Examining addends of the Ulam sequence

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How I came to the Ulam sequence?
I always asked questions.

I asked my brother what he was doing. I wanted to learn about his research.

And one day it led me to the Ulam sequence.

I asked him questions about the sequence, and then more.

I asked him what ended up as my research question, as there was no research on it yet.

And so here we are… I present to you my science fair project.

Ulam sequence explanation

U(1,2) = 1 , 2, 3, 4, ~~5~~, 6, …

1+2

1+3

1+4 & 2+3 = 5,
so the next smallest sum is next, which is 6. (2+4)

U(1, 2) = 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, …

Definition of the Ulam sequence

- The Ulam sequence U(1, n) is defined as the sequence starting with integers 1, n such that n > 1, and such that every subsequent term is the smallest integer that can be written as <u>the sum of distinct previous terms in exactly one way.</u>
- They are in honor of Stanislaw Ulam, who introduced the sequence U(1, 2) in 1964

U(1, 2) = 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, …

Main Discoveries- Literature Review

- Ulam sequences are <u>believed to grow close to linear</u>
- Ulam number <u>residues appear to be concentrated between 1/3 and 2/3</u>
  - Residue for number n modulo λ is defined as (n mod λ)/λ
  - λ has been determined as approximate 2.443443 (found by Fourier analysis)
- Generalizations→ Sequences starting with other two integers or polynomials have been analyzed with similar interesting results.

Figure 1: probability of being Ulam as a function of residue modulo λ.

Research Question

By definition, each Ulam number has two addends that are also Ulam numbers.

Define an addend sequence as number of times a particular addend has been used in the Ulam sequence.

What is the behavior of the addend sequence for different values of addends?

I expect there to be patterns in its behaviors, e.g. some of the sequences will grow much faster than others.

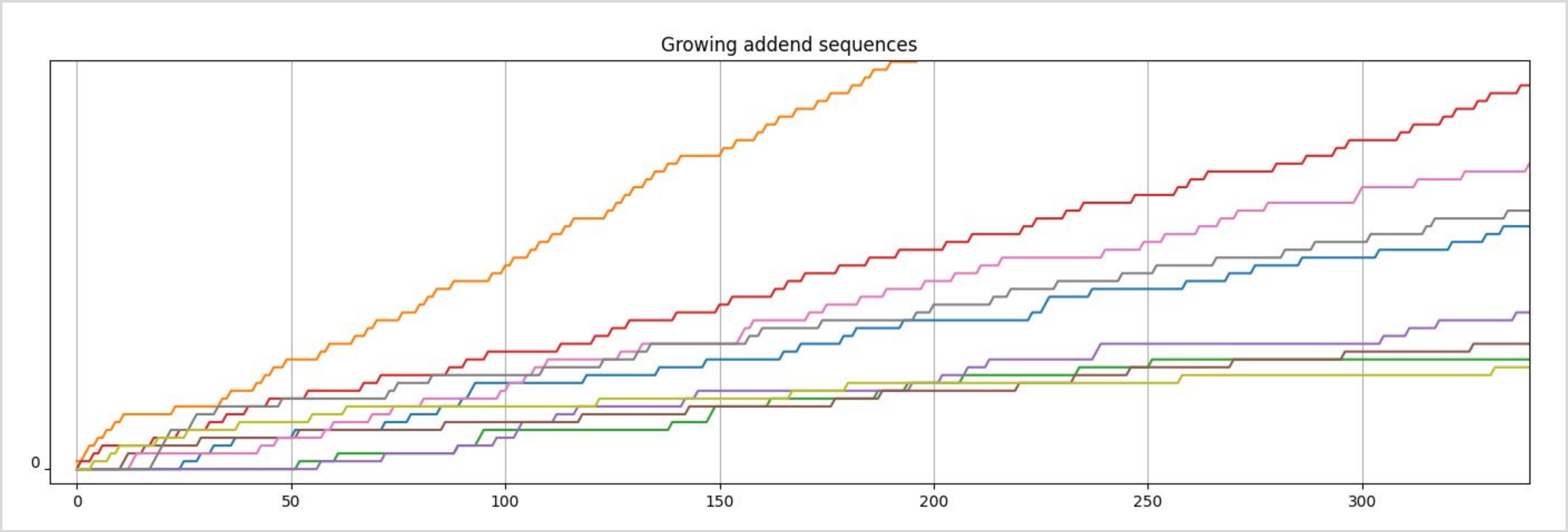
Method of Solving

Given files with Ulam numbers and their addends

- Compute sequences of addends' usage
- Identify which addend sequences grow and which are flat
- Plot growing addend sequences
- Compute residue of addends and plot their distribution

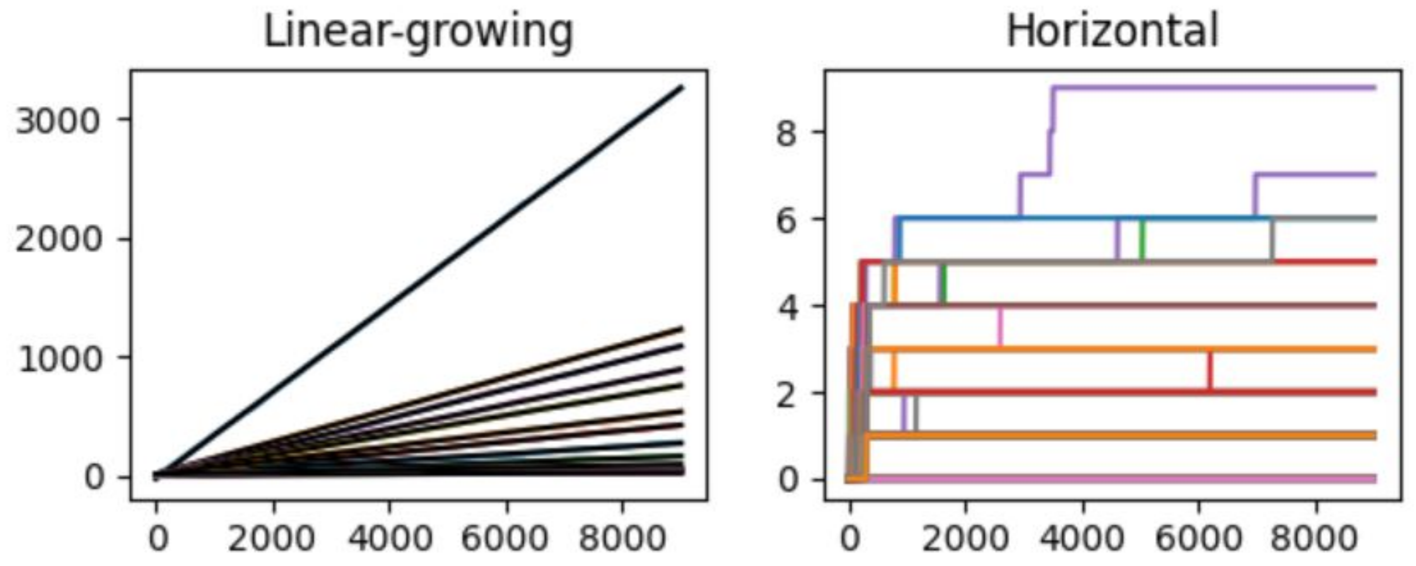
I used Python for my coding and for generating graphs. For the graphs, I also used Excel.

Quantitative Results



Addend 2 sequence: 1, 1, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 8, 9, 9, 10, 10, 10, 10, 10, 10…

At the very beginning of the addend sequences, they look like this.



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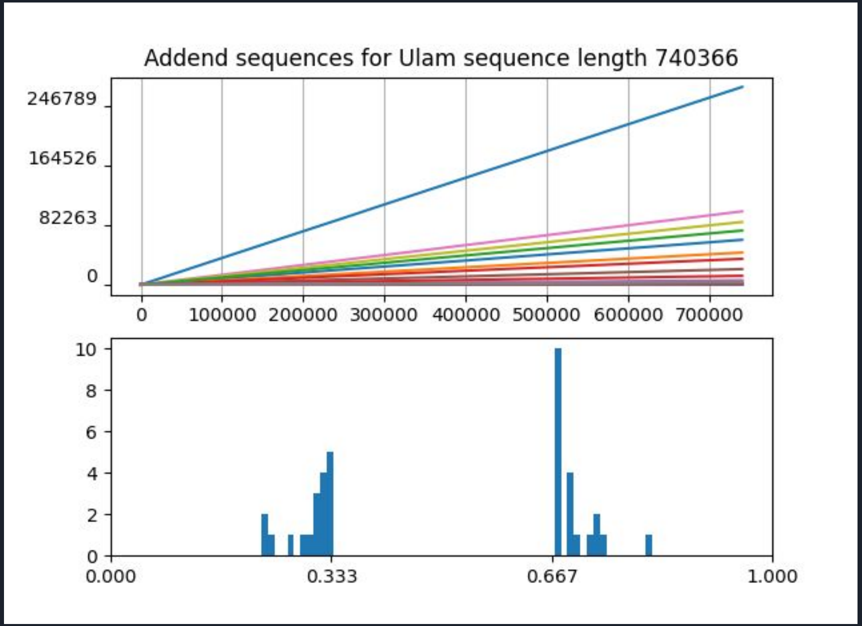
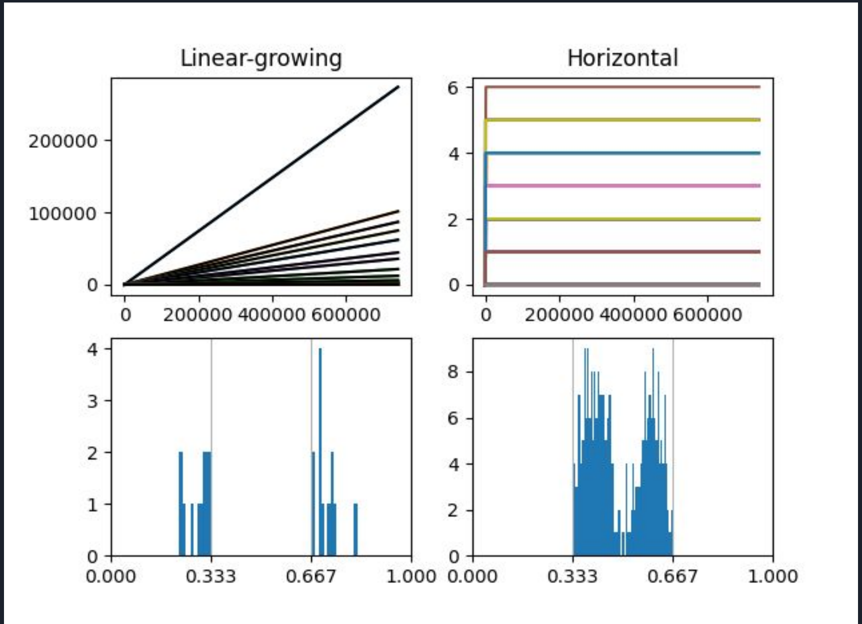
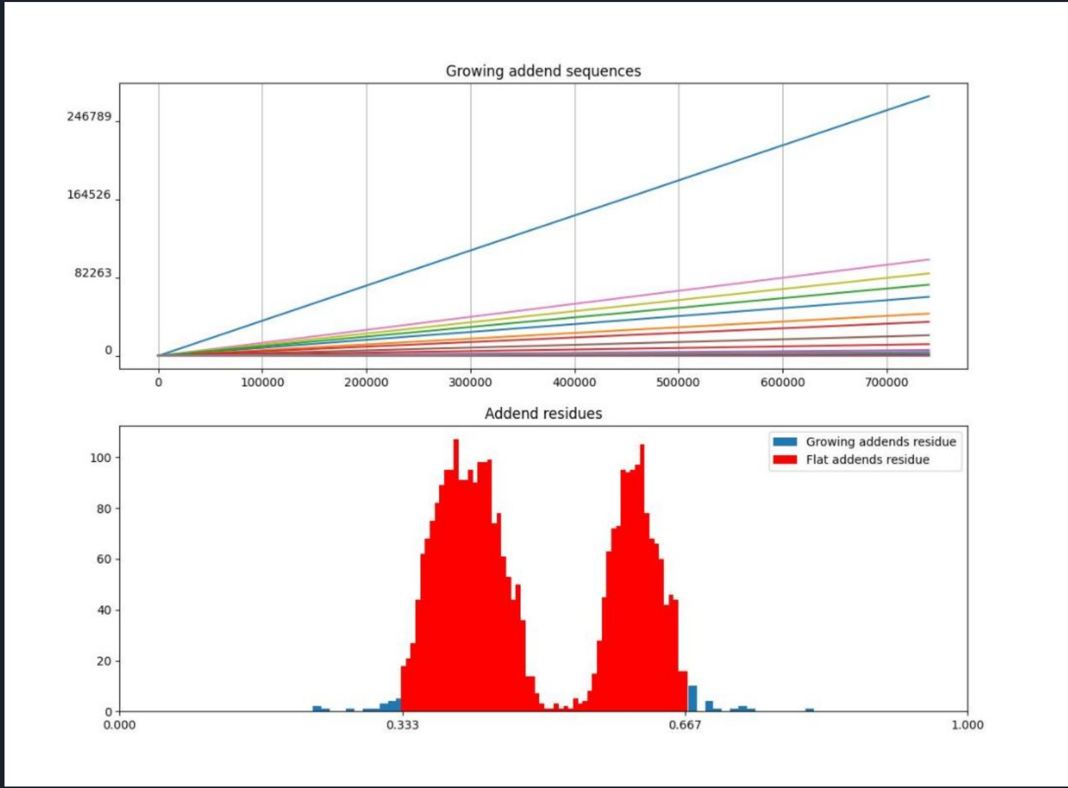


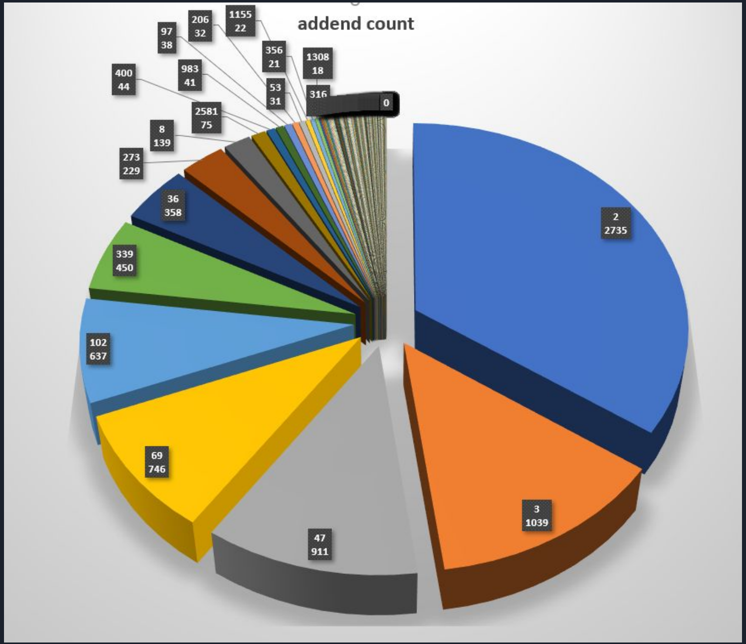


Data and graphs for Ulam sequence length 740,366, ulam numbers up to 10,000,000



Addend sequence graphs for the ulam sequence up to 1,000,000.

Statistical testing

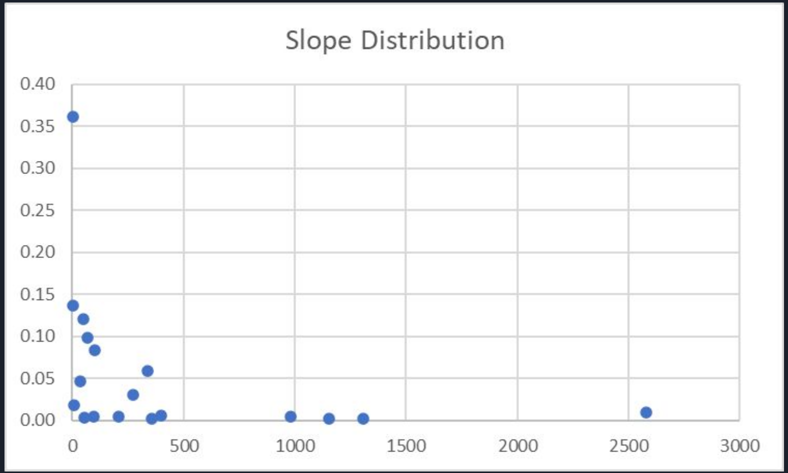


5.6% of the ulam numbers are addends 99.79% of the time up to ulam number 100,000.

	A	B	C
1	addend	slope	addend count
2	2	0.36	2735
3	3	0.14	1039
4	47	0.12	911
5	69	0.10	746
6	102	0.08	637
7	339	0.06	450
8	36	0.05	358
9	273	0.03	229
10	8	0.02	139
11	2581	0.01	75
12	400	0.01	44
13	983	0.01	41
14	97	0.01	38
15	206	0.00	32
16	53	0.00	31
17	1155	0.00	22
18	356	0.00	21
19	1308	0.00	18
20	316	0.00	9
21	13	0.00	7
22	1023	0.00	6
23	1462	0.00	6
24	3038	0.00	6
25	180	0.00	5
26	382	0.00	5
27	566	0.00	5
28	1646	0.00	5
29	2054	0.00	5
30	26	0.00	4
31	685	0.00	4
32	891	0.00	4
33	1514	0.00	4
179	3068	0.00	1
180	3090	0.00	1
181	3107	0.00	1
182	3217	0.00	1
183	3219	0.00	1
184	3261	0.00	1
185	38	0.00	0
186	72	0.00	0
296	2589	0.00	0
297	2618	0.00	0
298	2635	0.00	0
299	2650	0.00	0
300	2662	0.00	0
301	2696	0.00	0
302	2723	0.00	0
303	2750	0.00	0
304	2789	0.00	0
305	2816	0.00	0
306	2833	0.00	0
307	2899	0.00	0
308	2921	0.00	0
309	2987	0.00	0
310	3031	0.00	0
311	3070	0.00	0
312	3085	0.00	0
313	3092	0.00	0
314	3109	0.00	0
315	3131	0.00	0
316	3153	0.00	0
317	3207	0.00	0
318	3239	0.00	0
319	3263	0.00	0

More statistical testing

This is the distribution of the slopes of the commonly used addends up to ulam number 100,000.



Conclusions and Possible Future Research

Conclusions

It appears that addend sequences fall into two categories, those growing close to linear and those staying close to zero.

Also, residues of addends from the 2nd category fall between 1/3 and 2/3.

This is my repository:
https://github.com/sasha620/Ulam-sequence

Possible Future Research

The same computational technique can be done on longer Ulam(1,2) sequences as well on sequences starting with different two integers.

For example, U(1,3) could be analyzed or U(2,3).
Do similarly behaving ulam sequences have similar addend behavior?