Outer billiard outside the regular 7-gon

authors

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Abstract. Here is some abstract

1 Introduction

Let P be a regular 7-gon in $\mathbb{R}^2$ with vertices $\{p_1, ..., p_7\}$, let $X = \mathbb{R}^2 \setminus P$. Let $f : X \setminus B_0 \to X$ be a reflection in the rightmost vertex of P where B_0 is a ray continuation to the right for each side of the polygon (we denote them B_{01},...,B_{07} where the second index is an index of a base vertex of the ray from the polygon), on B_0 the map f is undefined. B_0 divides the domain X into domains $D_1, ..., D_7$ where subscript is an index of the rightmost vertex.

We say that on X a right outer billiard (outside P) is defined when such $f : X \setminus B_0 \to X$ is defined.

The set $O(x) = \{x, f(x), f^2(x), f^3(x), ...\}$ is the orbit of a point $x \in X \setminus P$(If we reach a point in B_0, we stop iteration).

$$\text{Let } B_1 = B_0 \cup f^{-1}(B_0),$$

$$B_n = B_0 \cup f^{-1}(B_0) \cup ... \cup f^{-n}(B_0) = B_{n-1} \cup f^{-n}(B_0),\ n \geq 0,$$
$$B_\infty = B_0 \cup f^{-1}(B_0) \cup ... \cup f^{-n}(B_0) \cup ...$$

Let us consider a letter sequence (a word) $l_0 l_1 ... l_n ...$ from an alphabet A, that is for all k $l_k \in A$. Let $FW(A)$ be a set of all finite words over the alphabet A (ex. $l_0 l_1 ... l_n \in FW(A)$), $LFW(A)$ be a set of all left finite words, but they can be not right finite (ex. $l_0 l_1 ... l_n ... \in LFW(A)$).

We define a concatenation operator $* : FW(A) \times LFW(A) \to LFW(A)$, $a * c := ac$, $a \in FW(A)$, $c \in LFW(A)$.

We define a length function $length : LFW(A) \to \mathbb{N}_0 \cup \{\infty\}$; $length(l_0...l_n) :=$ $n+1$; $length(l_0...l_n...) := \infty$; $length(ab) = length(a) + length(b)$.

We say that a letter sequence $l_0 l_1 ... l_n ...$ where for all k $l_k \in \{1, 2, ..., 7, b\} =:$ A is an itinerary of a point x $\in$ X ($it(x) := l_0 l_1 ... l_n ...$) when $l_k \neq b \Rightarrow$ $\Rightarrow f^k(x) \in D_{l_k}$, and $l_k = b \Rightarrow f^k(x) \in B_0$ ($f^0(x) := x$), the sequence ends on b or is infinite; for all k l_k is called a k-th address.

We say that a point $x \in X \setminus B_0$ is periodic if there is $k > 0$ s.t $f^k(x) = x$, k is called a period, $\min\{k_1, k_2, ...\}$ of such k_i is called a prime or least period.

We say that a point $x \in X \setminus B_0$ is aperiodic if there is no $k > 0$ s.t. $f^k(x) = x$
$f(B_k \setminus B_0) = B_{k-1}$, $k > 0$.

2 Itineraries

We consider $S_a = \{x \in X \setminus B_0 |\ it(x) = a...\}$ for some $a = a_0 a_1 ... a_k$ where $a_k \neq b$ that means that S_a is a set of all points s.t. their itineraries begin with a.

Lemma 1. *The map f^{k+1} is defined on S_a.*

Proof. For all $x \in S_a$ there exists $r(x)$ s.t. $it(x) = a_0 a_1 ... a_k r(x)$ by definition of S_a. By definition of an itinerary, $it(f^{k+1}(x)) = r(x)$. Hence, f^{k+1} is defined on S_a. □

Lemma 2. *The map f^{k+1} is undefined on ∂S_a (∂ denotes the boundary of a set).*

Proof. Let $S_a^k = f^k(S_a)$. We have $S_a^k = \{x | x \in D_{a_k}\}$. Hence, $\partial S_a^k \setminus \partial P =$ $B_{0\ (a_k - 1)\ mod\ 7} \cup B_{0 a_k}$(there can be a case when $\partial S_a^k \cup \partial P \neq \emptyset$). Also $f^k(\partial S_a \setminus$ $\partial P) = \partial S_a^k \setminus \partial P$. The map f is undefined on this set. Hence, f^{k+1} is undefined on ∂S_a. □

Corollary 1. *The boundary of S_a is a subset of $B_k \cup \partial P$(direct implication of the proof of the lemma 3).*

Corollary 2. *The set S_a is an open polygon.*

Lemma 3. *All components of $X \setminus B_k$ are convex.*

Proof. For two fixed points x, y we consider a closed interval $[x, y] : t =$ $= (1 - \alpha)x + \alpha y$, where $\alpha \in \mathbb{R}$ is continuously changing from 0 to 1.

Let us assume that there exists a component D that is not convex. That means that we can find two points $x, y \in D$ s.t. the interval $[x, y]$ will have a point $z = (1-\alpha_0)x + \alpha_0 y \notin D$.

The component $D \subset D_i$ for some i, hence, $[x, y] \subset D_i$, $f(x) = 2p_i - x$ for $x \in D$.

The map f takes any interval in D_i and returns an interval in X.

By the assumption we conclude that $[x, y] \cap B_{n_0} \setminus B_{n_0-1} \neq \emptyset$ for some $0 < n_0 \leq k$.

Generally there can be more intersections with other $B_{n_1}, B_{n_2}...$, for simplicity we will consider only n_0 and assume that it is minimal, further we will show that even with such single n_0 we get a contradiction. For each $n \leq n_0 - 1$ $f^n([x, y]) \cap B_{n_0-n} \setminus B_{n_0-1-n} \neq \emptyset$ and $f^n([x, y]) \cap B_{n_0-1-n} = \emptyset$ (because we chose minimal n_0). This statement implies that $f^n([x, y])$ is entirely inside one of the domains $D_1, ...D_7$, hence, $f^n([x, y]) = [f^n(x), f^n(y)]$ is an interval. In particular, $f^{n_0-1}([x, y]) = [f^{n_0-1}(x), f^{n_0-1}(y)]$. The interval $[f^{n_0-1}(x), f^{n_0-1}(y)]$ is placed inside one of the domains $D_1, ...D_7$ and has an intersection with B_1. And finally $f^{n_0}([x, y]) = [f^{n_0}(x), f^{n_0}(y)]$ is an interval.

That means $f^{n_0}(x) \in D_i,\ f^{n_0}(y) \in D_j,\ i \neq j$.

And we reach a contradiction because the n_0th-addresses are different, but we know that any two points of the same component of $X \setminus B_k$ will stay in the same component until f^k, hence, all the addresses until the kth-address must be the same.

□

Corollary 3. *The set S_a is convex.*

Theorem 1. *We suppose that k is even, that S_a has a point x_0 that has $it(x_0) = aac$ for some c. Then S_a has a periodic point with period k+1.*

Proof. So we have the point x_0 with the itinerary aac.
Since $x_0 \in S_a$ and $f^{k+1}(x_0) \in S_a$, S_a and $f^{k+1}(S_a)$ intersect.
The map f^{k+1} on S_a is a reflection:

$$f^{k+1}(x) = 2\sum_{i=0}^{k}(-1)^k(-1)^i p_{a_i} + (-1)^{k+1}x = 2\sum_{i=0}^{k}(-1)^i p_{a_i} - x,$$

$$p^* := \sum_{i=0}^{k}(-1)^i p_{a_i} = \text{const for } x \in S_a.$$

The intersection yields that there is a point $x \in S_a$ s.t.

$$f^{k+1}(x) = 2p^* - x = y \in S_a.$$

It implies that $p^* = \frac{x+y}{2}$. From the corollary about convexity of S_a it implies that $p^* \in S_a$. In that case, $f^{k+1}(p^*) = 2p^* - p^* = p^*$. That means that p^* has period $k+1$. $\square$

Corollary 4. *We suppose that k is even, that S_a has a point x_0 that has $it(x_0) = aac$ for some c. Then S_a contains a polygon with period $2k$.*

Example:

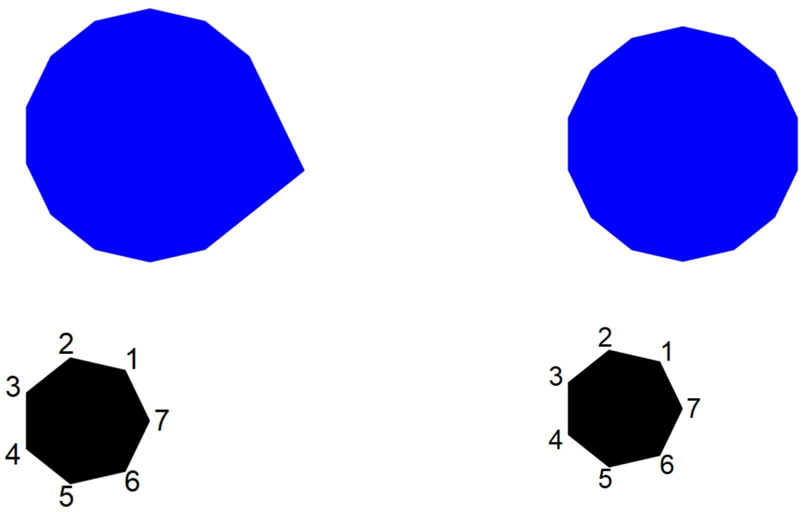

Figure 1: The left picture shows the polygon $S_{3625147}$, the right picture shows the polygon $S_{36251473625147}$, which is a periodic domain.

Point $(2,3)$ is in the $S_{36251473625147}$ and it has a period 36251473625147.

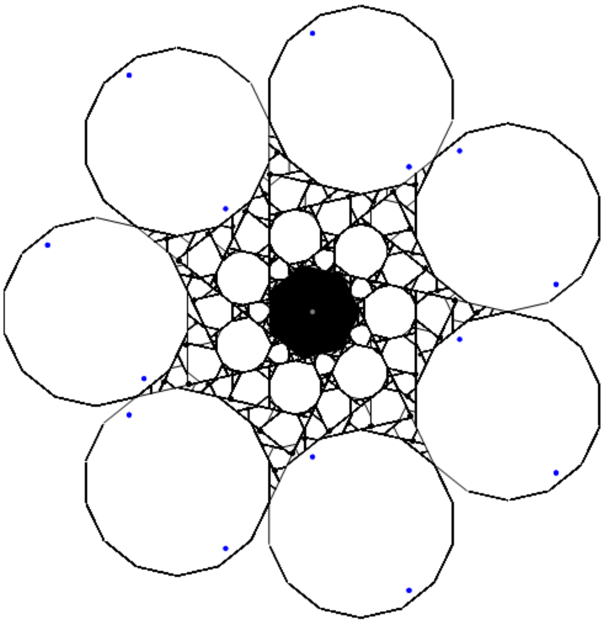

Figure 2: Blue points are the iterations of the point $(2,3)$.

3 Characterization of the set of aperiodic points

Lemma 4. *The set of aperiodic points is a subset of $Cl(B_\infty)$.*

Proof. Suppose that the statement of the lemma does not hold. Then there exists an aperiodic point x and an open ball $B(x,r)$ centered at x with radius $r>0$ such that $B(x,r)\cap B_\infty=\emptyset$. It is obvious by induction that for each $k\geq 0$ we have $f^k(B(x,r))=B(f^k(x),r)$, also, $f^k(B(x,r))$ does not intersect B_∞ and, in particular, for each $k\geq 0$ the set $B(f^k(x),r)$ is a subset of some $D_{i(k)}$, $1\leq i(k)\leq 7$. Hence, all points in $B(x,r)$ have the same itinerary.

It is a well-known fact (follows from [1], Prop.1) that each point of an outer billiard with respect to a regular polygon has a bounded trajectory. Hence, for some $R>0$ all the balls $f^k(B(x,r))$ are subsets of $B(0,R)$, which means that there exist $0\leq k<l$ such that

$$f^{2k}(B(x,r))\cap f^{2l}(B(x,r))\neq\emptyset.$$

As all points of $B(x,r)$ have the same itinerary, f^k is injective on $B(x,r)$ for each $k\geq 0$. Hence, $f^{2(l-k)}(B(x,r))\cap B(x,r)\neq\emptyset$. Moreover, for each $t\geq 0$,

$$f^{2(l-k)t}(B(x,r))\cap f^{2(l-k)(t+1)}(B(x,r))\neq\emptyset.$$

Combining this with the fact that for each $t \geq 0$ all points in $f^{2(l-k)t}(B(x,r))$ have the same itinerary, we conclude that each two points $x, y \in \bigcup_{t=0}^{\infty} f^{2(l-k)t}(B(x,r))$ have the same itinerary. Hence, $f^{2(l-k)}$ acts on $\bigcup_{t=0}^{\infty} f^{2(l-k)t}(B(x,r))$ as a parallel translation. So, if this parallel translation is not equal to the identity transform, then $f^{2(l-k)t}(x) = x + tv$ for all $t \geq 0$ and some vector v. If $v \neq 0$, then the set $\bigcup_{t=0}^{\infty} f^{2(l-k)t}(x)$ is unbounded, in particular, the orbit of x is unbounded, and we get a contradiction with [1], Prop.1. Hence, $v = 0$ and $f^{2(l-k)}(x) = x$, so x is periodic, which is also a contradiction. $\square$

On the other hand, as we have seen before, the set of periodic points is open, hence, it does not contain any points from $Cl(B_\infty)$. Combining this with Lemma 4, we conclude that the set of aperiodic points coincides with $Cl(B_\infty) \backslash B_\infty$.

In the future we are planning to get some additional information about the sets of periodic, aperiodic, and boundary points. In particular, two questions seem interesting:

1) Is it true that $Cl(B_\infty)$ is nowhere dense?

2) Is it true that $Cl(B_\infty)$ has Lebesgue measure 0?

In the case of a regular pentagon, the answer to these questions is positive, the proof is based on multiple similarities in the dynamics of some fragments of the outer billiard, but in the case of a 7gon it seems much harder to find analogous similarities.

References

[1] F. Vivaldi, A. Schaidenko, *Global stability of a class of discontinuous dual billiards*, Comm. Math. Phys. **110** (1987) 625-640