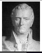
Duality of Decaying Turbulence to a Solvable String Theory

Alexander Migdal

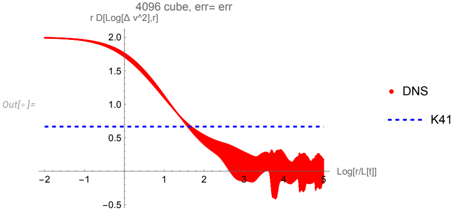
Institute for Advanced Study, Princeton, USA

London Presentation, January 2025

The Old Road to Turbulence

Discovery Year	Portrait	Key Contribution
1500s		Leonardo da Vinci: Documented early observations of turbulent flow.
1820s	 	Claude-Louis Navier and George Gabriel Stokes: Formulated the foundational NS equations.
1883		Osborne Reynolds: Characterized the transition to turbulence.
1941		Andrey Kolmogorov: Proposed scaling laws of turbulence (K41).
1944		Werner Heisenberg: Theoretical contributions to stability and dissipation models.
1946		Lars Onsager: Introduced insights into fractal velocity fields and vortex structures.
1956		Lev Landau: Advanced the understanding of hydrodynamic stability.
1960s		Richard Feynman: Explored turbulence through insights into chaotic systems.
1985	 	Giorgio Parisi and Uriel Frisch: Proposed multifractal models for turbulence.
2000-2022		K.R. Sreenivasan: DNS studies uncovering scaling law violations.

Scaling Law Violations: The Dead End of the Old Road



- Kolmogorov's K41 theory predicted universal scaling laws for velocity difference moments, such as an index of $2/3$ for the second moment's log-derivative.
- DNS simulations (4096^3 lattice) using a new method to extract effective indices from observed energy spectra [9] revealed significant deviations from K41 predictions.
- Multifractal models offered partial adjustments (e.g., shifted K41 index lines) but failed to capture the observed **nonlinear behavior** in DNS data.

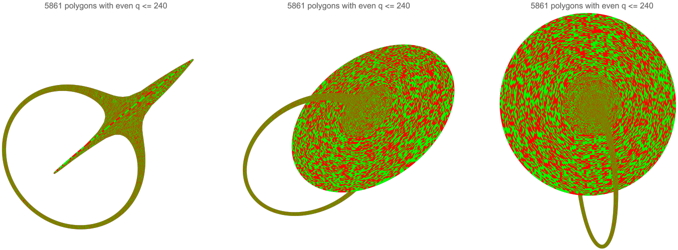
From Phenomenology to First Principles

- Kolmogorov's 1941 framework was a phenomenological model, built on simplifying assumptions and heuristic arguments.
- Experiments and DNS data exposed limitations of this model, showing violations of its scaling laws.
- A fundamental question remained: **How does randomness emerge from deterministic Navier-Stokes dynamics?**
- The only way forward: **Solve the Navier-Stokes equations exactly**, free of model assumptions or approximations.
- Progress began in the 1990s with **loop equations** [8], inspired by breakthroughs in QCD [6, 1].
- These advances culminated in a **first-principles microscopic theory** for decaying turbulence, leveraging quantum field theory methods.

Spontaneous Quantization of Classical Turbulence

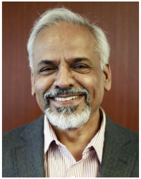
- A thirty-year effort led to an **exact, universal solution** for the Navier-Stokes equations in the turbulent regime [10].
- This solution introduces a **duality** between decaying turbulence and a solvable string theory with a discrete target space.
- Turbulent randomness arises from **spontaneous quantization**, where discrete parameters emerge from a manifold of algebraic solutions.
- Universal number-theoretic functions **quantitatively account** for DNS data violations of classical scaling laws [9].
- Collaboration with DNS experts [12] and mathematicians [4] validated this framework, **bridging theory and empirical data**.

Revealing a Hidden Identity



- We are not just computing turbulence statistics but **unveiling its hidden second identity as a discrete string theory**—a **symphony of primes beneath chaos**.
- While the mathematical structure is compelling, realization in Nature remains to be proven. (Consider the 40-year pursuit of quantum gravity via string theory.)
- Ultimately, **numerical and physical experiments** will determine whether this theory fully explains turbulence or is purely mathematical.

The Catalyst for Discovery: Sreeni's Observations



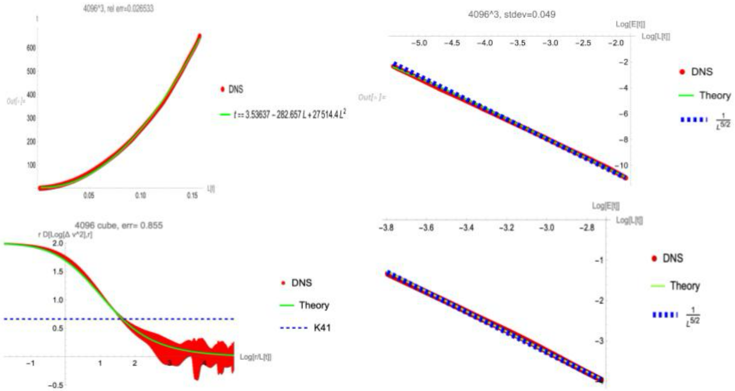
- In his 2023 ICTS talk, **K.R. Sreenivasan** presented DNS data on decaying turbulence, examining its alignment with **classical scaling laws (Kolmogorov-Saffman)**.
- He noted **persistent discrepancies**, remarking, "It is somewhat disappointing that the results are not more closely aligned with theoretical arguments."
- The key DNS dataset, briefly mentioned in a paper, remained unpublished but held clues to unresolved questions in turbulence theory.
- Sreeni went a step further by sharing the raw data from a private archive, enabling a deeper analysis and sparking productive collaboration.

The Breakthrough Analysis: Effective Index Method

- Using the **effective index method**, I magnified discrepancies in the DNS data, conclusively demonstrating that the **Kolmogorov scaling law (K41)** does not hold.
- The method introduced a **Fourier integral representation** [9] for the effective scaling index, **suppressing statistical noise** in the raw data.
- The analysis revealed a **remarkable agreement** between the DNS data and predictions of the Euler ensemble, confirming the absence of a **classical inertial range** in decaying turbulence.
- This represents a shift from **empirical models** to a **universal, number-theoretic framework** for turbulence.
- The persistence of the K41 scaling law in decaying turbulence reflects decades of fitting data to an assumed model. **Systematic biases and statistical limitations** may have reinforced its acceptance despite clear evidence of violations.

DNS Data and Theoretical Match

Verification by DNS (Sreenivasan et. al., 2024)



DNS Data Explained

- The lower-left panel shows the log derivative of the second velocity moment:
$$r\partial_r \log \left\langle (\vec{v}(\vec{r},t) - \vec{v}(0,t))^2 \right\rangle ,$$
comparing Euler ensemble predictions (green curve) and DNS data (red curve with error bars).
- **Key Insight:** Euler ensemble predictions match DNS results within error margins, while **Kolmogorov scaling** (dashed blue line) fails, confirming the absence of a classical inertial range.
- Multifractal scaling laws produce horizontal lines shifted relative to K41 but still fail to fit the data.
- Additional panels illustrate:
  - **Upper-left:** Relation between time and effective length scale.
  - **Upper- and lower-right:** Relationship between decaying kinetic energy and effective length.
- The Euler ensemble quantitatively describes all observed DNS features, offering a complete explanation.

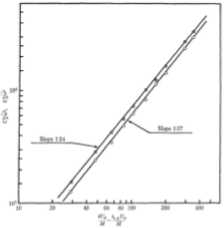
Decaying Energy Multi Scaling laws

$$
\begin{array}{|c|}
\Delta_p \\
\text{Leading} \rightarrow \; -\frac{5}{4} \\
-\frac{11}{4} \\
\boxed{-\frac{7}{2} \pm \frac{\imath}{2} t_n \text{ if } n \in \mathbb{Z}} \\
\boxed{-\frac{15}{4} - n \text{ if } n \in \mathbb{Z} \wedge n \geq 0} \\
\hline
\boxed{\frac{n}{2} \text{ if } n \in \mathbb{Z} \wedge n \geq 0}
\end{array}
$$

$$\zeta\left(\frac{1}{2} + \imath t_n\right) = 0$$

$$E(t) \propto \sum \Re A_p t^{\Delta_p}$$

Kolmogorov-Saffman model: 6/5 = 1.2



Comte-Bellot G, Corrsin S. data: 1.25

Experimental Data Explained

- Theoretical predictions of **decaying turbulent kinetic energy indices** from the Euler ensemble are shown on the left. The leading index is $-5/4$, with an **infinite series** of decreasing negative values.
- Experimental data [2, 3] for **decaying turbulence behind an oscillating grid** is shown on the right.
- Log-log plots depict $1/v_{\perp}^2$ and $1/v_{\|}^2$, the perpendicular and parallel velocity components.
- Total inverse kinetic energy, $K = 2/(v_{\perp}^2 + v_{\|}^2)$, has a slope of 1.255 ± 0.02, aligning closely with the Euler ensemble prediction $\frac{5}{4}$.
- Recent experiments in a **large tank** [5] confirm this **decay index**, further validating the theory.

Explicit formula for velocity correlation

The statistical limit of the Euler ensemble $N \to \infty, \nu \to 0, \tilde{\nu} = \nu N^2 =$ *const* can be computed in quadrature.

Here is the resulting **formula for the second moment of velocity difference** in decaying turbulence:

$$\left\langle \Delta \vec{v}^2 \right\rangle (r) = \frac{\tilde{\nu}^2}{\nu t} \int_{\epsilon - \imath\infty}^{\epsilon + \imath\infty} \frac{dp}{2\pi\imath} V(p) \left(\frac{|\vec{r}|}{\sqrt{\tilde{\nu} t}} \right)^p ; \qquad (1)$$

$$V(p) = -\frac{f(-1-p)\zeta\left(\frac{13}{2} - p\right)\csc\left(\frac{\pi p}{2}\right)}{16\pi^2 (p+1)(2p-15)(2p-5)\zeta\left(\frac{15}{2} - p\right)} \qquad (2)$$

Here $f(z)$ is an entire function computed using Mellin integrals of elementary functions. $V(p)$ is meromorphic. ν is physical viscosity and turbulent viscosity $\tilde{\nu}$ is a free parameter of our solution.

Spectrum of indices of velocity correlation

The spectrum of indices for velocity correlation is given by the poles of $V(p)$

$$\begin{array}{|c|}
\text{indexes of velocity correlation} \\
\hline
-1 \\
0 \\
\hline
\boxed{2n \text{ if } n \in \mathbb{Z} \wedge n \geq 1} \\
5/2 \\
11/2 \\
\boxed{\frac{15+4n}{2} \text{ if } n \in \mathbb{Z} \wedge n \geq 0} \\
\boxed{7 \pm \imath t_n \text{ if } n \in \mathbb{Z}}
\end{array} \quad (3)$$

where $\frac{1}{2} \pm \imath t_n$ are the **zeros of the Riemann ζ function**.

Quantum oscillations predicted

- As mentioned about the energy spectrum, there is no K41 scaling index $p = \frac{2}{3}$. This omission is not a contradiction, as K41 does not apply to decaying turbulence.
- Instead of pure scaling laws with single decay indexes, we found an **infinite spectrum of scaling indexes**, some of which come as complex conjugate pairs, which leads to **quantum oscillations** of the index at large $r/\sqrt{t}$.
- The imaginary parts of these complex scaling dimensions coincide with those of the **famous Riemann zeta zeros**, establishing an intriguing relation between Turbulence and Number Theory.
- These oscillations are inaccessible with the modern lattice sizes in the DNS, but rather would require lattices like $24K$, available in near future.

Quantum Oscillations Predicted

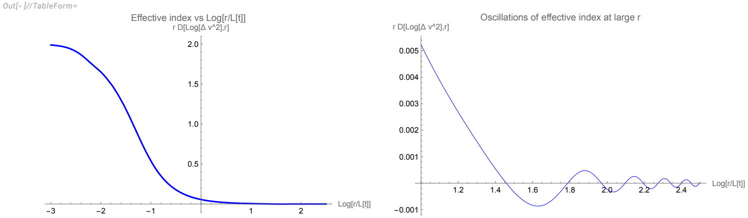


Figure: Oscillations of the effective index $\xi_2(r)$ at large $\log_{10} r$. This is a theoretical curve corresponding to the zoom into a region of large separations, currently inaccessible by DNS with the required accuracy.

Theory: Loop Average and Dimension Reduction

- The loop average is defined as the **characteristic function** for the distribution of velocity circulation:

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu}\Gamma_C\right)\right\rangle, \quad \Gamma_C = \oint d\vec{C}(\theta) \cdot \vec{v}(\vec{C}(\theta)).$$

- This is equivalent to the **Hopf functional** representation:

$$\Psi[\gamma, C] = \left\langle \exp\left(\int_{\vec{r}\in\mathbb{R}^3} \vec{J}_C(\vec{r}) \cdot \vec{v}(\vec{r})\, d^3r\right)\right\rangle.$$

- An imaginary source $\vec{J}(\vec{r})$ is concentrated on a fixed loop in space $\mathbb{R}_3$

$$\vec{J}_C(\vec{r}) = \frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta)\delta^3\left(\vec{r} - \vec{C}(\theta)\right)$$

Loop Equation as Quantum Mechanics in Loop Space

- The loop functional satisfies a key relation derived from the incompressible Navier-Stokes equation:

$$\imath\nu\partial_t\Psi[\gamma, C] = \left\langle \gamma \oint d\vec{C}(\theta) \cdot \left(-\nu\vec{\nabla} \times \vec{\omega} + \vec{v} \times \vec{\omega}\right) e^{\frac{\imath\gamma}{\nu}\Gamma_C} \right\rangle.$$

- This relation leads to the closed functional equation for the loop average [8]:

$$\imath\nu\partial_t\Psi[\gamma, C] = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta\vec{C}(\cdot)}\right]\Psi[\gamma, C].$$

- A crucial property: **lack of** C- **dependence** of $\vec{L}\left[\frac{\delta}{\delta\vec{C}(\cdot)}\right]$ in loop space enables an exact solution.

Key Insight: Plane Wave Solutions in Loop Space

- The loop equation maps to a **Schrödinger equation in loop space** with a Hamiltonian:

$$\hat{H}_C = \oint d\vec{C}(\theta) \cdot \vec{L}\left[\frac{\delta}{\delta \vec{C}(\cdot)}\right].$$

- A **plane wave solution** emerges naturally:

$$\Psi[\gamma, C] = \left\langle \exp\left(\frac{\imath\gamma}{\nu} \oint d\vec{C}(\theta) \cdot \vec{P}(t,\theta)\right)\right\rangle.$$

- The reduced momentum loop equations (MLE) describe the dynamics of $\vec{P}(t,\theta)$, with $\Delta\vec{P} = \vec{P}(\theta_+) - \vec{P}(\theta_-)$ being a discontinuity:

$$\nu \partial_t \vec{P} = -\gamma^2 (\Delta\vec{P})^2 \vec{P} + \Delta\vec{P}\left(\gamma^2 \vec{P} \cdot \Delta\vec{P} + \imath\gamma\left(\frac{(\vec{P} \cdot \Delta\vec{P})^2}{\Delta\vec{P}^2} - \vec{P}^2\right)\right).$$

The Euler Ensemble Solves MLE

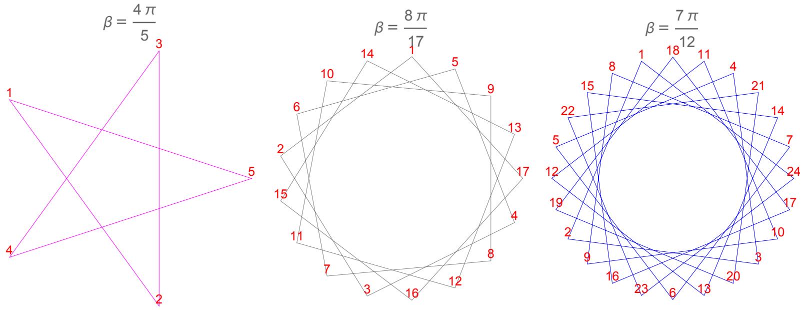
- The decaying solution [10] of MLE:

$$\vec{P}(t,\theta) = \sqrt{\frac{\nu}{2(t+t_0)}}\frac{\vec{F}(\theta)}{\gamma},$$

  involves a **fractal curve** $\vec{F}(\theta)$, defined as the limit $N \to \infty$ of the regular star polygon $\{q/p\}$ with unit side and radius $R = \frac{1}{2\sin \pi p/q}$. The plane of that polygon is rotated by an arbitrary matrix $\hat{\Omega} \in SO(3)$. A random walk takes place around this polygon with steps $\sigma_l = \pm 1$ $(l = 1,\dots,N)$, producing a winding number $w = rq$.
- The parameters $\hat{\Omega}, p, q, r, \sigma_0 \dots \sigma_N = \sigma_0$ are **random**, making $\vec{F}(\theta)$ a **fixed manifold** rather than a fixed point.
- **Validation:** The solution has been **rigorously validated** in collaboration with mathematicians [4] and cross-verified using analytic computations in *Mathematica*®notebooks [7].
- **Significance:** This solution quantitatively **links turbulence to number theory**, marking a paradigm shift in fluid dynamics.

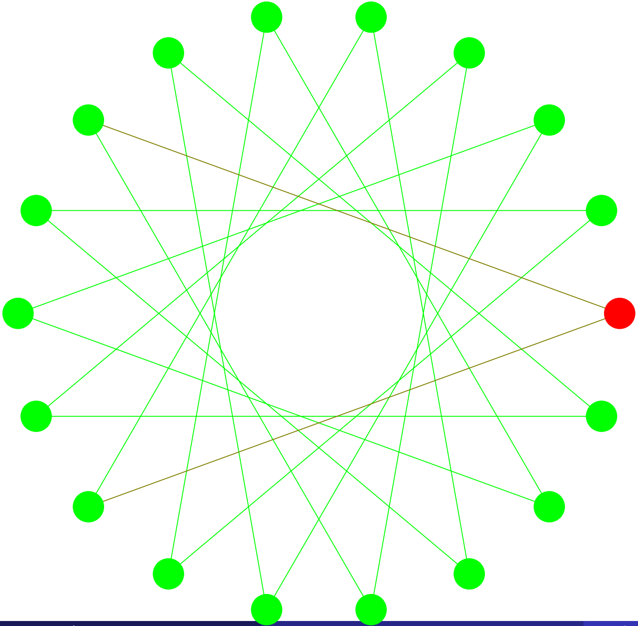
Regular Star Polygons for Euler Ensembles

Out[∘]=



Regular star polygons for Euler ensembles of various p, q.
Classified by **Thomas Bradwardine** (c. 1300 – 26 August 1349), the Archbishop of Canterbury.
The σ_k **variable** determines the direction of the random step along the link $k \leftrightarrow k+1$.
In general, the **random walk** can loop multiple times around the polygon, provided it returns to its starting point.

Random Walk on Star Polygon, $p = 7, q = 18, N = 100$



Visualization of a

The quantum ergodic hypothesis

- The statistical distribution of a solution to the nonlinear classical Navier-Stokes PDE is linked to a wave functional that satisfies a linear Schrödinger equation in loop space.
- The quantization mechanism mirrors that of ordinary quantum mechanics, relying on the **periodicity** of the solution.
- Each **distinct state** in a quantum system contributes to the partition function with unit weight, a principle adopted in this **quantum solution** of the nonlinear classical system.
- This **quantum statistical counting** is an **extra conjecture**, akin to the ergodic hypothesis in reversible Newtonian mechanics. While widely accepted in physics, such conjectures are mathematically proven only in specific cases.

Turbulence/String Duality: Insights

- **Duality phenomena** in physics connect strong coupling in one theory to weak coupling in another (e.g., **AdS/CFT correspondence**).
- Duality links **statistical averages** of two systems, without requiring a direct correspondence between their dynamical variables.
- For turbulence, the **target space** of the dual string theory is discrete, represented by **regular star polygons** with unit sides and rational angles $\beta = 2\pi \frac{p}{q}$.
- Periodicity conditions relate the **winding number** to q.
- Fermionic/Ising degrees of freedom $(\nu_k = 0, 1, \sigma_k = \pm 1)$ account for the random walk along polygon edges.
- The quantized radia of these polygons $R = \frac{1}{2 \sin \pi \frac{p}{q}}$ are distributed by the laws on number theory, involving Euler totients and Riemann zeta function [9].

Dual Amplitude and Loop Functional

- The loop functional in the Euler ensemble corresponds to the **dual amplitude** of string theory with discrete target space $\vec{F}(\theta)$ and distributed external momentum $\vec{Q}(\theta)$:

$$\Psi[C,t] \propto \left\langle \exp \imath \oint d\theta \vec{F}(\theta) \cdot \vec{Q}(\theta) \right\rangle_{F,\sigma}, \tag{4}$$

$$\vec{Q}(\theta) = \frac{\vec{C}'(\theta)}{\sqrt{2\nu(t+t_0)}}. \tag{5}$$

- Averaging over **string target space** $\vec{F}_k$ corresponds to summing over star polygons with unit side lengths and rational angles $\beta = 2\pi \frac{p}{q}$.
- Averaging over **fermionic/Ising degrees of freedom** leads to a random walk (Brownian motion in the continuum limit) on these polygons.

Discrete Symmetry and Target Space Integration

- **Contrast:** Turbulence features chaotic velocity fields, while the dual string theory exhibits **discrete symmetries** and harmonic fluctuations around a nonlinear classical solution for the fermion density = angle on a circle.
- Integration over the target space (star polygons) reduces to a **discrete sum** over Euler ensemble states:
  - Rational numbers $\frac{p}{q} \in \mathbb{Q}$,
  - Fermionic configurations $(\nu_k = 0, 1)$,
  - Winding number $w = \frac{p}{q} \sum (2\nu_k - 1)$.
- Visualizing polygons for fixed N , ordered by angle β, as a torus in 3D space, reveals the **world sheet** of a discrete string. Each polygon is a cross-section of this torus, with the thin part corresponding to small p, q and large cross-sections correspond to $q \to \infty$ at fixed p or $q - p$.
- Red/green coloring of polygon edges indicates fermionic occupation numbers (random walk directions).

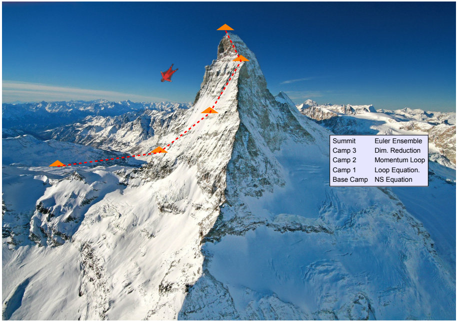
AI-driven Open Source Project

- The **momentum loop equation** and its solution via the Euler ensemble have been recently **mathematically verified** [4].
- However, computing the statistical limits of the Euler ensemble remains computationally intensive and impractical for manual methods.
- To address this, we are launching an **open source GitHub project**[11] dedicated to:
  - **Verifying and extending** the statistical limits of the Euler ensemble.
  - Building a **collaborative platform** for community feedback, enabling skeptics to voice concerns and contribute constructively.
- The project will feature:
  - An **interactive database** of research papers, *Mathematica*® notebooks, experimental data, and DNS results.
  - An **AI-driven interface** for seamless exploration and analysis.
  - **Heavy computations** will be handled by *Mathematica*®.

Epilogue: The Road Ahead

- The **equivalence between decaying turbulence and solvable string theory** marks a paradigm shift in **fluid mechanics**.
- This framework offers analytical tools for:
  - **Forced turbulence**,
  - **Magnetohydrodynamic (MHD) turbulence**,
  - And **other complex fluid regimes**.
- **Collaborations** with mathematicians, experimentalists, and DNS researchers are key to exploring new regimes and validating predictions.
- **Open Questions:**
  - **How does external forcing modify the turbulence-string theory duality?**
  - How to extend these findings to **compressible turbulence** or **MHD turbulence**?
  - Are there **other PDEs** exhibiting dimensional reduction?
  - What **generalizations of the Euler ensemble** exist for random walks on loop groups?

Closing Thoughts: The Ascent to Understanding



"Scaling the summit of turbulence theory is like climbing a formidable mountain. Each milestone—Navier-Stokes equations, loop equations, dimension reduction, and Euler ensemble —leads to the breathtaking view at the summit."

- The **Euler ensemble** not only explains decaying turbulence but offers a **new framework** for addressing fluid dynamics' most challenging puzzles.
- This climb has reached a significant peak, but the journey is far from over.
- I invite you to join this **expedition** as we unite theory, experiments, and simulations to unravel turbulence's enduring mysteries.

References I

References II

[1] A. Migdal. " Loop equations and $\frac{1}{N}$ expansion". In: *Physics Reports* 201 (1983).

[2] Geneviéve Comte-Bellot and Stanley Corrsin. "The use of a contraction to improve the isotropy of grid-generated turbulence". In: *Journal of Fluid Mechanics* 25.4 (1966), pp. 657–682. DOI: 10.1017/S0022112066000338.

[3] Geneviève Comte-Bellot and Stanley Corrsin. "Simple Eulerian time correlation of full-and narrow-band velocity signals in grid-generated, 'isotropic' turbulence". In: *Journal of Fluid Mechanics* 48.2 (1971), pp. 273–337. DOI: 10.1017/S0022112071001599.

[4] C. DeLellis E. Brue and A. Migdal. *On the mathematical foundation of duality of fluid mechanics*. "In preparation". 2025.

References III

[5] Jean-Baptiste Gorce and Eric Falcon. "Freely Decaying Saffman Turbulence Experimentally Generated by Magnetic Stirrers". In: *Phys. Rev. Lett.* 132 (26 June 2024), p. 264001. DOI: 10.1103/PhysRevLett.132.264001. URL: https://link.aps.org/doi/10.1103/PhysRevLett.132.264001.

[6] Yu.M. Makeenko and A.A. Migdal. "Exact equation for the loop average in multicolor QCD". In: *Physics Letters B* 88.1 (1979), pp. 135–137. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(79)90131-X. URL: https://www.sciencedirect.com/science/article/pii/037026937990131X.

[7] Alexander Migdal. *Decaying Turbulence Computations*. https://www.wo sasha.migdal/Published/DecayingTurbulenceComputations.nb. Sept. 2023.

References IV

[8] Alexander Migdal. "Loop Equation and Area Law in Turbulence". In: *Quantum Field Theory and String Theory*. Ed. by Laurent Baulieu et al. Springer US, 1995, pp. 193–231. DOI: 10.1007/978-1-4615-1819-8. URL: https://arxiv.org/abs/hep-th/9310088.

[9] Alexander Migdal. "Quantum solution of classical turbulence: Decaying energy spectrum". In: *Physics of Fluids* 36.9 (2024), p. 095161. DOI: 10.1063/5.0228660.

[10] Alexander Migdal. "To the Theory of Decaying Turbulence". In: *Fractal and Fractional* 7.10 (Oct. 2023), p. 754. ISSN: 2504-3110. DOI: 10.3390/fractalfract7100754. arXiv: 2304.13719 [physics.flu-dyn]. URL: http://dx.doi.org/10.3390/fractalfract7100754.

[11] Alexander Migdal. *TurbulenceDuality*. https://github.com/sashamigdal/ Open source project for study of Turbulencve Suality. Dec. 2025.

[12] John Panickacheril John, Diego A Donzis, and Katepalli R Sreenivasan. "Laws of turbulence decay from direct numerical simulations". en. In: *Philos. Trans. A Math. Phys. Eng. Sci.* 380.2218 (Mar. 2022), p. 20210089.

Energy Decay in the Euler Ensemble

- Solutions originating deep within the unit circle ($\Psi \neq 1$) can become turbulent and eventually decay toward $\Psi \to 1$, reflecting energy dissipation by micro-scale vortex structures.
- The decaying solution for $\vec{P}(\theta, t)$:

$$\vec{P} = \sqrt{\frac{\nu}{2(t+t_0)}}\frac{\vec{F}}{\gamma},$$

where $\vec{F}$ satisfies the fixed-point equation:

$$\left((\Delta\vec{F})^2 - 1\right)\vec{F} = \Delta\vec{F}\left(\gamma^2\vec{F}\cdot\Delta\vec{F} + \imath\gamma\left(\frac{(\vec{F}\cdot\Delta\vec{F})^2}{\Delta\vec{F}^2} - \vec{F}^2\right)\right).$$

Fractal Curve as the Euler Ensemble

- The equation for $\vec{F}(\theta)$ was solved and analyzed in [10, 9].
- The solution describes a **fractal curve**, the limit $N \to \infty$ of a polygon $\vec{F}_0 \dots \vec{F}_N = \vec{F}_0$, with vertices:

$$\vec{F}_k = \frac{\{\cos(\alpha_k), \sin(\alpha_k), \imath \cos(\frac{\beta}{2})\}}{2\sin(\frac{\beta}{2})},$$

where:

$$\theta_k = \frac{k}{N}, \quad \beta = \frac{2\pi p}{q}, \quad N \to \infty,$$
$$\alpha_{k+1} = \alpha_k + \sigma_k \beta, \quad \sigma_k = \pm 1, \quad \beta \sum \sigma_k = 2\pi p r.$$

- Parameters $\hat{\Omega}, p, q, r, \sigma_0 \dots \sigma_N = \sigma_0$ are **random**, making $\vec{F}(\theta)$ a **fixed manifold** instead of a fixed point.
- This manifold, called the **big Euler ensemble**, highlights the degeneracy of solutions.

Fixed Point and Degeneracy

- The fixed-point equation uses discrete versions of continuity and principal value:

$$\Delta\vec{F} \equiv \vec{F}_{k+1} - \vec{F}_k,$$
$$\vec{F} \equiv \frac{\vec{F}_{k+1} + \vec{F}_k}{2}.$$

- Both terms in the fixed-point equation vanish independently, leading to two scalar conditions:

$$(\Delta\vec{F})^2 = 1, \tag{6a}$$
$$\vec{F}^2 - \frac{\gamma^2}{4} = \left(\vec{F}\cdot\Delta\vec{F} - \frac{\imath\gamma}{2}\right)^2. \tag{6b}$$

- These equations ensure nonzero vorticity, avoiding $\vec{F} \parallel \Delta\vec{F}$.

Origin of Randomness and Integers

- Randomness in the Euler ensemble does not stem from infinitesimal stochastic forces but from the **degenerate fixed manifold** of nonlinear equations.
- This internal stochasticity resembles ergodic behavior in Newtonian dynamics, where trajectories uniformly cover the energy surface, leading to Gibbs-Maxwell distributions.
- Integer solutions ($\sigma_k = \pm 1$) emerge from:
  - Algebraic constraints in the fixed-point equations.
  - Rational quantization ($\beta = 2\pi p/q$) dictated by wave function periodicity.
- Remarkably, co-prime pairs of integers p and q define the quantized angles and radii of star polygons, reminiscent of quantum effects in magnetic fields.

The Euler Totient and the Ensemble

- The integers $1 \leq p < q$ are constrained to be co-prime with q and satisfy:
$$2 < q < N, \quad (N-q) \mod 2 = 0.$$
- The number of valid fractions $\frac{p}{q}$ for a fixed q is given by the **Euler totient function**:
$$\varphi(q) = \sum_{\substack{p=1 \\ (p,q)}}^{q-1} 1 = q \prod_{p|q} \left(1 - \frac{1}{p}\right).$$
- We named this set the **Euler ensemble** to honor Euler's contributions to both fluid dynamics and number theory.
- The inviscid Navier-Stokes equation is described by the Euler totient function rather than the Euler equation itself.