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# PO Assignment 3

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Ans 1-> $f(x_1, x_2) = x_1 - x_2 + 2x_1^2 + 2x_1x_2 + x_2^2$

$$x_1 = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix} \quad X_2 = \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} \quad X_3 = \begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$$

$\alpha = 1, \beta = 0.5, \gamma = 2$ & $\varepsilon = 0.2$

$$f_1 = f(X_1) = 4 - 4 + 2(4)^2 + 2(4)^2 + (4)^2$$
$$= 80$$
$$f_2 = f(X_2) = 5 - 4 + 2(5)^2 + 2(5)(4) + 4^2$$
$$= 107.$$
$$f_3 = f(X_3) = 4 - 5 + 2(4)^2 + 2(4)(5) + (5)^2$$
$$= 96$$

$$X_4 = X_2 = \begin{Bmatrix} 5 \\ 4 \end{Bmatrix} \quad f(X_4) = 107$$

$$X_L \Rightarrow X_1 = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \Rightarrow f(X_1) = 80$$

$\Rightarrow$ Centroid point $(X_0) = \dfrac{X_1 + X_3}{2} = \dfrac{1}{2}\begin{Bmatrix} 4+5 \\ 4+5 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 4.5 \end{Bmatrix}$

$$f(X_0) = 4 - 4.5 + 2(4)^2 + 2(4)(4.5) + (4.5)^2$$
$$= 87.75$$

$\Rightarrow$ Reflection point $(X_r) = 2X_0 - X_4 = \begin{Bmatrix} 8 \\ 9 \end{Bmatrix} - \begin{Bmatrix} 5 \\ 4 \end{Bmatrix}$

$$\Rightarrow X_r = \begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$$

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$f(X_r) = 3.5 + 2(3)^2 + 2(3)(3.5) + (3.5)^2$
$= 71$

Since $f(X_r) < f(X_L)$

We find $X_e = 2X_r - X_0$

$= \begin{Bmatrix} 6 \\ 10 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 4.5 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}$

$f(X_e) = 2 - 5.5 + 2(2)^2 + 2(2)(5.5) + (5.5)^2$
$= 56.75$

$f(X_e) < f(X_L)$ So we replace $X_4$ by $X_e$.

And new vertices are

$X_1 = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}, X_2 = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}, X_3 = \begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$

Convergence, $Q = \left\{ \sum_{i=1}^{n+1} \frac{[f(X_i) - f(X_0)]^2}{n+1} \right\}^{1/2} \leq \varepsilon$

$Q = \left[ \frac{(80 - 87.75)^2 + (56.75 - 87.75)^2 + (96 - 87.75)^2}{3} \right]^{1/2}$

$Q = 19.06$ not smaller than $0.2$

Iteration 2.

$X_1 = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}, X_2 = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}; X_3 = \begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$

$f(X_1) = 80 \quad f(X_2) = 56.75 \quad f(X_3) = 96$

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$X_4 = X_3 = \begin{Bmatrix} 4 \\ 5 \end{Bmatrix}$, $f(X_4) = 96$.

$X_L = X_2 = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}$, $f(X_L) = 56.75$

$X_0 = \frac{X_1 + X_2}{2} = \frac{1}{2}\begin{Bmatrix} 6 \\ 9.5 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 4.75 \end{Bmatrix}$ $f(X_0) = 67.31$

$X_r = 2X_0 - X_4 = \begin{Bmatrix} 6 \\ 9.5 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 4.5 \end{Bmatrix}$

$f(X_r) = 2 - 4.5 + 2(2)^2 + 2(2)(4.5) + (4.5)^2$
$= 43.75$

$f(X_r) < f(X_L)$

$X_e = 2X_r - X_0 = \begin{Bmatrix} 4 \\ 9 \end{Bmatrix} - \begin{Bmatrix} 3 \\ 4.75 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4.25 \end{Bmatrix}$

$f(X_e) = 1 - 4.25 + 2(1)^2 + 2(1)(4.25) + (4.25)^2$
$= 25.3125$

$f(X_e) < f(X_L)$, So, we replace $X_4$ by $X_e$.

New vertices $\Rightarrow$ $X_1 = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$, $X_2 = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}$, $X_3 = \begin{Bmatrix} 1 \\ 4.25 \end{Bmatrix}$

$Q = 26.1$ (not less than 0.2)

Iteration 3 $f(X_1) = 80$, $f(X_2) = 56.75$, $f(X_3) = 25.3125$

$X_4 = X_1 = \begin{Bmatrix} 4 \\ 4 \end{Bmatrix}$ $f(X_4) = 80$

$X_L = X_3 = \begin{Bmatrix} 1 \\ 4.25 \end{Bmatrix}$ $f(X_L) = 25.3125$

Scanned with CamScanner

$$X_0 = \frac{X_2 + X_3}{2} = \begin{bmatrix} 2+1 \\ 5.5+4.25 \end{bmatrix} \times \frac{1}{2}$$

$$= \begin{bmatrix} 1.5 \\ 4.875 \end{bmatrix}$$

$$f(X_0) = 1.5 - 4.875 + 2(1.5)^2 + 2(1.5)(4.875) + (4.875)^2$$
$$= 39.515$$

$$X_r = 2X_0 - X_4 = \begin{bmatrix} 3 \\ 9.75 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$X_r = \begin{bmatrix} -1 \\ 5.75 \end{bmatrix}$$

$$f(X_r) = -1 - 5.75 + 2(-1)^2 + 2(-1)(5.75) + (5.75)^2$$
$$= 16.8125$$

Since $f(X_r) < f(X_L)$, So, we find $X_e$.

$$X_e = 2X_r - X_0 = \begin{bmatrix} -2 \\ 11.5 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 4.875 \end{bmatrix}$$

$$X_e = \begin{bmatrix} -3.5 \\ 6.625 \end{bmatrix}$$

$$f(X_e) = -3.5 - 6.625 + 2(-3.5)^2 + 2(-3.5)(6.625) + (6.625)^2$$

$$= 11.89$$

$f(X_e) < f(X_L)$ ∴ So we replace $X_4$ by $X_e$.

$$X_1 = \begin{bmatrix} -3.5 \\ 6.625 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 2 \\ 5.5 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1 \\ 4.25 \end{bmatrix}$$

Convergence, $Q = \left[\dfrac{(11.89-39.5715)^2 + (56.75-39.5715)^2 + (25.3125-39.5715)^2}{3}\right]^{1/2}$

$\Rightarrow Q = 20.51$ (not less than $\varepsilon$)

Iteration 4

$X_1 = \begin{Bmatrix} -3.5 \\ 6.625 \end{Bmatrix}$, $X_2 = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}$, $X_3 = \begin{Bmatrix} 1 \\ 4.25 \end{Bmatrix}$

$f(X_1) = 11.89$, $f(X_2) = 56.75$, $f(X_3) = 25.3125$

$X_4 = X_2 = \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix}$ $f(X_4) = 56.75$

$X_L = X_1 = \begin{Bmatrix} -3.5 \\ 6.625 \end{Bmatrix}$ ; $f(X_L) = 11.89$

$X_0 = \dfrac{X_1 + X_3}{2} = \dfrac{1}{2}\begin{Bmatrix} -3.5+1 \\ 6.625+4.25 \end{Bmatrix} = \begin{Bmatrix} -1.25 \\ 5.4375 \end{Bmatrix}$

$f(X_0) = -1.25 - 5.4375 + 2(-1.25)^2 + 2(-1.25)(5.4375) + (5.4375)^2$

$f(X_0) = 12.41$

$X_r = 2X_0 - X_4 = \begin{Bmatrix} -2.5 \\ 10.875 \end{Bmatrix} - \begin{Bmatrix} 2 \\ 5.5 \end{Bmatrix} = \begin{Bmatrix} -4.5 \\ 5.375 \end{Bmatrix}$

$f(X_r) = -4.5 - 5.375 + 2(-4.5)^2 + 2(-4.5)(5.375) + (5.375)^2$
$= 11.14$

$f(X_r) < f(X_L)$ ; So we find $X_e$,

$X_e = 2X_r - X_0 = \cancel{-7.75}\ \begin{Bmatrix} -7.75 \\ 5.3125 \end{Bmatrix}$

$f(X_e) = -7.75 - 5.3125 + 2.(-7.75)^2 + 2(-7.75)(5.3125) + (5.3125)^2$

$= 52.9414$

$f(X_e) > f(X_r)$, so we replace $X_u$ by $X_r$

Iteration 5

$X_1 = \begin{bmatrix} -3.5 \\ 6.625 \end{bmatrix}$, $X_2 = \begin{bmatrix} -4.5 \\ 5.375 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 4.25 \end{bmatrix}$

$f(X_1) = 11.89$, $f(X_2) = 11.14$, $f(X_3) = 25.3125$

So after 5 iterations $f(X_2)$ is the minimum

Ans 2 ⇒ $f(x) = x_1^2 + x_1 x_2 + x_2^2 + 3x_1$

a) $\nabla f(x) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 + 3 \\ x_1 + 2x_2 \end{bmatrix}$

$\nabla f(x) = 0$ (for extremum points)

$\left.\begin{matrix} 2x_1 + x_2 + 3 = 0 \\ x_1 + 2x_2 = 0 \end{matrix}\right\} \rightarrow x^* = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

finding eigen values]

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

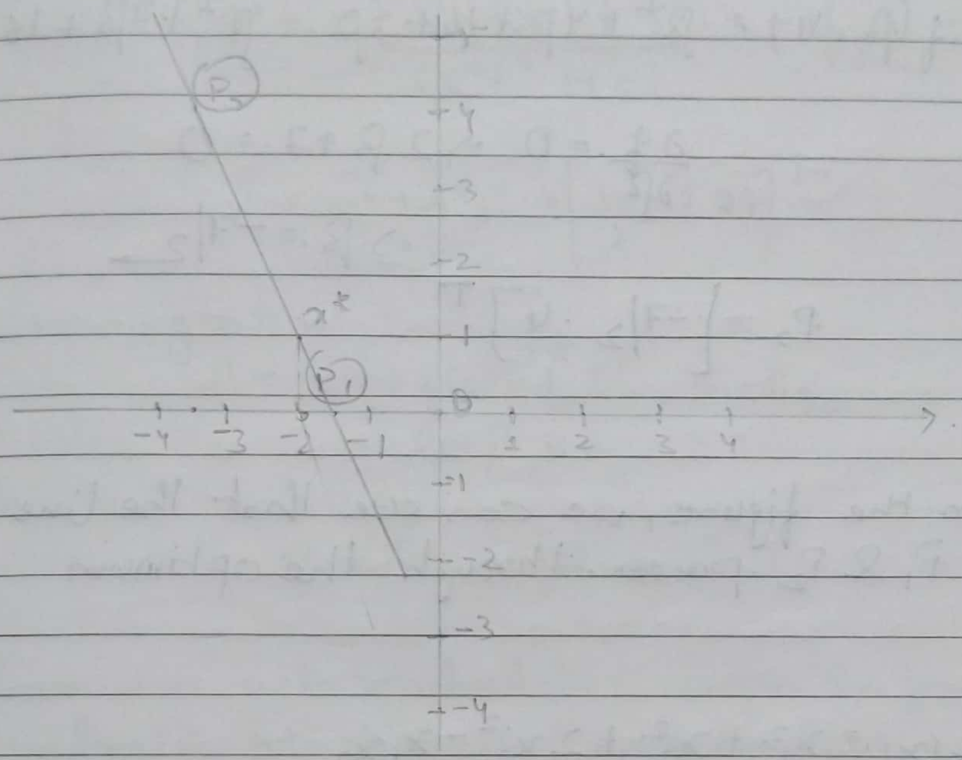
$\Rightarrow (2-\lambda)^2 - 1 = 0$. $2-\lambda = 1$ or $2-\lambda = -1$

$\lambda = 1$ $\lambda = 3$

So, $\lambda = (+ve)$ definite so, $x^*$ is a minimum.

(b) Since $f(x)$ has only one stationary point. So it is a global minimum.

(c)




(d) A univariate search will be a good method because the function is quadrate & well scaled. But the search direction must be chosen appropriately.

(e) Let $\lambda \rightarrow$ step size, So starting from $[0\ 0]^T$; the next point will be $[\lambda\ 0]^T$.

$$f(\lambda) = \lambda^2 + 3\lambda$$

$$\frac{\partial f}{\partial \lambda} = 0 \quad , \quad 2\lambda + 3 = 0$$

$$\Rightarrow \lambda = -3/2$$

$$P_1 = \begin{bmatrix} -3/2 & 0 \end{bmatrix}^T$$

for the point $[0\ 4]^T$, let $\beta$ be step size,
then $P_2 = [\beta\ 4]^T$.

$f(\beta, 4) = \beta^2 + 4\beta + 16 + 3\beta = \beta^2 + 7\beta + 16.$

$\frac{\partial f}{\partial \beta} = 0 \Rightarrow 2\beta + 7 = 0$

$\Rightarrow \beta = -7/2$

$P_2 = [-7/2\ \ 4]^T$

(f) In the figure, we can see that the line joining $P_1$ & $P_2$ passes through the optimum

Ans 2-) $f(x) = x_1^2 + x_2^2 + 2x_3^2 - x_1 x_2$

$\nabla f(x) = \begin{bmatrix} 2x_1 - x_2 \\ 2x_2 - x_1 \\ 4x_3 \end{bmatrix} \qquad x = \begin{bmatrix} +2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

H is (+ve) definite & stationary point $[0\ 0\ 0]^T$ is minimum

let $x^0 = [1\ 1\ 1]^T$ & $s^0 = [1\ 0\ 0]^T$.
If $s^1$ is conjugate to $s^0$, w.r.t H then

$(s^1)^T H s^0 = 0$

$$(s^1)^T \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0$$

$(s^1) = [1\ 2\ 0]^T$

Starting at $x^0$.

$$d^0 = -\frac{[\nabla f(x^0)]^T s^0}{(s^0)^T H s} = -1/2$$

$$x^1 = x^0 + d^0 s^0 = \left[\frac{1}{2}\ 1\ 0\right]^T$$

Starting at $x^1$

$$d^1 = -\frac{[\nabla f(x^1)]^T s^1}{(s^1)^T H s^1} = -1/2$$

$$x^2 = x^1 + d^1 s^1 = [0\ 0\ 1]^T.$$

Minimum is not reached in 2 steps. So, for a quadratic function of 3 independent variable, 3 steps will be required.

Ans 4- $f(x) = x_1^2 + x_1 x_2 + 16x_2^2 + x_3^2 - x_1 x_2 x_3$

$$s^{(1)} = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \qquad s^{(2)} = \begin{bmatrix} -1/\sqrt{3} \\ 2/\sqrt{3} \\ 0 \end{bmatrix}$$

$$\nabla f(x) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 - x_2 x_3 \\ 32x_2 + x_1 - x_1 x_3 \\ 2x_3 - x_1 x_2 \end{bmatrix}$$

$$H(x) = \begin{bmatrix} 2 & 1-x_3 & -x_2 \\ 1-x_3 & 32 & -x_1 \\ -x_2 & -x_1 & 2 \end{bmatrix}$$

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$s^1$ & $s^2$ will be conjugate w.r.t $H(x)$
when $(s^1)^T H(x) s^2 = 0$.

So, $|H| > 0$ or all the eigen values must be (+ve)

So. $\begin{bmatrix} -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} \end{bmatrix} \begin{bmatrix} 2 & 1-x_3 & -x_2 \\ 1-x_3 & 32 & -x_1 \\ -x_2 & -x_1 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt3 \\ 2/\sqrt3 \\ 0 \end{bmatrix} = 0$

on solving we get,

$$2x_1 - x_2 + 3x_3 + 63 = 0$$

And $H(x)$ has to be (+ve) definite
So, $x$ must lie on above plane and satisfy

$(1-x_3)^2 \leq 64$ or $-7 \leq x_3 \leq 9$

$x_2^2 \leq 4 \Rightarrow -2 \leq x_2 \leq 2$

$x_1^2 \leq 64 \Rightarrow -8 \leq x_1 \leq 8$

& $128 - 2x_1^2 - 32x_2^2 + 2x_1x_2(1-x_3) - 2(1-x_3)^2 > 0$

Ans 5 ⇒ $f(x) = 5x_1^2 + x_2^2 + \cancel{2x} 2x_1x_2 - 12x_1 - 4x_2 + 8$
Starting at $(0, -2)$

$\nabla f(x) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 10x_1 + 2x_2 - 12 \\ 2x_1 + 2x_2 - 4 \end{bmatrix}$

$H = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 x_2 \\ \partial^2 f/\partial x_2 x_1 & \partial^2 f/\partial x_2^2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix}$

$(10-\lambda)(2-\lambda)-4=0 \Rightarrow 20-12\lambda+\lambda^2-4=0$

$\Rightarrow \lambda^2-12\lambda+16=0$

$\lambda \rightarrow$ positive definite

If $S^1$ is conjugate to $S^0=[1\ 0]^T$, then

$$(S^1)^T \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 10S_1^1 + 2S_2^1 = 0$$

If $S_1^1 = 1$, $S_2^1 = -5$

So, $S^1 = [1\ -5]^T$

$$S^2 \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = -8S_2^2 = 0$$

$\Rightarrow S_2^2 = 0$

Second direction $S^2 = [1\ 0]^T$.

So, we got back at the initial search direction and this happens for quadratic function.

Ans 6). $f(x) = 10x_1^2 + x_2^2$ $\qquad x_0 = (1, 1)$

$$\nabla f(x) = \begin{bmatrix} 20x_1 \\ 2x_1 \end{bmatrix} \qquad x_0 = [1\ 1]^T$$

$$S^0 = -\nabla f(x^0) = [-20\ -2]^T$$

Iteration 1

$$x^1 = x^0 + \lambda_0 S^0 = [1-20\lambda_0\ \ 1-2\lambda_0]^T$$

$$f(x^1) = 10(1-20\lambda_0)^2 + (1-2\lambda_0)^2$$

$$\frac{\partial f}{\partial \lambda} = 0$$

$$800.8 d_0 - 404 = 0$$

$$d_0 = 0.05045$$

$$x^1 = [-8.991 \times 10^{-3} \quad 0.8991]^T$$

Iteration 2:

$$S^1 = -\nabla f(x^1) = [0.1798 \quad -1.798]$$

$$x^2 = x^1 + d_1 S^1 = \begin{bmatrix} -8.991 \times 10^{-3} + 0.179 d_1 & 0.8991 - 1.798 d_1 \end{bmatrix}$$

$$f(x^2) = 10(-8.991 \times 10^{-3} + 0.1798 d_1)^2 + (0.8991 - 1.798 d_1)^2$$

$$\frac{\partial f}{\partial d} = 0. \quad d = 0.459$$

$$x^2 = [0.07354 \quad 0.07382]^T$$

Optimum point is [0 0]
After two iteration, we didn't get the optimum point. So, we have to do more than 2 iteration.

Ex 7 (a) $f(x) = 3x_1^2 + x_2^2$ $\qquad x^0 = [1 \quad 1]^T$

$$\nabla f(x) = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 6x_1 \\ 2x_2 \end{bmatrix} = [6x_1 \quad 2x_2]^T$$

$$H = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \qquad x^0 = [1 \quad 1]^T$$

$$S^0 = -\nabla f(x^0) = [-6 \quad -2]^T$$

$$d_0 = \frac{-\nabla^T f(x^0) S^0}{(S^0)^T H S^0} = -0.1785$$

$$x_1 = x_0 + d_0 S^0$$
$$= [0.07142 \quad 0.6428]^T$$

$$\nabla f(x^1) = [-0.4285 \quad 1.2857]^T$$

$$\omega_0 = \frac{\nabla^T f(x^1) \nabla f(x^1)}{\nabla^T f(x^0) \cdot \nabla f(x^0)} = 0.04591$$

$$S^1 = -\nabla f(x^1) + \omega_0 S^0 = [0.1530 \quad -1.3775]^T$$

$$d_1 = \frac{-\nabla^T f(x^1) S^1}{(S^1)^T H S^1} = 0.4666$$

$$x^2 = x^1 + d_1 S^1 = [-1.203 \times 10^{-7} \quad -4.01 \times 10^{-8}]^T$$

Very close to minimum $x^* = [0 \quad 0]^T$

(b) $f(x) = 4(x_1 - 5)^2 + (x_2 - 6)^2$

$$\nabla f(x) = \begin{bmatrix} 8(x_1 - 5) \\ 2(x_2 - 6) \end{bmatrix} \qquad H = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$$

$$x^0 = [1 \quad 1]^T$$

$$\nabla f(x^0) = [-32 \quad -10]^T$$

$$S^0 = -\nabla f(x^0) = [32 \quad 10]^T$$

$$d_0 := \frac{-\nabla^T f(x^0) \cdot s^0}{(s^0)^T H s^0} = 0.1339$$

$$x^1 = x^0 + d_0 s^0 = [5.2859 \quad 2.3393]^T$$

$$w_0 := \frac{\nabla^T f(x^1) \cdot \nabla f(x^1)}{\nabla^T f(x^0) \cdot \nabla f(x^0)} = 0.05234$$

$$s^1 := -\nabla f(x^1) + w_0 s^0 = [0.6125 \quad 7.8446]^T$$

$$d_1 := \frac{-\nabla^T f(x^1) s^1}{(s^1)^T H s^1} = 0.4666$$

$$x^2 = x^1 + d_1 s^1 = [5.00012 \quad 6.00002]^T$$

which is very close to true minimum at $x^* = [5 \quad 6]^T$.

Ans 8:- $f(r,h) = \frac{1}{\pi r^2 h} + 2\pi r h + 10\pi r^2$

In order to apply newtons method the Hessian matrix should be positive definite

$$\nabla f(r,h) := \begin{bmatrix} \partial f / \partial r \\ \partial f / \partial h \end{bmatrix} = \begin{bmatrix} \frac{-2}{\pi h r^3} + 2\pi h + 20\pi r \\ \frac{-1}{\pi r^2 h^2} + 2\pi r \end{bmatrix}$$

$$H = \begin{bmatrix} \partial^2 f / \partial r^2 & \partial^2 f / \partial r \partial h \\ \partial^2 f / \partial h \partial r & \partial^2 f / \partial h^2 \end{bmatrix} = \begin{bmatrix} \frac{6}{\pi r^4} + 20\pi & \frac{2}{\pi r^3 h^2} + 2\pi \\ \frac{2}{\pi h^2 r^3} + 2\pi & \frac{2}{\pi r^2 h^3} \end{bmatrix}$$

$|H - \lambda I| > 0$ (for positive definite).

$$\left(\frac{6}{\pi r^4} + 20\pi - \lambda\right)\left(\frac{2}{\pi r^2 h^3} - \lambda\right) - \left(\frac{2}{\pi r^3 h^2} + 2\pi\right)^2 > 0$$

If we substitute (0.22, 2.16) in the above inequality, we will get

$$\left(\frac{6}{\pi (0.22)^4} + 20\pi - \lambda\right)\left(\frac{2}{\pi (0.22)^2 (2.16)^3} - \lambda\right) - \left[\frac{2}{\pi(0.22)^3 (2.16)^2} + 2\pi\right]^2$$

$$878.12 \times 1.305 - 878.12\lambda - 1.305\lambda - 364.78 + \lambda^2 = 0$$

$$\lambda^2 - 879.425\lambda + 781.166 = 0$$

$$\lambda = \underline{0.8891,\ 878.535}$$

(+ve) definite

So, Newton's method can be used here for convergence.

Ans 9 → $f(x) = 3x_1^2 + 3x_2^2 + 3x_3^2 \longrightarrow$ minimize.

$$x^0 = [10\ \ 10\ \ 10]^T$$

* Newton's method.

<u>Iteration (1)</u>

$$[H] = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 & \partial^2 f/\partial x_1 \partial x_3 \\ \partial f/\partial x_2 x_1 & \partial^2 f/\partial x_2^2 & \partial^2 f/\partial x_2 \partial x_3 \\ \partial^2 f/\partial x_3 x_1 & \partial^2 f/\partial x_3 x_2 & \partial^2 f/\partial x_3^2 \end{bmatrix}$$

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$$[H] = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$[H]^{-1} = \frac{1}{6^2}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}$$

$$\nabla f(x) = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \\ \partial f/\partial x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 \\ 6x_2 \\ 6x_3 \end{bmatrix}_{(10,10,10)} = \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$$

$$X_2 = X_1 - [J_1]^{-1} g_1 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/6 \end{bmatrix}\begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$$

$$X_2 = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{optimal point}$$

* **Steepest descent Method**

$$\nabla f(x) = \begin{bmatrix} 6x_1 \\ 6x_2 \\ 6x_3 \end{bmatrix} \qquad \nabla f(x_1) = \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix}$$

$$S_1 = -\nabla f_1 = \begin{Bmatrix} -60 \\ -60 \\ -60 \end{Bmatrix}$$

Minimise $f(X_1 + \lambda_1 S_1) = \left( \begin{Bmatrix} 10 \\ 10 \\ 10 \end{Bmatrix} + \begin{Bmatrix} -60\lambda \\ -60\lambda \\ -60\lambda \end{Bmatrix} \right)$

$$\cancel{S_1 = -\nabla f_1 = \begin{Bmatrix} -60 \\ -60 \\ -60 \end{Bmatrix}}$$

~~Minimize $f(X_1 +$~~

$$f = \begin{pmatrix} 10-60\lambda \\ 10-60\lambda \\ 10-60\lambda \end{pmatrix}$$

$$f(\lambda) = 3(10-60\lambda)^2 + 3(10-60\lambda)^2 + 3(10-60\lambda)^2$$
$$= 9(10-60\lambda)^2$$

$$\frac{\partial f}{\partial \lambda} = 0$$

$$(10-60\lambda) = 0$$
$$10 = 60\lambda$$
$$\lambda = 1/6$$

$$X_2 = X_1 + \lambda_1 S = \begin{Bmatrix} 10 \\ 10 \\ 10 \end{Bmatrix} + \frac{1}{6}\begin{Bmatrix} -60 \\ -60 \\ -60 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

So from above result we can conclude that both of them are equally fast as the no. of iterations is concerned, because search direction is same for both & both yield the optimum in one step.

Ans 10 → $F(x) = x_1^3 + x_1 x_2 - x_2^2 x_1^2$ → objective function

By newton's method, starting with $x^0 = [1\ 1]^T$

$$\nabla f(x) = \begin{bmatrix} 3x_1^2 + x_2 - 2x_1 x_2^2 \\ x_1 - 2x_1^2 x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix}$$

$$H(x) = \begin{bmatrix} \partial^2 f / \partial x_1^2 & \partial^2 f / \partial x_1 \partial x_2 \\ \partial^2 f / \partial x_2 \partial x_1 & \partial^2 f / \partial x_2^2 \end{bmatrix}$$

$$H(x) = \begin{bmatrix} 6x_1 - 2x_2^2 & 1 - 4x_1 x_2 \\ 1 - 4x_1 x_2 & -2x_1^2 \end{bmatrix}$$

at $x^0 = [1\ 1]^T$ : $H(x) = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$

Eigen value of $H(x^*) \Rightarrow \begin{vmatrix} 4 - \lambda & -3 \\ -3 & -2\lambda \end{vmatrix} = 0$

$$(4-\lambda)(-2-\lambda) - 9 = 0$$

$$-8 - 4\lambda + 2\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 - 2\lambda - 17 = 0$$

$\lambda = 5.2426, -3.24$ ∴ H is not positive definite which might be the reason why the code failed

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Ans 11- $f(x) = 2x_1^3 - 6x_1x_2 + x_2^2 \longrightarrow$ objective function

$$\nabla f(x) = \begin{bmatrix} \partial f / \partial x_1 \\ \partial f / \partial x_2 \end{bmatrix} = \begin{bmatrix} 6x_1^2 - 6x_2 \\ -6x_1 + 2x_2 \end{bmatrix}$$

$$H = \begin{bmatrix} \partial^2 f / \partial x_1^2 & \partial^2 f / \partial x_1 \partial x_2 \\ \partial^2 f / \partial x_2 \partial x_1 & \partial^2 f / \partial x_2^2 \end{bmatrix} = \begin{bmatrix} 12x_1 & -6 \\ -6 & 2 \end{bmatrix}$$

at $x = [1 \ 1]^T$, $H = \begin{bmatrix} 12 & -6 \\ -6 & 2 \end{bmatrix}$

eigen values $|H - \lambda I| = 0$

$$\begin{vmatrix} 12-\lambda & -6 \\ -6 & 2-\lambda \end{vmatrix} = 0$$

$(12-\lambda)(2-\lambda) - (36) = 0$

$\Rightarrow 24 - 12\lambda - 2\lambda + \lambda^2 - 36 = 0$

$\lambda^2 - 14\lambda - 12 = 0 \qquad \lambda = -0.81, 14.81$

So (H) is not positive definite.

By using Marquardt's method we have to make it (+ve) definite

Add $\begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix}$ to $H(1,1) \longrightarrow \begin{bmatrix} 12+\beta & -6 \\ -6 & 2+\beta \end{bmatrix}$

Now, $|H(1,1)| > 0$

$$(12+\beta)(2+\beta) - 36 > 0$$

$$\beta^2 - 14\beta - 12 > 0$$

$$\beta \in (-\infty, -0.81) \cup (14.81, \infty)$$

Ans 12) given,

$$\phi = \sum_{i=1}^{n} \left[ y_{observed} - y_{predicted} \right]_i^2$$

$$y_{predicted} = \frac{k_1}{k_1 - k_2} \left[ e^{-k_2 t} - e^{-k_1 t} \right]$$

$y_{observed} \rightarrow$

| $t$ | $y_{observed}$ |
| --- | --- |
| 0.5 | 0.263 |
| 1 | 0.455 |
| 1.5 | 0.548 |

$$y_{predicted} = \frac{k_1}{k_1 - k_2} \left[ e^{-k_2 t} - e^{-k_1 t} \right]$$

Using exponential expansion upto $t^2$ term -

$$y_p = \frac{k_1}{k_1 - k_2} \left[ \left(1 - k_2 t + \frac{k_2^2 t^2}{2}\right) - \left(1 - k_1 t + \frac{k_1^2 t^2}{2}\right) \right]$$

$$\Rightarrow y_p = \frac{k_1}{k_1-k_2}\left[(k_1-k_2)\cdot t + (k_2^2-k_1^2)\cdot\frac{t^2}{2}\right]$$

$$y_p = k_1 t - k_1(k_1+k_2)\cdot t^2/2$$

$$y_p = \left(k_1 t - \frac{k_1^2 t^2}{2} - k_1 k_2 \frac{t^2}{2}\right)$$

$$\phi = [y_o - y_p]^2_{t=0.5} + [y_o - y_p]^2_{t=1} + [y_o - y_p]^2_{t=1.5}$$

$$= [0.5k_1 - 0.125k_1^2 - 0.125k_1k_2 - 0.263]^2 + [k_1 - 0.5k_1^2 - 0.5k_1k_2 - 0.455]^2 + [1.5k_1 - 1.125k_1^2 - 1.125k_1k_2 - 0.548]^2$$

$$\Rightarrow (3.5k_1^2 + 1.53125k_1^4 + 1.53125k_1^2k_2^2 + 0.5765) + 2\begin{bmatrix} -2.25k_1^3 - 2.25k_1^2k_2 - 1.4085k_1 + 1.53125k_1^3\cdot k_2 + \\ 0.8768k_1^2 + 0.8768k_1k_2 \end{bmatrix}$$

$$\phi \Rightarrow 1.53125k_1^4 - 4.5k_1^3 + 3.0625k_1^3k_2 + 1.53125k_1^2k_2^2 + 5.25k_1^2 - 4.5k_1^2k_2 + 1.7536k_1k_2 - 2.817k_1 + 0.5765 = 0 \quad \text{— (A)}$$

differentiating $\phi$ w.r.t. $k_1$, we get.

$$\frac{d\phi}{dk_1} = 0$$

$$\Rightarrow 6.125k_1^3 - 13.5k_1^2 + 9.1875k_1^2k_2 + 3.0625k_1k_2^2 + 10.5k_1 - 9k_1k_2 + 1.7536k_2 - 2.817 = 0 \quad \text{— (I)}$$

differentiating $\phi$ w.r.t $k_2$ we get →

$$\frac{d\phi}{dk_2} = 0$$

$$\Rightarrow 3.0625 k_1^3 + 3.0625 k_1^2 k_2 - 4.5 k_1^2 + 1.75376 k_1 = 0 \quad -(2)$$

Since eq^n (1) & (2) are coupled eq^n so it is very difficult to solve $k_1$ & $k_2$ by Hand.

By calculation, with matlab → we get

$(k_1, opt) = 0.65 \quad k_{2opt} = 0.152$

Substituting $(k_1)_{opt}$ & $(k_2)_{opt}$ in eq^n (A) we get

$(\phi)_{opt} \simeq 0.000167$

$\Rightarrow (\phi)_{opt} \simeq 1.67 \times 10^{-4}$

Ans 13.) Minimize $f(x) = (x-1)^4$

(i) Newton method ~~for~~, $x^0 = -1$

$$f'(x) = 4(x-1)^3$$
$$f''(x) = 12(x-1)^2$$
$$f'''(x) = 24(x-1)$$

Iteration (1)

$$x^1 = x^0 - \frac{f'(x^0)}{f''(x^0)}$$

$$x^1 = (-1) - \frac{4(-1-1)^3}{12(-1-1)^2} \Rightarrow x^1 = -1/3$$

Iteration 2

$$x^2 = x^1 - \frac{f'(x^1)}{f''(x^1)}$$

$$= -\frac{1}{3} - \frac{1}{3}\left(-\frac{1}{3} - 1\right)$$

$$\Rightarrow x^2 = 1/9$$

Iteration (3) $\quad x^3 = x^2 - \frac{f'(x^2)}{f''(x^2)}$

$$x^3 = \frac{1}{9} - \frac{1}{3}\left(\frac{1}{9} - 1\right)$$

$$\Rightarrow x^3 = 11/27$$

Iteration 4 $\quad x^4 = x^3 - \frac{f'(x^3)}{f''(x^3)}$

$$= \frac{11}{27} - \frac{1}{3}\left(\frac{11}{27} - 1\right)$$

$$\Rightarrow x^4 = \frac{49}{81} \simeq 0.605$$

$x^0 = -0.5$

Iteration (1) $\quad x^1 = -0.5 - \frac{f'(x_0)}{f''(x_0)}$

$$\Rightarrow x^1 = -0.5 - \frac{1}{3}(-0.5 - 1) = 0$$

$x^0 = 0.$

Iteration (1). $\quad x^1 = 0 - 1/3(x - 1)$

$$= \frac{-1}{3}(-1) = 1/3$$

Iteration (2)

$x^2 = 0 - 1/3(0 - 1) = 1/3$

Iteration (3)

$x^3 = 1/3 - 1/3(1/3 - 1)$

$= 5/9$

Iteration (4)

$x^4 = 5/9 - 1/3(5/9 - 1)$

$= 19/27 \simeq 0.704$

Iteration (2), $x^2 = \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3} - 1\right) = 5/9$

Iteration (3) $x^3 = \frac{5}{9} - \frac{1}{3}\left(\frac{5}{9} - 1\right) = 19/27$

Iteration (4) $x^4 = \frac{19}{27} - \frac{1}{3}\left(\frac{19}{27} - 1\right) = \frac{65}{81} \simeq 0.802$

So from Newton's method after 4 iterations, $x^0 = 0$ gives more accurate result.

(ii) Quasi-Newton method

$f'(x) = 4(x-1)^3$
$f''(x) = 12(x-1)^2$

$x^p = -1 \quad x^q = 1$
$f'(-1) = -32 \quad f'(1) = 0$

iteration (1)

$$x^1 = 1 - \frac{0}{\left[0-(-32)/1-(-1)\right]} = 1$$

$x^p = 0.5 \quad , x^q = 1$

$f'(0.5) = -27/2 \quad , f'(1) = 0$

Iteration (1) $x^1 = 1 - \left[\frac{0}{\frac{0 + (27/2)}{1-(-1)}}\right] = 1$

$* \cdot x^p = 0 \quad , x^q = 1$

$f'(0) = -4 \quad f'(1) = 0$

$$x' = 1 - \frac{0}{\left[\frac{0-(-4)}{1-(-1)}\right]} = 1$$

So, here only in 1 iteration we reached the required optimum.

Ans 14- $f(x) = 2x^3 - 5x^2 - 8 \quad x \geq 1$

minimize

(1) Newton's method $(x^0 = 1)$

$f'(x) = 6x^2 - 10x \qquad f'(x^0) = -4$

$f''(x) = 12x - 10 \qquad f''(x^0) = 2$

$$x' = x^0 - \frac{f'(x^0)}{f''(x^0)} = 1 - (-4/2) = 3$$

$$x^2 = x' - \frac{f'(x')}{f''(x')} = 3 - \left(\frac{54-30}{36-10}\right) = 2.08$$

(2) Quasi-Newton Method

$x_A = 2 \Rightarrow f(2) = 2(2)^3 - 5 \times 2^2 - 8$

$= -12$ (-ve)

$x_B = 1.5$

$f(x_B) = 2 \times (1.5)^3 - 5 \times (1.5)^2 - 8$

$= -12.5$ (-ve)

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$x_B' = 3$

$f(x_B') = 2\times 3^3 - 5\times 3^2 - 8$

$= 54 - 45 - 8$

$= 1$

So range will be $[2, 3]$

$$x' = x_0 - \frac{f'(x'_0)}{\frac{f'(x_A') - f'(x_B')}{x_A - x_B'}}$$

$f'(x_A^0) = f'(2) = 6\times 2^2 - 10\times 2$

$= 24 - 20 = 4$

$f'(x_B^0) = f'(3) = 6\times 3^2 - 10\times 3$

$= 54 - 30 = 24$

$$x' = 3 - \frac{4}{\frac{4-24}{2-3}} = 3 - \frac{4}{20} = 2.8$$

$f'(2.8) \Rightarrow 6\times(2.8)^2 - 10\times 2.8 = 14.04$ (+ve)

$f(2.8) = -3.296$

$f(3) = 1$

Now, Range $= [2, 2.8]$

$x_A = 2 \quad x_B = 2.8$

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$$x^2 = 2.8 - \frac{f'(2.8)}{\frac{f'(2.8) - f'(2)}{2.8 - 2}}$$

$$= 2.8 - \frac{19.04}{\frac{19.04 - (-12)}{2.8 - 2}}$$

$$= 2.30927$$

~~(14) Polynomial approximation (quadratic)~~

~~Assuming 3 points as~~

~~Ans 15 → Todo~~

Ans 15 $x^2 - 6x + 3 \rightarrow$ minimize $f'(x) = 2x - 6$

$f''(x) = 2$

* Newton's method.

Assuming $x^0 = 1$

$$x^1 = x^0 - \frac{f'(x^0)}{f''(x^0)}$$

$$x^1 = 1 - \left(\frac{-4}{2}\right) = 3$$

2nd iteration $x^2 = x^1 - \frac{f'(x^1)}{f''(x^1)} = 3 - \frac{0}{2} = 3.$

So, by newton's method, we get the minimum value in one iteration itself.

⇒ finite difference newton method

$$x^{k+1} = x^k - \frac{\left[\frac{f(x+h) - f(x)}{h}\right]}{\frac{f(x+h) - 2f(x) - f(x-h)}{h^2}}$$

Assuming $x^0 = 1$, $h = 0.001$.

$$x^1 = x^0 - \left[\frac{f(1.001) - f(1)}{0.001}\right] \times \frac{0.001}{f(1.001) - 2f(1) + f(0.999)}$$

$$= 1 - \frac{-4 \times 10^{-6}}{2 \times 10^{-6}} = 3$$

It converged in one iteration to $x^* = 3$

⇒ Quasi Newton Method (Assuming $x^0 = 1$, $x^1 = 5$)

$f'(x^0) = 2(1) - 6 = -4$

$f'(x^1) = 2(5) - 6 = 4$

$$x^2 = 5 - \frac{4}{\frac{4-(-4)}{5-1}} = 3$$

Converged in one iteration to $x^* = 3$

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⇒ Quadratic interpolation

Assuming 3 points

$x_1 = 1 \quad x_2 = 2 \quad x_3 = 5$

$f(x_1) = -2 \quad f(x_2) = -5 \quad f(x_3) = -2$

$$x^* = \frac{1}{2} \frac{(x_2^2 - x_3^2) f(x_1) + (x_3^2 - x_1^2) f(x_2) + (x_1^2 - x_2^2) f(x_3)}{(x_2 - x_3) f(x_1) + (x_3 - x_1) f(x_2) + (x_1 - x_2) f(x_3)}$$

$$x^* = \frac{1}{2} \left[ \frac{42 + (-120) + (6)}{6 - 20 + 2} \right] = 3$$

Converged to $x^* = 3$ in 1 iteration

⇒ Cubic interpolation

Initial points $= x^1 = 1 \quad x^2 = 5$

$$x^* = x_2 - \left[ \frac{f_2' + w - z}{f_2' - f_1' + 2w} \right] (x_2 - x_1)$$

$$z = \frac{3(f_1 - f_2)}{(x_2 - x_1)} + f_1' + f_2'$$

$$w = (z^2 - f_1' \cdot f_2')^{1/2}$$

$f'(x) = 2x - 6 \quad f'(x^1) = -4 \quad \& \quad f'(x^2) = 4$

$f_1(x_1) = 1 - 6 + 3 = -2$

$f_2(x_2) = 25 - 30 + 3 = -2$

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$z=0, \quad w = \left(-(-4)\cdot \times 4\right)^{1/2} = 4$

$$x^{*1} = 5 - \left[\frac{4+4}{8+8}\right](4)$$

$= 3$

Converged to $x^* 3$ in one iteration

(b) $\sin x$ with $0 < x < 2\pi$

Rate of convergence

linear $= \left|\frac{x^{k+1} - x^*}{x^k - x^*}\right| \leq c \; (0 \leq c < 1)$

order $p = \frac{|x^{k+1} - x^*|}{(x^k - x^*)^p} < c \quad (c > 0, p \geq 1)$

superlinear $\Rightarrow \lim_{k\to\infty} \frac{\|x^{k+1} - x^*\|}{\|x^k - x^*\|} \to 0$

(i) Newtons method

$$x^1 = x^0 - \frac{f'(x^0)}{f''(x^0)}$$

Expansion of $\sin x = x - \frac{x^3}{3!}$

$f(x) = x - \frac{x^3}{3!} \quad f'(x) = 1 - \frac{3x^2}{3!}$

$f''(x) = \frac{-6x}{3!}$

$x_0 = 5$

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Iteration ①

$$x^1 = x^0 - \frac{f'(x^0)}{f''(x^0)}$$

$$= 5 - \left[ \frac{1 - (3(5)^2/3!)}{-\frac{6 \times 5}{3!}} \right] = 2.7$$

Iteration 2

$$x^2 = 2.7 - \frac{f'(x^1)}{f''(x^1)}$$

$$= 2.7 - \left[ \frac{1 - \frac{3(2.7)^2}{3!}}{-\frac{6(2.7)}{3!}} \right] = 1.72$$

ii) Finite difference method

$$x^{k+1} = x^k - \frac{\frac{f(x+h) - f(x)}{h}}{\left[\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}\right]}$$

$h = 0.001$, $x^0 = 4$

$$x^1 = 4 - \frac{\frac{f(4.001) - f(4)}{0.001}}{\frac{f(4.001) - 2f(4) + f(0.3999)}{(0.001)^2}}$$

$x^1 = 4.5$

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(c) $f(x) = x^4 - 20x^3 + 0.1x$ $\rightarrow f'(x) = 4x^3 - 60x^2 + 0.1$

$f''(x) = 12x^2 - 120x$

Newton method

Assuming $x^0 = 14$

$$x^1 = x^0 - \frac{f'(x^0)}{f''(x^0)}$$

Iteration (1) $x^1 = 14 - \left[\frac{4(14)^3 - 60(14)^2 + 0.1}{12(14)^2 - 120(14)}\right]$

$\Rightarrow x^1 = 15.166$

Iteration (2) $x^2 = 15.166 - \frac{f'(x^0)}{f''(x^0)}$

$x^2 = 15.166 - 0.16306$

$= 15.003$

Convergence is linear

$\Rightarrow$ Finite difference method.

$$x^{k+1} = x^k - \frac{[f(x+h) - f(x)]/h}{[f(x+h) - 2f(x) + f(x-h)]/h^2}$$

$\Rightarrow x^0 = 12$, $h = 0.001$

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Iteration (1)

$$x^1 = 12 - \frac{[f(12.001) - f(12)]/0.001}{[f(12.001) - 2f(12) + f(11.999)]/(0.001)^2}$$

$x^1 = 15.278$

Iteration (2)

$$x^2 = 15.278 - \frac{\frac{[f(15.278 + 0.001) - f(15.278)]}{0.001}}{\frac{f(15.279) - 2f(15.278) + f(15.277)}{(0.001)^2}}$$

$x^2 = 15.010$

Convergence is quadratic

⇒ Quasi Newton method

$x^0 = 14, \quad x^1 = 16$

$f'(x^0) = -783.9$

$f'(x^1) = 1024.1$

$$x^1 = 16 - \frac{1024.1}{\frac{1024.1 - (-783.9)}{16 - 14}} = 14.8671$$

$f(x^1) = (-ve)$ value $(-928.246.5627)$

So, we take new bracket as $(14.8671, 16)$

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Ans 16 - The total annual cost of operating a pump and motor

$$C = 500 + 0.9x + \frac{0.03}{x}(150{,}000)$$

Minimize → total cost
$x$ → size of motor

By using analytical method

$\nabla f(x) = 0$ ( to find the extremum points)

$$\frac{\partial C}{\partial x} = 0$$

$$\frac{\partial C}{\partial x} = 0.9 - \frac{0.03 \times 150{,}000}{x^2}$$

$$= 0.9 - \frac{4500}{x^2}$$

$$\frac{\partial C}{\partial x} = 0 \Rightarrow 0.9 = \frac{4500}{x^2}$$

$$x^2 = \frac{4500}{0.9} \qquad x = \sqrt{5000}$$

$x = 70.71$ hp.

$$\frac{\partial^2 C}{\partial x^2} = -(-2) \times \frac{4500}{x^3} = \frac{9000}{x^3} > 0$$

(as $x$ cannot be -ve, It represent the size of motor)

So, at $x = 70.71$ hp, then cost of operating motor & pump minimizes.

At $x = 70.71$

$$C = 500 + 0.9(70.71) + \frac{0.03}{70.71}(150000)$$

$$\boxed{C = \$627.28}$$

Minimized cost.