EQUIVARIANT METHODS IN REPRESENTATION THEORY
LECTURE 1

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1. Introduction

Let us begin by having a glimpse at the end goal of this course. If we had to distill representation theory to one meaningless phrase, it would be something like "the study of homomorphisms from an algebra A to $\mathsf{End}\, V$, where V is a vector space". Adding the adjective "geometric" would then translate to ending with "where V is the vector space of invariants of some geometric object X" instead. How can one produce such homomorphisms? The easiest setting is when we take X to be a finite set, and $V = \mathsf{Fun}(X, \mathbb{C})$ to be the set of $\mathbb{C}$-valued functions on X. It is an easy exercise to see that $\mathsf{End}\, V \simeq \mathsf{Fun}(X \times X, \mathbb{C})$, where the product is given by convolution:

$$(f * g)(x_1, x_2) = \sum_{y \in X} f(x_1, y) g(y, x_2).$$

If we have a group Γ acting on X, then we immediately have a homomorphism

$$\Gamma \to \mathsf{End}\, V, \quad g \mapsto \mathbf{1}_{\mathsf{Graph}(g)},$$

where $\mathsf{Graph}(g) = \{(x, gx) : x \in X\} \subset X \times X$. However, it is a very old observation that interesting algebras (e.g. Hecke algebras) appear inside such convolution algebras, but usually don't come from symmetries of the set X.

A souped up version of this picture is when X is a topological space/variety, and $V = H^*(X, \mathbb{C})$ the cohomology of X. Under some geometric assumptions, one can show that a subvariety $Z \subset X \times X$ defines an element $[Z] \in \mathsf{End}\, H^*(X)$. Thus a very natural thing to consider, given a collection of subvarieties $Z_i \subset X \times X$, the subalgebra of $\mathsf{End}\, H^*(X)$ generated by all $[Z_i]$'s. We are met with a question: how to compute such things?

A helping hand comes from symmetries. It turns out that the spaces one wants to consider often come equipped with an action of some Lie group G. Therefore it makes sense to consider a cohomology theory which takes into account the G-action; this is achieved by *equivariant* cohomology. It shares many properties of singular cohomology: it is functorial (for equivariant maps), has Chern classes (for equivariant vector bundles), and fundamental classes (of G-invariant subvarieties). However, one difference is that it is highly non-trivial even for $X = \mathrm{pt}$; in general, $\Lambda_G := H^*_G(\mathrm{pt}) \neq \mathbb{C}$! In particular, the pullback map $\Lambda_G := H^*_G(\mathrm{pt}) \to H^*_G(X)$

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endows G-equivariant cohomology of any X with the richer structure of a Λ_G-algebra.

We can then try to reduce our computations of convolution algebras by restricting to the fixed points X^G. This works best when $G = T$ is a torus. In nice situations, we have:

- The map $H_T^*(X) \to H^*(X)$ is *surjective*, and its kernel is generated by the kernel of $\Lambda_T \to \mathbb{C} = H^*(\mathrm{pt})$;
- The pullback $H_T^*(X) \to H_T^*(X^T)$ is *injective*, and becomes an isomorphism after inverting enough elements of Λ_T. Its image can be explicitly characterized;
- Pushforward along a proper T-equivariant map $X \to Y$ can be computed via restriction to fixed points.

All these properties can, and do, sometimes fail; for instance, the second property scarcely makes sense when the fixed point set is empty. Nevertheless, all of them hold true in many common situations; for example, when X is a nonsingular projective variety and X^T is finite. Theorems about when these properties hold constitute the *localization package*.

Returning to convolution algebras, going from X to X^T at a first glance brings us back to the simple situation of functions on finite sets. However, the presence of Λ_G-module structure unleashes combinatorial mayhem. Rich combinatorics of symmetric polynomials come into play, diagrammatics naturally appear, and representation theory becomes infinitely richer. We will explore some of these topics in the second half of the course.

2. Equivariant cohomology

Througout this course, we will work with algebraic varieties over $\mathbb{C}$. The original definition of equivariant cohomology by Borel requires using infinite dimensional topological spaces; because of that, we will slightly modify it in order to remain firmly in the algebraic realm. The advantage is that the same construction can be used verbatim with other cohomology theories. We consider cohomology with $\mathbb{Z}$-coefficients, unless otherwise stated.

2.1. G-**torsors.**

Let E be a complex vector bundle of rank n on a space Y. To it, we can associate the *frame bundle* $\mathsf{Fr}(E) \to Y$, whose fiber over a point $y \in Y$ is the set of all ordered bases $(v_1, \dots, v_n)$ of E_y.

There is a natural right GL_n-action on $\mathsf{Fr}(E)$:

$$(v_1, \dots, v_n) \cdot g = (w_1, \dots, w_n), \qquad w_j = \sum_i g_{ij} v_i,$$

which is transitive and free on each fiber $\mathsf{Fr}(E)_y$. Moreover, $\mathsf{Fr}(E)$ is an open of $E^{\oplus n}$, and thus a locally trivial fibration over Y.

Definition 2.1. Let $\mathbb{B}$ be a space, and G a Lie group. A *(right) G-torsor* over $\mathbb{B}$ is a map $p : \mathbb{E} \to \mathbb{B}$ with a free right G-action on $\mathbb{E}$, such that $\mathbb{B}$ is covered by opens U with G-equivariant isomorphisms $p^{-1}(U) \simeq U \times G$.

In particular, the frame bundle $\mathsf{Fr}(E) \to Y$ is a G-torsor, called the *associated principal bundle* (or torsor) to E.

Given a right G-action on Y, and a left G-action on X, we denote by $Y \times^G X$ the quotient of $Y \times X$ by the relation $(yg, x) \sim (y, gx)$. When Y is a G-torsor over B, this quotient is locally on B isomorphic to $U \times X$, and is therefore "nice" (separated etc) whenever B is.

Example 2.2. Consider the natural action of GL_n on $\mathbb{C}^n$. We have an isomorphism

$$\mathsf{Fr}(E) \times^{GL_n} \mathbb{C}^n \xrightarrow{\sim} E, \qquad (v_1, \dots, v_n) \times (z_1, \dots, z_n) \mapsto \sum_i z_i v_i.$$

In a similar way, we have

$$\mathsf{Fr}(E) \times^{GL_n} (\mathbb{C}^n)^\vee \simeq E^\vee, \quad \mathsf{Fr}(E) \times^{GL_n} \wedge^d \mathbb{C}^n \simeq \wedge^d E, \quad \mathsf{Fr}(E) \times^{GL_n} \mathsf{Sym}^d \mathbb{C}^n \simeq \mathsf{Sym}^d E.$$

Exercise 2.3. Let $d \leq n$, and consider the fiber bundle $\mathsf{Fr}(d, E) \to Y$, with

$$\mathsf{Fr}(d, E)_y = \left\{ (v_1, \dots, v_d) \,:\, v_i\text{'s are linearly independent in } E_y \right\}.$$

Show that $\mathsf{Fr}(d, E) \times^{GL_d} \mathbb{C}^d$ is naturally identified with the tautological rank d bundle S on $\mathsf{Gr}(d, E) = \mathsf{Fr}(E) \times^{GL_n} \mathsf{Gr}(d, \mathbb{C}^n)$.

2.2. **Borel construction.** A naive definition of equivariant cohomology would be simply $H^*(X/G)$. This has two immediate issues:

- It is not homotopy invariant. For example, compare $\mathrm{pt}/\mathbb{Z}$ with $\mathbb{R}/\mathbb{Z}$;
- The quotient X/G is typically very nasty; e.g. the quotient of $\mathbb{C}^2 \setminus \{0\}$ by $\mathbb{C}^*$, $t \cdot (x, y) = (tx, t^{-1} y)$ is not separated.

Both or these issues can be resolved by picking a G-torsor $\mathbb{E} \to \mathbb{B}$ with $\mathbb{E}$ contractible, and replacing X/G by $\mathbb{E} \times^G X$. The issue is that one cannot typically choose $\mathbb{E}$ to be algebraic, the classic example being $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^\infty$ for $G = \mathbb{C}^*$.

Exercise 2.4. Check that $\mathbb{C}^\infty \setminus \{0\}$ is contractible.

We sidestep this by approximating $\mathbb{E}$ by a sequence of G-torsors $\mathbb{E}_N \to \mathbb{B}_N$, such that $\mathbb{E}_N$ is path-connected and $H^i(\mathbb{E}_N) = 0$ for $0 < i < N$.

Definition 2.5. Let G be a Lie group, and X a G-variety. We define

$$H^i_G(X) := H^i(\mathbb{E}_N \times^G X) \qquad \text{for } i < N.$$

Of course, in order for this to make sense, we need to construct $\mathbb{E}_N$'s, and show that the definition is independent of choices. Existence is taken care of by the following lemma, which we will prove in **??**:

Lemma 2.6. *Let G be a complex linear algebraic group, and $N > 0$. We have a G-torsor $\mathbb{E} \to \mathbb{B}$ on a smooth algebraic variety $\mathbb{B}$, such that $\mathbb{E}_N$ is path-connected and $H^i(\mathbb{E}_N) = 0$ for $0 < i < N$.*

Onto the independence from choices:

Lemma 2.7. *If $\mathbb{E} \to \mathbb{B}$, $\mathbb{E}' \to \mathbb{B}'$ are two path-connected G-torsors with $H^i(\mathbb{E}) = H^i(\mathbb{E}') = 0$ for $0 < i < N$, then there are canonical isomorphisms*

$$H^i(\mathbb{E} \times^G X) \simeq H^i(\mathbb{E}' \times^G X)$$

for all $i < N$, compatible with cup product in this range.

Proof. Consider the product $\mathbb{E} \times \mathbb{E}'$ with the diagonal G-action. We have a commuting diagram

$$\begin{tikzcd}
\mathbb{E} \times X \arrow[d] & \mathbb{E} \times \mathbb{E}' \times X \arrow[l] \arrow[r] \arrow[d] & \mathbb{E} \times X \arrow[d] \\
\mathbb{E} \times^G X & (\mathbb{E} \times \mathbb{E}') \times^G X \arrow[l] \arrow[r] & \mathbb{E}' \times^G X
\end{tikzcd}$$

The horizontal maps are locally trivial fibrations, with fibers $\mathbb{E}$ and $\mathbb{E}'$. Recall Leray spectral sequence:

$$H^p(X, H^q(F)) \Rightarrow H^{p+q}(Y) \text{ for a fibration } Y \to X \text{ with fiber } F.$$

As easy consequence is that when F is path-connected and $H^i(F) = 0$ for $0 < i < N$, we have $H^i(X) \xrightarrow{\sim} H^i(Y)$, with the map being pullback. We obtain

$$H^i(\mathbb{E} \times^G X) \xrightarrow{\sim} H^i((\mathbb{E} \times \mathbb{E}') \times^G X) \xleftarrow{\sim} H^i(\mathbb{E}' \times^G X)$$

for $i < N$, and these maps are compatible with cup product because pullbacks are. $\square$

Any G-equivariant map $X \to Y$ determines a map $\mathbb{E} \times^G X \to \mathbb{E} \times^G Y$, so we get pullbacks

$$f^* : H^i_G(Y) \to H^i_G(X).$$

In particular, the pullback along $X \to \mathrm{pt}$ defines a ring homomorphism

$$\Lambda_G := H^*_G(\mathrm{pt}) \to H^*_G(X),$$

which makes $H^*_G(X)$ into a graded-commutative Λ_G-algebra.

Exercise 2.8. Check that the isomorphisms from Lemma 2.7 are functorial, i.e. commute with pullbacks.

Summarizing, we have constructed a functor

$$H^*_G : G\text{-spaces} \to \Lambda_G\text{-algebras}.$$

Let us consider the simple, and most important for us, example of $G = \mathbb{C}^*$. Take $\mathbb{E}_N = \mathbb{C}^N \setminus \{0\}$, with $\mathbb{C}^*$-action by scaling. This action is free, and the quotient is $\mathbb{B}_N = \mathbb{P}^{N-1}$. Since $\mathbb{E}_N$ is homotopic to $\mathbb{S}^{2N-1}$, and $H^i(\mathbb{S}^{2N-1}) = 0$ unless $i = 0, 2N-1$, for any $\mathbb{C}^*$-space X we have

$$H^i_{\mathbb{C}^*}(X) = H^i((\mathbb{C}^N \setminus \{0\}) \times^{\mathbb{C}^*} X) \quad \text{for } i < 2N-1.$$

In particular, $H^i_{\mathbb{C}^*}(\mathrm{pt}) = H^i(\mathbb{P}^{N-1})$ for $i < 2N-1$. Since we have a ring isomorphism $H^*(\mathbb{P}^{N-1}) \simeq \mathbb{Z}[t]/(t^N)$, $\deg t = 2$ for all N, this shows us that

$$\Lambda_{\mathbb{C}^*} \simeq \mathbb{Z}[t].$$

Similarly, for an algebraic torus $G = T = (\mathbb{C}^*)^m$, taking products of everything above we get $\Lambda_T \simeq \mathbb{Z}[t_1, \dots, t_m]$.

2.3. **Chern classes and fundamental classes.** Let G be an algebraic group, and X a G-space. A *G-equivariant vector bundle* on X is a vector bundle $E \to X$ with a G-action making the projection equivariant, such that for any $g \in G$, $x \in X$ the induced maps $g : E_x \to E_{gx}$ are linear. An equivariant vector bundle gives rise to an ordinary vector bundle $\mathbb{E} \times^G E \to \mathbb{E} \times^G X$. Choosing $\mathbb{E}$ appropriately, we can then define the equivariant Chern classes of E:

$$c_k^G(E) := c_k(\mathbb{E} \times^G X) \text{ in } H^{2k}_G(X) = H^{2k}(\mathbb{E} \times^G X).$$

Assume X is smooth. Then similarly, a G-invariant subvariety $V \subset X$ of codimension d gives rise to a subvariety $\mathbb{E} \times^G V \subset \mathbb{E} \times^G X$ of codimension d. We define the equivariant fundamental class by

$$[V]^G := [\mathbb{E} \times^G V] \text{ in } H^{2d}_G(X) = H^{2d}(\mathbb{E} \times^G X).$$

In the future, we will often drop the superscripts G.

Exercise 2.9. Show that these definitions are independent of $\mathbb{E}$.

Let us summarize some useful properties of Chern classes and fundamental classes; they are proved exactly in the way you can guess they are.

- Additivity: $c_1(L \otimes M) = c_1(L) \oplus c_1(M)$ for line bundles L, M;
- Whitney formula: $c(E) = c(E')c(E'')$ for an exact sequence $0 \to E' \to E \to E'' \to 0$;
- Let $E \to X$ be a vector bundle of rank r, $s : X \to E$ a G-equivariant section, and consider the zero locus $Z(s) \subset X$. If $\operatorname{codim} Z(s) = r$, then $[Z(s)] = c_r(E)$;
- Let $V, W \subset X$ be two invariant subvarieties with proper intersection. If $V \cdot W = \sum m_i Z_i$ as cycles, then all Z_i's are invariant as long as G is connected. Then $[V][W] = \sum m_i [Z_i]$ in $H^*_G(X)$. In particular, if the intersection is empty, then $[V][W] = 0$.

Exercise 2.10. Let $\mathbb{C}^*$ act on $\mathbb{C}$ in a standard way, and let $o \subset \mathbb{C}$ be the origin. Check that $[o]^2 \neq 0$ in $H^*_{\mathbb{C}^*}(\mathbb{C})$.

Let us now look at some examples, beginning with $X = \mathrm{pt}$. In this case a G-equivariant vector bundle is nothing else than a representation of G, and so each representation V has Chern classes $c_i(V) \in \Lambda^{2i}_G$.

Example 2.11. Let $G = \mathbb{C}^*$, and consider the 1-dimensional representations $\mathbb{C}_a$, $a \in \mathbb{Z}$. We have the isomorphisms

$$\begin{array}{ccc} (\mathbb{C}^* \setminus 0) \times^{\mathbb{C}^*} \mathbb{C}_1 & \xrightarrow{\sim} & \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ (\mathbb{C}^* \setminus 0) \times^{\mathbb{C}^*} \mathrm{pt} & \xrightarrow{\sim} & \mathbb{P}^{N-1} \end{array}$$

and so, taking $t = c_1(\mathbb{C}_1)$, we see that $\Lambda_{\mathbb{C}^*}$ is generated by the Chern class of the standard representation. This gives us a canonical choice of a generator for $\Lambda_{\mathbb{C}^*}$. Note that we have $c_1(\mathbb{C}_a) = at$ by additivity.

Example 2.12. Similarly, let $T = (\mathbb{C}^*)^n$ act on $V = \mathbb{C}^n$ by scaling coordinates. For $1 \le i \le n$, we have a 1-dimensional representation $\mathbb{C}_{t_i}$ of T, which only remembers the i-th component of T. Let us denote $t_i = c_1(\mathbb{C}_{t_i})$; then we have $\Lambda_T = \mathbb{Z}[t_1, \dots, t_n]$. By Whitney formula,

$$c_i(V) = e_i(t_1, \dots, t_n),$$

where e_i is the i-th elementary symmetric polynomial.

Now let V be a representation of G of dimension n. Then G acts on $\mathbb{P}(V)$, the tautological subbundle $\mathcal{O}(-1)$ and its dual $\mathcal{O}(1)$. Let $\zeta = c_1(\mathcal{O}(1)) \in H^2_G(\mathbb{P}(V))$.

Proposition 2.13. *We have a ring isomorphism*

$$H^*_G(\mathbb{P}(V)) = \Lambda_G[\zeta]/(\zeta^n + c_1\zeta^{n-1} + \dots + c_n),$$

where $c_i = c_i(V) \in \Lambda_G$ are the Chern classes.

Proof. Note that $\mathbb{E} \times^G \mathbb{P}(V)$ can be identified with the projective bundle $\mathbb{P}(\mathbb{E} \times^G V)$, in such a way that $\mathcal{O}(1)$ goes to $\mathcal{O}(1)$. Then the claim results from a general formula for cohomology of projective bundle in terms of cohomology of the base. □

For example, let $T = (\mathbb{C}^*)^n$ act on $V = \mathbb{C}^n$ in the standard way. Then

$$H^*_T(\mathbb{P}(V)) = \mathbb{Z}[t_1, \dots, t_n, \zeta]/\prod_i(\zeta + t_i).$$

Similarly, let $G = GL(V)$ act on V. We will see in the next section that $\Lambda_G = \mathbb{Z}[c_1, \dots, c_n]$, and so we get

$$H^*_G(\mathbb{P}(V)) = \mathbb{Z}[c_1, \dots, c_n, \zeta]/(\zeta^n + c_1\zeta^{n-1} + \dots + c_n).$$