dots, x^k) = a_{j_1 \dots j_m}^i x^{d_1} + \dots + a_{j_1 \dots j_m}^i x^{d_k} \dots x^{d_n}$, where $1 \leq i \leq n - k$, $1 \leq j_s \leq k$.

Evidently, x^1, \dots, x^n and $\{a_{j_1 \dots j_m}^i\}, \dots, \{a_{j_1 \dots j_m}^i\}$, where $a_{j_1 \dots j_m}^i$ is symmetric with respect to lower indices, form a local coordinate on \mathcal{F}_k^m . There are natural bundles $\pi_k^{m,m-1} : \mathcal{F}_k^m \rightarrow \mathcal{F}_k^{m-1}$ such that $\mathcal{K}^m = \pi_1^{m+1,m} \mathcal{K}^{m+1}$. The point $\mathcal{K}^{m+1} \in \mathcal{F}_k^{m+1}$ with coordinates $(x_0^1, \dots, x_0^n, \dots, {}^0 a^i)$ defines the k -subspace $\Pi(\mathcal{K}^{m+1}) \subset T_{\mathcal{K}^m} \mathcal{F}_k^{m+k}$ of the form

$$\begin{aligned} x^i &= a_{j_1}^i (x^{d_1} - x_0^{d_1}) + x_0^i \\ a_{j_1 \dots j_m}^i &= {}^0 a_{j_1 \dots j_{m+1}}^i (x^{d_{m+1}} - x_0^{d_{m+1}}) + {}^0 a_{j_1 \dots j_m}^i \end{aligned} \quad (1)$$

The following lemma is easy to verify.

Lemma 1.1. $\pi(\mathcal{K}^{m+1})$ is well-defined. At the point $(x_0^1, \dots, {}^0 a_{j_1 \dots j_m}^i)$ any subspace of the form (1), where ${}^0 a_{j_1 \dots j_{m+1}}^i$ is symmetric with respect to j_1, \dots, j_{m+1} , is obtained by the method above.

Let $\mathcal{F}_F^1 \subset \mathcal{F}^1$ be the submanifold that defines an F -structure on \mathcal{X} . Let $\psi : U \rightarrow \mathcal{F}_F^1$ be a section of \mathcal{F}_F^1 over \mathcal{X} . It is known that

$$T_{\psi(x)} \text{Gr}_k(T_x \mathcal{X}) \cong \text{Hom}(V_{\psi(x)}, T_x \mathcal{X}/V_{\psi(x)})$$

Let us consider $T_{\psi(x)} \mathcal{F}_F^1(x)$ as a subspace in $\text{Hom}(V_{\psi(x)}, T_x \mathcal{X}/V_{\psi(x)})$.

The point $\mathcal{K}^m \in \mathcal{F}_k^m$ is the m -jet of an integral manifold tangent to $V_{\psi(x)}$ if its lifting to \mathcal{F}^1 at $\psi(x)$ is $(m-1)$ -tangent to \mathcal{F}_F^1 .

2. The Frobenius form. A pair of subspaces $H_1, H_2 \subset T_{\psi(x)} \mathcal{F}_F^1$ with images isomorphic to $V_{\psi(x)}$ with respect to projection define the mapping

$$S_{H_1, H_2} : V_{\psi(x)} \rightarrow T_{\psi(x)} \mathcal{F}_F^1$$

as follows. If $h_1 \in H_1$, $h_2 \in H_2$ and $\pi^1(h_1) = \pi^1(h_2) = \mathcal{V}$, then $S_{H_1, H_2}(\mathcal{V}) = h_2 - h_1$.

Conversely, for a fixed H_1 the mapping S_{H_1, H_2} defines H_2 . Define $\delta_1(S_{H_1, H_2}) \in \text{Hom}(\Lambda^2 V_{\psi(x)}, T_x \mathcal{X}/V_{\psi(x)})$ by the formula

$$\delta_1(\delta_{H_1, H_2})(\mathcal{V}_1, \mathcal{V}_2) = S_{H_1, H_2}(\mathcal{V}_1)(\mathcal{V}_2) - S_{H_2, H_1}(\mathcal{V}_2)(\mathcal{V}_1).$$

Let $\mathcal{V}_i(x)$, for $i = 1, 2$, be a vector field on \mathcal{X} such that $\mathcal{V}_i(x) \in V_{\psi(x)}$.

Lemma 1.2. *The image of $[\mathcal{V}_1(x), \mathcal{V}_2(x)]|_{x=x_0}$ in $T_x \mathcal{X}/V_{\psi(x)}$ depends only on $\mathcal{V}_1(x_0)$, $\mathcal{V}_2(x_0)$ and $d\psi(V_{\psi(x)})$.*

Thus a 2-form $w_{d\psi}$ on $V_{\psi(x_0)}$ with values in $T_{x_0} \mathcal{X}/V_{\psi(x_0)}$ is well defined. It is natural to call this form *the Frobenius form* (of the 1-jet of a distribution on \mathcal{X}). For example, the curvature of a connection in a G -bundle is the Frobenius form of the horizontal distribution that defines the connection.

Lemma 1.3. $w_{d\psi} - w_{d\psi_1} = \delta_1(S_{d\psi_1, d\psi})$.

Proof of Lemmas 1.2 and 1.3. Let $\xi_1(x), \dots, \xi_n(x)$ be a set of vector fields generating $T_x \mathcal{X}$ at each point x so that $\xi_1(x), \dots, \xi_k(x)$ generate $V_{\psi(x)}$. Denote by $L_{\gamma_i} f(x)$ the derivative of $f(x)$ along $\mathcal{V}(x)$. For $\tilde{\mathcal{V}}_i(x) = v_i(x) + \gamma_i^d(x)\xi_j(x)$, where $i = 1, 2, j = 1, \dots, k$ and $\gamma_i^d(x_0) = 0$, we have

$$[\tilde{\mathcal{V}}_1(x), \tilde{\mathcal{V}}_2(x)]|_{x=x_0} = [\mathcal{V}_1(x), v_2(x)]|_{x=x_0} + (L_{\gamma_1} \gamma_2^j(x_0) - L_{\gamma_2} \gamma_1^j(x_0))\xi_j(x_0).$$

Furthermore, let $g(x) \in \text{Hom}(V_{\psi(x)}, T_x \mathcal{X})$ and $g(x_0) = id$. Put $V_{\psi(x)} = g(x)V_{\psi(x)}$. Then

$$w_{d\psi_1}(\mathcal{V}_1, \mathcal{V}_2) = [g(x)\mathcal{V}_1(x), g(x)\mathcal{V}_2(x)]|_{x=x_0} = [\mathcal{V}_1(x), \mathcal{V}_2(x)]|_{x=x_0} + (L_{\gamma_1} g(x_0))\mathcal{V}_2(x_0) - (L_{\gamma_2} g(x_0))\mathcal{V}_1(x_0). \blacksquare$$

Lemma 1.4. $w_{H_1} = 0$ for $H_1 \subset T_{\psi(x)} \mathcal{F}_F^1$ iff $H_1 = \Pi(\mathcal{K}^2)$, where $\mathcal{K}^2 \in \mathcal{F}^2(x)$.

The proof follows from Lemma 1.1 because if, in terms of coordinates, H_1 is of the form (1) for $n = 1$, then

$$w_{H_1}(\partial_{j_1} + a_{j_1}^i \partial_i, \partial_{j_2} + a_{j_2}^i \partial_i) = (a_{j_1 j_2}^i - a_{j_2 j_1}^i) \partial_i,$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

Thus, to each point $\psi(x) \in \mathcal{F}_F^1$ we have assigned the vector $\varphi_F^1(\psi(x))$ of

$$\text{Hom}(\Lambda^2 V_{\psi(x)}, T_x \mathcal{X}/V_{\psi(x)})/\delta_1(\text{Hom}(V_{\psi(x)}, T_{\psi(x)} \mathcal{F}_F^1)).$$

Lemmas 1.3 and 1.4 imply the following proposition.

Proposition 1.5. $\varphi_F^1(\psi(x_0)) = 0$ if and only if there is a 2-jet of the integral manifold tangent to $V_{\psi(x_0)}$.

§ 2. Higher obstructions to integrability of an F -structure.

1. The complex $C_{\psi(x)}^{k,l}$. In 2.2 we will construct obstructions to the existence of the integral manifold through $V_{\psi(x)}$. These obstructions are elements of the homology of $C_{\psi(x)}^{k,l}$.

First, let us construct the complex $C^{k,l}(T_f)$ connected with the linear subspaces $W_f \subset W$ and $T_f = \text{Hom}(W_f, W/W_f)$. Put

$$\begin{aligned} T_f^{-1} &= W/W_f; \quad T_f^0 = T_f; \\ T_f^s &= W_f^* \oplus T_f^{s-1} \cap S^2 W_f^* \oplus T_f^{s-2} \quad \text{for } s \geq 1; \\ C^{k,l}(T_f) &= \text{Hom}(\Lambda^l W_f, T^{k-1}). \end{aligned}$$

For $X \in C^{k,l}(T_f)$, define $\delta_l X \in C_f^{k-1,l+1}(T_f)$ by the formula $(\delta_l X)(\mathcal{V}_1, \dots, \mathcal{V}_{l+1}) = \sum_{0 \leq i \leq l} (-1)^i X(\mathcal{V}_1, \dots, \mathcal{V}_{i+1}, \dots, \mathcal{V}_{l+1-i}, \dots, \mathcal{V}_{l+1})$.

Lemma 2.1. $\delta_{l+1} \circ \delta_l = 0$.

Remark 2.2. If $T_f = \text{Hom}(W_f, W/W_f)$, then from the Koszul complex $(S(W_f) \oplus \Lambda(W_f), \delta)$ we recover the complex $(S(W_f) \oplus \Lambda(W_f) \oplus (W/W_f), \delta)$ by extending δ onto W/W_f by zero. Then $C^{k,l}(T_f)$ is a subcomplex in the new complex. Put

$$\begin{aligned} H^{k,l}(T_f) &= \text{Ker } \delta_l C^{k,l}(T_f) / \text{Im } \delta_{l-1} C^{k+1,l-1}(T_f); \\ C_{\psi(x)}^{k,l} &= C^{k,l}(T_{\psi(x)} \mathcal{F}_F^1(x)); \quad H_{\psi(x)}^{k,l} = H^{k,l}(T_{\psi(x)} \mathcal{F}_F^1(x)). \end{aligned}$$

All complexes $C_{\psi(x)}^{k,l}$ for different x 's are isomorphic to each other, since the triples $(V_{\psi(x_1)}, T_{x_1} \mathcal{X}, T_{\psi(x_1)} \mathcal{F}_F^1(x_1))$ and $(V_{\psi(x)}, T_x \mathcal{X}, T_{\psi(x)} \mathcal{F}_F^1(x))$ are linearly equivalent (cf. the remark in Introduction).

Let $F^1 \subset \text{Gr}_k(W \oplus T_f F)$ be a submanifold consisting of graphs of mappings $A: W_f \rightarrow T_f F$, where $A \in (T_f F)^1$. The manifold F^1 is the principal homogeneous space for the Abelian group $(T_f F)^1$. Assume that $\varphi_F^1(\psi) = 0$ for any $\psi \in \mathcal{F}_F^1|_U$. Then define the higher obstruction by putting $\mathcal{F}_F^2(\psi) = \{H \in T_4 \mathcal{F}_F^1|_{\pi^1} \text{ maps } H \text{ isomorphically onto } V_4 \text{ and } w_H = 0\}$. Therefore an F^1 -structure is defined on \mathcal{F}_F^1 . Call this structure the *first extension* of the F -structure on \mathcal{X} .

Let H be a point of \mathcal{F}_F^2 and the corresponding subspace. A priori $\varphi_{F^1}^1(H)$ belongs to

$$\text{Hom}(\Lambda^2 H, T_\psi \mathcal{F}_F^1/H)/\delta_1 \text{Hom}(H, T_H \mathcal{F}_{F^1|_\psi}^2)$$

for $H \in \mathcal{F}_F^2|_{\psi(x)}$. Since $\mathcal{F}_F^2|_\psi$ consists of sub-spaces with isomorphic projections onto V_ψ , then

$$T_H(\mathcal{F}_{F^1|_\psi}^2) \subset \text{Hom}(H, T_\psi \mathcal{F}_{F^1|_{x_0}}^1).$$

let us identify H with V_ψ with respect to the projection.

Then $w_H = 0$ if and only if

$$T_H \mathcal{F}_{F^1|_\psi}^2 \subset (T_\psi \mathcal{F}_{F^1|_{x_0}}^1)^1.$$

Furthermore, take 2 vector fields, $\mathcal{V}_1(\psi)$ and $\mathcal{V}_2(\psi)$, on \mathcal{F}_F^1 that at each point belong to a subspace which belongs to the F^1 -structure on \mathcal{F}_F^1 , so that $\mathcal{V}_i(\psi(x_0)) \in H$ for $i = 1, 2$. Evidently,

$$W_H = 0 \Leftrightarrow \pi^1([\mathcal{V}_1(\psi), \mathcal{V}_2(\psi)])|_{\psi=\psi(x_0)} = [\pi^1 \mathcal{V}_1, \pi^1 \mathcal{V}_2]|_{x=x_0}$$

Therefore, $[\mathcal{V}_1(\psi), \mathcal{V}_2(\psi)]|_{\psi=\psi(x_0)} \in H \oplus T_\psi(\mathcal{F}_F^1|_{x_0})$. Thus, $\varphi_{F^1}^1(H) \in C^{1,2}(T_\psi \mathcal{F}_F^1|_{x_0})/\delta_1 C^{2,1}(T_\psi \mathcal{F}_F^1|_{x_0})$.

Proposition 2.3. *If $H, \tilde{H} \in \mathcal{F}_F^2|_{x_0}$, then $\varphi_{F^1}^1(H) = \varphi_{F^1}^1(\tilde{H}) \in H^{1,2}(T_\psi \mathcal{F}_F^1|_{x_0})$.*

Proof. Suppose that $\pi^{2,1}$ maps $\tilde{H} \subset T_H \mathcal{F}_F^2$ isomorphically on H . If \tilde{H} is defined by equations (1) (where $m = 2$) we have

$$\tilde{H} \subset T_H \mathcal{F}_H^2 \Rightarrow a_{j,j_2}^i = a_{j,j_1}^i \Rightarrow {}^0 a_{j,j_2,j_3}^i = {}^0 a_{j_2,j_3}^i.$$

Moreover,

$$\begin{aligned} w_{\tilde{H}}(\partial_{j_2} + {}^0 a_{j_2}^i \partial_i, \partial_{j_3} + {}^0 a_{j_3}^i \partial_i)(\partial_{j_1} + {}^0 a_{j_1}^i \partial_i) &= ({}^0 a_{j_1,j_2,j_3}^i - {}^0 a_{j_2,j_3}^i) \partial_i \\ \delta_2 w_{\tilde{H}}(\partial_{j_1} + {}^0 a_{j_1}^i \partial_i, \partial_{j_2} + {}^0 a_{j_2}^i \partial_i, a_i \partial_{j_3} + {}^0 a_{j_3}^i \partial_i) &= \\ [({}^0 a_{j_3,j_2,j_1}^i - {}^0 a_{j_2,j_3,j_1}^i) - ({}^0 a_{j_2,j_1,j_3}^i - {}^0 a_{j_1,j_3,j_2}^i) + ({}^0 a_{j_1,j_2,j_3}^i - {}^0 a_{j_1,j_3,j_2}^i)] &= 0 \end{aligned} \quad (2)$$

If s_{j,j_2}^i are the coordinates of $S_{H,\tilde{H}} \in (T_\psi \mathcal{F}_F^1|_{x_0})^1$ in the basis $\{\partial_i, \partial_i + a_i^j \partial_j\}$, where $1 \leq i \leq n-k, 1 \leq j \leq k\}$ and \tilde{H} is of the form

$$\begin{aligned} x^i &= {}^0 a_j^i (x^j - x_0^j) + x_0^i; \\ a_i^j &= ({}^0 a_{j,j_2}^i + s_{j,j_2}^i)(x^j - x_0^j) + {}^0 a_i^j; \\ a_{j,j_2}^i &= b_{j,j_2,j_3}^i (x^{j_3} - x_0^{j_3}) + {}^0 a_{j,j_2}^i + g_{j,j_2}^i {}^0 a_j^i, \end{aligned}$$

then because of (2) $w_{\tilde{H}} = w_{\tilde{H}}$. ■

Therefore, if we put $\varphi_F^2(\pi^{2,1}(H)) = \varphi_{F^1}^1(H)$ the “function” $\varphi_F^2(\varphi(x_0))$ on \mathcal{F}_F^1 with values in $H^{1,2}(T_\psi \mathcal{F}_F^1|_{x_0})$ is well defined. This function is the 2-obstruction.

3. The $(m+1)$ -obstruction. Suppose the manifold $F^{(m)} \subset \text{Gr}_k(W \oplus (T_f F)^{(m-1)})$ consists of graphs of mappings $A: W_f \rightarrow (T_f F)^{(m-1)}$, where $A \in (T_f F)^{(m)}$. Let the m -obstruction vanish on $\mathcal{F}_F^1|_u$. The first extension of the $F^{(m-1)}$ -structure on \mathcal{F}_F^m defines the bundle $\pi^{m+1,m}: \mathcal{F}_F^{m+1} \rightarrow \mathcal{F}_F^m$, which defines an $F^{(m)}$ -structure on \mathcal{F}_F^m . This structure is the m -th extension of the F -structure on \mathcal{X} .

Proposition 2.5. *Suppose $h \in \mathcal{F}_F^{m+1}$ and that H is the corresponding subspace in $T_{\pi^{m+1,m}(h)} \mathcal{F}_F^m$. Let us identify H with $V_{\pi^{m+1,m}(h)}$ with respect to that projection. Then*

- (a) $\varphi_{F^{(m)}}^1(H) \in H^{m,2}(T_{\pi^{m+1,m}(h)} \mathcal{F}_F^1|_{\pi^{m+1,m}(h)})$;
- (b) $\varphi_{F^{(m)}}^1(H)$ depends only on $\pi^{m+1,m}(h)$.

Proof. Similar to that of Proposition 2.3. ■

Thus, if we put $\varphi_F^m(\pi^{m,1}(h)) = \varphi_{F^{m-1}}^1(h)$, the $(m+1)$ -obstruction $\varphi_F^m(\psi) \in H^{m-1,2}(T_\psi \mathcal{F}_F^1|_{\pi^1(\psi)})$ is well defined.

Lemma 2.6. *If $L \subset T_h \mathcal{F}_F^m$ and $w_L = 0$ then there is an $(m+1)$ -jet of the submanifold such that the lifting of the jet to \mathcal{F}^1 is the m -tangent to \mathcal{F}_F^1 .* ■

Thus, $\varphi_F^m(\psi) = 0$ if and only if there is an $(m+1)$ -jet of an integral manifold tangent to V_F . Recall that the existence of $\varphi_F^m(\psi)$ itself assumes the vanishing of all lower obstructions in a neighbourhood of ψ .

4. The main theorem. The F -structure is of (finite) type m if m is the minimal positive number such that $T_f^{(m)} = 0$.

Theorem A. (1) *The F -structure of finite type m is completely integrable in the domain U if and only if the first $m+1$ obstructions $\varphi_F^1, \dots, \varphi_F^{m+1}$ vanish on $\mathcal{F}_F^1|_U$.*

(2) *For a completely integrable F -structure there is a family of integral manifolds tangent to V_ψ of dimension $\sum_{1 \leq i \leq m} \dim H_F^{i,1}$.*

Proof. The condition $(T_{\psi(x)}(\mathcal{F}_F^1|_x))^{(m)} = 0$ means that the $(m+1)$ -th extension of the F -structure is a distribution on \mathcal{F}_F^m . Let us identify the $(m+1)$ -obstruction φ_F^{m+1} with the Frobenius form of this distribution (see Proposition 2.3). Therefore, by the classical Frobenius theorem, $\varphi_F^{m+1} = 0$ if and only if this distribution is completely integrable. This proves (1).

Let $S \in (\pi^{m,1})^{-1}(\psi)$ and \mathcal{V}_S be an integral manifold through S of the distribution considered. Then $\pi^m(\mathcal{V}_S)$ is an integral manifold of the F -structure tangent to V_ψ , and conversely, this is the way in which all integral manifolds tangent to V_ψ arise. It remains to note that

$$\dim (\pi^{m,1})^{-1}(\psi) = \sum_{1 \leq i \leq m} \dim H_F^{i,1}. \quad \blacksquare$$

In the category of analytical spaces a more precise theorem is valid.

Theorem A'. *The F -structure is completely integrable in the domain U if and only if all obstructions vanish on $\pi^{-1}(U)$.*

Remark. The F -structure is of infinite type if and only if the family of integral manifolds tangent to V_ψ is infinite-dimensional.

Example. (The \mathcal{P} -structure of Bernstein and Gindikin [12]). Let $E = \sum_{1 \leq i \leq s} k_i \mathcal{O}(i)$ and $W = \Gamma(\mathbb{C}P^1, E) = \bigoplus_{1 \leq i \leq s} \Gamma(\mathbb{C}P^1, \mathcal{O}(i)) \otimes \mathbb{C}^{k_i}$. Let us realize $\Gamma(\mathbb{C}P^1, \mathcal{O}(i))$ as the space of polynomials of degree not greater than i on the line; then put $W_f = \sum_{1 \leq i \leq s} \{a_1 x + \dots + a_i x^{i-1}\} \otimes \mathbb{C}^{k_i}$. Clearly, $T_f = 1$. Let

$T \in T_f$ be such that $T(a(x) \otimes e_r) = \frac{\partial a}{\partial x} \Big|_{x=0} \otimes e_r$, where $\{e_r\}_{1 \leq r \leq k_i}$ is a basis in \mathbb{C}^{k_i} .

Let $E = \mathcal{O}(2)$. Then F is the set of planes tangent to the quadratic cone in \mathbb{C}^3 . This F -structure on 3-dimensional manifold defines a conformal structure and vice versa.

Lemma 2.7. *Let $E = \mathcal{O}(2)$. Then $\dim T_f^{(m)} = 1$, and $H_f^{m,2} = 0$ for $m = 0, 1, 2, \dots$*

Proof. $W_f = \{a_1x + a_2x^2\}$. Let x^*, x^{*2} be the dual basis. Let $T^1 \in T_f^{(1)}$ and $T^1(x) = A\partial_x$, $T(x^2) = B\partial_x$. Then $0 = T^1(x)(x^2) - T^1(x^2)(x) = -B$, hence $\dim T_f^{(1)} = 1$ and so on. We have $x^* \otimes \dots \otimes 1 \in T_f^{(m)}$ and $\dim C_f^{m,2} = 1$, $\dim \delta_1 \text{Hom}(W_f, T_f^{(m)}) = L$ hence $H_f^{m,2} = 0$. ■

By Theorem A, this F -structure is always completely integrable and for any plane V_ψ there is an infinite-dimensional family of integral manifolds tangent to V_ψ . For example, for the flat family of cones in \mathbb{R}^3 the surfaces $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 0$ are integral manifolds.

Notice (see the Remark in the Introduction) that the same is true for an F -structure such that F is a curve in Gr_2^3 .

Lemma 2.8. *If $\sum_{1 \leq i \leq j} k_i > 1$ then $T_f^{(1)} = 0$.*

Proof. Similar to that of Lemma 2.7. ■

Lemma 2.9. *There is a unique integral manifold through each subspace of this completely integrable F -structure.*

This lemma, which is a reformulation of the “Desargues theorem” from [12], is a corollary of Theorem A.

Chapter II

Generalized Conformal Structures

§3. Geometry of the cone $K(X)$

1. CHSS's. Suppose that the Dynkin diagram for the Lie algebra \mathfrak{g} is obtained by discarding the vertex $\bar{\gamma}$ of the Dynkin diagram of the Lie algebra \mathfrak{s} . The root γ corresponding to $\bar{\gamma}$ enters with coefficient 1 into the decomposition of the minimal root with respect to simple roots and, conversely, every such simple root is connected with CHSS ([17,18]). The following table collects the information about CHSS; CHSS is presented in the form $S_{\mathbb{R}}/G_{\mathbb{R}}$, where $S_{\mathbb{R}}$ and $G_{\mathbb{R}}$ are maximal compact subgroups in S and G , respectively. By E_6 and E_7 we denote compact groups of type E_6 and E_7 . The vertex $\bar{\gamma}$ is enlarged. Denote by Y_α the root vector corresponding to the root α .

N	CHSS	rank	The Dynkin diagram
I	$SU(p+q)/S(U(p) \times U(q))$	$\min(p, q)$	
II	$SO(2n)/U(n)$	$[n/2]$	
III	$Sp(2n)/U(n)$	n	
IV	$SO(n+2)/SO(2) \times SO(n)$	2	
V	$E_6/SO(10) \times U(1)$	2	
VI	$E_7/E_6 \times U(1)$	3	

2. The cone $K(X)$. Let $P_x = G_x N_x$ be the Levi decomposition of the stabilizer of $x \in X$ in S .

Let K_x be the cone of highest weight vectors in the G_x -module $T_x X$, i.e. each element in K_x is highest with respect to a Borel subgroup in G_x .

Clearly, $s \in S$ transforms K_x into K_{sx} ; therefore the cone $K(X) \subset V$ is associated with X and $K(X)$ is linearly equivalent to all cones K_x . It will be convenient for us to identify V with $\mathfrak{n}_- \cong T_e X$ and $K(X)$ with K_e .

3. The structure of the G -module $S(\mathfrak{n}_-)$. The results of this subsection

elucidate various facts on G -orbits in \mathfrak{n}_- . However, all statements in the proof where we make use of these results may be proved by case-by-case checking.

Let \mathfrak{h} be the Cartan subalgebra in \mathfrak{s} which is at the same time a Cartan subalgebra in \mathfrak{g} , and Δ the root system for $(\mathfrak{h}, \mathfrak{s})$.

Let Δ^+ be positive roots, Δ_N^+ positive roots corresponding to \mathfrak{n} and N_G the maximal nilpotent subalgebra in G corresponding to roots $\Delta^+ \setminus \Delta_N^+$.

Roots α and β are called **strictly orthogonal** if $\alpha - \beta \notin \Delta$ and $\alpha + \beta \notin \Delta$ and α is orthogonal to β .

The *system of strictly orthogonal Harish-Chandra roots* $\gamma_1 < \dots < \gamma_r$, where $r = rk X$, is defined by induction as follows: $\gamma_1 = \gamma$ and γ_i is the unique minimal root in Δ_N^+ strictly orthogonal to $\gamma_1, \dots, \gamma_{i-1}$ ([17]).

In what follows, V_λ is the irreducible G -module with the highest weight λ and $(V_\lambda, S^k(\mathfrak{n}_-)) = \dim \text{Hom}_G(V_\lambda, S^k(\mathfrak{n}_-))$.

Theorem 3.1. (Schmid [24]). $(V_\lambda, S^k(\mathfrak{n}_-)) = 1(0)$ if and only if λ is (im)possible to present as $\lambda = -m_1\gamma_1 - \dots - m_r\gamma_r$, where $m_1 \geq \dots \geq m_r \geq 1$ and $m_1 + \dots + m_r = k$.

Denote by $'V_\lambda$ a submodule in $S(\mathfrak{n}_-)$ isomorphic to V_λ and put $\alpha_i = \gamma_1 + \dots + \gamma_i$. Let $f_\lambda \in 'V_\lambda$ be a vector of highest weight with respect to N_G .

Corollary 3.2. The set $S(\mathfrak{n}_-)^{N_G}$ is an algebra isomorphic to the algebra of polynomials in $f_{\alpha_1}, \dots, f_{\alpha_r}$.

Lemma 3.3. If $(V_\beta, V_\alpha \otimes V_\beta)_G > 0$, then $|\beta - \gamma| \leq \alpha$.

Lemma 3.3 is an easy corollary of the Steinberg formula for the tensor product [25].

Let $'V_{\alpha_1} \circ 'V_{\alpha_1}$ be the product in $S(\mathfrak{n}_-)$.

Proposition 3.4. $(V_{\alpha_{i+1}}, 'V_{\alpha_i} \circ 'V_{\alpha_i})_G = 1$.

Proof. $S^i(\mathfrak{n}_-) \circ 'V_{\alpha_1} = S^{i+1}(\mathfrak{n}_-)$. Let $'V_\beta \subset S^i(\mathfrak{n}_-)$ and $\beta \neq \alpha_1$. Then $|\alpha_{i+1} - \beta| \geq |\gamma_i - \gamma_j| > |\alpha_j| = |\gamma_j|$. (This follows from the fact that $|\gamma_i| = |\gamma_j|$, see [21]), hence by Lemma 3.3, $('V_\beta, 'V_{\alpha_i} \circ 'V_{\alpha_i}) = 0$. ■

Let $'V_\lambda^*$ be a submodule in $S(\mathfrak{n}_-)^*$ isomorphic to V_λ^* . Let Z'_{i-1} be the set of common zeros of polynomials of $'V_{\alpha_i}^* \subset S(\mathfrak{n}_-)^*$. Proposition 3.4 implies that $0 = Z'_0 \subset Z'_1 \subset \dots \subset Z'_{r-1}$. Put $Z'_i = Z'_i \setminus Z'_{i-1}$ and $Z_r = \mathfrak{n}_- \setminus Z'_{r-1}$. Let f_ψ^* be a vector of highest weight in $'V_\psi^*$.

Proposition 3.5. Let \mathcal{O} be a nonzero G -orbit in \mathfrak{n}_- . Then $\mathcal{O} = Z_i$ for some i , such that $1 \leq i \leq r$.

Proof. There is an i such that $1 \leq i \leq r$ and $\mathcal{O} \subset Z_i$. Suppose that $\mathcal{O} \neq Z_i$. Then there is a G -orbit $\mathcal{O}' \subset Z_i$ such that $\bar{\mathcal{O}}' \neq Z_i$. Let $x \in \mathcal{O}'$ and $f \in \sum_{1 \leq j \leq n} 'V_{\beta_j}^*$, so that $f|_{Gx} = 0$. By the Bernside density theorem and Corollary 3.2 we have $f_\beta|_{\mathcal{O}'} = 0$ and $f_{\beta_j}^* = f_{\alpha_{j_1}}^* \dots f_{\alpha_{j_m}}^*$ for $j_1 < \dots < j_m$. From the

definition of Z'_i we get $f_{\alpha_j}^*|_{Gx} = 0$ for $j < i$. Hence, $j_m \geq i$. Thus, if $f|_{e'} = 0$, then $f|_{Z_i} = 0$. But if $y \in Z_i$ and $y \notin \bar{\mathcal{O}}'$, there is a polynomial g such that $g|_{e'} = 0$ and $g(y) \neq 0$. Contradiction. ■

Proposition 3.6. (a) Put $Y_k = Y_{-\gamma_1} + \dots + Y_{-\gamma_k}$. Then the G -orbits of vectors Y_k are mutually distinct.

(b) If $\rho_\gamma: L \rightarrow \mathrm{GL}(W)$ is a fundamental representation corresponding to the simple root γ and v_γ is its highest vector of weight χ_γ , then $d\rho(Y_k)^k v_\gamma \neq 0$ and $d\rho_\gamma(Y_k)^{k+1}(v_\gamma) = 0$.

Proof. Clearly (b) implies (a). Let $H_{\gamma_i} = [E_{\gamma_i}, E_{-\gamma_i}]$. Since $\gamma_i + \gamma_j$ is not a root, then $[E_{\gamma_i}, E_{\gamma_j}] = 0$ and $\chi_\gamma(H_{\gamma_i}) = 1$. Restricting the representation $d\rho_\gamma$ to the Lie algebra $\{E_{\gamma_i}, H_{\gamma_i}, E_{-\gamma_i}\}$ we have $d\rho_\gamma(Y_{-\gamma_i})v_\gamma \neq 0$ and $d\rho_\gamma(Y_{-\gamma_i})^2 v_\gamma = 0$. This immediately implies $d\rho_\gamma(Y_k)^{k+1}(v_\gamma) = 0$. Furthermore, for the sake of simplicity let us assume $k = 2$. Then $d\rho_\gamma(Y_{-\gamma_1} + Y_{-\gamma_2})(v_\gamma) = (d\rho_\gamma(Y_{-\gamma_1})^2 + d\rho_\gamma(Y_{-\gamma_2})^2 + 2d\rho_\gamma(Y_{-\gamma_1})d\rho_\gamma(Y_{-\gamma_2}))(v_\gamma) = 2d\rho_\gamma(Y_{-\gamma_1})d\rho_\gamma(Y_{-\gamma_2})(v_\gamma)$. If we consider the Lie algebra $\{Y_{-\gamma_2}, H_{\gamma_1}, Y_{\gamma_2}\}$, we get $d\rho_\gamma(E_{-\gamma_2})(v_\gamma) \neq 0$. The weight of this vector is $-\chi_\gamma - \gamma_2$. But $(\chi_\gamma - \gamma_2, H_{\gamma_1}) = (\chi_\gamma, H_{\gamma_1}) = 1$, therefore, considering $\{Y_{-\gamma_1}, H_{\gamma_1}, Y_{\gamma_1}\}$ we get $d\rho_\gamma(Y_{-\gamma_1})d\rho_\gamma(Y_{-\gamma_2})(v_\gamma) \neq 0$. ■

Corollary 3.7. The number of non-zero G -orbits in \mathfrak{n}_- equals $\mathrm{rk} X$. The orbit $K(X) \setminus 0$ is minimal. $K(X)$ is defined by a system of quadratic equations.

4. Examples. (I) $X = \mathrm{Gr}_m^{m+n}$. Then K_x is the cone induced in $T_x X$ by the set of subspaces with an $(m-1)$ -dimensional intersection with x .

Another reformulation: $K(X) = \{v \otimes w \in \mathbf{C}^m \otimes \mathbf{C}^n\}$.

One more reformulation: $K(X)$ is the cone of matrices of rank 1 in the set of matrices of size $m \times (n-m)$. This cone plays an important role in integral geometry (see [9]). Other orbits are matrices of higher ranks.

(II) $X = SO(2m)/U(n)$. In \mathbf{C}^{2n} define a non-degenerate complex metric. Then X is the orthogonal Grassmannian, i.e. the connected component of the manifold of maximal isotropic subspaces in \mathbf{C}^{2n} and $K(X) = \{v \wedge w \in \Lambda^2 \mathbf{C}^n\}$.

Other $\mathrm{GL}(n)$ -orbits in $\Lambda^2 \mathbf{C}^n$ are 2-vectors θ such that $\Lambda^r \theta \neq 0$ and $\Lambda^{r+1} \theta = 0$ for $r = 1, 2, \dots, [n/2]$.

In $T_x X$ the G_x -orbits are induced by the set of maximal isotropic subspaces having an $(n - \mathcal{D}_r)$ -dimensional intersection with x . For $r = 1$ we get the cone K_x .

(III) $X = Sp(2n)/U(n)$. In \mathbf{C}^{2n} define a symplectic structure. Then X is the Lagrange Grassmannian, i.e. the manifold of Lagrange subspaces in \mathbf{C}^{2n} and $K(X) = \{v \cdot v \in S^2 \mathbf{C}^n\}$.

Other $\mathrm{GL}(n)$ -orbits in $S^2 \mathbf{C}^n$ are quadratic forms of rank $r \leq n$ on \mathbf{C}^{n*} . Each G_x -orbit in $T_x X$ is induced by the set of Lagrange subspaces having $(n - r)$ -dimensional intersection with x . For $r = 1$ we obtain the cone K_x .

(IV) $X = SO(n+2)/SO(n) \times SO(2)$. This is a non-degenerate quadratic in \mathbf{CP}^{n+1} , and $K(X)$ is a quadratic cone in \mathbf{C}^n , while K_x is the inter-

section of $X \subset \mathbf{C}P^{n+1}$ and a hyperplane in $\mathbf{C}P^{n+1}$ tangent to X at x .

Another G_x -orbit is $T_x X \setminus (K_x \cup 0)$.

(V) $X = E_6/SO(10) \times U(1)$ is the complexification of the projective plane over octaves. $K(X)$ is a cone of simple semispinors [7] in the semispinor representation of $SO(10)$. The manifold X may be realized as the set of irreducible idempotents in the Jordan algebra recovered from $M_3^+(K) \otimes_{\mathbb{R}} \mathbf{C}$, i.e. the complexification of 3×3 Hermitian matrices over octaves ([8]).

Another G_x -orbit is $T_x X \setminus (K_x \cup 0)$.

(VI) $X = E_7/E_6 \times U(1)$; then $K(X)$ is a cone spanned by the set of irreducible idempotents in $M_3^+(K) \otimes_{\mathbb{R}} \mathbf{C}$.

Let us describe the G -orbits. Let $A \circ B = (AB + BA)/2$ be the Jordan product in $M_3^+(K) \otimes_{\mathbb{R}} \mathbf{C}$ and

$$\det(A, B, C) = \frac{1}{3} \text{tr } A \circ B \circ C + \frac{1}{2} \text{tr } A \cdot \text{tr } B \cdot \text{tr } C - \frac{1}{6} [\text{tr } A \cdot \text{tr } B \circ C + \text{tr } B \cdot \text{tr } A \circ C + \text{tr } C \cdot \text{tr } A \circ B].$$

$E_6 \otimes \mathbf{C}$ is the group of \mathbf{C} -linear transformations in $M_3^+(K) \otimes_{\mathbb{R}} \mathbf{C}$, preserving $\det(A, B, C)$ (cf. [8]) and $K(X) = \{A \in M_3^+(K) \otimes_{\mathbb{R}} \mathbf{C} | \det(A, A, B) = 0 \text{ for any } B\}$.

Other orbits: $\{A | \det(A, A, A) = 0\} \setminus K(X)$ and $\{A | \det(A, A, A) \neq 0\}$.

5. Isotropic subspaces in $K(X)$ and families. In the fundamental representation $\rho_\gamma: S \rightarrow \text{GL}(W)$ corresponding to γ , consider the cone of vectors K_γ of highest weight. For example, for Gr_m^{m+n} this cone consists of simple m -vectors in $\Lambda^m \mathbf{C}^n$ (corresponding to the Plücker embedding of Gr_m^{m+n} into $P(\Lambda^m \mathbf{C}^n)$) and for the space \mathbf{C}^{2n} with a complex metric and a maximal isotropic subspace $x \in SO(2n)/U(n)$ in \mathbf{C}^{2n} , the simple semispinor that E . Cartan assigned to x is the eigenvector for P_x ([7]).

Proposition 3.8. (a) K_γ is a cone over X .

(b) The intersection of K_γ with the subspace tangent to K_γ along the generatrix $\tilde{v} = \mathbf{C}^* v$ is a cone over K_x .

Proof. Statement (a) is evident. Let us assume that $P_x = P$, i.e. $v = v_\gamma$. Each vector $T_{v_\gamma} K_\gamma$ is uniquely presentable in the form $d\rho_\gamma(Y)v_\gamma$, where $Y \in \mathfrak{n}_- \oplus$ Center G . Points of the cone K_γ in the neighbourhood of v_γ are expressed in the form

$$\exp(t \cdot d\rho_\gamma(Y))v_\gamma = v_\gamma + t \cdot d\rho_\gamma(Y)v_\gamma + \frac{t^2}{2} d\rho_\gamma(Y)^2 v_\gamma + \dots$$

Therefore $\exp(t \cdot d\rho_\gamma(Y)) \in T_{v_\gamma} K_\gamma$ iff $d\rho_\gamma(Y)^2 v_\gamma \equiv 0$. It remains to make use of Proposition 3.6(b). ■

Corollary 3.9. G is the general group of linear automorphisms of $K(X)$.

Sketch of the proof. First let us verify that a linear transformation inducing the identical transformation of a base of the cone is a homothety. Furthermore, note that a base of the cone $K(X)$ is a (degenerate) flag variety of the group G_0 ; hence the group of holomorphic automorphisms of the base is G_0 . ■

It turns out that $K(X)$ is always a cone over CHSS, and we obtain a series of CHSS's such that each subsequent term is the base of the cone corresponding to the preceding term:

- (0) $\mathbf{CP}^n, \mathbf{CP}^{n-1}, \dots, \mathbf{CP}^1$.
- (I) $\mathrm{Gr}_n^{m+n}, \mathbf{CP}^{m-1} \times \mathbf{CP}^{n-m-1}, \dots, \mathbf{CP}^1$.
- (II) $SO(2n)/U(n), \mathrm{Gr}_2^{n+2}, \mathbf{CP}^{n-3} \times \mathbf{CP}^1, \mathbf{CP}^{n-4}, \dots, \mathbf{CP}^1$.
- (III) $Sp(2n)/U(n), \mathbf{CP}^{2n-1}, \dots, \mathbf{CP}^1$.
- (IV) $SO(n+2)/(SO(n) + SO(2)), SO(n)/(SO(n-2) \times SO(2)), \dots \begin{cases} SO(3)/SO(2) \\ SO(4)/SO(2) \times SO(2) \end{cases}$
- (E) $E_7/(E_6 \times U(1)), E_6/(SO(10) \times U(1)), SO(10)/U(5), \mathrm{Gr}_2^7, \mathbf{CP}^2 \times \mathbf{CP}^1, \mathbf{CP}^1$.

A subspace that belongs to $K(X) \cup 0$ will be called *isotropic*. Observe that $T_x X$ is the space of a fundamental representation of G_x except for $X = Sp(2n)/U(n)$. Therefore, Proposition 5.3(b) implies that if X_1 and X_2 are neighboring terms of any series except III) then each isotropic subspace in $K(X_1)$ is a cone over an isotropic subspace in $K(X_2)$. Connected components of the manifold of isotropic subspaces will be called *families*. The preceding discussion implies the following description of families.

Proposition 3.10. *There is a bijection of families in $K(X)$ with subgraphs in the Dynkin diagram of \mathfrak{s} that contain γ and are isomorphic to A_m (chains). The dimension of the isotropic subspace is equal to the number of vertices in a chain. G is transitive on each family. If μ_1, \dots, μ_k are simple roots that form a chain, then the subspace spanned by $Y_{-\mu_1}, Y_{-(\mu_2+\mu_1)}, \dots, Y_{-(\mu_1+\dots+\mu_k)}$ is isotropic and belongs to the family corresponding to the chain μ_1, \dots, μ_k .*

§4. The equivalence problem for generalized conformal structures

1. Flat structures. At each point $v \in V$ define the cone K_v which is the parallel transport of $K(X)$. The generalized conformal structure is *locally flat* if it is locally equivalent to the structure in V .

Proposition 4.1. Cones K_x define a locally flat structure on X .

Proof. The reason is the commutativity of \mathfrak{n}_- . The mapping $f: \mathfrak{n}_- \rightarrow (\exp \mathfrak{n}_-)P$ defines an isomorphism of \mathfrak{n} onto a neighbourhood of \bar{e} in S/P . If $v \in K(X) \subset \mathfrak{n}_-$ then

$$\frac{d}{dt} \exp(n) \exp(tv) P|_{t=0} = \frac{d}{dt} \exp(tv) \exp(n) P|_{t=0} = \exp(n+v)P. \blacksquare$$

Choose a G -orbit \mathcal{O}_G in the set of frames for V . Then by Corollary 3.9 a G -structure may be introduced on a manifold with a generalized conformal structure. In fact, $A_x(\mathcal{O}_G)$ defines the reduction of a bundle of frames to the structure group G , i.e. a G -structure. Note that although a G -structure on \mathcal{X} depends on \mathcal{O}_G it is of no importance for the equivalence problem.

2. The Cartan continuation. Put

$$\begin{aligned} g^{(-1)} &= V; & g^{(0)} &= \mathfrak{g} \subset \mathrm{gl}(V); \\ g^{(i)} &= S^2 V^* \otimes \mathfrak{g}^{(i-2)} \cap V^* \otimes \mathfrak{g}^{(i-2)} \quad \text{for } i > 0. \end{aligned}$$

In $\mathfrak{g}^{(*)} = \bigoplus_{i \geq -1} \mathfrak{g}^{(i)}$ we introduce the Lie algebra structure, which is called the

Cartan continuation of the pair (V, \mathfrak{g}) . It is the Lie algebra of infinitesimal automorphisms of the flat G -structure ([26]).

Proposition 4.2. If $\mathrm{rk} X > 1$ then

$$\begin{aligned} g_0^{(1)} &= 0; & g^{(1)} &\cong V^*, & g^{(2)} &= 0; \\ \mathfrak{s} &= g^{(-1)} \oplus g^{(0)} \oplus g^{(1)}; & \mathfrak{g} &= g^{(0)} \oplus g^{(1)}. \end{aligned}$$

Proof. This follows easily from the Borel-Weil-Bott theorem (see below), and D. V. Alexeyevsky has told me that it is also a corollary of results in [20], where it is proved that if $\mathfrak{g}' \subset \mathrm{gl}(V)$ is an irreducible Lie algebra, then $\mathfrak{g}'^{(1)} \neq 0$ implies either $\mathfrak{g}' = \mathfrak{g}$ for some CHSS or $\mathfrak{g}' = \mathrm{sp}(V)$ or $\mathfrak{g}' = \mathrm{sp}(V) \oplus \mathbf{C}$. ■

Remark. 4.3 This result and Corollary 3.7 imply that if $\mathfrak{g}' \subset \mathrm{gl}(V)$ is an irreducible linear reductive Lie algebra, then $\mathfrak{g}'^{(1)} \neq 0$ if and only if $\exp \mathfrak{g}'$ acts on V with a finite number of orbits and $\mathfrak{g}'^{(2)} \neq 0$ if and only if there is exactly one non-zero orbit.

Propositions 4.1 and 4.2 enable us to recover the whole CHSS from a domain U where a flat generalized conformal structure of type X with $\mathrm{rk} X > 1$, is defined. In fact, the Lie algebra of a local Lie group of diffeomorphisms that preserves a family of curves is isomorphic to \mathfrak{s} . (For $\mathfrak{s} = \mathrm{co}(n)$, where $n \geq 3$, this is exactly the Liouville theorem). \mathfrak{p} is a subalgebra in \mathfrak{s} that preserves a point and $X = \exp \mathfrak{s}/\exp \mathfrak{p}$. Therefore, U has a canonical compactification, for example, in the Penrose twistor theory Gr_2^4 is a conformal compactification of the complexified Minkowski space.

3. Structure functions. Put $C_G^{k,1}(V) = \text{Hom}(\Lambda^1 V, g^{(k-1)})$ and

$$\delta f(v_1, \dots, v_{l+1}) = \sum_{0 \leq i \leq l} (-1)^i f(v_1, \dots, v_{l+1-i}, \dots, v_{l+1}) \quad (v_{l+1-i})$$

for $f \in C_G^{k,1}(V)$. Then $\delta_L f \in C^{k-1,l+1}(V)$ and $\delta_l \circ \delta_{l-1} = 0$. Let

$$H_G^{k,1}(V) = \text{Ker } \delta_l \cap C_G^{k,1}(V) / \text{Im } \delta_{l-1} \cap C_G^{k,1}(V). \quad (*)$$

The (k) -th order structure functions (see [16], [20], [26]), provided that structure functions of lesser orders vanish, constitute an obstruction to the identification of a $(k+1)$ -th infinitesimal neighborhood of a point on the manifold with the G -structure with the $(k+1)$ -th neighborhood of the G -flat manifold. We will interpret k -th order structure functions as sections of the bundle $\mathcal{H}_G^{k-1,2}$ with the fiber $H_G^{k-1,2}(T_x \mathcal{X})$ over x . For example, for $G = SO(n)$ the Riemannian curvature is the section of $\mathcal{H}_{SO(n)}^{1,2} = \text{Hom}(\Lambda^2 \mathcal{F} \mathcal{X}, \mathcal{E}nd_{SO(n)} \mathcal{F} \mathcal{X})$, where $\mathcal{E}nd_{SO(n)} \mathcal{F} \mathcal{X}$ is the bundle of endomorphisms that preserve the metric.

Example 4.4 For the conformal structure ($G = \mathcal{O}(n)$) the 1st order structure functions constitute the *torsion*. It is equal to 0, since $H_{\mathcal{O}(n)}^{0,2} = 0$. The 2nd order structure functions constitute the *Weil tensor* if $n \geq 4$ while $H_{\mathcal{O}(n)}^{1,2} = 0$. Instead, we have $H_{\mathcal{O}(3)}^{2,2} \neq 0$.

For the Riemannian manifolds 2nd order structure functions constitute the Riemann tensor that splits into the Weil tensor, the scalar curvature and the traceless Ricci tensor. For $n = 4$ the Weil tensor is reducible. Its components are called the *self-dual* and *anti-self-dual* parts.

Suppose $R: \Lambda^2 T_x \rightarrow SO(T_x \mathcal{X})$ is the Riemann tensor at the point x , i.e. $R \subset C_{SO(n)}^{1,2}(T_x \mathcal{X})$. The identity $R(a, b)C + R(c, a)b + R(b, c)a = 0$, which implies all other linear relations on the Riemann tensor, is the unfolding of the condition $\delta_2 R = 0$.

We have

$$\begin{aligned} \Lambda^2((\mathbf{C}^m)^* \otimes (\mathbf{C}^{n-m})^*) &= S^2(\mathbf{C}^m)^* \otimes \Lambda^2(\mathbf{C}^{n-m})^* \oplus \\ &\quad \Lambda^2(\mathbf{C}^m) \otimes S^2(\mathbf{C}^{n-m})^* = \Lambda_+^2 \oplus \Lambda_-^2. \end{aligned}$$

Forms of $\Lambda_+^2(\Lambda_-^2)$ are called *(anti)self-dual*.

Theorem 4.5. (a) If $X \neq \text{Gr}_m^{m+n}$ then the G -module $\bigoplus_k H^{k,2}$ is irreducible. More precisely,

$$H_G^{0,2} \neq 0 \iff G \neq O(n)$$

and

$$H_G^{1,2} \neq 0 \iff G = O(n) \text{ for } n > 4;$$

$$H_G^{2,2} \neq 0 \iff G = O(3).$$

Let $H^{k,2} \neq 0$. Then the highest weight of $H^{k,2}$ equals the sum of the highest weights of $\Lambda^2 V^*$ and of $\mathfrak{g}^{(k-1)}$, both modules being irreducible for CHSS's.

(b) If $X = \text{Gr}_m^{m+n}$, then $\bigoplus_k H^{k,2} = H_+ \oplus H_-$, i.e. splits into self-dual and anti-self-dual parts, and both H_+ and H_- are irreducible.

$H_+^{0,2} = 0$ (resp. $H_-^{0,2} = 0$) if and only if $m = 2$ (respectively, $n - m = 2$) and

$$H_{\pm}^{0,2} = 0 \Leftrightarrow H_{\pm}^{1,2} \neq 0.$$

The highest weight of $H_+(H_-)$ is the sum of the highest weights of $\Lambda_+^2(\Lambda_-^2)$ and of $\mathfrak{g}^{(k-1)}$, where k is such that $H_+^{k,2} \neq 0$ ($H_-^{k,2} \neq 0$).

Proof. When \mathfrak{n}_- is identified with V , the complex $\bigoplus_k C_G^{k,l}(V)$ is identified with the complex of cochains on \mathfrak{n}_- with coefficients in \mathfrak{s} . Thus, $\bigoplus_k H_G^{k,l} \cong H^l(\mathfrak{n}_-; \mathfrak{s})$. The proof of Theorem 4.5 now follows easily from the Borel-Weil-Bott theorem (see [6]) for S/P . One of its variants runs as follows.

Theorem 4.6. (Borel-Weil-Bott). Let V_χ be a finite-dimensional \mathfrak{s} -module with the lowest weight χ . Let B be the set of sample roots for \mathfrak{s} and Δ^+ the set of positive roots. Let ρ be the half-sum of positive roots; W , the Weil group; and $W_\gamma = \{w \in W \mid w(\Delta \setminus \gamma) \subset \Delta^+\}$. Let $w_1^i, \dots, w_{k(i)}^i$ be elements of W_γ of length i . Then the lowest weights of irreducible components of the G -module $H^i(\mathfrak{n}_-, V_\chi)$ are $-w_1^i(\chi + \rho) + \rho, \dots, -w_{k(i)}^i(\chi + \rho) + \rho$.

Observe that since the number of irreducible components equals $k(i)$ and does not depend on χ , then the G -module $H^2(\mathfrak{n}_-; 1) = \Lambda^2(\mathfrak{n}_-)$ is irreducible iff $H^2(\mathfrak{n}_-; \mathfrak{s}) = \bigoplus_k H_G^{k,2}$ is irreducible.

A G_0 -structure subordinate to the G -structure is a reduction of the principal G -bundle that defines the G -structure on the group G_0 .

For $G = \mathcal{O}(n)$ this reduction means that we single out a metric of a conformal class.

Theorem 4.7. (a) $H_G^{k,2} = H_{G_0}^{k,2}$ for $k \neq 1$.

(b) There is a natural decomposition $H_G^{1,2} = \hat{H}_G^{1,2} \oplus H_{\text{Ricci}}^{1,2}$ so that, as G -modules, $\hat{H}_G^{1,2} \cong H_G^{1,2}$ and $H_{\text{Ricci}}^{1,2} \cong S^2 V^*$ splits into two irreducible components. One of these is isomorphic to a submodule in $S^2 V^*$ consisting of functions that vanish on $K(X) \subset V$.

For a Riemannian metric the structure functions corresponding to this component constitute the scalar curvature, and the structure functions corresponding to the other component constitute the traceless Ricci tensor.

Proof. Statement (a) is an evident corollary of the definitions and the following properties:

$$\mathbf{g}^{(1)} \cong V^*; \quad \mathbf{g}^{(2)} = 0;$$

$$\mathbf{g} = \mathbf{g}_0 \oplus \text{Center } \mathbf{g}, \quad \dim_{\mathbb{C}} \text{Center } \mathbf{g} = 1.$$

$$(b) \quad H_{G_0}^{1,2} = \text{Hom}(\Lambda^2 V, \mathbf{g})/\delta_1 \text{Hom}(V, \mathbf{g}^{(1)});$$

$$H_{G_0}^{1,2} = \text{Hom}(\Lambda^2 V, \mathbf{g}_0);$$

$$\text{Ker } \delta_1 \text{Hom}(V, \mathbf{g}^{(1)}) = \mathbf{g}^{(2)} = 0;$$

$$\text{Hom}(V, \mathbf{g}^{(1)}) \cong V^* \otimes V^* \cong S^2 V^* \oplus \Lambda^2 V^*.$$

Let ' $S^2 V^* \subset \text{Hom}(V, \mathbf{g}^{(1)})$ ' be a component isomorphic to $S^2 V^*$. Put $H_{\text{Ricci}}^{1,2} = \delta_1 S^2 V^*$. Then $(S^2 V^*, \Lambda^2 V^*)_G = 0$ implies that $H_{\text{Ricci}}^{1,2} \subset \text{Hom}(\Lambda^2 V, \mathbf{g}_0)$. The Steinberg formula [25] easily implies that if $(V, \Lambda^2 V^*)_G = 0$, then $(V, \Lambda^2 V^* \otimes \mathbf{g}_0)_G \leq 1$. In particular, $(S^2 V^*, \Lambda^2 V^* \otimes \mathbf{g}_0)_G = 1$. Therefore, there is a canonical G -invariant decomposition $H_{G_0}^{1,2} = \hat{H}_G^{1,2} \oplus H_{\text{Ricci}}^{1,2}$. Then the G -module isomorphism $\hat{H}_G^{1,2} \cong H_G^{1,2}$ is an evident corollary of the definitions (*) of $H_G^{k,l}$. The facts on the decomposition of $S^2 V^*$ follow from the Schmid theorem and Corollary 3.7. ■

4. Connections and O-geodesics. Let $\pi: E \rightarrow \mathcal{X}$ be a principal G -bundle that defines the G -structure on \mathcal{X} and $\pi_0: E_0 \rightarrow \mathcal{X}$ its reduction to G_0 that defines the G_0 -structure on \mathcal{X} subordinate to the G -structure (recall that E_0 is a sub-bundle in the bundle of frames and the point $e \in E$ is interpreted as the isomorphism $\tilde{e}: V \rightarrow T_{\pi(e)} \mathcal{X}$).

Choose a decomposition

$$\text{Hom}(\Lambda^2 V, V) = C \oplus \delta_1 \text{Hom}(V, \mathbf{g}_0). \tag{*}$$

Since $V^* \cong \mathbf{g}^{(1)} = \text{Ker } \delta_1 \text{Hom}(V, \mathbf{g})$ and $0 = \mathbf{g}_0^{(1)} = \text{Ker } \delta_1 \text{Hom}(V, \mathbf{g}_0)$, then dimension considerations imply that $\delta_1 \text{Hom}(V, \mathbf{g}) = \delta_1 \text{Hom}(V, \mathbf{g}_0)$ and $\text{Hom}(\Lambda^2 V, V) = C \oplus \delta_1 \text{Hom}(V, \mathbf{g})$. Because of the reductiveness of \mathbf{g} the complement C may be chosen to be G -invariant.

The property $\mathbf{g}_0^{(1)} = 0$ means exactly that the choice of C in (*) enables us to introduce the canonical G_0 -connection in $\pi_0: E_0 \rightarrow \mathcal{X}$ (see [26]). If $G_0 = SO(n)$, it is exactly the statement of the Levi-Civita theorem that states that on a manifold with metric there is a canonical connection that preserves this metric.

Let us recall briefly the construction of the connection. To each horizontal subspace $H \subset T_e E_0$ an element $t_H \in \text{Hom}(\Lambda^2 V, V)$ ("torsion") is assigned. Fixing H , we obtain a bijection between $\text{Hom}(V, \mathbf{g}_0)$ and horizontal subspaces in $T_e E_0$. Let $b_{H,H_1} \in \text{Hom}(V, \mathbf{g}_0)$ correspond to H_1 . Then $t_{H_1} = t_H + \delta_1 b_{H,H_1}$. Hence it is always possible to find a horizontal subspace \hat{H} such that $t_{\hat{H}} \in C$, so that these subspaces are indexed by elements of $\text{Ker } \delta_1 \text{Hom}(V, \mathbf{g}_0)$, which is trivial in our case. As for any connection in the bundle of frames, geodesics are defined for this connection.

Since $\mathfrak{g}^{(1)} \neq 0$ there is no canonical linear connection in E . Nevertheless, on a manifold with a G -structure an isotropic plane may be transported along vectors that belong to this plane. More precisely, let δ be a chain of the Dynkin diagram of \mathfrak{s} and $I(\delta)$ the corresponding (by Proposition 3.10) family of isotropic subspaces. G is transitive on $I(\delta)$.

Let $\pi_\delta: E_\delta \rightarrow \mathcal{X}$ be the bundle whose fiber over x is the manifold of isotropic subspaces in \mathcal{H}_x of $I(\delta)$. For a G_0 -structure consider the connection in E_δ induced by the canonical G_0 -connection.

Theorem 4.8. *For any $e \in E_\delta$ a horizontal lifting in e of the isotropic subspace corresponding to e does not depend on the choice of a G_0 -structure subordinate to the G -structure.*

The proof is a consequence of the following lemma.

Lemma 4.9. *Let W_δ be an isotropic subspace in $K(X)$ of $I(\delta)$ and $g \in \mathfrak{g}^{(1)}$. Then $g(W_\delta)(W_\delta) \subset W_\delta$.*

Proof. Let $\gamma = \mu_1, \mu_2, \dots, \mu_k$ be simple roots that form the chain δ and $\psi_j = \mu_1 + \dots + \mu_j$ for $1 \leq j \leq k$. Since G is transitive on $I(\delta)$ we may assume that W_δ is spanned by $Y_{-\psi_1}, \dots, Y_{-\psi_k}$. We must verify that if $Y_\beta \in \mathfrak{n}$ then $[[Y_\beta, Y_{-\psi_1}], Y_{-\psi_k}] \subset W_\delta$, but if $[[Y_\beta, Y_{-\psi_1}], Y_{-\psi_k}] \neq 0$, then $[[Y_\beta, Y_{-\psi_1}], Y_{-\psi_k}] = \lambda Y_{\beta - \psi_1 - \psi_k}$. Let $n_{\beta, \gamma}$ be a coefficient of γ in the decomposition of β with respect to simple roots. Then $n_{\beta - \psi_1 - \psi_k} = -1$ and $n_{\psi_j, \gamma} = 1 = n_{\psi_{j+1}, \gamma}$ imply $n_{\beta, \gamma} = 1$. Moreover, if μ is a simple root that does not belong to the chain, then $n_{\beta, \mu} = 0$. Hence $\beta = \psi_r$ for some r , $1 \leq r \leq k$. This easily implies the rest of the proof. ■

In particular, an isotropic line can be transported along itself. In other words the following statement holds.

Corollary. 4.10. *A curve in \mathcal{X} which is a geodesic of a G_0 -structure and tangent at some (hence at any) point x of \mathcal{H}_x does not depend on the choice of a G_0 -structure subordinate to the G -structure.*

These curves will be called O -geodesics of the G -structure.

§5. Connection of F-structures and G-structures

1. The G_F -structure connected with the F-structure. Let $G_F \subset \mathrm{GL}(W)$ be the group of all transformations preserving F and \mathfrak{g}_F the Lie algebra of G_F . Choose a frame in W ; we obtain a G_F -structure on the manifold x with the F -structure. (The choice of a frame will not affect our constructions).

Put $C_{G_F}^k(x)$ for a k -th order structure function at $x \in \mathcal{X}$. Recall that $C_{G_F}^k(x) \in \mathcal{H}_{G_F}^{k-1,2}|_x$ (see 4.3). Making use of projections $p_f: V \rightarrow V/V_f$ and $q_f: \mathfrak{g}_F \rightarrow T_f F$, where $q_f(A)(v) = P_f(Av)$ for $A \in \mathfrak{g}_F$ and $v \in V_f$, we naturally define mappings $h_f^k: H_{G_F}^{k-1,2} \rightarrow H^{k-1,2}(T_f F)$. For example the mapping

$$h_p^1: \mathrm{Hom}(\Lambda^2 V, V)/\delta_1 \mathrm{Hom}(V, \mathfrak{g}_F) \rightarrow \mathrm{Hom}(\Lambda^2 V_f, V/V_f)/\delta_1 \mathrm{Hom}(V_f, T_f F)$$

is defined as follows. Let $A \in \text{Hom}(\Lambda^2 V, V)$ and $v_1, v_2 \in V_f$. Put $'h_f^1(A)(v_1, v_2) = p_f(A(v_1, v_2))$ and $'h_f^1(A) \in \text{Hom}(\Lambda^2 V_f, V/V_f)$. Let $B \in \text{Hom}(V, g_F)$, then $'h_f^1(B)(v_1) = q_f(B(v_1))$ and $'h_f^1(B) \in \text{Hom}(V_f, T_f F)$.

Proposition 5.1. *Let $k = 1, 2$. The k th obstruction to integrability of an F -structure vanishes if and only if $h_f^k(C_{G_F}^1(x)) = 0$ for every $f \in F$.*

The proof follows easily from the definitions.

2. F -structures on a generalized conformal manifold. Let δ be a chain in the Dynkin diagram of \mathfrak{s} with at least 2 vertices. In case $X = Sp(2n)/U(n)$ there is no such chain. Therefore, in this subsection $X \neq Sp(2n)/U(n)$. On a generalized conformal manifold the F -structure, where $F = I(\delta)$, is induced ($\mathcal{F}(x)$) consists of isotropic subspaces of \mathcal{K}_x of the family $I(\delta)$.

Theorem 5.2. *Let X be irreducible, $\text{rk } X > 1$ and $X \neq Sp(2n)/U(n)$.*

(a) *If $X \neq \text{Gr}^{m+n}$ then the generalized conformal structure is flat if and only if for any isotropic subspace of the family $I(\delta)$ there is an integral manifold tangent to it.*

(b) *For $X = \text{Gr}_m^{m+n}$ the (anti)self-dual part of the structure function vanishes if and only if for any isotropic subspace of the given family of subspaces of type $\beta(\alpha)$ there is an integral submanifold tangent to it. If both parts of the structure function vanish, then the generalized conformal structure is flat.*

Proof. (a) Make use of Proposition 5.1. The set $A^k = \{A \in H_{G_F}^{k-1,2} | h_f^k(A) = 0 \text{ for any } f \in F\}$ is a G_F -module. By Theorem 4.6 $\bigoplus H_G^{k-1,2}$ is irreducible and $H_G^{k-1,2} = 0$ for $k \geq 3$ if $X \neq \text{Gr}_k^n, Sp(4)/U(2)$. Corollary 3.9 implies that $G_F = G$. It is easy to see that in case considered $A^k \neq H_G^{k-1,2}$. Therefore, $A^k = 0$, proving the sufficiency. The necessity is evident.

The proof of (b) is similar to that of (a), making use of Lemma 4.5. ■

§6. A construction of representations of Weil type for several simple Lie algebras

1. Preliminaries. Let V be a finite-dimensional linear space over \mathbb{C} and \langle , \rangle a pairing of V and V^* . For $v \in V$ define the differentiation ∂_v in the ring $S(V^*)$ by putting $\partial_v(v^*) = \langle v, v^* \rangle$ for $v^* \in V^*$. The linear function on V^* defined by v will be denoted by ψ_v . Put $\mathcal{D}(V)$ for the Lie algebra of regular differential operators on V .

Define the Lie algebra isomorphism $F: \mathcal{D}(V) \rightarrow \mathcal{D}(V^*)$ by the formulas $F(\partial_v) = i\psi_v$ and $F(\psi_{v*}) = i\partial_{v*}$. The isomorphism F is well defined, since

$$F([\partial_v, \psi_{v*}]) = \langle v, v^* \rangle = [i\psi_v, i\partial_{v*}] = [F(\partial_v), F(\psi_{v*})].$$

The mapping F is called the *Fourier transform*.

Let K_x^* be the cone of the vectors of the highest weight in the G_x -module T_x^*X . Put $K^* = K_e^* \subset \mathfrak{n}_+^*$. In what follows, \mathfrak{n}_+^* and \mathfrak{n} are identified with respect to the Killing form.

Let I be the ideal in $S(\mathfrak{n}^*)$ that defines $K^* \cup 0$ and $I^{(k)}$ the space in I of homogeneous polynomials of degree k .

Proposition 6.1. (a) $I^{(2)}$ is an irreducible G -module. Its multiplicity in $\mathfrak{n}^* \otimes \mathfrak{n}^*$ equals 1.

(b) $I^{(2)} \cdot S(\mathfrak{n}^*) = I$

Proof. (a) Making use of Schmid's theorem (3.1) we have $I^{(2)} = V_{\gamma_1}^*$ and $(V_{\gamma_1}^*, S^2 \mathfrak{n}^*)_G = 1$ in the notation of 3.3. The fact that $(V_{\gamma_1}^*, \Lambda^2 \mathfrak{n}^*)_G = 0$ has a straightforward verification. Another method is to make use of the fact that $\Lambda^2 \mathfrak{n}^*$ is reducible if and only if $G = \mathrm{GL}(k)X_C \mathrm{GL}(m)$.

(b) Suppose that $f' \in \sum_{1 \leq i \leq n} V_{\alpha_i}^*$ and $f'|_{K(X)} = 0$. Then $uf'|_{K(X)} = 0$ for $u \in U(\mathfrak{s})$. Using the Burnside density theorem and the simplicity of the spectrum of G in $S\mathfrak{n}^*$, we have that for any i , $1 \leq i \leq m$, there is an $u_i \in U(\mathfrak{s})$ such that $uf' = f_{\alpha_{j_1}}^* \cdots f_{\alpha_{j_k}}^*$, where $j_1 < \dots < j_k$. By the Hilbert Nullstellensatz there is an N such that $(uf')^N = (f_{\alpha_{j_1}}^* \cdots f_{\alpha_{j_k}}^*)^N \in V_{\gamma_1}^* S\mathfrak{n}^*$ and $(uf')^N$ is the vector of highest weight with respect to G . Hence, by Corollary 3.2, $(uf')^N = f_{\gamma_1}^* \varphi$ implies that $\alpha_{j_1} = \gamma_1$.

Let L_t be a vector field on \mathfrak{n}_- , where $t \in \mathfrak{s}$, induced by the S -action on X . Recall that \mathfrak{n}_- is embedded in X as in the Introduction. Consider $S\mathfrak{n}_-^* \otimes \mathfrak{n}_-$ as the space of vector fields on \mathfrak{n}_- . In what follows, $n \in \mathfrak{n}$ and $y_1, y_2 \in \mathfrak{n}_-$.

Proposition 6.2. (a) $L_n \in S^2 \mathfrak{n}_-^* \otimes \mathfrak{n}_-$.

(b) $L_n([y_1, y_2]) = [[n, y_1], y_2]/2$.

Proof. (a) $[L_{y_1}, L_n] = L_{[y_1, n]} \in \mathfrak{n}_-^* \otimes \mathfrak{n}_-$ because $[y_1, n] \in \mathfrak{g} \subset \mathfrak{n}_+^* \otimes \mathfrak{n}_-$.

(b) By the Jacobi identity and the commutativity of \mathfrak{n}_- the right-hand side in (b) defines an element of $S^2 \mathfrak{n}_-^* \otimes \mathfrak{n}_-$. It suffices to verify that taking the bracket of both parts first with L_{y_1} and then with L_{y_2} give the same result.

2. The main construction. Denote by \mathcal{D}_k^l the space of differential operators of $\mathcal{D} = \mathcal{D}(\mathfrak{n}_-)$ of degree $\leq k$ and homogeneity l . For example, $L_{y_1} = \partial_{y_1} \in \mathcal{D}_1^{-1}$. We will write an operator $D \in \mathcal{D}$ in the form $D = \sum \partial_{y_1}^{\alpha_1} \cdots \partial_{y_k}^{\alpha_k} a(y)$. The homogeneity considerations imply that

$$[L_n, f] = D_2^{-1} + D_1^{-1}, \quad (1)$$

where $f \in F(I^{(2)})$.

Proposition 6.3. $D_2^{-1} \in F(I^{(2)})$. \mathcal{D} .

Proof. Let $\sigma_2(D)$ be the 2-symbol of D . The symbol is a function on $T^*\mathfrak{n}_-$. Put L'_n for the vector field on $T^*\mathfrak{n}_-$ corresponding to L_n on \mathfrak{n}_- . Let $K^*(y)$ be a shift of $K^* \subset T^*_0\mathfrak{n}_- \cong \mathfrak{n}$ at the point $y \in \mathfrak{n}_-$. It is possible to verify that $K^*(y)$ is identified with K_y^* under the embedding of \mathfrak{n}_- into X . Therefore, $g(K_{gx}^*) = K_x^*$ implies that $\sigma_2([L_n, f]) = L'_n(\sigma_2(f))$ vanishes on each cone $K^*(y)$. ■

Put $\mathcal{D}(K^*) = \text{Norm } (I\mathcal{D})/I\mathcal{D}$, where $\text{Norm } (I\mathcal{D}) = \{D \in \mathcal{D} | DI \subset I\mathcal{D}\}$. Let $n \in \mathfrak{n}$, $y \in \mathfrak{n}_-$, $\psi_n \in \mathfrak{n}_-^*$. Put

$$\mathcal{L}_n = L_n + \lambda\psi_n, \quad \mathcal{L}_{[n,y]} = L_{[n,y]} - \lambda \langle n, y \rangle, \quad \mathcal{L}_y = L_y. \quad (2)$$

For $s \in \mathfrak{s}$ the mapping $s \mapsto \mathcal{L}_s$ is a Lie algebra isomorphism (see Introduction).

Theorem 6.4. *There is a $\lambda \in \mathbb{C}$ such that $F(\mathcal{L}_y) \in \mathcal{D}(K^*)$ (see (2)).*

Proof. To L_n and $f \in F(I^{(2)})$, assign the operator D^{-1} . We get the G -module morphism $A: \mathfrak{n} \otimes F(I^{(2)}) \rightarrow \mathfrak{n}_-$. Consider the morphism of the same G -modules $A': \mathfrak{n} \otimes f \mapsto [\psi_n, f]$. Proposition 6.1(a) implies that $\dim \text{Hom}_G(\mathfrak{n} \otimes F(I^{(2)}), \mathfrak{n}_-) = 1$. Since $A' \neq 0$ we find that there is a $\lambda \in \mathbb{C}$ such that $A = \lambda A'$. This is just the required λ since $[\mathcal{L}_n, F(I^{(2)})] \subset F(I^{(2)})\mathcal{D}$ and $F(I) = F(I^{(2)}) \cdot F(S\mathfrak{n}^*)$ for this λ . ■

Proposition 6.5. $\dim \mathcal{O}_G = \dim K^*$

Proof. Let Y_ψ be a vector of highest weight of the \mathfrak{s} -module \mathfrak{s} . Then Y_ψ is evidently the vector of highest weight of the G -module \mathfrak{n} . Now the proof follows from the commutativity of \mathfrak{n} and the following facts:

- (a) $[Y_\mu, Y_\psi] \neq 0$ for $Y_\mu \in \mathfrak{g} \iff \mu + \psi = \chi$
is a root $\iff -\mu = \psi - \chi$ is a root $\iff [Y_{-\chi}, Y_\psi] \neq 0$;
- (b) $[Y_{-\psi}, Y_\psi] \in \mathfrak{h}$;
- (c) $[[Y_{-\psi}, Y_\psi], Y_\psi] = \alpha Y_\psi$, where $\alpha \in \mathbb{C}^*$. ■

Examples. In these examples $(E_{s,t})_{kl} = \delta_{sk}\delta_{tl}$. We define only the embedding $G: \mathfrak{n}_+ \rightarrow \mathcal{D}(K^*)$, because the rest is evident.

(I) $\mathfrak{s} = \mathfrak{sl}(n+m) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ where } \text{tr } A + \text{tr } D = 0, \text{ and } A = n, \text{ ord } \mathcal{D} = m \right\}$. Then

$$\mathfrak{n}_+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{n}_- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\}, \quad \mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\};$$

$$K^* = \{B \in \text{Mat}(m \times n) | \text{rk } B = 1\}.$$

Let $x_{ij} \in \mathfrak{n}_+^*$ be such that

$$(x_{ij}, E_{n+s,t}) = \delta_{is}\delta_{jt}.$$

Put

$$G: E_{i,m+j} \mapsto \sum_{k,t} \frac{\partial}{\partial x_{it}} \frac{\partial}{\partial x_{ik}} x_{tk} - (n+m) \frac{\partial}{\partial x_{ij}}$$

for $1 \leq i \leq m$, $1 \leq j \leq n$.

(II) $\mathfrak{s} = O(2n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, C = -C^t \right\}$. Then

$$\mathfrak{n}_+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{n}_- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\}, \quad \mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \right\};$$

$$K^* = \{B \in \mathfrak{n}_+ \mid b_{i_1 i_2} b_{i_3 i_4} - b_{i_1 i_3} b_{i_2 i_4} + b_{i_1 i_4} b_{i_2 i_3} = 0\}.$$

Put

$$G: E_{i,n+j} \mapsto \sum_{k,t} \left(\frac{\partial}{\partial x_{it}} \frac{\partial}{\partial x_{jk}} - \frac{\partial}{\partial x_{jt}} \frac{\partial}{\partial x_{ik}} \right) x_{tk} - 2(n-1) \frac{\partial}{\partial x_{ij}}.$$

(III) $\mathfrak{s} = \mathfrak{sy}(2n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, C = C^t \right\}$. Then

$$\mathfrak{n}_+ = \left\{ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{n}_- = \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\}, \quad \mathfrak{g} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix} \right\};$$

$$K^* = \{B \in \mathfrak{n}_+ \mid b_{i_1 i_2} b_{i_3 i_4} - b_{i_1 i_3} b_{i_2 i_4} = 0\}.$$

Put

$$G: E_{i,n+j} \mapsto \sum_{k,t} \left(\frac{\partial}{\partial x_{it}} \frac{\partial}{\partial x_{jk}} + \frac{\partial}{\partial x_{jt}} \frac{\partial}{\partial x_{ik}} \right) x_{tk} - 2n \frac{\partial}{\partial x_{ij}}.$$

(IV) (a) $\mathfrak{s} = o(2n+2)$; $\mathfrak{g} = o(2) \oplus o(2n) = co(2n)$. Then

$$K^* = \{(z_1, \dots, z_n, w_1, \dots, w_n) \mid \sum x_i w_i = 0\}.$$

Put

$$\Delta = \sum \frac{\partial}{\partial z_i} \frac{\partial}{\partial w_i}, \quad J_z = \sum z_i \frac{\partial}{\partial z^i}, \quad J_w = \sum W_i \frac{\partial}{\partial w_i}.$$

The 2nd order differential operators acting in $\mathcal{D}(K^*)$ are

$$W_i = w_i \Delta - (J_z + J_w - n + 1) \frac{\partial}{\partial z_i}$$

and

$$Z_i = z_i \Delta - (J_z + J_w - n + 1) \frac{\partial}{\partial w_i},$$

where $1 \leq i \leq n$.

(b) $\mathfrak{s} = o(2n + 1)$; $\mathfrak{g} = o(2) \oplus o(2n - 1) = co(2n - 1)$. Then

$$K^* = \{(z_1, \dots, z_n, w_1, \dots, w_n, t) | z_i w_i + \frac{1}{2}t^2 = 0\}.$$

Put

$$\Delta = \sum \frac{\partial}{\partial z_i} \frac{\partial}{\partial w_i} + \frac{1}{2} \left(\frac{\partial}{\partial t} \right)^2, J_z \text{ and } J_w \text{ as in IV (a).}$$

The 2nd order differential operators acting in $\mathcal{D}(K^*)$ are

$$W_i = w_i \Delta - \left(J_z + J_w + t \frac{\partial}{\partial t} - n + \frac{1}{2} \right) \frac{\partial}{\partial z_i},$$

$$Z_i = z_i \Delta - \left(J_z + J_w + t \frac{\partial}{\partial t} - n + \frac{1}{2} \right) \frac{\partial}{\partial w_i}$$

for $1 \leq i \leq n$ and

$$T = t \Delta - \left(J_z + J_w + t \frac{\partial}{\partial t} - n + \frac{1}{2} \right) \frac{\partial}{\partial t}.$$

§7. Manifolds with quaternionic structure

1. Definitions. Let \mathbf{H}^n be a left n -dimensional quaternionic space, $\mathrm{GL}(n; \mathbf{H})$ the group of \mathbf{H} -linear transformations of \mathbf{H} , i.e. the group of invertible quaternionic $n \times n$ matrices (acting from the right on n rows of quaternions) with the *unusual composition* $A \circ B = BA$, where juxtaposition denotes the matrix product. Put \mathbf{H}^* for the group of invertible quaternions acting on \mathbf{H} from the left. Put $\mathbf{H} \times {}_{\mathbf{R}}\mathrm{GL}(n; \mathbf{H})$ for the subgroup of $\mathrm{GL}_{\mathbf{R}}(\mathbf{H}^n)$ generated by \mathbf{H}^* and $\mathrm{GL}(n; \mathbf{H})$.

Example 7.1. $n = 1$. Define the norm of $q = a + bi + cj + dk$ by the formula $|q| = a^2 + b^2 + c^2 + d^2$. The group $SO(4)$ is the connected component of the unit of the subgroup of $\mathrm{GL}_{\mathbf{R}}(\mathbf{H})$ that preserves the norm. The formula $q \mapsto s_1 q s_2$, where $s_1, s_2 \in \mathbf{H}^*$, induces the isomorphism $\mathbf{H}^* \times {}_{\mathbf{R}}\mathrm{GL}(n; \mathbf{H}) \cong \mathbf{R}^* \cdot SO(4)$.

A *left quaternionic structure* on the n -dimensional manifold \mathcal{X} is a $\mathbf{H}^* \times {}_{\mathbf{R}}\mathrm{GL}(n; \mathbf{H})$ -structure on \mathcal{X} .

Another definition. Let \mathbf{H}' be a subalgebra in $\mathrm{End}_{\mathbf{R}}(\mathbf{H}^n)$, whose elements are quaternions acting on \mathbf{H}^n from the left. On \mathcal{X} , a *left quaternionic structure* is defined if a subalgebra $h_x \subset \mathrm{End}_{\mathbf{R}}(T_x \mathcal{X})$ is given at each point $x \in \mathcal{X}$ so that there is an \mathbf{R} -linear isomorphism $A_x: \mathbf{H}' \rightarrow T_x \mathcal{X}$ that transforms \mathbf{H}' into h_x .

These definitions are equivalent because $\mathbf{H}^* \times {}_{\mathbf{R}}\mathrm{GL}(n; \mathbf{H})$ is the group of all \mathbf{R} -linear transformations of \mathbf{H}^n that transform \mathbf{H}' into itself ($\mathbf{H}^*/\mathbf{R}^*$ is the group of all automorphisms of the subalgebra \mathbf{H}' and $\mathrm{GL}(n; \mathbf{H})$ preserves the set \mathbf{H}' elementwise).

Similarly, making use of the right-quaternionic space we define a *right quaternionic structure*.

Example 7.2. To define a left quaternionic structure on a 4-dimensional manifold is the same as defining an orientation and a conformal class of a Riemannian metric on the manifold. In fact, $\mathbf{H}^* \times {}_{\mathbf{R}}\text{GL}(1; \mathbf{H}) \cong \mathbf{R}^* \cdot SO(4)$. The change of orientation is equivalent to the shift from the left quaternionic structure to the right one. In fact, on one hand, the quaternionic conjugation $q \mapsto \bar{q}$ shifts an orientation in $\mathbf{R}^4 \cong \mathbf{H}$ and, on the other hand, makes a left quaternionic space into a right one and vice versa by the formula $\overline{\lambda q} = \bar{q}\bar{\lambda}$.

Remark 7.3. The space $T_x \mathcal{X}$ is not quaternionic because there is no canonical isomorphism between h_x and \mathbf{H} . For example, $S^4 \cong \mathbf{H}P^1$ has no almost complex structure, though it possesses a quaternionic structure. If there were a canonical isomorphism between h_x and \mathbf{H} , we could have taken the image of i for the almost complex structure on S^4 .

Example 7.4. Put $\mathbf{H}P^n = \mathbf{H}^* \setminus (\mathbf{H}^{n+1} \setminus 0)$. Then \mathbf{H}_x is a line in \mathbf{H}^{n+1} corresponding to $x \in \mathbf{H}P^n$. There is a canonical isomorphism $T_x \mathbf{H}P^n \cong \text{Hom}_{\mathbf{H}}(\mathbf{H}_x, \mathbf{H}^{n+1}/\mathbf{H}_x)$. Let $v \in \mathbf{H}_x$ and $f \in \text{Hom}_{\mathbf{H}}(\mathbf{H}_x, \mathbf{H}^{n+1}/\mathbf{H}_x)$. Define an action (depending on v) of $q \in \mathbf{H}$ on f by the formula $(T_v(q)f)(\lambda v) = \lambda q f(v)$. Since

$$(T_{\beta v}(q)f)(\lambda \beta v) = \lambda q f(\beta v) = \lambda q \beta f(v)$$

we have $T_{\beta v}(q) = T_v(\beta^{-1} q \beta)$. Thus, the algebra generated by operators $T_v(q)$, where $q \in \mathbf{H}$, does not depend on v and defines a left quaternionic structure on $\mathbf{H}P^n$.

When $\mathbf{H}P^n$ is realized as the set of hyperplanes in \mathbf{H}^{n+1} , we have $T_x \mathbf{H}P^n \cong \text{Hom}_{\mathbf{H}}(\mathbf{H}_x^*, \mathbf{H}^{n+1}/\mathbf{H}_x^*)$ and $\mathbf{H}P^n$ is endowed with the right quaternionic structure. In fact, if $v \in \mathbf{H}^{n+1}/\mathbf{H}_x$ and $f \in \text{Hom}_{\mathbf{H}}(\mathbf{H}_x, \mathbf{H}^{n+1}/\mathbf{H}_x)$, where $f(h) = vt_f(h)$, then $(T_v(q)f)(h) = vq t_f(h)$.

If the initial space \mathbf{H}^{n+1} is a right \mathbf{H} -module then $(\mathbf{H}^{n+1} \setminus 0)/\mathbf{H}^*$ has a right structure, whereas hyperplanes in \mathbf{H}^{n+1} have a left one.

Observe that $\mathbf{H}P^1$ possesses both right and left quaternionic structures regardless of realization.

2. The cone in $T_x \mathcal{X} \otimes {}_{\mathbf{R}} \mathbf{C}$. Put $C_x = \{J \in h_x | J^2 = -1\}$. Then $C_x \cong \mathbf{C}P^1$. In fact, it suffices to verify this for \mathbf{H} , when $C = \{\alpha i + \beta j + \gamma k | \alpha^2 + \beta^2 + \gamma^2 = 1\}$ and the inner automorphisms of \mathbf{H} act on C as elements of $SO(3)$ act on S^2 . Therefore, $C_x \cong \mathbf{C}P^1$.

Elements of C_x are called **complex structures subordinate to the quaternionic structure h_x** . For $J \in C_x$ put $V_J = \{y + \sqrt{-1} Jy | y \in T_x \mathcal{X}\}$. The space V_J is a \mathbf{C} -linear subspace in $T_x \mathcal{X} \otimes {}_{\mathbf{R}} \mathbf{C}$ because $\sqrt{-1}(y + \sqrt{-1} Jy) = -Jy + \sqrt{-1} J(-Jy)$.

Let $\mathcal{K}_x = \bigcup_{J \in C_x} V_J$. Let w give another description of \mathcal{K}_x . Clearly, the

complexification of the $\mathbf{H}^* \times {}_{\mathbf{R}}\text{GL}(n; \mathbf{H})$ -module \mathbf{H}^n is isomorphic to the $\text{GL}(2) \times \text{GL}(2n)$ -module $\mathbf{C}^2 \otimes \mathbf{C}^{2n}$. Let $A_x^{\mathbf{C}}: \mathbf{H}^n \otimes {}_{\mathbf{R}}\mathbf{C} \rightarrow T_x\mathcal{X} \otimes {}_{\mathbf{R}}\mathbf{C}$ be a complexification of A_x and $K = \{v \otimes w \in \mathbf{C}^2 \otimes \mathbf{C}^{2n}\}$.

Lemma 7.5. $\mathcal{K}_x = A_x^{\mathbf{C}}(K)$.

Proof. It suffices to verify this lemma for \mathbf{H}^n . The group $\mathbf{H}^* \times {}_{\mathbf{R}}\text{GL}(n; \mathbf{H})$ preserves $\bigcup_{J \in C_x} V_J$ because $(y + \sqrt{-1} Jy)A = yA + \sqrt{-1} JyA$ and $q(y + \sqrt{-1} Jy) = qy + \sqrt{-1} J(qJq^{-1})qy$. But the manifold \mathcal{K}_x is a complex manifold; therefore it is preserved by $\text{GL}(2) \times \text{GL}(2n)$. Since $\mathcal{K}_x \neq 0$, $\mathcal{K}_x \neq T_x\mathcal{X} \otimes {}_{\mathbf{R}}\mathbf{C}$ and $\text{GL}(2) \times {}_{\mathbf{C}}\text{GL}(2n)$ has three orbits in $\mathbf{C}^2 \otimes \mathbf{C}^{2n}$, it follows that $\mathcal{K}_x = A_x^{\mathbf{C}}(K)$. ■

Subspaces of V_J will be called α -subspaces. The bundle $\pi: \mathcal{P} \rightarrow \mathcal{X}$ with the fiber C_x over x may be interpreted as a bundle of α -subspaces. Observe that since α -subspaces are transversal to $\sqrt{-1} T_x\mathcal{X}$, then their projection in $T_x\mathcal{X}$ parallel to $\sqrt{-1} T_x\mathcal{X}$ is an isomorphism that enables us to introduce a complex structure in $T_x\mathcal{X}$ is an isomorphism that enables us to introduce a complex structure in $T_x\mathcal{X}$.

3. The main example: $\mathcal{X} = \mathbf{H}P^n$. Let us identify \mathbf{C} with the subfield in \mathbf{H} generated by 1 and i . Expressing quaternions in the form $z_1 + z_2j$ we identify \mathbf{H} with \mathbf{C}^2 and \mathbf{H}^{n+1} with \mathbf{C}^{2n+2} and obtain an embedding of $\mathbf{H}P^n$ in Gr_2^{2n+2} . Assigning to a complex line in \mathbf{C}^{2n+2} the quaternionic line that contains this complex line we obtain a bundle $\pi: \mathbf{C}P^{2n+1} \rightarrow \mathbf{H}P^n$ with the fiber $\mathbf{C}P^1$. The space $T_x\mathbf{H}P^n \otimes {}_{\mathbf{R}}\mathbf{C}$ is identified with $T_x\text{Gr}_2^{2n+2}$.

Let \mathbf{H}_x be the line corresponding to $x \in \mathbf{H}P^n$. The α -subspaces in \mathcal{K}_x are parameterized by \mathbf{C} -lines in \mathbf{H}_x . In fact, the tangent space to the manifold of 2-planes in \mathbf{C}^{2n+2} intersecting \mathbf{H}_x along the fixed line is an α -subspace and all α -subspaces arise this way. Therefore, $\mathbf{H}P^n$ is identified with the bundle of α -subspaces.

Recall that a 2-form is *self-dual* if and only if it vanishes on each α -subspace.

Proposition 7.6. A 2-form in $T_x\mathcal{X}$ is self-dual if and only if it is of type $(1, 1)$ for all complex structures subordinate to the quaternionic structure in $T_x\mathcal{X}$.

Proof. In fact, the lack of $(2, 0)$ (resp. $(0, 2)$) components in the decomposition $w = w^{2,0} + w^{1,1} + w^{0,2}$ connected with the complex structure $J \in C_x$ is equivalent to the vanishing of w on V_J (respectively, V_{-J}). ■

4. A complex structure on the twistor manifold. Theorem 4.9 implies that the choice of the complement C in $\text{Hom}(\Lambda^2 V, V) = C \oplus \partial_1(\text{Hom}(V, \text{gl}(2)) \otimes_{\mathbf{C}} \text{gl}(2n))$ defines at each point $J \in \mathcal{P}$ a subspace $W_J \subset T_J\mathcal{P} \otimes {}_{\mathbf{R}}\mathbf{C}$ isomorphic to V_J with respect to the projection.

Let R_J be a $(1, 0)$ -part of the complexified tangent space to the fiber at J . The space $R_J \oplus W_J$ is a $(1, 0)$ -part in $T_J \mathcal{P} \otimes_{\mathbb{R}} \mathbb{C}$ for the complex structure in $T_J \mathcal{P}$. This means that \mathcal{P} possesses an almost complex structure.

Theorem 7.7. *An almost complex structure on \mathcal{P} is integrable if and only if an antiselfdual part of the structure function of the quaternionic structure vanishes.*

Proof. This follows from the integrability theorem in [2, section 3] and the results of 4.3 above. ■

The complex manifold \mathcal{P} is an analogue of the twistor manifold of R. Penrose.

Remark. If $\dim \mathcal{X} = 4n > 4$ then the torsion is an antiselfdual part of the structure function. When $\dim \mathcal{X} = 4$ there is no torsion at all. Therefore the lack of torsion is a necessary condition for the integrability of an almost complex structure on \mathcal{P} . But, when there is no torsion, the choice of C is redundant and, there is a canonical complex structure on \mathcal{P} .

Observe that example 7.3 actually shows that \mathbf{HP}^n is the twistor manifold for \mathbf{CP}^{n+1} .

5. Self-dual bundles over an antiselfdual manifold and holomorphic bundles over \mathcal{P} . A vector bundle E over \mathcal{X} with the connection ∇ is *self-dual* if its curvature form F_∇ is self-dual, i.e. F_∇ vanishes on α -subspaces.

Let $(\pi^* E, \pi^* \nabla)$ be a lifting of (E, ∇) to \mathcal{P} and $\pi^* F_\nabla$ the curvature form of $\pi^* \nabla$.

Lemma 7.8. *(E, ∇) is a self-dual if and only if $\pi^* F_\nabla$ vanishes on $R_J \oplus W_J$.*

Proof. In fact, since $\pi^* F_\nabla$ is a lifting from the base, $\pi^* F_\nabla$ vanishes if at least one of two vectors belongs to R_J and $\pi^* F_\nabla|_{W_J} = 0$ is equivalent to the self-duality of F_∇ . ■

Lemma 7.8 implies that an almost holomorphic structure may be introduced on $\pi^* E$. In fact, $s \in \Gamma(\mathcal{P}, \pi^* E)$ is holomorphic if and only if $\pi^* \nabla_v(s) = 0$ for every $v \in \Gamma(\mathcal{P}, R_J \oplus W_J)$.

Following [2] (see also [4]), this result may be somewhat generalized. Define the involution $\tau: \mathcal{P} \rightarrow \mathcal{P}$, putting $\tau(J) = -J$.

Theorem 7.9. *Let E be a Hermitian vector bundle with a self-dual connection on an anti-selfdual manifold \mathcal{X} . Then*

- (1) $\pi^* E$ is a holomorphic bundle on \mathcal{P} .
- (2) $\pi^* E$ is holomorphically trivial on each fiber $\pi^{-1}(x)$.
- (3) There is a holomorphic isomorphism $\sigma: \tau^* \pi^* E \rightarrow (\pi^* E)^*$, which induces a positive definite Hermitian structure on the space of holomorphic sections of $\pi^* E$ on each fiber.

Conversely, each such bundle on \mathcal{P} is the inverse image of a Hermitian bundle with a self-dual connection.

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