REVIEWS OF TOPICAL PROBLEMS

Stochastic quantization of field theory

Stochastic quantization of field theory

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A new quantization procedure based on the analogy between quantum theory and nonequilibrium statistical mechanics is described. Stochastic quantization is particularly useful for gauge theories, including lattice theories. Both the theoretical basis of the method and its computer implementation are discussed. A detailed discussion is given of applications to reduced Eguchi-Kawai models, which correspond to lattice quantum chromodynamics in the limit of an infinite number of colors. The complex stochastic equation corresponding to quantum theory in the ordinary Minkowski space is also examined. The equations are discussed in curved field space, as are the difference equations that replace the stochastic equation with a high degree of precision.

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1. INTRODUCTION

The language of quantum field theory is evolving by adapting to new concepts arising in new problems. The operator language has become obsolete and has been replaced by the functional language. Quantum processes are increasingly described by a set of alternative classical processes, and the alternative amplitudes are summed by functional integration.

A practical method of functional integration is provided by the Monte Carlo procedure which reduces to the determination of an equilibrium sequence of alternative probable states. The states are counted with allowance for their weight: the more probable states are more frequent.

It is possible to consider a continuous analog of this discrete process of counting of states by introducing a fifth coordinate τ, i.e., the stochastic or computer time. The transition from classical to quantum theory is then especially general and simple. In particular, to quantize an arbitrary Bose system (Fermi systems are somewhat more complicated, but can still be fitted into the scheme), it is sufficient to add the stochastic time and introduce two new terms into the equation of motion, namely, a random force and the force of friction. The variance of the random force is proportional to the Planck constant and is responsible for quantum fluctuations. The force of friction is proportional to the derivative of the corresponding dynamic variable with respect to the stochastic time. This force is responsible for the conversion of the "energy" supplied by the random force into "heat."

We shall use the thermodynamic analogy: in quantum field theory, action plays the part of the energy of the four-dimensional statistical system and Planck's constant plays the part of temperature. The stochastic time then plays the part of the physical time necessary to establish thermodynamic equilibrium.

The stochastic equation is well-known in statistical mechanics. It is the so-called Langevin equation, describing the kinetics of weakly-nonequilibrium systems. We emphasize that, although, in statistical mechanics, the Langevin equation is not equivalent to the original dynamics (some of the

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degrees of freedom are simulated by random forces), the state of thermodynamic equilibrium is nevertheless reproduced exactly by it. In particular, it can be shown that the Langevin equation preserves detailed balancing between different states, i.e., the probabilities of mutual transitions between any two states are in the ratio of their Boltzmann factors. Hence, the probability distribution in the state of thermodynamic equilibrium, if it is reached, is identical with the Boltzmann distribution.

In quantum field theory, this means that the averages calculated with the aid of the Langevin equation will be exactly the same as the quantum averages calculated by functional integration. The evaluation of averages with the aid of the Langevin equation reduces simply to the averaging of the corresponding variable over a large interval of stochastic time. Equivalence to functional integration, i.e., averaging over an ensemble of systems, signifies the ergodicity of the motion described by the Langevin equation.

The foregoing discussion applies to the Euclidean variant of quantum field theory, i.e., to purely imaginary time $t = ix_4$.[1] When we pass to real time, the Feynman amplitude exp(-action) becomes complex and can no longer be interpreted as the Boltzmann factor. The Monte Carlo method in its literal form is then invalid because random numbers cannot be counted with complex probability.

However, the Langevin equation, which implements detailed balancing in a somewhat different form, does admit of analytic continuation into Minkowski space. The random force remains a real Gaussian quantity, but the force of friction becomes imaginary.

The equation is then complex, like the Schrödinger equation, and is nonlinear. The corresponding dynamic variable moves stochastically in the complex plane. Quantum mechanics can thus be interpreted as the statistical mechanics in complex space. Instead of the usual complex amplitude of the real coordinate, we can then introduce a real positive probability for the complex coordinate.

Interference effects will arise after averaging over stochastic time. The corresponding integral will converge because of the oscillations in the complex integrand, and so will the Feynman integral. It is expected that these oscillations will not be as fast as in the Feynman case, so that it will be possible to simulate them in numerical experiments.

The first such experiments have already been successfully performed in single-particle quantum mechanics. Of course, this is not enough to enable us to say with confidence that there is a more adequate scheme than that of the usual Schrödinger equation. However, it is hoped that the complex Langevin equation will enable us to see quantum mechanics in a new light and to demonstrate by computer simulation all its surprising properties which are not accessible to direct observation.

As far as applications to Euclidean quantum field theory are concerned, quite a lot has already been done, and this will be the main concern of the present review. The application of the Langevin equation to quantum field theory was initiated by Nelson, Parisi, and Wu.[1] In particular, Parisi and Wu recognized that this was a way of avoiding the problem of having to fix the gauge in gauge theories.

Many papers using and developing the stochastic quantization procedure have appeared since then. In many respects, this procedure has been found to be more convenient than the Monte Carlo method, and is capable of greater precision. It also enables us to prove various general relationships more simply than in other quantization procedures.

In this review, we shall discuss the applications of stochastic quantization to ordinary and gauge field theories without fermions. The structure of our review is as follows. The general scheme of stochastic quantization is introduced and discussed in Sections 2–4, and the problems examined above are discussed in greater detail.

Sections 5–8 are more technical in character. They consider the stochastic perturbation theory and the problem of defining the gauge. Stochastic perturbation theory is examined in its simplest form, namely, against the background of the trivial classical vacuum. Section 14 briefly discusses perturbation theory against the background of the quantum vacuum of reduced models with an infinite number of colors.

In Sections 9–11, we introduce and discuss reduced models in both ordinary and gauge theories. These models correspond to spatially homogeneous (to within unitary transformations) matrix fields in the limit as the size of the matrix tends to infinity. In this limit, reduced models reproduce the results of ordinary theories with inhomogeneous fluctuating fields.

Reduction is examined in greater detail in Sections 12–14, including computational aspects. Specific computational schemes are introduced for continuous and lattice theories. A brief discussion is given of stochastic perturbation theory against the background of reduced models.

Finally, the situation is summarized, and the possibilities for the future are discussed, in Section 15. The final section also considers the complex Langevin equation together with possible schemes lying outside the framework of quantum field theory.

Some of the details missing from the main text can be found in the appendices. Appendix A establishes the connection between the Langevin equation and the principle of detailed balancing. Appendix B gives a derivation of the Fokker-Planck equation, used to investigate the gauge problem. A derivation of the difference analog of the Langevin equation, which preserves detailed balancing, is given in Appendix C. This leads to a numerical algorithm that differs from the Metropolis algorithm. The reduction formulas are proved in Appendix D.

In writing this review, we have drawn extensively on the results published with M. A. Bershadskiĭ and T. A. Kozhamkulov in previous papers.[24–28,32] I am indebted to them for their help and to V. A. Novikov for constructive criticism. I also acknowledge useful discussions with V. A. Kazakov, A. M. Polyakov, and Ya. G. Sinaĭ.

2. QUANTUM MECHANICS AND STATISTICS

The idea of stochastic quantization is based on the analogy between quantum theory and statistical mechanics. It is

well-known that, if we introduce the replacement $t \to -i\beta$, the operator describing the evolution of the quantum-mechanical system is replaced by the Bloch matrix of the statistical system:

$$\exp(-itH) \to \exp(-\beta H), \quad t \to -i\beta. \tag{2.1}$$

Interference effects then vanish but, if we are interested in statistical properties such as the energies of the ground and first excited states, the magnetic moments, and so on, these can be extracted directly from the Bloch matrix by allowing $\beta \to \infty$.

To describe interference effects, we must return to real time by introducing the analytic continuation $\beta \to it$. The advantage of the Bloch matrix as compared with the S matrix from the computational point of view is that the matrix elements decrease and do not oscillate as in quantum mechanics. This means that the sums over intermediate states converge.

This applies, above all, to the functional integral for the trace of the S matrix. The famous Feynman-Katz formula was obtained forty years ago for this quantity. We reproduce it here in the Euclidean variant for a scalar field φ:

$$\operatorname{tr}\exp(-\beta H) = \int D\varphi \exp(-S_E); \tag{2.2}$$

where

$$S_E = \int d^4x \left[\frac{1}{2}(\nabla\varphi)^2_4 + U(\varphi)\right] \tag{2.3}$$

is the Euclidean action and $x_4 = -it$ is assumed real.

The following periodic boundary condition is implied:

$$\varphi(0, x) = \varphi(\beta, x). \tag{2.4}$$

This condition has arisen because we have taken the trace tr. A detailed derivation of (2.2) can be found, for example, in the book of Slavnov and Faddeev.[5]

In contrast to ordinary action, which contains the difference between the kinetic energy $\frac{1}{2}(\partial\varphi^2/\partial\tau)$ and the potential energy $U(\varphi) + \frac{1}{2}(\nabla\varphi)^2$, the Euclidean action (2.3) contains a sum of positive terms and is therefore positive definite. The integral in (2.2) converges both for large fields and for high field gradients. Of course, this does not remove the problem of ultraviolet divergences. The functional integral must be defined as the limit of a multiple integral, say, by dividing the space x into the cells of a hypercubic lattice.

The problem of ultraviolet divergences is that the parameters of the potential $U(\varphi)$ must be varied as the lattice constant a is reduced in order to ensure that observed quantities remain finite. In particular, in the case of a scalar field, the potential U is characterized by a bare mass m and bare constant λ_0:

$$U = \frac{m_0^2\varphi^2}{2} + \frac{\lambda_0\varphi^4}{4!}. \tag{2.5}$$

The required functions $m_0(a)$ and $\lambda_0(a)$ are given by the renormalization group equation,[6] which will not be discussed here. We confine ourselves to noting that, in the case of a scalar field in four-dimensional space, it has not been possible to find functions $m_0(a)$, $\lambda_0(a)$ that would ensure that the physical constant λ_{ph} remains finite for $a \to 0$. Evidently, $\lambda_{ph} = 0$ always for $a \to 0$ (the case of zero charge).

This concludes our brief excursion into the theory of functional integration. We shall not need it again.

To proceed to stochastic quantization, let us examine in detail the analogy with statistical mechanics. So far, we have confined our attention to the quantum statistical mechanics of the original three-dimensional system with Hamiltonian H. The Bloch matrix (2.1) describes the quantum Gibbs distribution for the original three-dimensional system. This analogy lies on the surface.

However, there is another, less trivial, analogy. The functional integral (2.2) may be looked upon as the partition function of a four-dimensional system with potential energy S_E. Actually, the Boltzmann distribution $\exp(-U/kT)$ is formally analogous to the distribution $\exp(-S_E)$ in Euclidean field theory. The analogy can be strengthened by isolating the factor λ^{-1} from the Euclidean action, where λ is the coupling constant. In the case of a scalar field, it is then sufficient to redefine φ so that $\varphi \to \varphi/\lambda^{1/2}$. It then becomes clear that classical field theory, obtained by solving the equation

$$\frac{\delta S_E}{\delta\varphi_{cl}} = 0, \tag{2.6}$$

corresponds to the state of static equilibrium of the analogous four-dimensional system. Quantum fluctuations correspond to thermal fluctuations due to interaction with the thermostat. The coupling constant, and not the time, now plays the part of temperature.

This treatment of Euclidean field theory is relativistically invariant, and is convenient in both analytic and numerical calculations. The thermodynamics of black holes, instantons, and the Monte Carlo method—in a word, all the recent achievements of quantum field theory, are associated with precisely this statistical approach.

3. THE LANGEVIN EQUATION

Stochastic quantization is also based on the four-dimensional statistical treatment of quantum field theory. The next natural step is to introduce into (2.6) an explicit random force applied to the field by the thermostat, which brings the field to the state of thermodynamic equilibrium. The work done by the random force is converted into heat by the forces of friction, which is also added to the classical equation. The thermodynamic equilibrium is reached gradually, over a long period of "time." We emphasize that this is not the physical time $t = ix_4$, but a new independent variable. Let us examine this important point in greater detail.

Consider the usual three-dimensional statistical system, e.g., a gas in an external field. The corresponding Boltzmann distribution is

$$\prod_i dp_i\, dq_i \exp\left(-\frac{H}{kT}\right), \tag{3.1}$$

where

$$H = \sum_i \frac{p_i^2}{2m} + U(q_1, \ldots, q_N). \tag{3.2}$$

We can then eliminate the momenta p_i by Gaussian integration and obtain the effective distribution for the coordinates

$$\prod_i dq_i \exp\left(-\frac{U}{kT}\right). \tag{3.3}$$

The question is: how is this distribution realized in nature? We observe a vessel containing a gas for a period of time that is longer than the relaxation time, and measure, say, the pressure, density, and so on. The averages determined in this way will be the same as if we were dealing with an ensemble of vessels, weighted in accordance with the Boltzmann distribution, and then taking the average not over time but over the ensemble.

The achievement of classical workers in statistical mechanism was that they were able to eliminate from the discussion the complicated problem of the dynamics (temporal evolution) of the system. If, nevertheless, we are to take up the question of relaxation phenomena, we still need not consider the complete dynamics.

The Langevin equation is approximately valid for small deviations from thermodynamic equilibrium, i.e.,

$$\Gamma\frac{\partial q_i}{\partial t}+\frac{\partial U}{\partial q_i}=f_i, \tag{3.4}$$

where f_i are random forces obeying the Gaussian distribution

$$\exp\left(-\int\frac{dt}{\Gamma}\frac{1}{4kT}\sum_i f_i^2\right)df_t. \tag{3.5}$$

The quantity Γ is proportional to the relaxation time, T in (3.5) is the usual temperature, and k is Boltzmann's constant.

The equivalence of the Boltzmann distribution (3.3) and the Langevin equation (3.4) must be understood as being valid for the averages. The average evaluated over the Boltzmann ensemble is identical with the time average evaluated over the Langevin equation.

The fact that these averages are equal can be written down in terms of the probability distributions (this is the ergodic theorem):

$$Z^{-1}\exp\left(-\frac{U(q)}{kT}\right)=\overline{\delta(q-q(t))}, \tag{3.6}$$

where

$$\overline{F(t)}=\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\,F(t) \tag{3.7}$$

is the time average. The function $q(t)$ in (3.6) is to be understood as the solution of the Langevin equation.

A proof of the validity of the ergodic theorem (3.7) in the case of the Langevin equation is given in Appendix A. This appendix also gives the covariant generalization of the Langevin equation which will be used below in lattice gauge theories.

For a particle gas in a static gravitational field with metric $g_{\alpha\beta}$, this covariant Langevin equation assumes the form

$$\Gamma e_\alpha(i)\frac{\partial r^\alpha}{\partial t}+e^\beta(i)\frac{\partial U}{\partial r^\beta}=f(i), \tag{3.8}$$

where $f(i)$ represents random forces with the Gaussian distribution

$$dW(f)=\exp\left(-\int\frac{dt}{\Gamma}\frac{1}{4kT}\sum_i f^2(i)\right)\prod_i df(i); \tag{3.9}$$

and $e_\alpha(i)$ are tetrads corresponding to the metric

$$g_{\alpha\beta}=\sum_i e_\alpha(i)\,e_\beta(i)\,. \tag{3.10}$$

4. STOCHASTIC QUANTIZATION IN FIELD THEORY

The analogy between Euclidean quantum field theory and four-dimensional statistical mechanics leads directly to the Langevin field equation

$$\frac{\partial\varphi}{\partial\tau}+\frac{\delta S}{\delta\varphi}=f. \tag{4.1}$$

We have assumed that the "relaxation time" is $\Gamma\equiv 1$ because the scale of the auxiliary variable τ is arbitrary. The derivative $\partial U/\partial q$ of the potential energy will obviously be identical with the variational derivative $\delta S/\delta\varphi$. The subscript E will be omitted from the action S as long as we confine our attention to the Euclidean variant of field theory.

The analog of the ergodic theorem states [$F(\varphi)$ is an arbitrary functional of the field]:

$$\langle F(\varphi)\rangle_S=\lim_{t\to\infty}\int_0^t\frac{d\tau}{t}F(\varphi(\tau)). \tag{4.2}$$

The Langevin equation (4.1) enables us to view field quantization from a new standpoint. It turns out that the quantum theory equation differs from the classical equations only by the presence of two terms, i.e., the force of friction and the random force.

The question is: are these the hidden parameters that were the subject of so much argument among the creators of quantum mechanics? The answer is: definitely not. The fact is that we are dealing here not with hidden parameters but a hidden variable, namely, the Langevin time τ. In quantum mechanics, a system can occupy simultaneously many alternative states but, in classical mechanics, it can occupy only one. The special variable τ is introduced in stochastic quantization to assist with the enumeration of alternative states. For fixed τ, we have a classical system, and quantum effects arise after averaging over τ. This averaging is equivalent to the Feynman averaging with the weight $\exp(-S)$, but the counting of states is performed nonrandomly.

The Feynman recipe prescribes that we must consider an infinite set of classical states, each with weight $\exp(-S)$. Stochastic quantization provides the recipe for passing from one state to another in such a way that the probability of the appearance of each state is proportional to $\exp(-S)$. The difference between the two is that states of low probability do not appear in stochastic quantization. In other words, stochastic quantization reveals the uncertainty corresponding to an infinite number of infinitesimal contributions to the functional integral of Euclidean theory.

For example, in the classical limit, when $S\to\infty$, equation (4.1) describes small fluctuations of the field around the classical configuration $\varphi=\varphi_{cl}+\delta\varphi$.

Let us consider these fluctuations in greater detail. In the linear approximation, the equation for the fluctuations is

$$\dot{\delta\varphi} + S'' \delta\varphi = f; \tag{4.3}$$

where S'' is the operator corresponding to the second variation of the action S:

$$S'' \delta\varphi(x) = \int \frac{\delta^2 S}{\delta\varphi(x)\,\delta\varphi(y)} \delta\varphi(y)\, d^4y. \tag{4.4}$$

Expanding $\delta\varphi$ and f over the complete orthonormal set of eigenfunctions ψ of the operator S'', we obtain the corresponding coefficients $\delta\varphi_\lambda$:

$$\delta\varphi = \sum \psi_\lambda \delta\varphi_\lambda, \quad S''\psi_\lambda = \mu_\lambda \psi_\lambda, \tag{4.5}$$

$$\delta\varphi_\lambda = \int_0^t d\tau \exp(-\mu_\lambda \tau) f_\lambda(t-\tau). \tag{4.6}$$

It is clear that the random force leads to deviations from equilibrium, i.e., from classical behavior. The force of friction results in delay. The deviation $\delta\varphi$ is a superposition of a large number of random forces at previous instants of time.

If we were to turn off the random force at a particular instant of time, the system would relax to the classical state (because all the $\mu_\lambda \geqslant 0$). If the force of friction were to be turned off in precisely the same way, the delay would vanish, and the deviation would be proportional to the instantaneous value of the random force.

We shall show that the Langevin equation correctly describes the correlations between fluctuations in the field φ. These correlations can be readily found from (4.6):

$$\begin{aligned}\langle \varphi^*_\lambda \varphi_{\lambda'} \rangle &= \Big\langle \int_0^t d\tau\, e^{-\mu_\lambda \tau} \int_0^t d\tau'\, e^{-\mu_{\lambda'}\tau'} f_{\lambda'}(t-\tau) f_{\lambda'}(t-\tau') \\ &\to \int_0^t d\tau \int_0^t d\tau'\, e^{-\mu_\lambda \tau - \mu_{\lambda'}\tau'\delta}(\tau-\tau')\cdot 2\delta_{\lambda\lambda'} \\ &= 2\delta_{\lambda\lambda'} \frac{1-e^{-2\mu_\lambda t}}{2\mu_\lambda}.\end{aligned} \tag{4.7}$$

It is clear that a finite limit is reached for nonzero modes as $t \to \infty$. This limit corresponds to the usual perturbation theory in the functional integral:

$$\langle \delta\varphi(x)\, \delta\varphi(y) \rangle \to \sum_\lambda \psi^*_\lambda(x)\, \psi_\lambda(y)\, \mu_\lambda^{-1} = (S'')^{-1}_{xy}. \tag{4.8}$$

For example, if $\varphi_{cl} = \text{const}$, the eigenfunctions $\psi_\lambda(x)$ are the plane waves $\exp(ikx)$. We then have $\mu_k = k^2 + m^2$ and (4.8) becomes the Green's function for a Klein-Gordon particle. This is examined in greater detail in the next section, where a correspondence is found between stochastic and ordinary perturbation theories.

A nontrivial situation arises when there is a zero mode as, for example, in gauge theories. The Green's function (4.8) does not then exist, and this usually means that we have to fix the gauge by adding to the action further terms that fix the gauge, as well as ghost terms. However, as we shall see below, this can be avoided in the Langevin equation. In fact, the Langevin time plays the part of the gauge parameter. For a fixed time τ, the propagator (4.7) is finite even for zero modes:

$$\langle \varphi^*_\lambda \varphi_{\lambda'} \rangle = 2\delta_{\lambda\lambda'} t \quad (\mu_\lambda = 0). \tag{4.9}$$

If we first fix the time, evaluate the propagators, and then pass on to observable variables, we would expect that the growing terms (4.9) would mutually cancel out in the same way as terms depending on the gauge parameters are found to cancel in the usual approach. This will be examined in greater detail below.

Of course, these technical conveniences are not the whole story. The Langevin equation has attracted the attention of researchers because it reveals new ways of going beyond perturbation theory. The first steps in this direction have already been taken and will be discussed in this review.

5. SCALAR FIELD THEORY. DIAGRAM TECHNIQUE

Let us now consider the theory of the field φ^4. Action now takes the form

$$S = \int d^D x \left[\frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 \right] \tag{5.1}$$

and the Langevin equation becomes

$$\Gamma \partial_t \varphi + (-\Delta + m^2)\, \varphi = -\lambda \varphi^3 + \eta. \tag{5.2}$$

The coefficient Γ determines the rate of relaxation of the system (or the time scale). It must be taken into account in the analysis of the renormalization group properties of the Langevin equation. For convenience, we shall set $\Gamma = 1$ when we construct the perturbation theory. Equation (5.2) is amenable to all the well-known methods of the physics of nonequilibrium phenomena (see the book by Patashinskiĭ and Pokrovskiĭ[7]).

We shall solve this equation by iteration in the coupling constant λ. We begin by defining the propagator of the theory:

$$(\partial_t + m^2 - \Delta)\, G(t-t') = \delta(t-t'), \tag{5.3}$$

where $G(t-t') = 0$ for $t < t'$.

The solution of this equation is

$$G(t-t', x) \doteq \theta(t-t') \exp[-(p^2 + m^2)(t-t')]. \tag{5.4}$$

Our strategy in developing perturbation theory will be as follows. Equation (5.2) is inhomogeneous (the inhomogeneity will be represented by $J = -\lambda\varphi^3 + \eta$). The solution of (5.2) will be written in the form

$$\varphi(x, t) = \int_0^t G(t-t', x-y)\, J(y, t')\, dy\, dt'. \tag{5.5}$$

However, this is essentially an integral equation because J depends on the field φ. The zero-order term of the perturbation theory is obtained by taking $J^0 = \eta$ instead of J. We then have

$$\varphi^{(0)}(x, t) = \int_0^t G(t-t', x-y)\, \eta(t', y)\, dt'\, dy. \tag{5.6}$$

We shall represent this term graphically. The propagator G will be represented by a straight line and the random force

$\eta(x, t)$ by a "cross." Thus:

$$\varphi^{(0)}(x,t) = \longrightarrow\!\!\times\!\!\bullet \tag{5.7}$$

The next order of perturbation theory is obtained by replacing J with $J^{(1)}$ [$J^{(1)} = -\lambda(\varphi^{(0)})^3$] which can be expressed in terms of the zero-order approximation to the field φ alone. In general, we can write down the following set of equations:

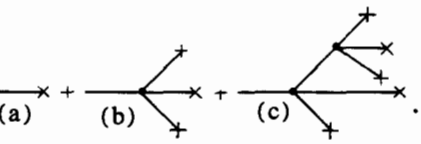
$$J^{(n)} = \eta\delta_{n,0} - \lambda \sum_{l,\, s,\, k\geqslant 0} \varphi^{(k)}\varphi^{(l)}\varphi^{(s)}\,\delta_{n-1,\, l+s+k}, \tag{5.8}$$

$$\varphi^{(n)}(x,\, t) = \int_0^t G(t-t',\, x-y)\, J^{(n)}(y,\, t')\, dy\, dt'. \tag{5.9}$$

It is readily seen that $J^{(n)}$ is expressed in terms of $\varphi^{(s)}$ alone, where $s < n$. Hence, by solving (5.9) for $n = 0$ and then substituting in (5.8) with $n = 1$, we find $J^{(1)}$. Substituting $J^{(1)}$ in (5.9) and solving the equation, we obtain $\varphi^{(1)}$. Once we know $\varphi^{(0)}$ and $\varphi^{(1)}$, we can find $J^{(2)}$, and so on. This procedure is conveniently written down in graphical form. Let

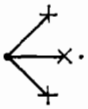
$(-\lambda) =$  (5.10)

We then readily see that the solution of (5.8) and (5.9) takes the form of the series

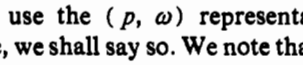
$\varphi(x,t) =$ 

(5.11)

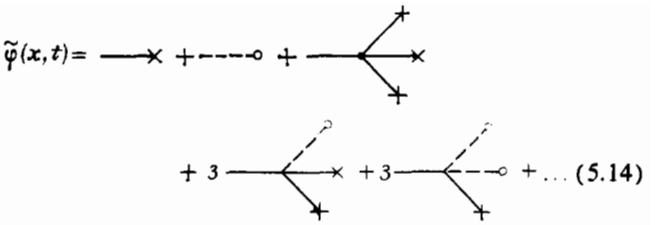
The term (b) corresponds to first-order perturbation theory, where

$J^{(1)}(x,t) =$  (5.12)

Similar graphical expressions can be written down for all the $J^{(n)}$. For convenience, all the subsequent expressions will be written in the (p, t) representation. In this representation, the propagator is given by (5.4). Occasionally, it is more convenient to use the (p, ω) representation and, whenever this is done, we shall say so. We note that, by writing the Langevin equation in the integral form given by (5.5), we have imposed an initial condition on the field $\varphi(x, t)$. In particular, it is implied that $\varphi(x, 0) = 0$. The zero-order approximation of the theory must be modified somewhat when nonzero initial conditions have to be employed. Thus, suppose that we demand that $\tilde{\varphi}(x, 0) = \Phi(x)$. Let us then add to (5.7) the solution of (5.2) with $J = 0$, i.e., the solution of the corresponding homogeneous equation, satisfying the initial conditions

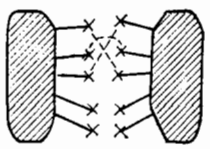
$\varphi^{(0)}(x,t) \sim$  $+ \exp[-(p^2+m^2)t]\Phi(p).$ (5.13)

The subsequent procedure used to construct the solution of (5.2) will remain exactly the same as before. The additional propagator will be denoted by a broken line, and the initial conditions by a circle. We can then write for $\tilde{\varphi}$

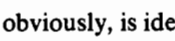
$\tilde{\varphi}(x,t) =$  (5.14)

It is readily seen that, because of the presence of the factor $\exp[-(p^2 + m^2)t]$, the initial conditions are rapidly forgotten (in a time $\lesssim 1/m^2$), and the limit as $t \to \infty$, in which we are interested, will not depend on these conditions.

Let us now examine the behavior of the two-point correlator. We shall show that it actually tends to its quantum-mechanical average. However, to begin with, let us consider averaging over the field $\eta(x, t)$. We shall express the solution of the Langevin equation in terms of $\eta(x, t)$ (in the form of a perturbation-theory series), and we shall do the same for all polynomial functionals of φ. The random current $\eta(x, t)$ is a Gaussian random variable, which means that Wick's theorem applies, i.e., we must pair up the random currents in all possible ways. This can be written graphically in the form



Since the fields $\eta(x, t)$ correlate to a δ-function, we have, in effect, a new distribution function

 (5.15)

which, obviously, is identical with

$$\langle\varphi^{(0)}(p,\ t)\,\varphi^{(0)}(-p,\, t')\rangle. \tag{5.16}$$

Let us now evaluate

$$\langle\varphi^{(0)}(p,\, t)\,\varphi^{(0)}(-p,\, t')\rangle.$$

Since

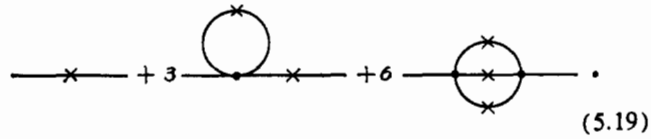
$$\langle\eta(x,\, t)\,\eta(y,\, t')\rangle \doteq 2\delta(p+q)\,\delta(t-t')\,(2\pi)^D, \tag{5.17}$$

we have

$$\int_0^{\min(t,\, t')} G(p,\, t-\tau)\,G(-p,\, t'-\tau)\cdot 2d\tau$$

$$= \frac{1}{p^2+m^2}\{\exp[-(p^2+m^2)\,|\,t-t'\,|] - \exp[-(p^2+m^2)(t+t')]\} \tag{5.18}$$

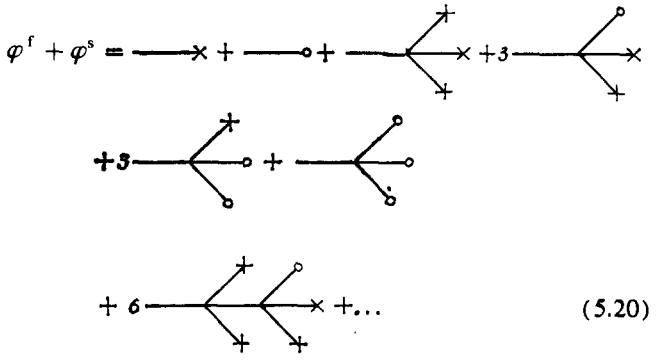
As $t = t' \to \infty$, the second term in (5.18) tends to zero and the first gives the familiar expression for the free-particle propagator. The two-point correlator of the field $\varphi(x, t)$ is conveniently represented graphically as follows:

 (5.19)

This shows the terms up to the third order in λ. The dia-

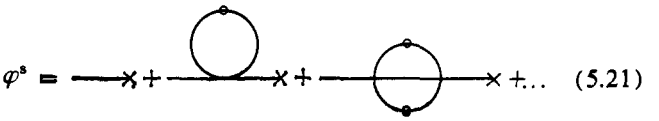
grams have the same form as the usual Feynman diagram in the φ^4 theory. However, the combinatorial coefficients in stochastic perturbation theory are different from those in the Feynman theory. The fact that, for $t = \infty$, the sum of stochastic diagrams is equal to the sum of the Feynman diagrams in each order can be proved by using nontrivial combinatorial identities.[8] This proof will not be reproduced here because the proof based on the kinetic equations and given in Appendices A and B is much simpler and physically clearer. When the series (5.19) is evaluated, we naturally encounter the problem of regularization and renormalization. Details of this can be found in the book by Patashinskiĭ and Pokrovskiĭ.[7] We shall now give a brief account of the basic ideas of renormalization.

The renormalization procedure consists of averaging over small-scale fluctuations. Suppose that the maximum momentum is Λ. All the integrations over loops in the diagrams are performed with respect to the momenta p, where $|p| < \Lambda$. We shall also assume that the fields φ, η contain only the Fourier components with momenta less than Λ. We now subdivide the random force $\eta(x, t)$ and $\varphi(x, t)$ into two parts, namely, a fast and a slow part. The fast part contains only the Fourier harmonics with momenta in the interval between $\Lambda - \delta\Lambda$ and Λ, and the slow part contains only the Fourier harmonics with $|p| < \Lambda - \delta\Lambda$. The average over the small-scale fluctuations is none other than the average of all the quantities in which we are interested, including the Langevin equation itself, over the rapidly oscillating current $\eta^f(x, t)$. We now require the effective equations for the slowly-varying part of $\varphi(x, t)$, i.e., $\varphi^s(x, t)$. The renormalizability of the theory signifies that the effective equation for $\varphi^s(x, t)$ has precisely the same form as the original Langevin equation but, in general, with different constants (Γ, λ depend on the cutoff radius Λ). We shall represent the fast oscillations in the current η^f by a "small circle" but shall use the previous notation for the slowly-varying part of the current. When both the fast and slow components are taken into account, expansion (5.14) assumes the form



(5.20)

Averaging over the rapidly-oscillating current signifies the usual mutual "closure" of the small circles. Terms including an odd number of "small circles" automatically yield zero (including $\langle\varphi^f\rangle = 0$). Let us now group these terms in a special way. We are interested in diagrams with one "cross" at the end. Their sum corresponds to the renormalized propagator



(5.21)

Relatively simple manipulation (which can be found, for example, in Ref. 7) yield the usual formulas for mass and charge renormalization. The kinetic coefficient is also renormalized and need not therefore be taken into account in scalar field theory.

This concludes our analysis of the scalar field, which we shall use mostly for illustrative purposes. We see that the diagram technique, corresponding to iterations of the Langevin equation in the coupling constant, differs from the usual diagram technique, but leads to the same result in each order. We have also verified that, in the massive case, corrections due to the finite time τ decrease exponentially.

6. STOCHASTIC QUANTIZATION OF MAXWELL'S EQUATIONS

Let us now consider a purely electromagnetic field and leave aside the problem of a consistent analysis of fermions within the framework of stochastic quantization. In the absence of fermions, electrodynamics becomes a theory without interaction:

$$S = \frac{1}{4}\int (F_{\mu\nu})^2\, d^4x, \tag{6.1}$$

where $F_{\mu\nu}$ is, as usual, $[\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Langevin equation takes the usual form: the Maxwell equation is augmented by the random force and the force of friction:

$$\partial_t A_\mu = (\Box\delta_{\mu\nu} - \partial_\mu\partial_\nu)\, A_\nu + \eta_\mu. \tag{6.2}$$

As can be seen, the random force can now be interpreted as a random vacuum current. This equation is actually two equations: one for the longitudinal and the other for the transverse components of the field A_μ. It will be convenient to transform to the Fourier components of the field A_μ. Separating out the longitudinal and transverse components, we can transform (6.2) into two equations:

$$\partial_t A_\nu^\perp = -k^2 A_\nu^\perp + \eta_\nu^\perp, \tag{6.3}$$

$$\partial t A_\nu^\parallel = \eta_\nu^\parallel. \tag{6.4}$$

This also separates out the two components of the random force:

$$\eta_\mu^\perp(k) = \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\eta_\nu(k),$$
$$\eta_\mu^\parallel(k) = \frac{k_\mu k_\nu}{k^2}\,\eta_\nu(k). \tag{6.5}$$

We note that the random forces $\eta^\perp$ and $\eta^\parallel$ may be looked upon as independent. Actually,

$$\langle\eta_\mu^\perp(k)\,\eta_\nu^\parallel(p)\rangle = \left(\delta_{\mu\alpha} - \frac{k_\mu k_\alpha}{k^2}\right)\frac{k_\nu k_\beta}{k^2}\,\langle\eta_\alpha(k)\,\eta_\beta(p)\rangle$$
$$= 2\delta(p+k)\,\delta_{\alpha\beta}\left(\delta_{\mu\alpha} - \frac{k_\mu k_\alpha}{k^2}\right)\frac{k_\nu k_\beta}{k^2} = 0, \tag{6.6}$$

whereas

$$\langle \eta_\mu^\perp(k)\,\eta_\nu^\perp(p)\rangle = 2\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\delta(p+k), \tag{6.7}$$

$$\langle \eta_\mu^\parallel(k)\,\eta_\nu^\parallel(p)\rangle = 2\,\frac{k_\mu k_\nu}{k^2}\,\delta(p+k). \tag{6.8}$$

As expected, the fields $A_\mu^\parallel$ and $A_\mu^\perp$ do not interact:

$$\langle A_\mu^\parallel A_\nu^\perp\rangle = 0.$$

Let us now evaluate the two-point correlator of the field A_μ. Solving (6.6) and (6.4), we obtain

$$A_\mu^\perp(p,\ t) = \int_0^t \exp\left[-p^2(t-\tau)\right]\eta^\perp(p,\ \tau)\,d\tau, \tag{6.9}$$

$$A_\mu^\parallel(p,\ t) = \int_0^t \eta^\parallel(p,\ \tau)\,d\tau. \tag{6.10}$$

We can now readily find $\langle A_\mu^\perp A_\nu^\perp\rangle$ and $\langle A_\mu^\parallel A_\nu^\parallel\rangle$:

$$\begin{aligned}&\langle A_\mu^\perp(p,\ t)\,A_\nu^\perp(-p,\ t')\rangle\\ &= \int_0^{\min(t,\,t')} d\tau \exp\left[-p^2(t+t'-2\tau)\right]\cdot 2\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\\ &= \frac{1}{p^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\{\exp\left[-p^2|t-t'|\right]\\ &\qquad -\exp\left[-p^2(t+t')\right]\},\end{aligned} \tag{6.11}$$

$$\langle A_\mu^\parallel(p,\ t)\,A_\nu^\parallel(-p,\ t)\rangle = \frac{2p_\mu p_\nu}{p^2}\min(t,\ t'). \tag{6.12}$$

$\langle A_\mu^\perp A_\nu^\perp\rangle$ tends to the well-known transverse propagator as $t\to\infty$. The longitudinal field correlator increases with time. This corresponds to the zero modes examined above. Fortunately, the longitudinal component of A_μ is not physically meaningful; it is not gauge invariant. All the physically interesting quantities are gauge invariant and can be expressed in terms of $A^\perp$. Hence, (6.4) can be discarded, and we can confine our attention to the transverse component of the field A_μ, which satisfies (6.3). The complete two-point correlator is

$$\langle A_\mu(p,\ t)\,A_\nu(-p,\ t)\rangle = \left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}+2t\,\frac{p_\mu p_\nu}{p^2}+\ldots \tag{6.13}$$

The dots at the end represent the term that tends to zero as $\exp(-2p^2t)$. If we now wish to evaluate some gauge-invariant quantity, we do not need to consider the dependence on the correlator of the longitudinal components (and, correspondingly, the time). This means that, for gauge-invariant quantities, we can everywhere replace $\langle A_\mu A_\nu\rangle$ with $\langle A_\mu^\perp A_\nu^\perp\rangle$.

7. THE LANGEVIN EQUATION IN NON-ABELIAN GAUGE THEORIES

We shall now consider U(N) gauge theories. The functional integral assumes the form

$$Z = \int DA\exp\left(-\frac{S(A)}{g_0^2}\right), \tag{7.1}$$

$$S(A) = -\int d^4x\,\frac{1}{4}\,\mathrm{tr}\,F_{\mu\nu}^2, \tag{7.2}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu,\ A_\nu]; \tag{7.3}$$

where the fields A_μ, $F_{\mu\nu}$ are anti-Hermitian $N\times N$ matrices.

As can be seen, the Euclidean action $S(A)$ is analogous to energy in the Gibbs distribution and the bare constant g_0^2 is analogous to temperature.

The Langevin equation can be written immediately by analogy with statistical mechanics:

$$\frac{\partial A_\mu}{\partial\tau}+\frac{\delta S}{\delta A_\mu} = J_\mu, \tag{7.4}$$

$$\langle J_\mu^{ij}(x,\ \tau)\,J_\nu^{kl}(x',\ \tau')\rangle = 2g_0^2\delta_{il}\delta_{jk}\delta(x-x')\,\delta(\tau-\tau'). \tag{7.5}$$

Here, the Gaussian random current J_μ^{ij} is an anti-Hermitian matrix like A_μ:

$$J_\mu^{ij} = -J_\mu^{ji}.$$

The variable τ will be referred to as intrinsic time in order to distinguish it from the time $t = ix_4$.

By analogy with statistical mechanics, we may suppose that the averages $\langle\Phi\rangle$, calculated with the aid of the probability distribution

$$P(A) = \exp\left(-\frac{S(A)}{g_0^2}\right)Z^{-1}, \tag{7.6}$$

$$\langle\Phi(A)\rangle = \int DA\,P(A)\,\Phi(A), \tag{7.7}$$

are identical with the averages over a long interval of intrinsic time along the trajectories of (7.4):

$$\overline{\Phi} = \lim_{T\to\infty}\frac{1}{T}\int_0^T d\tau\,\Phi(A(\tau)). \tag{7.8}$$

This averaging over intrinsic time automatically leads to averaging over random noise, i.e., fluctuations in the average are inversely proportional to $T^{1/2}$.

Thus, it is natural to suppose that

$$\langle\Phi\rangle = \overline{\Phi}. \tag{7.9}$$

It is at this point that new possibilities inherent in the Langevin equation are revealed. The point is that the usual probability distributions (7.7) are poorly defined. It is well-known that the gauge invariance of this distribution complicates the development of a perturbation theory. The classical vacuum

$$A_\mu^{cl} = S^{-1}(x)\,\partial_\mu S(x),\quad S^+(x)\,S(x) = 1 \tag{7.10}$$

depends on the arbitrary unitary matrix $S(x)$, i.e., it is degenerate.

To construct a perturbation theory, we must fix the gauge by the well-known Faddeev-Popov procedure. This procedure is defined only for small field fluctuations, and the well-known Gribov ambiguities arise for large fields.[9] All the perturbation-theory terms are correctly reproduced by the Faddeev-Popov procedure, but the nonperturbative effects are distorted.

On the other hand, the Langevin equation is nondegenerate. It is invariant only under the ordinary τ-independent gauge transformations

$$A_\mu(x,\ \tau)\to S^{-1}(x)\left[\partial_\mu + A_\mu(x,\ \tau)\right]S(x), \tag{7.11}$$

$$J_\mu(x,\ \tau)\to S^{-1}(x)\,J_\mu(x,\ \tau)\,S(x). \tag{7.12}$$

Hence, the dynamics, i.e., the dependence on τ of the solution of the Cauchy problem

$$A_\mu(x, 0) = A_\mu(x) \tag{7.13}$$

does not contain ambiguities. The initial conditions (7.13) are equivalent to fixing the gauge.

Of course, the physical gauge problem does not vanish, but the mathematical aspects become simpler. Analysis of the Langevin equation in the gauge theory shows that the vector potential fills uniformly the entire gauge orbit in the course of intrinsic time, i.e., on average, fields differing by a gauge transformation are encountered equally frequently. This occurs after a long enough intrinsic time so that the initial conditions are "forgotten." Gauge-invariant variables then tend to their averages corresponding to the Gibbs-Yang-Mills distribution.

8. FIXING THE GAUGE

As noted above, the gauge need not be fixed in the Langevin equation of the gauge theory. The terms that grow with τ mutually cancel out in each perturbation-theory order for any gauge-invariant quantity.

However, the Langevin equation can be transformed[10] so that these terms will vanish. This transformation of the Langevin equation is equivalent to fixing the gauge. It will now be described because it is convenient in practical calculations.

We note that, to evaluate the average $\langle\Phi(A)\rangle$ of a gauge-invariant functional, we can replace the current A with gauge-transformed A^M, with the time-dependent transformation matrix $M(x, \tau)$. Of course, this field satisfies a different equation because the Langevin equation is not invariant under such transformations.

It is readily verified that the transformation adds the following additional term to the equation:

$$\frac{\partial B_\mu}{\partial\tau} + [D_\mu a] + \frac{\delta S}{\delta B_\mu} = K_\mu, \tag{8.1}$$

where

$$a = M^{-1}\frac{\partial M}{\partial\tau}. \tag{8.2}$$

The quantity a is arbitrary. For example, we can take it in the form

$$a = \frac{1}{\alpha}\partial_\nu B^\nu. \tag{8.3}$$

When $\alpha = 1$ in (8.1), the terms $\partial_\mu\partial_\nu B^\nu$ cancel out in the linear approximation.

Substituting for a in (8.1) in the form given by (8.3), we have

$$\frac{\partial B_\mu}{\partial\tau} + \frac{1}{\alpha}[D_\mu\partial_\nu B^\nu] + [D_\nu F^\nu_\mu] = K_\mu. \tag{8.4}$$

It is clear that the linear part of (8.4) is nondegenerate:

$$\frac{\partial B_\mu}{\partial\tau} + \Box B_\mu + O(B^2) = K_\mu. \tag{8.5}$$

At first sight, this exhausts the problem of fixing the gauge. This viewpoint has often been advanced in the literature.[10]

There is, however, a subtle detail that is often ignored. The point is that a depends on B_μ and thus on the random force J_μ in accordance with (8.4). Hence, the matrix M, defined by (8.2),

$$M = T\exp\int_0^t a(\tau)\,d\tau, \tag{8.6}$$

will also depend on J_μ.

But this means that, in contrast to J_μ, we find that K_μ is not a random Gaussian quantity. For example, the correlator

$$\langle K_\mu(x, \tau), K_\nu(y, \tau)\rangle = \langle M^{-1}J_\mu(x, \tau)\,MM^{-1}J_\nu(y, \tau)\,M\rangle, \tag{8.7}$$

will, in general, contain additional terms due to the pairing of M and J_μ. Nevertheless, it may be shown (see Appendix B) that, on the right-hand side of (8.4), we can replace the random force K_μ with the usual random Gaussian quantity J_μ, correlated to a δ-function. In general, this leads to the new equation

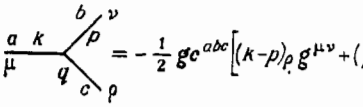
$$\frac{\partial B_\mu}{\partial\tau} + \frac{1}{\alpha}[D_\mu\partial_\nu B_\nu] + [D_\nu F_{\mu\nu}] = J_\mu. \tag{8.8}$$

However, the averages of gauge-invariant quantities, evaluated with the aid of this, will be precisely the same as for (8.4).

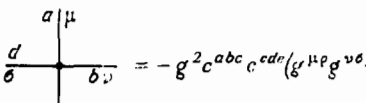
We now turn to studying the stochastic equation written in the form of (8.8). The parameter α will be considered to be arbitrary for the moment. When $\alpha = \infty$, equation (8.8) is gauge invariant. Gauge invariance is destroyed only by the initial conditions imposed on the field B_μ. If, on the other hand, $\alpha \neq \infty$, then gauge invariance is destroyed explicitly. The parameter α essentially fixes the gauge and is analogous to the α-parameter in the Feynman gauge. The perturbation theory can be constructed for (8.8) in the usual way (it is precisely analogous to the perturbation theory for φ^4). The propagation function for (8.8) has the form

$$G_{\mu\nu}(p, t, t') = \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\exp[-p^2(t-t')] + \exp\left[-\frac{p^2}{\alpha}(t-t')\right]\frac{p_\mu p_\nu}{p^2}. \tag{8.9}$$

In addition to the usual Yang-Mills vertexes



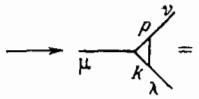
$$= -\frac{1}{2}gc^{abc}\left[(k-p)_\rho g^{\mu\nu} + (p-q)_\mu g^{\rho\nu} + (q-k)_\nu g^{\mu\rho}\right], \tag{8.10}$$



$$= -g^2 c^{abc}c^{cd\sigma}(g^{\mu\rho}g^{\nu\delta} - g^{\mu\delta}g^{\nu\rho}) + \text{permutations}, \tag{8.11}$$

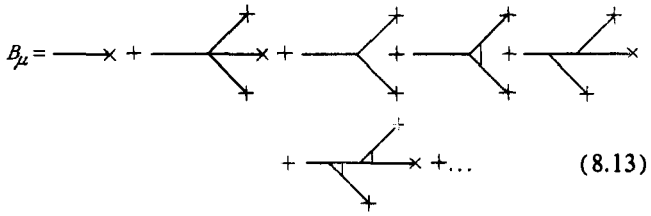
the term (8.8) leads to the additional vertex

$$\frac{1}{\alpha}D_\mu\partial_\nu B^\nu = \frac{1}{\alpha}(\partial_\mu\partial_\nu B^\nu - ig[B_\mu\partial^\nu B_\nu])$$



$$= \frac{1}{\alpha}\frac{g}{2}(\delta_{\mu\nu}k_\lambda - \delta_{\mu\lambda}p_\nu)c^{abc}. \tag{8.12}$$

The perturbation-theory series for the field B_μ is



(8.13)

Evaluation of the averages gives rise to a new propagation function [cf. (5.15)]:

$$\langle B_\mu(x,\ t)\,B_\nu(y,\ t')\rangle \doteq \int_0^t \int_0^{t'} G_{\mu\alpha}(t-\tau,\ p)\,G_{\nu\beta}(t'-\tau,\ q) \times \langle h_\alpha(p,\ \tau)\,h_\beta(q,\ \tau')\rangle\,dq\,d\tau\,d\tau'$$
$$= P^{\perp}_{\mu\nu}\frac{1}{p^2}\left\{\exp\left[-p^2\,|t-t'|\right]-\exp\left[-p^2\,(t+t')\right]\right\} + P^{\parallel}_{\mu\nu}\frac{\alpha}{p^2}\left\{\exp\left[-\frac{p^2}{\alpha}\,|t-t'|\right]-\exp\left[-\frac{p^2}{\alpha}\,(t+t')\right]\right\}. \tag{8.14}$$

The projectors onto the transverse and longitudinal components are denoted by $P^{\perp}_{\mu\varphi}$ and $P^{\parallel}_{\mu\nu}$, respectively. When $\alpha\to\infty$, the expression given by (8.14) yields

$$\langle B_\mu(x,\ t)\,B_\nu(y,\ t')\rangle \doteq P^{\perp}_{\mu\nu}\frac{1}{p^2}\left\{\exp\left[-p^2\,|t-t'|\right]-\exp\left[-p^2\,(t+t')\right]\right\} + P^{\parallel}_{\mu\nu}\cdot 2\min(t,\ t'). \tag{8.15}$$

For $\alpha = 1$, we have the simplest form, namely,

$$\langle B_\mu(x,\ t)\,B_\nu(y,\ t')\rangle \doteq \delta_{\mu\nu}\frac{1}{p^2}\left\{\exp\left[-p^2\,|t-t'|\right]-\exp\left[-p^2\,(t+t')\right]\right\}. \tag{8.16}$$

The dependence on α disappears when the gauge-invariant quantities are evaluated. Ghost-type fields need not then be introduced. It is convenient to set $\alpha = 1$ in numerical calculations

The problem of regularization arises in the stochastic approach, as it does in ordinary field theory. A frequency cutoff equivalent to the momentum cutoff in φ^4 theory is inappropriate in gauge theories. As usual, it is reasonable to employ a gauge-invariant regularization, i.e., dimensional regularization.

We shall now reproduce the results of calculations of the Wilson average (to within the fourth order in g). The Wilson average is defined as the P-exponential of the vector potential along a closed contour C (see Ref. 11):

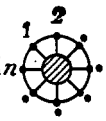
$$W(C) = \left\langle \operatorname{tr} P\exp\oint_C igA_\mu\,dx^\mu\right\rangle = \lim\left\langle \operatorname{tr}\prod_\mu\left(1+igA_\mu(x)\,\Delta x^\mu\right)\right\rangle. \tag{8.17}$$

The renormalizability of the Wilson average was demonstrated by Dotsenko and Vergeles[12] for the usual Yang-Mills theory. In our case, the Wilson average depends on the "fictitious" time. As $t\to\infty$,

$$W(C,\ t)\to W(C). \tag{8.18}$$

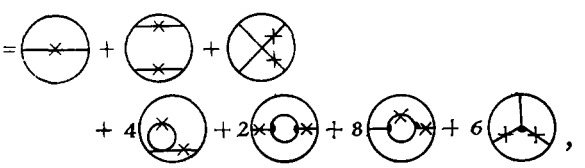
If we evaluate (8.17), we find that

$$W(C) = \sum_n (ig)^n \oint_0^{x_n} dx_n^{\mu_n}\int_0^{x_n} dx_{n-1}^{\mu_{n-1}}\cdots \times\int_0^{x_2} dx_1^{\mu_1}\langle A_{\mu_1}(x_1)\cdots A_{\mu_n}(x_n)\rangle, \tag{8.19}$$

$$W(C) = \sum_n n$$  . (8.20)

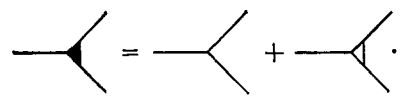
We have introduced a new notation for the nth term of the series in the sum (8.19). The theory of loop functionals of this type is investigated in detail in the review article of Ref. 13.

To calculate $W(C,\ t)$, we must substitute into (8.19) the value of A_μ expressed in terms of random currents with the aid of (8.19). The quantum-mechanical average $\langle\ldots\rangle$ is replaced by the average over the random field J_μ. The result is

$W(C,t) =$ 

(8.21)

where the effective three-tail is defined by



(8.22)

It can be shown that

$$W(C,t) = \left(1+\frac{22}{3}\,g^2\,\frac{C_2}{16\pi^2\varepsilon}\right)+O(g^4);$$

where C_2 is the eigenvalue of the Casimir operator for the group SU(N) and ε is the space dimension defect ($D = 4-\varepsilon$). The expression in parentheses is none other than the well-known expression for the renormalized charge, i.e., only the charge needs to be averaged in order to remove the divergences from $W(C,\ t)$. This means that the kinetic coefficient is not renormalized:

$$\Gamma(\varepsilon) = \Gamma(0) + O(g^2).$$

We can set it equal to unity right from the outset and then ignore it. This is precisely how it has been treated so far.

9. LARGE N MATRIX THEORIES

Let us begin by considering a U(N) gauge theory. When $N\to\infty$, the U(N) and SU(N) theories become equivalent. Actually, SU(N) fields and U(N) fields differ from one another by one Abelian field component:

$$\mathrm{SU}(N) = \frac{N}{(Ng^2)}\int \operatorname{tr}(F^a_{\mu\nu}T^a + \mathcal{F}_{\mu\nu})^2\,d^4x = S_{\mathrm{SU}(N)} + \frac{N^2}{Ng^2}\int(\mathcal{F}_{\mu\nu})^2\,d^4x. \tag{9.1}$$

where T^a are the SU(N) generators. We have two noninteracting theories: the SU(N) gauge theory and the free electromagnetic field. If the theory contains quarks, then photons and gluons will interact. We then have the well-known zero-charge situation, i.e, the interaction between gluons and photons is suppressed by the cutoff radius. On the other hand, the interaction between gluons and photons is also suppressed according to $1/N$.

To investigate the behavior of the U(N) gauge theory for large N, it is convenient to reformulate the diagram technique. We shall need to separate the color indices explicitly. The gluon-field propagator can be written in the form

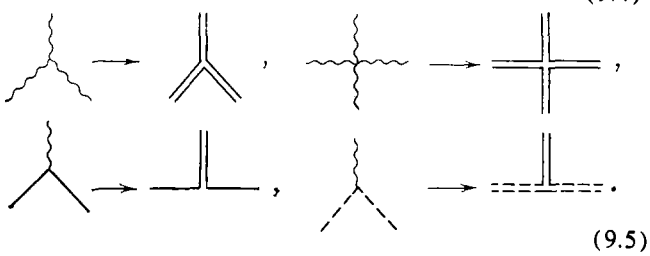
$$\langle A_\mu^{ij}(x)\, A_\nu^{kl}(y)\rangle = (G^a)^{ij}\,(G^b)^{kl}\,\langle A_{a\mu}(x)\, A_{b\nu}(y)\rangle$$
$$= (G^a)^{ij}\,(G^b)^{kl}\,\delta_{ab}G_{\mu\nu}(x-y) = \delta^{il}\delta^{jk}G_{\mu\nu}(x-y), \qquad (9.2)$$

where G^a are the U(N) generators. We have used the result $\Sigma_a (G^a)^{ij}\times(G^b)^{kl} = \delta^{il}\delta^{jk}$. We note that such a simple relationship is not available for SU(N) [for this group, we have $\Sigma_a (T^a)^{ij}(T^a)^{kl} = \delta^{il}\delta^{jk} - (1/N)\delta^{ij}\delta^{kl}$]. It is now clear that the gluon-field propagator can be represented graphically by two lines, with the color index propagating along each:

$$\left\langle A_\mu^{ij}(x)\, A_\nu^{kl}(y)\right\rangle \sim \mu \overset{i}{\underset{j}{\rightrightarrows}} \overset{l}{\underset{k}{\rightrightarrows}} \nu. \qquad (9.3)$$

Similarly, we can redefine the quark and ghost propagators and vertices:

$$\left\langle \bar\Psi^i(x)\Psi^j(y)\right\rangle \sim \overset{i}{\longrightarrow}\overset{j}{}, \quad \left\langle C^{ij}(x)C^{kl}(y)\right\rangle \sim \overset{i}{\underset{j}{\rightrightarrows}} \overset{l}{\underset{k}{\rightrightarrows}}, \qquad (9.4)$$



(9.5)

This reformulation of the Feynman diagrams was proposed in the pioneering work by 't Hooft.[14] In this notation, we can readily estimate the order of any Feynman diagram in N. It is none other than the number of index cycles in the diagram. For a singlet quantity, normalized to a constant for $N\to\infty$, e.g., for the Wilson average, the Feynman diagram is proportional to

$$N^{2-2H-L}\,(g^2N)^k, \qquad (9.6)$$

where $k = \frac{1}{2}V_3 + V_4$, V_3 is the number of three-tails, V_4 is the number of four-tails, H is the number of handles on the surface spanning the diagram, and L is the number of holes in this surface. The holes are related to quark loops.

Thus, only plane (without handles) diagrams without internal quark loops remain when $N=\infty$. The effective coupling constant for $N\to\infty$ is $\lambda = Ng^2$ and not g^2, i.e., as $N\to\infty$, the charge g should tend to zero so that the product Ng^2 remains finite.

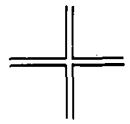
Precisely the same considerations can be used in the theory of the Hermitian matrix field with action

$$S[\Phi] = \mathrm{tr}\int \mathrm{d}^4x\left(\frac{1}{2}(\partial_\mu\Phi)^2 + \frac{m^2}{2}\Phi^2 + \frac{\tilde g}{4}\Phi^4\right); \qquad (9.7)$$

where Φ is assumed to be an $N\times N$ Hermitian matrix. This theory will be useful later as a convenient example illustrating the idea of reduction. In the matrix model, the propagator is represented by the two lines [see (9.3)]

$$\overset{i}{\underset{k}{\rightrightarrows}}\overset{j}{\underset{l}{\rightrightarrows}} \sim \delta^{ij}\delta^{kl}, \qquad (9.8)$$

and the vertex by [see (9.5)]



(9.9)

In the gauge theory, the diagram is proportional to

$$\sim N^{2-2H-L}(g^2N)^{(1/2)V_3+V_4}. \qquad (9.10)$$

The formula for the order of a diagram in the theory of the Hermitian matrix field is obtained from (9.6) by simply setting $L=0$ and $V_3=0$. Hence, the diagram is proportional to

$$\sim N^{2-2H}\,(\tilde gN)^{V_4}. \qquad (9.11)$$

Again, as in the theory of gauge fields, we see that the principal contribution is provided by planar graphs, and the effective coupling constant is

$$\lambda = \tilde gN. \qquad (9.12)$$

We shall show in the next section how the methods of stochastic quantization can be used to obtain reduced models for $N\to\infty$, which reproduce the diagrams of the orginal (not reduced) theories for $N\to\infty$.

10. REDUCED HERMITIAN FIELD

We shall now illustrate the method of reduction by taking a Hermitian field as an example. The action is then given by (9.7), and the Langevin equation can be written in the form

$$\partial_t\Phi_{ij} = (\Box - m^2)\,\Phi_{ij} - \frac{\lambda}{N}(\Phi^3)_{ij} + \eta_{ij}, \qquad (10.1)$$

where η_{ij} is a random Gaussian matrix field:

$$\langle\eta_{ij}(x,t)\,\eta_{kl}(y,t')\rangle = 2\delta_{ij}\delta_{ik}\delta(x-y)\,\delta(t-t'). \qquad (10.2)$$

The theory of the Hermitian matrix field is invariant under global U(N) transformations and a change in the sign of the field Φ:

$$\Phi\to U\Phi U^+, \quad \Phi\to -\Phi. \qquad (10.3)$$

We shall consider the traces of the $2k$-point correlators [the $(2k+1)$-point correlators are obviously equal to zero]

$$\langle \mathrm{tr}\Phi(x_1,t)\ldots\Phi(x_{2k},t)\rangle. \qquad (10.4)$$

The perturbation theory developed in the last few sections is suitable for (10.1). Substituting the expression for the field $(\Phi)_{ij}$ in (10.4) in the form of a series, we see from the perturbation theory that the correlators of the trace of the product of random matrices (10.4) is expressed exclusively in

terms of the correlators of the traces of the product of random matrices η_{ij}. The matrix field $\eta_{ij}(x, t)$ constitutes white noise. The basic idea of reduction is that we do not necessarily use white noise to evaluate the correlator

$$\langle \operatorname{tr} \eta(x_1, t_1) \ldots \eta(x_n, t_n) \rangle \tag{10.5}$$

for $N \to \infty$, but can use some other specially constructed random quantity. For example, the same white-noise effect can be achieved by introducing for each component η_{ij} the single harmonic

$$\eta_{ij}(x, t) = \bar{\eta}_{ij}(t) \exp[i(p_i - p_j)x], \tag{10.6}$$

where $\bar{\eta}_{ij}$ is a random Gaussian quantity correlated to a δ-function, and the p_i (the so-called frozen momenta) are random vectors, distributed uniformly in momentum space. For the ultraviolet cutoff, the range of the momenta can be confined to the Brillouin zone, i.e., a cube of side $2\pi\Lambda$:

$$dW(p) = \prod_i \frac{d^D p_i}{(2\pi\Lambda)^D} \prod_\mu \theta(\pi\Lambda - |p_i^\mu|). \tag{10.7}$$

It is readily verified that, if we average (10.5) over the random force $\eta_{ij}(t)$, and then average over momenta with the weight $dW(p)$, we obtain the same result for $N \to \infty$ as would be obtained by averaging $\eta_{ij}(x, t)$ over white noise. This is proved in detail in Appendix D. In particular,

$$\begin{aligned}\frac{1}{N^2}\langle \operatorname{tr} \eta(x, t)\, \eta(y, t')\rangle &= 2\delta(t-t')\,\delta(x-y) - \frac{2}{N}\delta(t-t')\,\delta(x-y) \\ &+ \frac{2}{N}\delta(t-t')\frac{\Lambda^D}{(2\pi)^D} = 2\delta(t-t')\,\delta(x-y) + O\left(\frac{1}{N}\right).\end{aligned} \tag{10.8}$$

As $N \to \infty$, the first term is the only dominant one, and gives the correct answer. Similarly, it may be shown (see Appendix D) that the leading order in $1/N$ is obtained by evaluating $\langle \operatorname{tr} \eta(x_1, t_1) \ldots (x_n, t_n)\rangle$ with the aid of (10.6), and gives the correct answer.

The random current (10.6) enables us to dispose of the coordinate in an explicit manner in (10.1). Let us substitute the following ansatz for the field in (10.1):

$$\Phi_{ij}(x, t) = \exp[i(p_i - p_j)x]\,\bar{\Phi}_{ij}(t). \tag{10.9}$$

It is then readily seen that $\bar{\Phi}_{ij}(t)$ satisfies the equation

$$\partial_t \bar{\Phi}_{ij} = -[(p_i - p_j)^2 + m^2]\,\bar{\Phi}_{ij} - \frac{\lambda}{4}(\bar{\Phi}^3)_{ij} + \eta_{ij}(\Lambda)^{D/2}. \tag{10.10}$$

It may seem surprising that the Fourier harmonics do not multiply in the nonlinear equation since, usually, nonlinearity leads to the interaction between different harmonics. This "paradox" can be explained by the special form of the equation and the ansatz (10.9). The ansatz can be written as the unitary transformation

$$\Phi(x) = e^{iPx}\Phi e^{-iPx}, \quad P = \operatorname{diag}(P_1, \ldots, P_N). \tag{10.11}$$

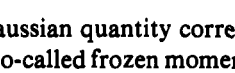
The factors $\exp(iPx)$ cancel out, so that the harmonics do not, in fact, interact.

Formula (10.11) has a clear physical meaning. Instead of the momentum operator P in Hilbert space, we use the diagonal matrix P in the space of internal symmetry of the system. In mathematical language, we represent a translation algebra in the Cartan subalgebra of the group $U(\infty)$. Of course, this is valid only in the limit as $N \to \infty$.

However, let us return to the reduced Langevin equation (10.10). The propagation function for this equation is

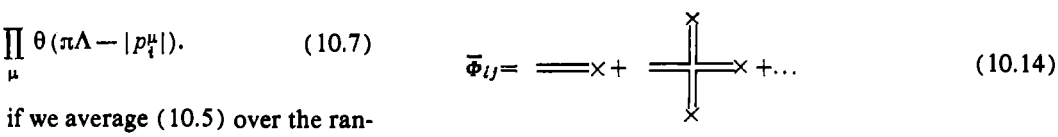
$$\begin{aligned}&G_{jl}^{ik}(p_i - p_j;\ t - t') \\ &= \delta_{ik}\delta_{jl} \exp[-[(p_i - p_j)^2 + m^2](t - t')]\,\theta(t - t').\end{aligned} \tag{10.12}$$

This can be represented graphically by the two lines



(10.13)

As usual, the solution of (10.10) is given by the sum of a series of tree diagrams



(10.14)

The two-point correlator of the fields $\bar{\Phi}_{ij}(t)$ is

$$\begin{aligned}\langle \bar{\Phi}_{ij}(t)\,\bar{\Phi}_{ik}(t')\rangle = \ = \times \ = \ &= \delta_{ik}\delta_{jl}(\Lambda)^D \frac{1}{(p_i - p_j)^2 + m^2} \\ &\times [\exp\{-[p_i - p_j]^2(t - t')\} \\ &- \exp\{-[p_i - p_j]^2(t + t')\}].\end{aligned} \tag{10.15}$$

When $t = t' \to \infty$, the expression given by (10.15) is proportional to the usual quantum-mechanical propagator. They differ only by a factor. The original average (6.4) can be expressed with the aid of the relation

$$\begin{aligned}&\langle \operatorname{tr} \varphi_{ij}(x_1, t) \ldots \varphi_{kl}(x_n, t)\rangle \\ &= \int \prod_{i,j} \frac{dP}{(2\pi\Lambda)^D} \operatorname{tr} \exp[i(p_i - p_j)x_1] \\ &\ldots \exp[i(p_i - p_j)x_n]\,\langle \varphi_{ij}(t_1) \ldots \varphi_{kl}(t_n)\rangle.\end{aligned} \tag{10.16}$$

Integration with respect to the momenta p_i in (10.16) will now be denoted by $\langle \ldots \rangle_P$. As $t \to \infty$, the expression $\langle \varphi_{ij}(t) \ldots \varphi_{kl}(t)\rangle$ tends to the average of a theory whose action is readily guessed:

$$S_{\text{red}}[\Phi_E, P] = \sum_{ij} \frac{1}{2}[(p_i - p_j)^2 + m^2]\,\Phi_{ij}\Phi_{ji} + \frac{\lambda}{4N} \operatorname{tr} \Phi^4. \tag{10.17}$$

This is, in fact, the action of the reduced model. We can now formulate the recipe for evaluating the averages

$$\langle \varphi(x_1) \ldots \varphi(x_n)\rangle \tag{10.18}$$

without using the stochastic equation (10.10). We fix the set of momenta p_i (this is called the freezing of momenta). Next, we consider the theory with the action given by (10.17). Free energy in the reduced theory is obtained as

$$F_{\text{red}}(J) = -\ln Z_{\text{red}}(J), \tag{10.19}$$

where J is the external matrix current and

$$Z_{\text{red}} = \Big[\int \prod_{i,j} d\Phi_{ij} \exp\left(-S_{\text{red}} + \operatorname{tr} J\varphi\right)\Big] \times \Big[\int \prod_{i,j} d\Phi_{ij} \exp\left(-S_{\text{red}}\right)\Big]^{-1}. \quad (10.20)$$

The average (10.16) is then written in the form

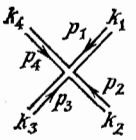
$$\langle \operatorname{tr} \varphi_{ij}(x_1, t) \ldots \varphi_{kl}(x_n, t)\rangle = \Big\langle \operatorname{tr} \exp\left[i(p_i - p_j)x_1\right] \ldots \exp\left[i(p_k - p_l)x_n\right] \frac{\delta^4 F_{\text{red}}}{\delta J_{ij} \ldots \delta J_{kl}} \Big\rangle_p. \quad (10.21)$$

In actual fact, when $N \to \infty$, we need not explicitly integrate with respect to the momenta. Since the momenta p_i are uniformly distributed in a cube of side $2\pi\Lambda$, we can initially impose a distribution on them, and then freeze it, i.e., this essentially means that we are evaluating the integral by the Monte Carlo method.

Let us now examine the connection between the usual Feynman diagrams and the diagrams of the reduced model (10.17). The reduced-model propagator is identical with the limit of (10.15) as $t \to \infty$

$$\overset{i}{\underset{j}{\rightrightarrows}}\overset{i}{\underset{j}{\rightrightarrows}} = \frac{1}{(p_i - p_j)^2 + m^2} (\Lambda)^D. \quad (10.22)$$

The "double line" (propagator) carries the momentum $k_{ij} = p_i - p_j$. In the vertex



$k_1 = p_2 - p_1$,
$k_2 = p_3 - p_2$,
$k_3 = p_3 - p_4$,
$k_4 = p_4 - p_1$

the momentum is then automatically conserved:

$$k_1 + k_2 = k_3 + k_4. \quad (10.23)$$

Suppose the diagram contains k lines. When $k \ll N$, we may suppose that all the momenta are uncorrelated, i.e., averaging over the random momenta $\langle \ldots \rangle_p$ in the reduced model is none other than integration over the internal momenta of the diagram. When $k \sim N$ or $K > N$, we clearly obtain an incorrect result because the momenta carried by the lines are highly correlated, and we cannot assume that integration over them takes place independently. For finite N, the left-hand side of (10.21) reproduces correctly the diagrams in which the number of internal lines is much less than N. Since we are interested in the case $N \to \infty$, the expression given by (10.21) remains valid. We note that the additional factor $(\Lambda)^D$ in (10.22) is required in order to achieve the correct normalization of the integral with respect to p.

11. REDUCED GAUGE FIELDS

A reduced gauge theory can be constructed in an analogous manner. We start with the Langevin equation in the form given by (7.4). We then substitute [see (10.9)]

$$\hat{A}_\mu(x, t) \to e^{i\hat{p}x} \hat{A}_\mu(t) e^{-i\hat{p}x}, \quad \hat{J}_\mu(x, t) \to e^{i\hat{p}x} \hat{J}_\mu(t) e^{-i\hat{p}x}, \quad (11.1)$$

where we have introduced the diagonal momentum matrix

$$(\hat{p}_\mu)_{ij} = p_\mu^i \delta_{ij}. \quad (11.2)$$

With this change of variables, the covariant derivative is transformed into a commutator:

$$D_\mu = \partial_\mu + A_\mu \to \tilde{D}_\mu = ip_\mu + A_\mu, \quad (11.3)$$

$$F_{\mu\nu} = [D_\mu D_\nu] \to \tilde{F}_{\mu\nu} = [ip_\mu + A_\mu,\ ip_\nu + A_\nu], \quad (11.4)$$

$$D_\mu F_{\mu\nu} \to \tilde{D}_\mu \tilde{F}_{\mu\nu}. \quad (11.5)$$

We can now formulate the stochastic equation for the field $A_\mu(t)$:

$$\partial_\tau A_\mu(t) = \tilde{D}_\nu \tilde{F}_{\mu\nu} + J_\mu(t). \quad (11.6)$$

In the reduced theory, action is uniquely given by

$$S_{\text{red}}[p_\mu A_\mu] = \left(\frac{1}{\Lambda}\right)^D \cdot \frac{1}{4} \operatorname{tr}_a |F_{\mu\nu}|^2. \quad (11.7)$$

Equation (11.6) is invariant under the reduced gauge transformation

$$A_\mu(t) \to S(A_\mu + ip_\mu) S^+ - ip_\mu = SA_\mu S^+ + S[ip_\mu S^+] \quad (11.8)$$

with arbitrary but t-independent unitary matrix S.

Let us now examine in greater detail the gauge stochastic equation (11.6). It can be verified (see, for example, Ref. 4) that direct iteration of this equation in the coupling constant gives, in the limit as $N \to \infty$, the same perturbation-theory terms as the usual Langevin equation (7.4) in the case of the gauge-invariant traces, for example, the Wilson average $W(C)$. In the limit as $t \to \infty$, this procedure reproduces the results of the Faddeev-Popov diagram technique.

We note, however, that the limits $t \to \infty$ and $N \to \infty$ do not commute (this question is examined in detail by Alfaro[8]). Actually, if these limits were to commute, the dependence on the frozen momenta p_μ would vanish altogether because these momenta can be transformed by the shift $A_\mu = B_\mu - ip$ from (11.6) to the initial conditions $B_\mu(0) = ip_\mu$.

The phenomena occurring in the reduced models for $N \to \infty$ are similar to those occurring in quantum mechanics as we pass from matrices of finite size to operators in Hilbert space [here, the Hilbert space is inscribed into the SU(∞) manifold, so that the operator $\hat{P}$ is represented by a diagonal matrix].

We recall that, for example, for matrices of finite size, the commutator $[pq]$ is a traceless matrix, whereas, for the coordinate and momentum operators, this commutator is proportional to the unit operator.

We shall now show more specifically how the limit $\lim_{t\to\infty} \lim_{N\to\infty}$ is reached, and why the frozen momenta do not vanish.

For fixed N, t, the hazardous terms that do not decrease with t are due to the zero modes discussed above. Let us separate out the zero modes by introducing the change of variables

$$A_\mu(t) = S(t) B_\mu(t) S^+(t) + S(t)[ip_\mu, S^+(t)], \quad \eta(t) \to S(t)\eta(t)S^+(t): S(0) = 1. \quad (11.9)$$

It is readily verified that the equation for B_μ now has an additional term:

$$\frac{\partial B_\mu}{\partial t} - [D_\mu \alpha] + D_\nu F_{\mu\nu} = \eta, \tag{11.10}$$

where α is related to S by

$$\frac{\partial S}{\partial t} = S\alpha. \tag{11.11}$$

We choose

$$\alpha = [ip_\mu + A^c_\mu,\ A_\mu - A^c_\mu], \tag{11.12}$$

where A^c_μ is the solution of the classical equations, and we construct the quasiclassical expansion in the neighborhood of this solution (in the simplest case, $A^c_\mu = 0$).

As in the case of the Langevin field equation, the linear part of the above equation is then nondegenerate because the zero modes vanish.

Having taken the initial conditions in the form

$$A_\mu(0) = A^c_\mu \tag{11.12'}$$

and integrating with respect to the constant g, we have disposed of the growing terms. Clearly, they are all collected in the matrix

$$S = T \exp \int_0^t \alpha\, dt. \tag{11.13}$$

This matrix cancels out in gauge-invariant quantities, which proves the existence of the limit as $t \to \infty$. On the other hand, the dependence of p_μ is now clear because these quantities cannot be removed from the equation by a transformation.

A detailed account of how $\langle \mathrm{tr}\, F\mu^2\nu \rangle$ can be evaluated by this method can be found in Alfaro's paper.[8] Without going into technical details, the essential point is that the contributions of zero modes that grow with t contain the correction terms t/N, which do not cancel out in observed quantities for finite N, but do vanish in the limit as $N \to \infty$ at constant t.

Thus, if we take the correct limit (first, $N \to \infty$, and then $t \to \infty$), the Langevin equation can be written in the gauge theory in a very simple form:

$$\frac{\partial B_\mu}{\partial t} + [B_\nu [B_\nu B_\mu]] = J_\mu, \tag{11.14}$$

$$B_\mu(0) = ip_\mu + A^c_\mu. \tag{11.15}$$

In this form of the equation, the frozen momenta are introduced through the initial conditions and influence the solution for $t \to \infty$ because of the zero modes.

This limiting procedure will be less surprising if we recall that, for finite N, the reduced theory contains a finite number of freedom ($N \times N$ matrix independent of x) and cannot therefore be a serious contender for a quantum-field theory.

We note that more complicated reduced gauge equations have been examined in the literature[4,8,18] by introducing some particular limitations on the matrix A_μ in order to prevent the possibility of the removal of momenta by the shift $A_\mu \to A_\mu - iP_\mu$. We shall not reproduce these equations because, when the limits are correctly taken, these limitations can be avoided.

12. NUMERICAL METHODS IN THE CONTINUOUS CASE

Most calculations involving reduced models have been performed within the framework of lattice regularization. However, before we proceed with these calculations, we must say a few words about the alternative method which, so far, has not been implemented, but can, in principle, be carried out.

The procedure we have in mind here is the Gaussian regularization of the reduced Langevin equation. The point is that, for fixed frozen momenta p_i and fixed N, the ultraviolet divergences do not in general arise, since there is no integration with respect to momenta in the Feynman diagrams. These integrations effectively arise as a result of the summation of color indices, as discussed above in the example of the scalar fields. The typical sum is:

$$G(x-y) = \frac{1}{N^2} \sum_{a,b} \exp[i(x-y)(p_a - p_b)] \times \frac{1 - \exp[-2(p_a-p_b)^2 t]}{(p_a-p_b)^2}$$
$$= \int \rho(q)\, dq \exp[iq(x-y)] \frac{1-\exp(-2q^2t)}{q^2} \to \frac{2t}{N} + \int \frac{dq}{(2\pi)^D} \frac{\exp[iq(x-y)]}{q^2}; \tag{12.1}$$

where

$$\rho(q) = \frac{1}{N^2} \sum \delta(q - p_a + p_b)$$
$$= \frac{1}{N}\delta(q) + \frac{1}{N^2} \sum_{a,b}{}' \delta(q - p_a + p_b). \tag{12.2}$$

The term $(1/N)\delta(q)$ provides a contribution $\sim t/N$ which vanishes in the limit as $N \to \infty$. The second term in (12.2) has a finite limit independent of q if p_a and p_b are uniformly distributed in an infinite volume.

If we cut off the distribution over the momenta p_a by placing them inside a unit cube or unit sphere, the density $\rho(q)$ will decrease for large q, so that there will be no propagator singularities for $x \to y$. Obviously, this also applies to the other ultraviolet divergences.

They all arise because of the sums over the color indices, and vanish if the probability distribution for the momenta p_a is cut off, so that, on average, the sums converge.

The simplest distribution is Gaussian:

$$dW(p_a) = \prod_a \frac{d^D p_a}{(2\pi\Lambda^2)^{D/2}} \exp\left(-\sum \frac{p_a^2}{2\Lambda^2}\right). \tag{12.3}$$

In this case, ρ is also Gaussian:

$$\rho(q) \xrightarrow[N\to\infty]{} \int \frac{d^D p_1 d^D p_2}{(2\pi\Lambda^2)^D} \exp\left(-\frac{p_1^2}{2\Lambda^2} - \frac{p_2^2}{2\Lambda^2}\right) \delta(q - p_1 + p_2)$$
$$= \int \frac{dP}{(2\pi\Lambda^2)^D} \exp\left(-\frac{2p^2}{2\Lambda^2}\right) \exp\left(-\frac{q^2}{4\Lambda^2}\right)$$
$$= (2\Lambda\pi^{1/2})^{-D} \exp\left(-\frac{q^2}{4\Lambda^2}\right). \tag{12.4}$$

The numerical realization of the Gaussian quantities $p_{a\mu}$ is

exceedingly simple. The four components p_1, p_2, p_3, p_4 must be divided into pairs:

$$p_{a1} + ip_{a2} = \Lambda\,(-2\ln\xi)^{1/2}\exp(2\pi i\eta'),$$
$$p_{a3} + ip_{a4} = \Lambda\,(-2\ln\xi')^{1/2}\exp(2\pi i\eta'), \tag{12.5}$$

where ξ, ξ', η, η' are random numbers that are uniformly distributed between 0 and 1.

Having specified $p_{a\mu}$, we can solve numerically the Langevin equation, e.g., in the nonsingular form (11.10). The precision can be improved by using higher-order difference schemes (see Appendix C). The main problem arising in this procedure is related to the frozen momenta. As mentioned above, all the normalized matrix traces are independent of the realizations of the random quantities $p_{a\mu}$ in the limit as $N = \infty$. We have here the so-called self-averaging, well-known in solid state physics (where the part of frozen variables is played by the coordinates and spins of impurity atoms).

However, a large number of degrees of freedom is necessary for self-averaging to occur with sufficient precision. For finite N, the corrections due to fluctuations in averages such as (12.1) are of the order of $1/N$, i.e., of the same order as the corrections due to diagonal terms. To take these corrections down to, say, the 1% level, we must take $N \sim 100$. The corresponding matrices will have 10^4 elements, which is just about the limit of the capability of modern computers. Relatively few results (see Ref. 15) have therefore been obtained by the method of frozen momenta.

13. REDUCED LATTICE GAUGE THEORIES

An interesting alternative to frozen momenta has been found in lattice gauge theories, for which the corrections decrease as N^{-2} and not N^{-1}. This is the so-called TEK model (Twisted Eguchi-Kawai model; see Ref. 17).

We shall start with the usual Eguchi-Kawai model,[16] describing the high-temperature phase of lattice gauge theories. Like TEK, this model was obtained with the aid of loop equations (see the review paper of Ref. 13). However, the conclusion based on the Langevin equation[8] seems physically clearer and enables us to advance further by including the terms discarded in the Langevin equation and using the stochastic perturbation theory (see Ref. 25 and below).

First, we must derive the Langevin equation in the lattice gauge theory. It is shown in Appendix A (see also Ref. 4) that, in curved space of the fields, the Langevin equation has the form

$$e^a_\alpha(\varphi)\frac{\partial\varphi^\alpha}{\partial t} + e^{a\alpha}(\varphi)\frac{\delta S}{\delta\varphi^\alpha} = f^a, \tag{13.1}$$

where $e^a_\alpha(\varphi)$ are tetrads in the space of the fields, which have the usual properties

$$e^a_\alpha e^{b\alpha} = \delta^{ab}, \quad e^a_\alpha e^\beta_a = g_{\alpha\beta}. \tag{13.2}$$

As far as the random forces f^a are concerned, they have a Gaussian distribution because, in flat space,

$$dW(f) = Df\exp\left(-\frac{1}{4}\int dt\,d^4x\,(f^a)^2\right). \tag{13.3}$$

In the case of the $N\times N$ unitary matrices U_l, in which we are interested here, and which are assigned to the edges of the lattice,[2] the parameters φ^α can be taken to be, say, the components of the anti-Hermitian matrix $R = (U-1)(U^+-1)^{-1}$. For the moment, we shall not specify φ^α further because, in general, the equation is covariant.

In group space, the part of the tetrads e^a_α is played by the quantities

$$e^a_\alpha = \frac{1}{2t}\operatorname{tr}\left(T^aU^{-1}\frac{\partial U}{\partial\varphi^\alpha}\right), \tag{13.4}$$

where T^a are the group generators.

It is readily verified that the e^a are invariant for left shifts $U\to AU$, $A = \text{const}$, but for right shifts $U\to UB$, $B = \text{const}$, the e^a_α are subjected to the linear transformation

$$e^a_\alpha \to \Omega^{ab}e^b_\alpha, \quad \Omega^{ab} = \frac{1}{2}\operatorname{tr}(T^aB^{-1}T^bB), \tag{13.5}$$

which conserves the metric

$$g_{\alpha\beta} = -\frac{1}{2}\operatorname{tr}\left(U^{-1}\frac{\partial U}{\partial\varphi^\alpha}U^{-1}\frac{\partial U}{\partial\varphi^\beta}\right). \tag{13.6}$$

Consequently, the volume element (Haar measure) is

$$(dU) = g^{1/2}\,d\varphi = (\det e)\,d\varphi. \tag{13.7}$$

To obtain the Langevin equation (13.1) in a convenient form, we use a locally flat coordinate system, i.e., we take U in the form

$$U = U_0(1 + i\textstyle\sum\varphi_a T^a), \tag{13.8}$$

where φ_a are infinitesimal functions of e. In this coordinate system, $e^a_\alpha = \delta^a_\alpha$ and (3.1) is trivial:

$$\frac{\partial\varphi_a}{\partial t} + \frac{\partial S\,|U(1+i\sum\varphi_aT^a)|}{\partial\varphi_a} = f^a. \tag{13.9}$$

All that remains is to express $\partial U/\partial t$ in terms of $\partial\varphi_a/\partial t$, and then substitute $\varphi = 0$:

$$\frac{\partial U}{\partial t} = iU\sum_a T^a\left(f^a - \frac{\partial S\,|U(1+i\sum\varphi_aT^a)|}{\partial\varphi_a}\right)_{\varphi=0}. \tag{13.10}$$

This equation could have been guessed as the only equation that is invariant (covariant) under the left (right) shift, and transforms into the usual equation as $U\to 1$.

It is convenient to look upon the action S as a function of the independent variables U, U^+. Using the completeness condition

$$\sum_a (T^a)^{ij}(T^a)^{kl} = \delta^{il}\delta^{jk}, \tag{13.11}$$

we can then transform (13.10) to the matrix form

$$\frac{\partial U_l}{\partial t} = U_l(\mathcal{F}^{(l)} + J^{(l)} - J^{(l)+}), \tag{13.12}$$

where

$$\mathcal{F} = i\sum_a T^af^a$$

is an anti-Hermitian random Gaussian matrix and

$$J^{(l)}_{ik} = \frac{\partial S}{\partial U^{(l)}_{ri}}U^{(l)}_{rk}. \tag{13.13}$$

The equation we have obtained is valid in all theories

with U(N) matrices, namely, chiral, lattice gauge, and so on, theories.

In lattice gauge theories, the action is

$$S = -\beta N \sum_{\square} \operatorname{tr}(U_1 U_2 U_3 U_4 + \text{H.c.}), \tag{13.14}$$

where U_1, U_2, U_3, U_4 correspond to the edges of a plaquette and the sum is evaluated over all the plaquettes on the lattice. This expression is bilinear in U, U^+. The current J is readily found:

$$J^{(1)} = -\beta N \sum_{\square} U_2 U_3 U_4 U_1. \tag{13.15}$$

It is implied here that the variable U_1 is varied. The sum is evaluated over all plaquettes of edge 1 [the number of such plaquettes is $2(D-1)$, where D is the dimension of the space].

It is readily verified that the current J is gauge-invariant, i.e., it transforms like

$$J^{(1)} \to S_1^{-1} J^{(1)} S_1, \tag{13.16}$$

where the matrices U are subjected to the gauge transformations

$$U_k \to S_{k-1}^{-1} U_k S_k \quad (k = 1, 2, 3, 4), \quad S_0 = S_4. \tag{13.17}$$

We must now apply the reduction procedure to the matrix Langevin equation. It will be convenient to write U_l and $U_\mu(x)$, where x corresponds to a finite point on an edge and $\mu = \pm 1, \pm 2, \ldots, \pm D$ describes its direction. Similarly, $\mathscr{F}^{(1)}$, $J^{(1)}$ will be written as $\mathscr{F}_\mu(x)$ and $J_\mu(x)$.

Exactly as above, the random white noise $\mathscr{F}$ can be simulated with the aid of the frozen momenta p_μ:

$$\mathscr{F}_\mu(x) = e^{-iPx} f_\mu e^{iPx}. \tag{13.18}$$

All correlators of the form

$$N^{-2} \langle \operatorname{tr} \mathscr{F}(x_1) \ldots \mathscr{F}(x_n) \rangle \tag{13.19}$$

will be correctly reproduced as $N \to \infty$. But, on the other hand, (13.18) is a gauge transformation. Hence, the frozen momenta can be removed by introducing corresponding transformation of the fields

$$U_\mu(x) = e^{-iP(x-\hat{\mu})} W_\mu e^{iPx}; \tag{13.20}$$

where $x - \hat{\mu}$ is the origin of an edge and x its end. The equation for W_μ is independent of the frozen momenta:

$$\frac{\partial W_\mu}{\partial t} = W_\mu (f_\mu + J_\mu - J_\mu^+), \tag{13.21}$$

$$J_\mu = -\beta N \sum_\nu (U_\nu U_\mu^+ U_\nu^+ U_\mu + \text{H.c.}). \tag{13.22}$$

As above, it is, in principle, possible to introduce frozen momenta through the initial conditions at $t = 0$. However, it was noted by Okawa and Gonzalves-Arroyo[17] that there was another method that was much more effective and attractive.

In particular, we note that, prior to reduction, the following field transformation could be made in the lattice theory:

$$U_l \to Z_l U_l, \tag{13.23}$$

where Z_l are the elements of the center $\mathbb{Z}_N$ of the groups SU(N), i.e.,

$$Z_l = \exp\left(\frac{2\pi i n_l}{N}\right), \quad n_l \text{ — integer.} \tag{13.24}$$

The product of matrices over a plaquette will then also acquire a phase factor:

$$(U_1\, U_2\, U_3\, U_4) \to (U_1\, U_2\, U_3\, U_4)\, Z(\square). \tag{13.25}$$

We can therefore introduce the phase factor $Z(\square)$ into the action (13.14), without altering the essence of the theory. The Bianchi identity

$$\prod_{\text{cube}} Z(\square) = 1, \tag{13.26}$$

is a criterion for whether a phase factor $Z(\square)$ specified in advance can be removed by a field transformation, the product in (13.26) being evaluated over all the faces of the elementary cube (taking their orientation into account).

Since each face appears together with another face of oppostie orientation, we can take, for example,

$$Z_{\mu>\nu} = \exp\left(\frac{2\pi i n}{N}\right), \qquad Z_{\mu<\nu} = Z_{\mu>\nu}^* \tag{13.27}$$

and independent of the plaquette coordinate.

If we now perform the reduction, i.e., replace the matrices $U(N)$ with constants, the four matrices of a plaquette will no longer be independent because opposite edges are obtained from one another by a parallel translation and a change of orientation

$$U_3 = U_1^+, \; U_4 = U_2^+, \qquad \operatorname{tr}(U_1 U_2 U_3 U_4) = \operatorname{tr}(U_1 U_2 U_1^+ U_2^+). \tag{13.28}$$

However, this means that the phase factor $Z(\square)$ cannot be removed after reduction by a transformation of the matrices U_1, U_2. In other words, there is a whole family of reduced models with arbitrary phase factors $Z(\square)$ satisfying the Bianchi identity. The original EK model corresponds to $Z_{\mu\nu} = 1$.

Which choice of $Z_{\mu\nu}$ is correct? Careful analysis has shown[10] that the best properties are obtained by solving (13.21) with

$$n = N^{1/2} \text{ — integer}. \tag{13.29}$$

When this is so, the reduced model reproduces both the strong coupling expansion (series in powers of β) and the Faddeev-Popov perturbation theory (series in powers of β^{-1}).

Perturbation theory is reproduced in a very nontrivial manner.[18] The part of the momenta p_μ^i is played by the parameters of the degenerate vacuum of the TEK model, i.e., the solutions of its classical equations

$$U_\mu^{\text{cl}} U_\nu^{\text{cl}} = Z_{\mu\nu} U_\nu^{\text{cl}} U_\mu^{\text{cl}}. \tag{13.30}$$

It then turns out that not only is the limit of infinite volume reproduced in the limit of infinite N, but the lattice system of finite size L is reproduced for finite $N = L^2$. Details of this

beautiful construction can be found in Ref. 19.

Numerical calculations,[20] obtained for TEK with the Langevin equation (13.21), are in good agreement with, and in some respects better than, the Monte Carlo calculations[21] (see also Refs. 22 and 23).

14. STOCHASTIC PERTURBATION THEORY

The Langevin equation presents us with new ways of advancing beyond perturbation theory. In particular, the zero-order approximation can be taken to be not the classical solution but the stochastic solution corresponding to the reduced model. In the zero-order approximation, this procedure takes into account important group-compactness effects which we believe are responsible for quark confinement. As $N \to \infty$, the zero-order approximation becomes exact, so that all the corrections tend to zero. We may therefore expect that the stochastic perturbation theory will be successful for finite N, including the $N = 3$ case in which we are interested.

We shall not go into the technical details of the stochastic perturbation theory and will confine ourselves to a few words about the strategy of this approach. The idea is to divide the random force into two terms, the first of which corresponds to the reduced model. It is well-known that the sum of independent Gaussian random quantities is itself a Gaussian, whose variance is equal to the sum of the variances, so that the second term in the random force may be looked upon as an independent Gaussian quantity with variance equal to the variance difference between the exact and reduced models. All the correlators of this component of the random force are then correctly reproduced because there are no cross terms (both terms are Gaussian).

Now that the right-hand part of the Langevin equation is written as the sum of a leading term and a perturbation, it is quite clear how the perturbation theory is to be constructed. The field must be represented by a superposition of a main term and a perturbation, and the left-hand side must be expanded into a sum of terms of different order. In the leading order, we obtain the linearized equation, and the higher-order terms are expressed in terms of the Green's function of the linearized equation. The result is a generalized diagram technique.[25]

15. PROBLEMS AND PROSPECTS

Each time we manage to look at the fundamentals of the theory in a new way, we expect some interesting applications. Stochastic quantization has now been available for a few years, and applications to lattice gauge theories have been found during that time. What else can be expected of this approach?

1. First, let us compare this method with the well-known Monte Carlo technique in the Metropolis variant.[2] In principle, the two methods give the same result, namely, a random sequence of states satisfying the principle of detailed balancing, which reproduces, on average, the Feynman distribution of probabilities exp(-action) of Euclidean theory.

However, the realizations of detailed balancing are different, so that the two methods augment rather than duplicate one another. The main difference is that, in the Metropolis method, the states are not always accepted[2)]: the system may be stuck in a particular state for an arbitrarily long time, if states of low transition probability are being offered. This phenomenon is particularly hazardous in the quasiclassical region when the states can be separated by a barrier. One way of combating this is to vary the interval $\Delta\varphi$ of variation of the field φ per step. It is advantageous to choose the interval $\Delta\varphi$ large in the quantum region and small in the classical region.

On the contrary, in the method of stochastic quantization, the states are always accepted. From each state we continue in one step to a neighboring state which differs by the amount $\Delta\varphi$, which is calculated and not specified as an entry parameter. This is a great advantage, especially in the quasiclassical region. In particular, in contrast to the Metropolis method, stochastic quantization leads directly to perturbation theory. As far as instantons and other subbarrier transitions are concerned, they are difficult to detect by numerical methods in cases when they introduce small corrections. However, when they are dominant, as, for example, in the creation of chiral spinors, they can be reproduced in any method by suitably choosing the boundary conditions.

Until recently, the Langevin equation was considered to suffer from the defect that an exact difference scheme could not be devised for it: when the derivatives are replaced literally by finite differences, the precision is low because the random force is discontinuous. However, this difficulty can be overcome. Appendix C (see also Refs. 26 and 28) presents difference schemes for the Langevin equations.

Thus, as in the Metropolis method, it is possible to use finite (although small) steps but, in contrast to this method, to accept new states each time.

This explains the greater efficiency of the Langevin equation, noted in the literature.[20]

2. The Langevin equation is usually written in Euclidean space, in which it is real. However, generalization and numerical implementation of the Langevin equation to the Minkowski space is exceedingly important. Actually, when this is done, it is possible to simulate processes in real time, including the phenomenon of interference.

For example, consider the scalar field theory. When we change the variables so that $x_4 = -it$, $\tau = i\zeta$, the Langevin equation

$$\frac{\partial\varphi}{\partial\tau} + \left(m^2 - \frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}\right)\varphi + \lambda\varphi^3 = \eta, \qquad (15.1)$$

$$\langle\eta(x, \tau)\,\eta(x', \tau')\rangle = 2\delta^4(x - x')\,\delta(\tau - \tau') \qquad (15.2)$$

becomes the complex equation

$$-i\frac{\partial\varphi}{\partial\zeta} + \left(m^2 - \Delta + \frac{\partial^2}{\partial t^2}\right)\varphi + \lambda\varphi^3 = \eta, \qquad (15.3)$$

$$\langle\eta(r, t, \zeta)\,\eta(r', t', \zeta')\rangle = 2\delta^3(r - r')\,\delta(t - t')\,\delta(\zeta - \zeta'). \qquad (15.4)$$

For the integrals with respect to ζ to converge in perturbation theory, we must make the Feynman replacement $m^2 \to m^2 - i\varepsilon$. It is not difficult to see that ζ plays the same part as the intrinsic time in the scalar particle propagator.

As far as the white noise η is concerned, it remains unal-

tered because the factors $\pm i$ cancel out.

A remarkable property of the complex Langevin equation (15.3) is that it describes interference-type phenomena that are inaccessible to the conventional Monte Carlo method. In fact, the Metropolis algorithm does not enable us to generate states with complex probability $\exp(iS)$. Equation (15.3) obviates this difficulty because the white noise remains as the usual Gaussian noise with positive probability. The complex unit factor in front of the term $\partial\varphi/\partial\zeta$ is, in fact, responsible for interference phenomena. This can be verified by linearizing the equation near the classical solution.

While, previously, the deviations $\delta\varphi$ from the classical solution were due to the addition of a large number of contributions due to the real terms corresponding to the proper modes ψ_λ of the linearized equation, the sum

$$\delta\varphi(r,\ t_1\ \zeta)=\sum_\lambda \psi_\lambda(r,\ t)\, i\int_0^\zeta d\zeta' \exp\left[-i\lambda(\zeta-\zeta')\right]\eta_\lambda(\zeta') \tag{15.5}$$

is now complex. Moreover, while the deviations were then restricted by convergence, i.e., the old values of the random force η ($\zeta' > \zeta$) were "forgotten", this now occurs as a result of interference between old and new terms.

It would be interesting to analyze, in terms of the complex Langevin equation, the oscillations in K-meson beams, the interference of photons, and other quantum-mechanical phenomena. In principle, it should be possible to deduce them, but it is not clear how this can be done.

3. The Langevin equation offers us a new possibility of generalizing quantum field theory so that it can be "derived" in some approximation from more fundamental laws. Of course, here we enter the world of fantasy but, after the advances made in quantum field theory during the last few years, this fantasy may serve as a stimulus to the creation of realistic models.

The basic immutable law of quantum theory is the principle of superposition, according to which each process is characterized by a complex amplitude that is the sum of the amplitudes of the alternative histories of the process.

On the other hand, in nature, linear processes are usually idealizations, and actual phenomena are nonlinear. Is this the case in quantum theory? What if quantum theory is a linearization of the equations of some more fundamental theory, acting over differences of the order of the Planck length?

We shall never know unless we try to construct at least a rough model of this type of theory. If fact, the Langevin equation presents us with a natural basis for this. In particular, we may suppose that this equation is an approximation to a nonlinear dynamic system in which τ (or ζ in Minkowski space) plays the part of time.

This system should involve two types of variable, namely, variables such as η, which tend to white noise over large scales, and variables such as φ, which become fields.

The fields φ are then *not* small, so that the equation in terms of them is nonlinear, while the fields η *are* small, so that the equation for φ can be linearized in η. Schematically,

$$\frac{\partial\varphi}{\partial\tau}=F(\varphi,\ \eta)\to A(\varphi)+B(\varphi)\,\eta, \tag{15.6}$$

$$\frac{\partial\eta}{\partial\tau}=G(\varphi,\ \eta). \tag{15.7}$$

The coefficient $B(\varphi)$ in front of η in the above equation can be removed by transforming the fields φ, η. Of course, the essential point is that this should lead to a set of equations such as (3.4), i.e., $A(\varphi)$ should be a gradient of some functional $S(\varphi)$.

The dynamic mechanism responsible for white noise is the most complicated aspect of this construction. In the second equation, the variable η must be a random variable over small scales, so that there are no correlations at large scales. This is readily achieved in the case of ergodicity, i.e., the solutions of the equation for η at constant φ (which varies much more slowly) are distributed with a certain probability distribution.

The Gaussian density

$$dW(\eta)\sim \exp\left(-\int C\eta^2\, d^Dx\right) d\eta$$

can arise naturally as in the central limit theorem of probability theory. For this, it is sufficient to have symmetry that excludes terms linear in η. Terms involving gradients and time derivatives of η will be significant only at ultrasmall scales, since they are of a higher order in η.

These hypotheses become less strange when we recall how white noise is generated by a computer. This can only be done with a dynamic system, but one using discrete time. The Gaussian numbers are expressed in terms of uniformly distributed random numbers, produced recurrently using formulas of the form

$$\zeta_{n+1}=\{a\zeta_n+b\}, \tag{15.8}$$

where $\{...\}$ represents the fractional part. A linear expression has been chosen in order to economize on machine time. Stochastic behavior in a nonlinear equation can be generated just as simply. This question has been examined in many publications in the course of the last few years,[30] but we shall not discuss them here.

A further, nontrivial point is the averaging over the Langevin time τ (or ζ). The scale τ (which is arbitrary!) is usually chosen in the Langevin equation to be of the same order as the scale of physical time. However, this may not be appropriate to the matter at hand. In order to elucidate why observable quantities in quantum field theory correspond to averages over τ, we must assume that the true scale of τ (the relaxation time) is exceedingly small. It is possible that this is the Planck scale. We emphasize that, so long as we are interested in averages, the scale τ is arbitrary, so that there is no conflict between the computer time and the Planck relaxation time.

It has not been our aim to construct a realistic model of a nonlinear system that would generalize, say, quantum gravity, at Planck scales. Our goal was much more modest, namely, to draw the attention of researchers to the Langevin equation which offers new prospects in quantum field theory.

In conclusion, a few words about the literature devoted

to stochastic quantization. Our list of references does not claim to be exhaustive, especially since the subject is undergoing rapid development. A more complete reference list can be found in Ref. 31, which also discusses the quantization of spinor fields.

APPENDICES

A. The Langevin equation and the Boltzmann distribution

Consider a transition from the state $q_i = q_i(t)$ to $q_i' = q_i(t+\delta t)$ in time δt. Since the random force has the Gaussian distribution (3.5), the transition probability is proportional to

$$dW(q \to q') \sim \exp\left(-\frac{\delta t}{\Gamma}\sum_i \frac{f_i^2(q, q')}{4kT}\right)\prod_i df_i(q, q'). \tag{A.1}$$

Substituting from (3.4)

$$f_i(q, q') = \Gamma\frac{q_i'-q_i}{\delta t} + \frac{\partial U}{\partial q_i} \tag{A.2}$$

and transforming to the variable q_i' (the Jacobian is a constant for $\delta t \to 0$), we find that

$$dW(q \to q') \sim \exp\left[-\frac{\delta t}{4\Gamma kT}\sum_i\left(\Gamma\frac{q_i'-q_i}{\delta t}+\frac{\partial U}{\partial q_i}\right)^2\right]\prod_i dq_i'. \tag{A.3}$$

Now consider the ratio

$$\frac{dW(q \to q')/dq'}{dW(q' \to q)/dq}, \tag{A.4}$$

i.e., the ratio of direct and inverse transition probabilities.

In the leading order in $q'-q \sim \delta t$, we have only the cross term in f_i^2 and we find that

$$\frac{dW(q \to q')/dq'}{dW(q' \to q)/dq} = \exp\left[-\frac{1}{kT}\sum_i (q_i'-q_i)\frac{\partial U}{\partial q_i}\right] \to \exp\left(-\frac{U(q_i')-U(q_i)}{kT}\right). \tag{A.5}$$

This is the essence of the situation. The probability ratio is equal to the ratio of the Boltzmann factors. This is the so-called principle of detailed balancing, which guarantees that the ergodic theory is valid.

In fact, consider the change in the number $dN(q)$ of particles with coordinates in the range $q_i + dq_i$ in a time δt. Taking into account the arrival and departure of particles, we have

$$\delta dN(q) = \int dW(q' \to q)\, dN(q') - dW(q \to q')\, dN(q). \tag{A.6}$$

This is the well-known transport equation. It is clear from the transport equation that the equilibrium distribution of the particle number is related to the transition probabilities $dW(q \to q')$ by the equation of detailed balancing

$$\frac{dW(q \to q')}{dW(q' \to q)} = \frac{dN(q')}{dN(q)}. \tag{A.7}$$

When the probability distribution is known *a priori*, the transition probabilities must be taken on the basis of detailed balance. We can then be sure that the asymptotic distribution will be "as ordered."

It is interesting to note that there are not one but many solutions for the transition probabilities corresponding to the same equilibrium distribution. This is clear from (A.7) because only the antisymmetric part in the exponent of $dW(q \to q')$ is bounded. The symmetric part cancels out in the equation of detailed balancing.

The last remark is related to the covariant generalization of the Langevin equation and the Boltzmann distribution. Let us suppose that we have a gas in curved space with Riemann metric. It is convenient to work with the tetrads $e_\alpha(i)$:

$$g_{\alpha\beta} = \sum_i e_\alpha(i)\, e_\beta(i). \tag{A.8}$$

We also introduce the reciprocal tetrads $e^\alpha(i)$ such that

$$\sum_\alpha e^\alpha(i)\, e_\alpha(j) = \delta_{ij}. \tag{A.9}$$

Instead of $q_i(t)$, we shall write x_μ, which is more familiar in the theory of gravitation. We recall that the "time" t is unrelated to x_4.

It is clear from the derivation of the equation of detailed balancing that the exponential in (A.5) will again converge to $U(x') - U(x)$ because of the orthogonality relation (A.9).

At first sight, it would appear that the $e_\alpha(i)$ are arbitrary in the Langevin equation. However, a more careful analysis reveals that the Jacobian of the transformation from f to x

$$\det\frac{\partial f_i}{\partial x_\mu'} \sim \det e = (\det g_{\alpha\beta})^{1/2} \tag{A.11}$$

gives the correct volume element in $dW(x \to x')$, i.e.,

$$dV = (\det g_{\alpha\beta})^{1/2}\, d^4x \tag{A.12}$$

but only when these tetrads are related to the metric (A.8).

Thus, we have derived the covariant Langevin equation, namely, the limit of (A.10) when $\delta t = 0$:

$$\Gamma e_\alpha(i)\frac{dx^\alpha}{dt} + e^\beta(i)\frac{\partial U}{\partial x^\beta} = f_i, \tag{A.13}$$

$$dW(f_i) \sim \exp\left(-\int dt \sum_i \frac{f_i^2}{4kT\Gamma}\right)\prod_i df_i. \tag{A.14}$$

B. Investigation of the Fokker-Planck equation

Consider the stochastic differential equation

$$\Gamma\frac{\partial q_i}{\partial t} + C_i(q) = f_i, \tag{B.1}$$

where f_i are random forces with a Gaussian distribution, which are correlated to a δ-function. If $C_i(q)$ is the partial derivative of some function U with respect to the ith coordinate [$C_i(q) = \partial U/\partial q_i$], we have the usual Langevin equation (3.4). We shall now find the equation for the probability distribution

$$P(x, t) = \left\langle\prod_{i=1}^N \delta(x_i - q_i(t))\right\rangle_f, \tag{B.2}$$

where $q_i(t)$ is the solution of (B.1). For simplicity, we confine our attention to the case of one degree of freedom. Differentiating $P(x,t)$ with respect to time, and using (B.1), we obtain

$$\Gamma\frac{\partial P}{\partial t}=\left\langle -\Gamma\dot{q}(t)\frac{\partial}{\partial x}\delta(x-q(t))\right\rangle$$
$$=-\frac{\partial}{\partial x_i}\langle f_i-C_i(q(t))\,\delta(x-q(t))\rangle$$
$$=\frac{\partial}{\partial x_i}C_i(x)\,P(x,t)-\frac{\partial}{\partial x_i}\langle f_i\delta(x-q(t))\rangle. \qquad (B.3)$$

We have replaced all the derivatives $\partial/\partial q(t)$ with the derivative $-\partial/\partial x$ because the δ-function depends on the difference $x-q(t)$. We must now express $\langle f\delta[x-q(t)]\rangle$ in terms of known quantities. Since f_i is white noise, we can use Wick's theorem

$$\langle fF(f)\rangle_f=2\Gamma\left\langle\frac{\delta F}{\delta f}\right\rangle_f. \qquad (B.4)$$

This is a somewhat unusual form of Wick's theorem. However, the validity of (B.4) can be readily verified directly, e.g., by expanding the functional F into a power series in f. We can then write

$$\langle f\delta(x-q(t))\rangle_f=2\Gamma\left\langle\frac{\delta q(t)}{\delta f(t)}\left(-\frac{\partial}{\partial x}\right)\delta(x-q(t))\right\rangle$$
$$=-2\Gamma\frac{\partial}{\partial x}\left\langle\frac{\delta q(t)}{\delta f(t)}\delta(x-q(t))\right\rangle. \qquad (B.5)$$

The variational derivative of $q(t)$ with respect to the random force $f(t')$ is $(1/\Gamma)\theta(t-t')+0(t-t')$. In the standard regularization procedure (in the sense of the Fourier expansion), this expression is equal to $(1/\Gamma)\theta(0)=\frac{1}{2}\Gamma$ for $t=t'$. This normalization is the same, as will be seen below, as the normalization of f obtained in Appendix A on the basis of detailed balancing. As a result, we have the following equation (so far, for a single variable)

$$\Gamma\frac{\partial P}{\partial t}=\frac{\partial^2}{\partial x^2}P(x,t)+\frac{1}{2}\frac{\partial}{\partial x}C(x)\,P(x,t).$$

This equation is readily generalized to a system with a finite or infinite number of variables:

$$\Gamma\frac{\partial P}{\partial t}=-L^+P(q,t), \qquad (B.6)$$

where

$$L^+=-\sum\frac{\partial}{\partial q_i}\left[\frac{\partial}{\partial q_i}+C_i(q)\right].$$

The original stochastic equation given by (B.1) can then be written in the form

$$\Gamma\frac{\partial q_i}{\partial t}+Lq_i=f_i, \qquad (B.7)$$

where

$$L=-\sum_i\left[\frac{\partial}{\partial q_i}-C_i(q)\right]\frac{\partial}{\partial q_i}.$$

The operators L and L^+ are conjugates.

We now consider the special case where $C_i(q)=\partial U/\partial q_i$ [this corresponds to the usual Langevin equation (3.4)]. It is convenient to substitute $P(q,t)=\exp(-U/2)\tilde{P}(q,t)$ in (B. 6). The function $\tilde{P}(q,t)$ then satisfies the equation

$$\Gamma\frac{\partial\tilde{P}}{\partial t}=-H\tilde{P}(q,t), \qquad (B.8)$$

where

$$H=\frac{1}{2}\sum_i Q_i^+Q_i,\quad Q_i=\frac{\partial}{\partial q_i}+\frac{1}{2}\frac{\partial U}{\partial q_i},$$
$$Q_i^+=-\frac{\partial}{\partial q_i}+\frac{1}{2}\frac{\partial U}{\partial q_i};$$

and H is a self-adjoint positive-definite operator. Equation (B.8) is similar to the Schrödinger equation. The effective potential energy is

$$V(q)=\sum_i\frac{1}{8}\left(\frac{\partial U}{\partial q_i}\right)^2-\frac{1}{4}\frac{\partial^2U}{\partial q_i\,\partial q_i}.$$

The quantity $\tilde{P}(q,t)$ can be expanded into a series in terms of the eigenfunctions of the operator H:

$$\tilde{P}(q,t)=\sum_n C_ne^{-\lambda_nt}\varphi_n(q), \qquad (B.9)$$

where the φ_n satisfy the equation $H\varphi_n=\lambda_n\varphi_n$. Only the zero-order eigenvalue survives as $t\to\infty$:

$$P(q,t)\xrightarrow[t\to\infty]{}\varphi_0(q)\exp\left(-\frac{U}{2}\right).$$

It is possible to produce an explicit expression for the zero mode:

$$\varphi_0(q)=\exp\left(-\frac{U}{2}\right). \qquad (B.10)$$

In principle, there can be other zero modes satisfying the equations

$$Q\varphi_0\neq 0,\quad Q^+Q\varphi_0=0.$$

For a finite number of degrees of freedom, quantum mechanics shows that the vacuum is nondegenerate, which means that the zero mode $\varphi_0=\exp(-U/2)$ is unique. We have thus shown that, in the case of a finite number of degrees of freedom, the limit of the probability distribution function for $t\to\infty$ is identical with the Boltzmann distribution $\exp(-U)$.

Vacuum can be degenerate for an infinite number of degrees of freedom and this is, in fact, the case in gauge theories. The limit of the distribution function is then found to depend on the initial conditions. This is precisely as expected since the integration measure (distribution function) in gauge theories depends explicitly on the chosen gauge.

Let us now examine in greater detail the properties of stochastic quantization in gauge theories.

It is reasonable in gauge theories to consider the averages not of arbitrary functionals but only of the gauge-invariant quantities $\Phi[a]$:

$$\hat{G}^a\Phi=0, \qquad (B.11)$$

where $\hat{G}^a$ are the generators of gauge transformations:

$$\hat{G}^a(x)=\int dy\,[-\partial_\mu\delta^{ab}\delta(x-y)$$
$$+f^{abc}A_\mu^c(x)\,\delta(x-y)]\frac{\delta}{\delta A_\mu^c(y)}.$$

The average of a gauge-invariant functional can be written in the form

$$\langle\Phi\rangle(t) \sim \int (e^{-L^+ t} P(A, 0))\,\Phi(A)\,DA. \tag{B.12}$$

We have explicitly shown the dependence of the distribution function on time by "solving" the Fokker-Planck equation for $P(A, t)$; the quantity $P(A, 0)$ is the initial probability distribution. The effect of the operator L^+ can be "transferred" to the functional $\Phi(A)$ by integrating by parts:

$$\langle\Phi\rangle(t) = \int P(A, 0)\, e^{-Lt}\, \Phi(A)\, DA. \tag{B.13}$$

We now use the fact that $\Phi[A]$ is a gauge-invariant quantity, so that

$$\langle\Phi\rangle(t) = \int P(A, 0)\, e^{-t(L+\alpha\hat{G})}\Phi(A)\, DA$$
$$= \int [e^{-t(L+\alpha\hat{G})^+} P(A, 0)]\,\Phi(A)\, DA. \tag{B.14}$$

This means that, instead of the usual Langevin equation (7.4), we can write

$$\frac{\partial A_\mu}{\partial t} + (LA)_\mu + ((\alpha\hat{G})\,A)_\mu = J_\mu. \tag{B.15}$$

In terms of the original notation, we then have

$$\frac{\partial A_\mu}{\partial t} + \frac{\delta S}{\delta A_\mu} + [D_\mu a] = J_\mu. \tag{B.16}$$

This equation is of the same form as (8.1) except that J_μ is a random current with a Gaussian distribution, correlated to a δ-function, whereas the field correlator K_μ has a nontrivial form [see (8.2)].

Equations (8.1) and (B.16) are different, but the averages of the gauge-invariant quantities evaluated with them are identical. We thus encounter a situation that is familiar in the quantum theory of gauge fields: there is an infinite number of different integration measures in the functional integral (corresponding to different ways of fixing the gauge), but the averages of the gauge-invariant quantities are equal.

C. Difference Langevin equation

It was shown in Appendix A that the differential Langevin equation follows from detailed balancing in the leading order in the time step Δt. The question is: is it possible to find the higher-order difference equations from the relation for detailed balancing?

To begin with, we shall seek the difference Langevin equation in an implicit form:

$$\varphi' - \varphi - \Delta t F(\varphi', \varphi) = \eta,$$
$$\varphi = \varphi_n;\ \varphi' = \varphi_{n+1}; \tag{C.1}$$

where η is the Gaussian noise with variance $2\Delta t$ and $F(\varphi', \varphi)$ is an unknown function which we shall assume to be symmetric under the interchange of the arguments.

The Gaussian distribution

$$dW(\eta) = \exp\left(-\frac{\eta^2}{4\Delta t}\right) d\eta \tag{C.2}$$

can be treated as the transition probability $dW(\varphi \to \varphi')$ by expressing η in terms of φ' using (C.1):

$$dW(\varphi \to \varphi') = \exp\left[-\frac{(\varphi' - \varphi - \Delta t F(\varphi', \varphi))^2}{4\Delta t}\right]\left|\frac{\partial\eta}{\partial\varphi'}\right| d\varphi'. \tag{C.3}$$

Thus,

$$dW(\varphi \to \varphi') = \exp\left[-\frac{(\varphi' - \varphi - \Delta t F(\varphi', \varphi))^2}{4\Delta t} + \ln\left|1 - \Delta t \frac{\partial F}{\partial\varphi'}\right|\right] d\varphi'. \tag{C.4}$$

Detailed balancing (A.5) (in our notation) now assumes the form

$$\Delta\varphi F(\varphi', \varphi) = -\Delta S(\varphi) + \Delta\Phi(\varphi', \varphi), \tag{C.5}$$

$$\Phi(\varphi, \varphi') = \ln\left(1 - \Delta t \frac{\partial F}{\partial\varphi'}\right), \tag{C.6}$$

where

$$\left.\begin{aligned} \Delta\varphi &= \varphi' - \varphi, \\ \Delta S(\varphi) &= S(\varphi') - S(\varphi), \\ \Delta\Phi(\varphi', \varphi) &= \Phi(\varphi', \varphi) - \Phi(\varphi, \varphi'). \end{aligned}\right\} \tag{C.7}$$

The right-hand side of (C.5) can be rewritten in the form of the integral

$$\int_{-1}^{1} d\lambda \frac{\Delta\varphi}{2}\left[-\frac{\partial S(\varphi_\lambda)}{\partial\varphi_\lambda} + \left(\frac{\partial}{\partial\varphi_\lambda} - \frac{\partial}{\partial\varphi_{-\lambda}}\right)\Phi(\varphi_\lambda, \varphi_{-\lambda})\right], \tag{C.8}$$

$$\varphi_\lambda = \bar{\varphi} + \lambda\frac{\Delta\varphi}{2}, \quad \bar{\varphi} = \frac{\varphi + \varphi'}{2}.$$

Cancelling $\Delta\varphi$, we then obtain

$$F(\varphi', \varphi) = \frac{1}{2}\int_{-1}^{1} d\lambda\left[-S'(\varphi_\lambda) + \left(\frac{\partial}{\partial\varphi_\lambda} - \frac{\partial}{\partial\varphi_{-\lambda}}\right)\Phi(\varphi_\lambda, \varphi_{-\lambda})\right]$$
$$= \frac{1}{2}\int_{-1}^{1} d\lambda\left[-S'(\varphi_\lambda) + \Delta t\left(1 - \Delta t\frac{\partial F}{\partial\varphi_\lambda}\right)^{-1} \times\left(\frac{\partial^2 F}{\partial\varphi_\lambda\,\partial\varphi_{-\lambda}} - \frac{\partial^2 F}{\partial\varphi_\lambda\,\partial\varphi_\lambda}\right)\right]. \tag{C.9}$$

This equation can be iterated in Δt and $\Delta\varphi$. It must be remembered in these iterations that $\Delta\varphi \sim (\Delta t)^{1/2}$ because the variance of the random force is $\sim\Delta t$.

In the first order in Δt and, correspondingly, in the second order in $\Delta\varphi$, we find that [terms $\sim\Delta t$ cancel out in (C.9)]

$$F(\varphi, \varphi') = -S'(\bar{\varphi}) - \frac{1}{24}(\Delta\varphi)^2 S'''(\bar{\varphi}) + O((\Delta t)^2). \tag{C.10}$$

To the same degree of precision, we have

$$F(\varphi, \varphi') = -\frac{1}{6}(S'(\varphi) + 4S'(\bar{\varphi}) + S'(\varphi')). \tag{C.11}$$

The discrete Langevin equation (C.1) can now be reduced to the equation for $\Delta\varphi$ with fixed φ:

$$\Delta\varphi = \eta - \frac{\Delta t}{6}\left(S'(\varphi) + 4S'\left(\varphi + \frac{\Delta\varphi}{2}\right) + S'(\varphi + \Delta\varphi)\right) + O((\Delta t)^3). \tag{C.12}$$

We now introduce the four successive iterations:

$$\left.\begin{aligned}
&\Delta\varphi_1 = \eta - \Delta t S'(\varphi),\\
&\Delta\varphi_2 = \Delta\varphi_1 - \Delta t\left[S'\left(\varphi+\frac{\Delta\varphi_1}{2}\right) - S'(\varphi)\right],\\
&\Delta\varphi_3 = \Delta\varphi_2 - \Delta t\left[S'\left(\varphi+\frac{\Delta\varphi_2}{2}\right) - S'\left(\varphi+\frac{\Delta\varphi_1}{2}\right)\right],\\
&\Delta\varphi_4 = \Delta\varphi_3 - \frac{\Delta t}{6}\left[S'(\varphi) + S'(\varphi+\Delta\varphi_2)\right.\\
&\qquad\qquad \left. - 2S'\left(\varphi+\frac{\Delta\varphi_2}{2}\right)\right].
\end{aligned}\right\} \quad (C.13)$$

By construction, $\Delta\varphi_4$ satisfies the Langevin equation to within $(\Delta t)^2 \sim (\Delta\varphi)^2\Delta t \sim (\Delta\varphi)^4$, inclusive. It is not difficult to construct higher-order schemes by iterating the detailed balancing equation (C.9).

We note that the usual Runge-Kutta scheme cannot be used for the stochastic equation because it presupposes the existence of a certain number of derivatives of the right-hand side of the equation.

The above equations can be readily generalized to an arbitrary number of degrees of freedom φ^1 and the field $\varphi^i(x)$. It is sufficient to replace the derivatives in (C.9) with the corresponding partial (functional, in the case of fields) derivatives, and the logarithm with the trace of the logarithm of a matrix (operator, in the case of a field). In particular, the equations in (C.13) remain unaffected when $S'(\varphi)$ is understood to be $\delta S/\delta\varphi$.

Generalization to a curved φ-space is somewhat less trivial. In this case, instead of (C.1), we must write the following covariant equation

$$\Delta^i\varphi - \Delta t F^i(\varphi', \varphi) = \eta^i, \quad (C.14)$$

$$\Delta^i\varphi = \int_{\varphi}^{\varphi'} d\varphi^\alpha e^i_\alpha(\varphi), \quad (C.15)$$

where $e^i_\alpha(\varphi)$ are tetrads, η^i is a Gaussian random variable with variance $2\Delta t$, and F^i is the symmetric function of φ', φ which is being sought. The integral in (C.15) is evaluated over a geodesic between φ and φ'.

Equation (C.5) transforms to

$$\Delta^i\varphi F^i(\varphi, \varphi') = S(\varphi) - S(\varphi') + \Phi(\varphi', \varphi) - \Phi(\varphi, \varphi'), \quad (C.16)$$

$$\Phi(\varphi', \varphi) = \operatorname{tr}\ln\left\|\delta_{ij} - \Delta t\frac{\partial F^i}{\partial\varphi'^\alpha}e^{\alpha j}(\varphi')\right\|. \quad (C.17)$$

The equation can be reduced to a form convenient for iterations by differentiating with respect to φ^α and multiplying by $-e^\alpha_k$. Since

$$e^\alpha_k\frac{\partial}{\partial\varphi_\alpha}\Delta^i\varphi = \delta^k_i + O(\Delta\varphi), \quad (C.18)$$

the term $F^k(\varphi', \varphi)$ is found to appear. Taking the other terms onto the right-hand side, we find that $[\partial_i = \varepsilon^\alpha_i(\varphi)\partial/\partial\varphi_i, \partial'_i = e^\alpha_i(\varphi')\partial/\partial\varphi'_i]$

$$\begin{aligned}
F^k(\varphi', \varphi) = &-\partial_k S(\varphi) + \Delta^i\varphi\,\partial_k F^i\\
&+ \Delta t(\delta - \Delta t\,\partial' F)^{-1}_{ji}\partial'_j\partial_k F^i\\
&- \Delta t(\delta - \Delta t\,\partial F)^{-1}_{ji}\partial_j\partial_k F^i. \quad (C.19)
\end{aligned}$$

To within terms $\sim\Delta t \sim (\Delta\varphi)^2$ inclusive, all that remains is the first row

$$F^k = -\partial_k S(\varphi) + \Delta^i\varphi\partial_k F^i + O((\Delta t)^2). \quad (C.20)$$

Differentiating (C.19), we obtain the equation for the derivative $\partial_l F^k \equiv F^k_{,l}$:

$$F^k_{,l} = -\partial_l\partial_k S - F_{,k}{}^l + \Delta^i\varphi\partial_l F^i_{,k}. \quad (C.21)$$

Taking the term $-F'_{,k}$ into the left-hand side, we find in the leading order

$$F^k_{,l} = -\frac{1}{2}\partial_k\partial_l S \ldots, \quad (C.22)$$

where the terms containing $[\partial_k\partial_l]$ have the highest order of small quantities. In (C.20), we find that

$$F^k = -\left(1 + \frac{1}{2}\Delta^i\varphi\partial_i\right)\partial_k S \ldots = -\partial_k S(\bar\varphi) + \ldots, \quad (C.23)$$

$$\bar\varphi^\alpha = \varphi^\alpha + \frac{1}{2}e^\alpha_i(\varphi)\Delta^i\varphi + \ldots. \quad (C.24)$$

The geometric interpretation of this formula is that the geodesic between φ and φ' contains the midpoint $\bar\varphi$, which is equally distant from φ and φ' and is the covariant analog of $(\varphi + \varphi')/2$. The symmetry between φ and φ' means that, with this choice of $\bar\varphi$, the next correction begins with $(\Delta\varphi)^2$, as in the flat case (C.10). In principle, this correction is not difficult to calculate but we shall not do so here.

D. Reduced white noise

Averaging over the traces of a product of random forces using the ansatz (10.6) leads to the same results as averaging with ordinary white noise to within terms of the order of $1/N$.

As an example, consider the normalized two-point correlator

$$\begin{aligned}
&\frac{1}{N^2}\langle\operatorname{tr}\eta(x, t)\eta(y, \tau)\rangle = \sum_{i,j}\frac{1}{N^2}\langle\eta^{ij}(x, t)\eta^{ji}(y, \tau)\rangle\\
&= \sum_{ij}\frac{1}{N^2}\langle\exp[i(p_i - p_j)(x-y)]\,\eta^{ij}(t)\,\eta^{ji}(\tau)\rangle_{\eta,\,p}. \quad (D.1)
\end{aligned}$$

The momenta p_i are uniformly distributed in a cube of size $2\pi\Lambda(\Lambda \equiv 1)$. Consider the terms with $i \neq j$ in (D.1):

$$\begin{aligned}
&\frac{1}{N^2}\langle\operatorname{tr}\eta(x, t)\eta(y, \tau)\rangle\\
&\quad = 2\delta(t-\tau)\frac{1}{N^2}\int\prod_i\frac{dp_i}{(2\pi)^D}\left\{\sum_{i\neq j}\exp[i(p_i - p_j)\right.\\
&\qquad\qquad \left.\times(x-y)] + N\right\}\\
&\quad = 2\delta(t-\tau)\frac{1}{N^2}[(N^2 - N)\delta(x-y) + N]\\
&\quad = 2\delta(x-y)\delta(t-\tau) + O\left(\frac{1}{N}\right). \quad (D.2)
\end{aligned}$$

For the four-point correlator of random forces, we have

$$\begin{aligned}
&\left(\frac{1}{N}\right)^3\langle\operatorname{tr}\eta(x, t_1)\eta(y, t_2)\eta(z, t_3)\eta(\omega, t_4)\rangle\\
&\quad = \sum_{i,j,k,l}\frac{1}{N^3}\langle\eta^{ij}(t_1)\eta^{jk}(t_2)\eta^{kl}(t_3)\eta^{li}(t_4)\\
&\times\exp\{i[p_j(y-x) + p_k(z-y) + p_l(\omega - z) + p_i(x-\omega)]\}\rangle. \quad (D.3)
\end{aligned}$$

The average of Gaussian random forces η can be written

down with the aid of the Wick theorem:

$$\sum_{i,j,k,l} \frac{1}{N^3} \langle \exp\{i[p_j(y-x)+p_k(z-y) + p_l(\omega-z)+p_i(x-\omega)]\} \times [\langle \eta^{ij}(t_1)\eta^{jk}(t_2)\rangle \langle \eta^{kl}(t_3)\eta^{li}(t_4)\rangle + \langle \eta^{ij}(t_1)\eta^{kl}(t_3)\rangle \times \langle \eta^{jk}(t_2)\eta^{li}(t_4)\rangle + \langle \eta^{ij}(t_1)\eta^{li}(t_4)\rangle \langle \eta^{jk}(t_2)\eta^{kl}(t_3)\rangle]\rangle. \quad (D.4)$$

By evaluating this expression, we obtain

$$\begin{aligned}
&\frac{1}{N_3} 4\delta(t_1-t_2)\delta(t_3-t_4) \\
&\quad \times \sum_{ijl} \int \frac{dp}{(2\pi\Lambda)^D} [\exp\{i[p_i(x-\omega+z-y)] \\
&\qquad + ip_l(\omega-z)\}\exp[ip_j(y-x)]] \\
&\quad + \frac{1}{N^3} 4\delta(t_1-t_3)\delta(t_2-t_4) \sum_{i,j,k,l} \delta_{il}\delta_{jk}\delta_{kl}\delta_{jl} \int \frac{dp}{(2\pi)^D} \\
&\quad + \frac{1}{N^3} 4\delta(t_1-t_4)\delta(t_2-t_3) \\
&\quad \times \sum_{j,i,k} \int \frac{dp}{(2\pi)^D} [\exp\{i[p_j(y-x+\omega-z)] \\
&\quad + ip_i(x-\omega)\}\exp[ip_k(z-y)]] \\
&\quad = 4\delta(t_1-t_2)\delta(t_3-t_4)\delta(x-y)\delta(z-\omega) \\
&\quad + 4\delta(t_1-t_4)\delta(t_2-t_3)\delta(x-\omega)\delta(z-y) + O\left(\frac{1}{N}\right).
\end{aligned} \quad (D.5)$$

The δ-functions appear, as usual, as a result of integration with respect to the momenta. When the indices of the momenta in the exponential are different, we obtain the leading contribution to the expression for the average. When, on the other hand, some of the indices are equal, the resulting terms are of the order of $1/N$. A similar discussion can be given for the average of an arbitrary (even) number of random forces.

[1)]This will be examined in greater detail below.

[2)]The problem of acceptance of offered states is solved with the aid of a random number generator.

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