Duality of Decaying Turbulence to a Solvable String Theory

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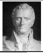
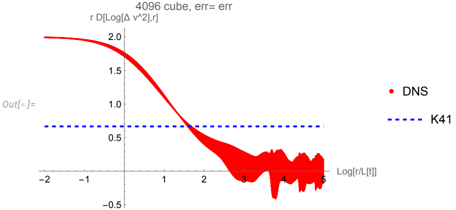
The Old Road to Turbulence

Discovery Year	Portrait	Key Contribution
1500s		Leonardo da Vinci: Early observations of turbulent flow.
1820s	 	Claude-Louis Navier and George Gabriel Stokes: Formulation of the NS equations.
1883		Osborne Reynolds: Study of the transition to turbulence.
1941		Andrey Kolmogorov: Proposed scaling laws of turbulence (K41).
1944		Werner Heisenberg: Contributions to stability and theoretical insights.
1946		Lars Onsager: Insights into fractal velocity and vortex structures.
1956		Lev Landau: Contributions to hydrodynamic stability.
1960s		Richard Feynman: Proposed insights into chaotic systems.
1985	 	Giorgio Parisi and Uriel Frisch: Suggested multifractal models for turbulence.
2000-2022		K.R. Sreenivasan: DNS evidence revealing violations of scaling laws.

Scaling Law Violations: The Dead End of the Old Road



- Kolmogorov's K41 theory predicted universal scaling laws for velocity difference moments (e.g., an index of $2/3$ for the second moment's log-derivative).
- Direct Numerical Simulations (DNS, red dots with error bars, 4096^3 lattice) exposed violations of K41 predictions, showing a nonlinear effective index curve as a function of $r/\sqrt{t}$.
- The methodology for extracting effective indices from observed energy spectra was introduced in [8].
- Lattice artifacts affected the data for $\log r/L(t) > 2$, but the left-side data reliably invalidated universal scaling laws.
- Multifractal models provided partial corrections (e.g., shifted K41 scaling), but failed to capture the observed nonlinear behavior.

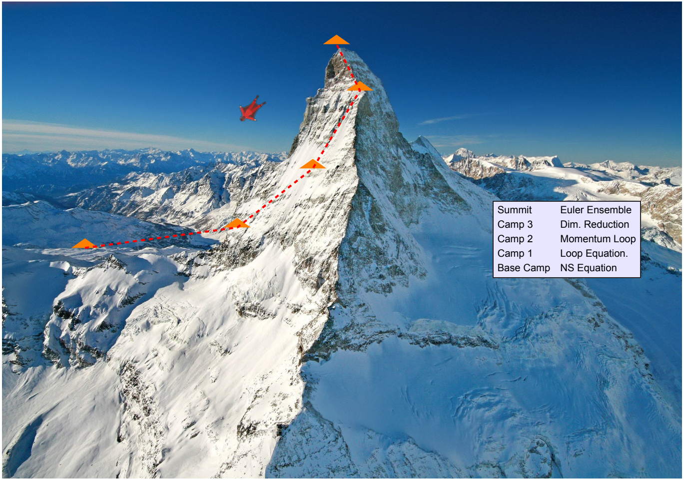
Get Back to Kolmogorov and Take the Turn You Missed

- Kolmogorov's 1941 framework laid the foundation for turbulence theory, but DNS and experiments exposed critical flaws in its assumptions.
- Scaling law violations (Sreenivasan et al.) highlighted the need for a new theoretical approach.
- The key mystery remained: How does randomness emerge from the deterministic Navier-Stokes equation?
- The new path began in the 1990s with **loop equations** [7], inspired by advances in QCD [6, 1].
- Subsequent progress in loop equations [10] and string theory paved the way for an exact solution.
- Collaboration with DNS experts [11] and mathematicians [4] validated the emerging theory, linking it to empirical observations.

How Duality Solved the Turbulence Problem

- A thirty-year journey culminated in a **universal, exact solution** to the Navier-Stokes equations in the turbulent regime [9].
- Reformulating the Navier-Stokes equations in **loop space** revealed the true origin of **spontaneous stochasticity**, arising from the degeneracy of discrete solution manifolds rather than external noise.
- The dual description employed mathematical frameworks from high-energy physics, rendering turbulence analytically tractable with tools from number theory.
- This approach resolves longstanding puzzles and predicts experimental and numerical outcomes with unprecedented accuracy.
- DNS data violations of scaling laws are now quantitatively described by universal number-theoretic functions [8].

The Ascent to the Summit



Like scaling a challenging peak, the journey to this solution demanded persistence, creativity, and collaboration.

Loop Average and Dimension Reduction

- The loop average is defined as the **characteristic function** for the distribution of velocity circulation:

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle, \quad \Gamma_C = \oint d\vec{C}(\theta) \cdot \vec{v}(\vec{C}(\theta)).$$

- This is equivalent to the **Hopf functional** representation:

$$\Psi[\gamma, C] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}^3} \vec{J}_C(\vec{r}) \cdot \vec{v}(\vec{r})\, d^3 r\right)\right\rangle.$$

- An imaginary source $\vec{J}(\vec{r})$ is concentrated on a fixed loop in space $\mathbb{R}_3$

$$\vec{J}_C(\vec{r}) = \frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\delta^3\left(\vec{r} - \vec{C}(\theta)\right)$$

Loop Equation as Quantum Mechanics in Loop Space

- The loop functional satisfies a key relation derived from the incompressible Navier-Stokes equation:

$$\imath\nu\partial_t\Psi[\gamma, C] = \left\langle \gamma \oint d\vec{C}(\theta) \cdot \left(-\nu\vec{\nabla} \times \vec{\omega} + \vec{v} \times \vec{\omega}\right) e^{\frac{\imath\gamma}{\nu}\Gamma_C} \right\rangle.$$

- This relation leads to the closed functional equation for the loop average [7]:

$$\imath\nu\partial_t\Psi[\gamma, C] = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta\vec{C}(\cdot)}\right]\Psi[\gamma, C].$$

- A crucial property: **translation invariance** of $\vec{L}\left[\frac{\delta}{\delta\vec{C}(\cdot)}\right]$ in loop space enables an exact solution.

Key Insight: Plane Wave Solutions in Loop Space

- The loop equation maps to a **Schrödinger equation in loop space** with a Hamiltonian:

$$\hat{H}_C = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta \vec{C}(\cdot)}\right].$$

- A **plane wave solution** emerges naturally:

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta) \cdot \vec{P}(t,\theta)\right)\right\rangle.$$

- The reduced momentum loop equations (MLE) describe the dynamics of $\vec{P}(t,\theta)$:

$$\nu \partial_t \vec{P} = -\gamma^2 (\Delta \vec{P})^2 \vec{P} + \Delta \vec{P}\left(\gamma^2 \vec{P} \cdot \Delta \vec{P} + \imath\gamma \left(\frac{(\vec{P} \cdot \Delta \vec{P})^2}{\Delta \vec{P}^2} - \vec{P}^2\right)\right).$$

The Euler Ensemble Solves MLE

- The decaying solution of MLE:

$$\vec{P} = \sqrt{\frac{\nu}{2(t+t_0)}}\frac{\vec{F}}{\gamma},$$

involves a **fractal curve** $\vec{F}(\theta)$, defined as the limit $N \to \infty$ of the polygon $\vec{F}_0 \dots \vec{F}_N = \vec{F}_0$, with vertices:

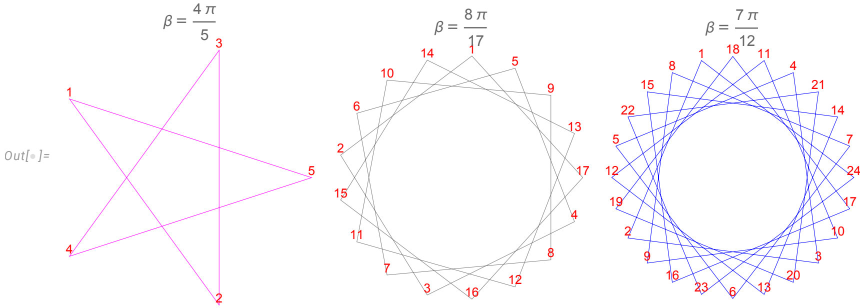
$$\vec{F}_k = \frac{\left\{\cos(\alpha_k), \sin(\alpha_k), \imath \cos\left(\frac{\beta}{2}\right)\right\}}{2\sin\left(\frac{\beta}{2}\right)}, \tag{1}$$

$$\theta_k = \frac{k}{N}, \quad \beta = \frac{2\pi p}{q}, \quad N \to \infty, \tag{2}$$

$$\alpha_{k+1} = \alpha_k + \sigma_k \beta, \quad \sigma_k = \pm 1, \quad \beta \sum \sigma_k = 2\pi p r. \tag{3}$$

- The parameters $\hat{\Omega}, p, q, r, \sigma_0 \dots \sigma_N = \sigma_0$ are **random**, making $\vec{F}(\theta)$ a **fixed manifold** rather than a fixed point.

Regular Star Polygons for Euler Ensembles



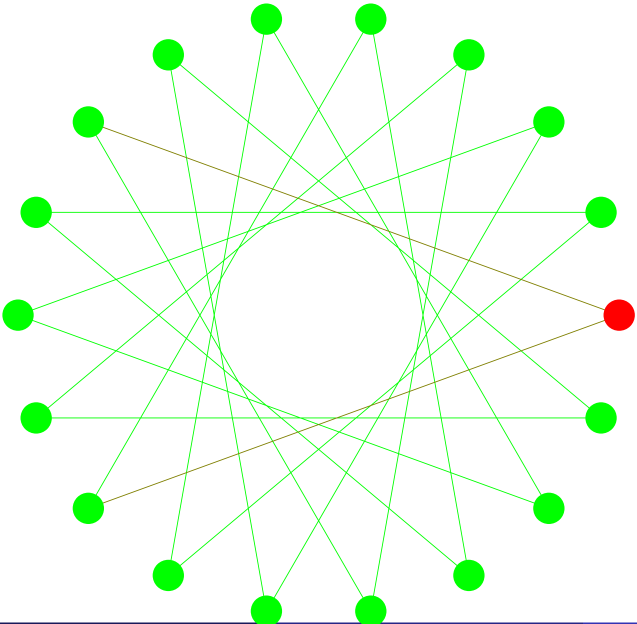
Regular star polygons for Euler ensembles of various p, q.

The σ_k **variable** determines the direction of the random step along the link $k \leftrightarrow k+1$.

In general, the **random walk** can loop multiple times around the polygon, provided it returns to its starting point.

The **periodicity** of the sequence $\vec{f}_k$ enforces that the angular step be a fraction of 2π, forming the Euler ensemble described by (1).

Random Walk on Star Polygon, $p = 7, q = 18, N = 100$



Visualization of a

Statistical Distribution and Quantization

- The statistical distribution of a solution to the nonlinear classical Navier-Stokes PDE is linked to a wave functional that satisfies a linear Schrödinger equation in loop space.
- The quantization mechanism mirrors that of ordinary quantum mechanics, relying on the **periodicity** of the solution.
- Each **distinct state** in a quantum system contributes to the partition function with unit weight, a principle adopted in this **quantum solution** of the nonlinear classical system.
- This **quantum statistical counting** is an **extra conjecture**, akin to the ergodic hypothesis in reversible Newtonian mechanics. While widely accepted in physics, such conjectures are mathematically proven only in specific cases.

Euler Ensemble as Dual of Turbulence

- The **Euler ensemble** serves as a **dual theory** to Navier-Stokes turbulence, providing a groundbreaking framework.
- Turbulence is reformulated in terms of discrete **star polygons**, capturing the momentum loop-space dynamics of fluid motion.
- This reformulation replaces Kolmogorov's empirical framework with exact **universal decay laws** and energy spectra derived from number theory.
- In the dual theory, the strong turbulence regime corresponds to a **weak coupling** phase, allowing an **analytic solution** in quadrature.
- This is reminiscent of **asymptotic freedom in QCD**, where weakly interacting quarks describe the strong interactions of hadrons.
- **Validation:** Numerical NS simulations (4096^3 resolution) and experimental data confirm these predictions, revealing hidden structures within chaotic fluid motion.

Turbulence/String Duality: Insights

- **Duality phenomena** in physics connect strong coupling in one theory to weak coupling in another (e.g., **AdS/CFT correspondence**).
- Duality links **statistical averages** of two systems, without requiring a direct correspondence between their dynamical variables.
- For turbulence, the **target space** of the string theory is discrete, represented by **star polygons** with unit sides and rational angles $\beta = 2\pi \frac{p}{q}$.
- Periodicity conditions relate the **winding number** to q.
- Fermionic degrees of freedom $(\nu_k = 0, 1)$ account for the directional properties of motion along polygon edges.

Dual Amplitude and Loop Functional

- The loop functional in the Euler ensemble corresponds to the **dual amplitude** of string theory with discrete target space $\vec{F}(\theta)$ and distributed external momentum $\vec{Q}(\theta)$:

$$\Psi[C,t] \propto \left\langle \exp \imath \oint d\theta \vec{F}(\theta) \cdot \vec{Q}(\theta) \right\rangle_{F,\sigma}, \tag{4}$$

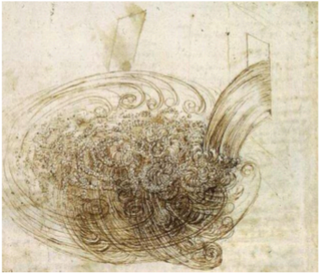
$$\vec{Q}(\theta) = \frac{\vec{C}'(\theta)}{\sqrt{2\nu(t+t_0)}}. \tag{5}$$

- Averaging over **string target space** $\vec{F}_k$ corresponds to summing over star polygons with unit side lengths and rational angles $\frac{p}{q}$.
- Averaging over **fermionic/Ising degrees of freedom** leads to a random walk (Brownian motion in the continuum limit) on these polygons.

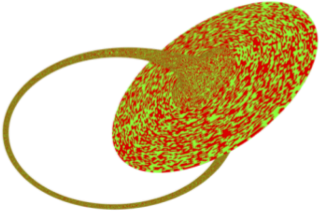
Turbulence = Random Walk on Star Polygons

Duality of Turbulence to String Theory

Chaotic Vortex Motion



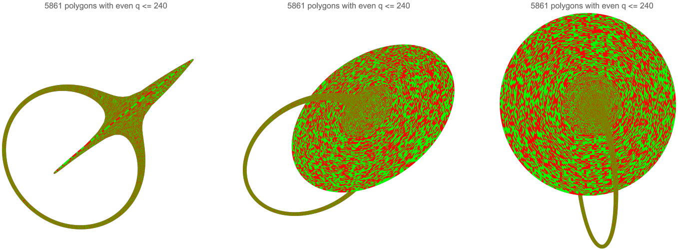
Random Walk on Star Polygons



Discrete Symmetry and Target Space Integration

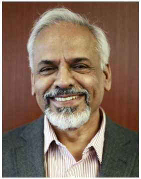
- **Contrast:** Turbulence features chaotic velocity fields, while the dual string theory exhibits **discrete symmetries** and frozen fluctuations.
- Integration over the target space (star polygons) reduces to a **discrete sum** over Euler ensemble states:
  - Rational angles $\frac{p}{q}$,
  - Fermionic configurations $(\nu_k = 0, 1)$,
  - Winding number $w = \frac{p}{q} \sum (2\nu_k - 1)$.
- Visualizing polygons for fixed N in 3D space, ordered by angle β, reveals the **world sheet** of a discrete string. Red/green coloring of polygon edges indicates fermionic occupation numbers (random walk directions).

Revealing a Hidden Identity



- We are not merely calculating turbulence correlation functions; we are uncovering its **hidden second identity as a discrete string theory**, exposing the **beauty of primes under the mask of chaos**.
- Discovering a beautiful mathematical structure is compelling, but it does not guarantee realization in Nature (consider the 40-year pursuit of quantum gravity via string theory).
- Ultimately, **real and numerical experiments** must decide whether this theory is purely mathematical or a true physical description of turbulence.

The Catalyst for Discovery: Sreeni's Observations

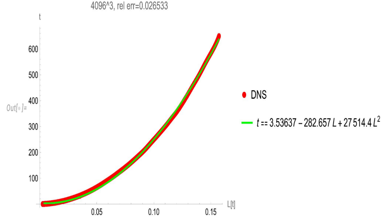


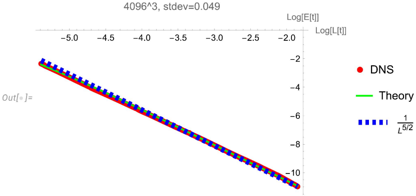
- In his 2023 ICTS talk, **K.R. Sreenivasan** presented DNS data on decaying turbulence, examining its alignment with **classical scaling laws (Kolmogorov-Saffman)**.
- He acknowledged **persistent discrepancies**, stating, "It is somewhat disappointing that the results are not more closely aligned with theoretical arguments."
- Despite these observations, no **alternative theoretical framework** was available to explain the data.
- Sreeni's **generosity in sharing raw, unpublished DNS data** enabled a deeper analysis and further insights.
- His **commitment to scientific rigor** laid the foundation for this breakthrough discovery.

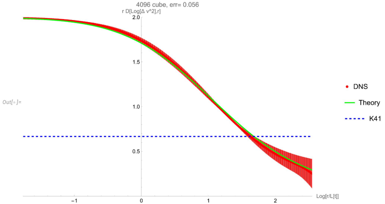
The Breakthrough Analysis: Effective Index Method

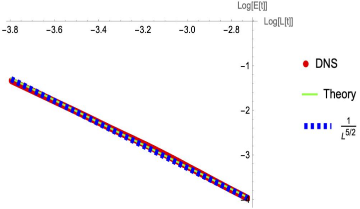
- Using my **effective index method**, I magnified the discrepancies in the DNS data, showing that the **Kolmogorov scaling law (K41)** was decisively ruled out.
- The method derived a **Fourier integral representation** for the effective scaling index from the energy spectrum, **suppressing statistical noise** in raw data.
- Analysis revealed remarkable agreement between the DNS data and predictions of the **Euler ensemble**, confirming the absence of a **classical inertial range**.
- Sreeni's **awareness of this data's importance** and his support were instrumental to this breakthrough.
- This transition marks a shift from **empirical corrections** to a **universal, number-theoretic framework** for turbulence.

Verification by DNS (Sreenivasan et. al., 2024)









DNS Data Explained

- The lower-left panel shows the log derivative of the second velocity moment:
$$r\partial_r \log \left\langle (\vec{v}(\vec{r},t) - \vec{v}(0,t))^2 \right\rangle ,$$
comparing Euler ensemble predictions (green curve) and DNS data (red curve with error bars).
- **Key Insight:** Euler ensemble predictions match DNS results, while **Kolmogorov scaling** (dashed blue line) fails, confirming the absence of a classical inertial range.
- Multifractal scaling laws produce horizontal lines shifted relative to K41 but still fail to fit the data.
- Additional panels illustrate:
  - **Upper-left:** Relation between time and effective length scale.
  - **Upper- and lower-right:** Relationship between decaying kinetic energy and effective length.
- The Euler ensemble quantitatively describes all observed DNS features, offering a complete explanation.

Decaying Energy Multi Scaling laws

$$\Delta_p$$

Leading $\;-\frac{5}{4}$

$$-\frac{11}{4}$$

$$\boxed{-\frac{7}{2} \pm \frac{\imath}{2} t_n \text{ if } n \in \mathbb{Z}}$$

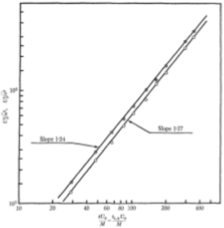
$$\boxed{-\frac{15}{4} - n \text{ if } n \in \mathbb{Z} \land n \geq 0}$$

$$\boxed{\frac{n}{2} \text{ if } n \in \mathbb{Z} \land n \geq 0}$$

$$\zeta\left(\frac{1}{2} + \imath t_n\right) = 0$$

$$E(t) \propto \sum \Re A_p t^{\Delta_p}$$

Kolmogorov-Saffman model: 6/5 = 1.2



Comte-Bellot G, Corrsin S. data: 1.25

Experimental Data Explained

- Theoretical predictions of decaying turbulent kinetic energy indices from the Euler ensemble are shown on the left. The leading index is $-5/4$, with an infinite series of decreasing negative values.
- Experimental data [2, 3] for decaying turbulence behind an oscillating grid is shown on the right.
- Log-log plots depict $1/v_{\perp}^2$ and $1/v_{\parallel}^2$, the perpendicular and parallel velocity components.
- Total inverse kinetic energy, $K = 2/(v_{\perp}^2 + v_{\parallel}^2)$, has a slope of 1.255 ± 0.02, aligning closely with the Euler ensemble prediction $\frac{5}{4}$.
- Recent experiments in a large tank [5] confirm this decay index, further validating the theory.

AI-driven Open Source Project

- The momentum loop equation and its solution via the Euler ensemble have been recently **mathematically verified** [4].
- However, computing the statistical limits of the Euler ensemble remains computationally intensive and impractical for manual methods.
- To address this, we are launching an **open source GitHub project** dedicated to:
  - Verifying and extending the statistical limits of the Euler ensemble.
  - Building a collaborative platform for community feedback, enabling skeptics to voice concerns and contribute constructively.
- The project will feature:
  - An **interactive database** of research papers, *Mathematica*® notebooks, experimental data, and DNS results.
  - An AI-driven interface for seamless exploration and analysis.
- **Heavy computations** will be handled by AI, leveraging *Mathematica*® notebooks to ensure speed and accuracy.

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Epilogue: The Road Ahead

- The **equivalence between decaying turbulence and solvable string theory** marks a paradigm shift in **theoretical physics** and **fluid mechanics**.
- This framework offers analytical tools for:
  - **Forced turbulence**,
  - **Magnetohydrodynamic (MHD) turbulence**,
  - And **other complex fluid regimes**.
- **Collaborations** with mathematicians, experimentalists, and DNS researchers are key to exploring new regimes and validating predictions.
- **Open Questions:**
  - **How does external forcing modify the turbulence-string theory duality?**
  - Can these findings extend to **compressible turbulence** or **MHD turbulence**?
  - Are there **other PDEs** exhibiting dimensional reduction?
  - What **generalizations of the Euler ensemble** exist for random walks on loop groups?

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Energy Decay in the Euler Ensemble

- Solutions originating deep within the unit circle ($\Psi \neq 1$) can become turbulent and eventually decay toward $\Psi \to 1$, reflecting energy dissipation by micro-scale vortex structures.
- The decaying solution for $\vec{P}(\theta, t)$:

$$\vec{P} = \sqrt{\frac{\nu}{2(t+t_0)}}\frac{\vec{F}}{\gamma},$$

where $\vec{F}$ satisfies the fixed-point equation:

$$\left((\Delta\vec{F})^2 - 1\right)\vec{F} = \Delta\vec{F}\left(\gamma^2\vec{F}\cdot\Delta\vec{F} + \imath\gamma\left(\frac{(\vec{F}\cdot\Delta\vec{F})^2}{\Delta\vec{F}^2} - \vec{F}^2\right)\right).$$

Fractal Curve as the Euler Ensemble

- The equation for $\vec{F}(\theta)$ was solved and analyzed in [9, 8].
- The solution describes a **fractal curve**, the limit $N \to \infty$ of a polygon $\vec{F}_0 \dots \vec{F}_N = \vec{F}_0$, with vertices:

$$\vec{F}_k = \frac{\{\cos(\alpha_k), \sin(\alpha_k), \imath \cos(\frac{\beta}{2})\}}{2 \sin(\frac{\beta}{2})},$$

where:

$$\theta_k = \frac{k}{N}, \quad \beta = \frac{2\pi p}{q}, \quad N \to \infty,$$
$$\alpha_{k+1} = \alpha_k + \sigma_k \beta, \quad \sigma_k = \pm 1, \quad \beta \sum \sigma_k = 2\pi p r.$$

- Parameters $\hat{\Omega}, p, q, r, \sigma_0 \dots \sigma_N = \sigma_0$ are **random**, making $\vec{F}(\theta)$ a **fixed manifold** instead of a fixed point.
- This manifold, called the **big Euler ensemble**, highlights the degeneracy of solutions.

Fixed Point and Degeneracy

- The fixed-point equation uses discrete versions of continuity and principal value:

$$\Delta\vec{F} \equiv \vec{F}_{k+1} - \vec{F}_k,$$
$$\vec{F} \equiv \frac{\vec{F}_{k+1} + \vec{F}_k}{2}.$$

- Both terms in the fixed-point equation vanish independently, leading to two scalar conditions:

$$(\Delta\vec{F})^2 = 1, \tag{6a}$$
$$\vec{F}^2 - \frac{\gamma^2}{4} = \left(\vec{F} \cdot \Delta\vec{F} - \frac{\imath\gamma}{2}\right)^2. \tag{6b}$$

- These equations ensure nonzero vorticity, avoiding $\vec{F} \parallel \Delta\vec{F}$.

Origin of Randomness and Integers

- Randomness in the Euler ensemble does not stem from infinitesimal stochastic forces but from the **degenerate fixed manifold** of nonlinear equations.
- This internal stochasticity resembles ergodic behavior in Newtonian dynamics, where trajectories uniformly cover the energy surface, leading to Gibbs-Maxwell distributions.
- Integer solutions ($\sigma_k = \pm 1$) emerge from:
  - Algebraic constraints in the fixed-point equations.
  - Rational quantization ($\beta = 2\pi p/q$) dictated by wave function periodicity.
- Remarkably, co-prime pairs of integers p and q define the quantized angles and radii of star polygons, reminiscent of quantum effects in magnetic fields.

The Euler Totient and the Ensemble

- The integers $1 \leq p < q$ are constrained to be co-prime with q and satisfy:

$$2 < q < N, \quad (N - q) \mod 2 = 0.$$

- The number of valid fractions $\frac{p}{q}$ for a fixed q is given by the **Euler totient function**:

$$\varphi(q) = \sum_{\substack{p=1 \\ (p,q)}}^{q-1} 1 = q \prod_{p|q} \left(1 - \frac{1}{p}\right).$$

- We named this set the **Euler ensemble** to honor Euler's contributions to both fluid dynamics and number theory.
- The inviscid Navier-Stokes equation is described by the Euler totient function rather than the Euler equation itself.