consider the play $1[453]^\omega$. The sequence of weights is $4[3, 2, -5]^\omega$. The average for $n \equiv 1 \mod 3$ is $4/n$ which tends to 0. Similarly for the others. So the mp is 0.

Consider the play $[1453]^\omega$. The sequence of weights is $[4, 3, 2, 2]^\omega$. The average for $n \equiv 0 \mod 4$ is $11/4$. Similarly the others tend to $11/4$. so the mp is $11/4 = 2.75$.

Theorem 5. *MPG are positionally determined, using uniform strategies, and solving them is in* NP *and co-*NP*.*

Proof is postponed till next lecture.

Meanwhile:

Proposition 1. *Solving Parity games can be reduced to mean-payoff games in polynomial time.*

Proof. WLOG, colors are in $[d-1]$. Define weight of (v, w) to be $(-1)^i n^i$ where $c(v) = i$ and $n = |V|$. In particular, supposing d is even, the weights are from the set $\{-n^{d-1}, n^{d-2}, ..., -n, 1\}$.

It is sufficient to show (*) that for every pair of memoryless strategies, the same strategy wins both games: indeed, suppose pl i has a ws in PG. Then by mem det of PG this can be taken to be a memless strategy. If pl i has no ws in MPG then by memdet pl $1-i$ does, and this can be taken to be memless. By (*) for this pair of memless strategies, the same strategy wins both games. but this is impossible.

To prove (*), fix a pair of memless strategies, and note that a cycle C is formed which governs the winner. Note that $avg(C) \geq 0$ iff $sum(C) \geq 0$. We show that the max col c on C is even iff $sum(C) \geq 0$. Let x be weight with largest absolute value.

Suppose max col on C is even. Then $x > 0$, and so

$$sum(C) \geq x - (n-1)x/n = x/n > 0$$

since there are at most $n-1$ other edges on the cycle, each of whose weight is at most x/n. The other direction is symmetric. □

6 First Cycle Games

We already saw a game that was not positionally determined. Can we characterise those games that are? We introduce FCG. To motivate them notice that if players can win positionally, then fixing positional strategies, the resulting play is a lasso, and who wins (for prefix independent games) depends only on this cycle.

Definition 10. *A* Edge-colored arena *is a tuple* $(A, \lambda : E \to \mathbb{U})$.

A cycle property Y *is a subset of* $\mathbb{U}^*$.

A first-cycle objective wrt Y, written $FC(Y)$ *consists of all plays whose first cycle is in* Y.

A first-cycle game (FCG) *is a game with a first-cycle objective.*

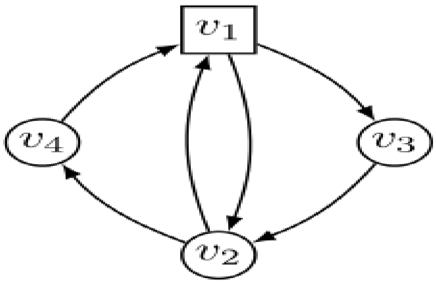
Remark: we transform vertex labeled games into edge labeled games by labeling an edge (u, v) by the label of its source v.

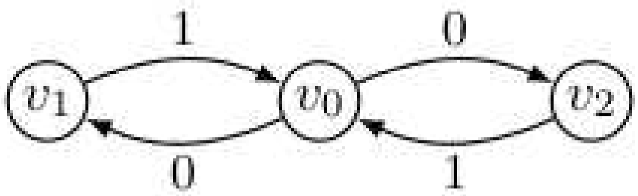
Finitise the following objectives: Buchi, Parity, MP.

Example 3. • *the cycle should contain an edge from T (i.e., $u \in Y$ iff $u_i \in T$ for some i)*

- *the largest priority occuring on the cycle is even (i.e., $u \in Y$ iff $\max_i c(u_i)$ is even)*
- *the average of the weights on the cycle is non-negative (i.e., $u \in Y$ iff $avg(u) \geq 0$).*

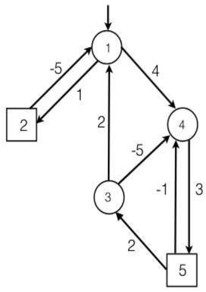
Fact 1. first-cycle games are determined (why?).
Fact 2. Not necc. positionally determined (use even-length)
Fact 3. Even if positionally determined, not necc. uniform positionally determined.





Definition 11. *An* edge-path *is a sequence of edges $e_1e_2\cdots$ such that $last(e_i) = start(e_{i+1})$ for all i. An edge-path is* simple *if $last(e_j) = start(e_i)$ implies $i = j + 1$.*

The *cycles-decomposition of an edge-path* takes as input a (usually empty) simple path s, which is treated as the initial contents of a stack, and a path π (finite or infinite). At step $j \geq 1$, the edge π_j is pushed onto the stack and if, for some k, the top k edges on the stack form a cycle, this cycle is output, then popped, and the procedure continues to step $j + 1$.



Example 4. *What are the sequence of cycles output by the cycles decomposition of*

$$(1,2)(2,1)(1,2)(2,1)(1,4)[(4,5)(5,3)(3,4)]^{\omega}$$

Algorithm 1 Cycles-Decomposition $CD(s,\pi)$

Require: s is a finite (possibly empty) simple path ▷ initial stack content
Require: π is a finite or infinite path $\pi_1\pi_2\cdots$ ▷ the path to decompose
Require: If s is non-empty then $\mathsf{end}(s) = \mathsf{start}(\pi)$ ▷ $s\pi$ must form a path

```
  step = 1
  while step ≤ |π| do                                  ▷ Start a step
      Append π_step to s                               ▷ Push current edge into stack
      Say s = e_1 e_2 ··· e_m
      if ∃i : e_i e_{i+1} ··· e_m is a cycle then      ▷ If stack has a cycle
          Output e_i e_{i+1} ··· e_m                   ▷ output the cycle
          s := e_1 ··· e_{i-1}                         ▷ Pop the cycle from the stack
      end if
      step := step + 1                                 ▷ advance to next input edge
  end while
```

Definition 12. *Let Y be a cycle property. The* first-cycle objective based on Y, written *$FC(Y)$ is the set of plays such that the labelling of the first cycle output by CD is in Y.*

Define $N_z(\pi) \in \mathbb{N} \cup \{\infty\}$ to be the index of the first edge that starts with z, if one exists. Define $head_z(\pi)$ to be the prefix of π before $N_z(\pi)$, and $tail_z(\pi)$ to be the suffix of π starting at $N_z(\pi)$.

Formally, $N_z(\pi) := \infty$ if z does not occur on π, and otherwise $N_z(\pi) := \min\{j : \mathsf{start}(\pi_j) = z\}$. Also, $head_z(\pi) := \pi[1, N_z(\pi) - 1]$ and $tail_z(\pi) := \pi[N_z(\pi), |\pi|]$. By convention, if $N_z(\pi) = \infty$ then $head_z(\pi) = \pi$ and $tail_z(\pi) = \epsilon$.

We now define a game, that is very similar to the first-cycle game, except that one of the nodes of the arena is designated as a "reset" node:

Definition 13. *Fix an arena A, a vertex $z \in V$, and a cycle property Y. Define the objective $FC_z(Y)$ to consist of all plays π satisfying the following property: if $head_z(\pi)$ is not a simple path then $first(\pi) \in Y$, and otherwise $first(tail_z(\pi)) \in Y$.*

Playing the game with objective $FC_z(Y)$ is like playing the first-cycle game over Y, however, if no cycle is formed before reaching z for the first time, the prefix of the play up to that point is ignored. Thus, in a sense, the game is reset. Also note that if play starts from z, then the game reduces to a first-cycle game. It turns out that we may assume that a strategy of $(A, FC_z(Y))$ makes the same move every time it reaches z:

Definition 14 (Forgetful at z from v)**.** *For an arena A, a vertex $v \in V$, a Player $i \in \{0, 1\}$, and a vertex $z \in V_i$ belonging to Player i, we call a strategy T for Player i* forgetful at z from v *if there exists $z' \in V$ such that $(z, z') \in E$ and for all $\pi \in plays(T, v)$, and all $n \in \mathbb{N}$, if $\mathsf{start}(\pi_n) = z$ then $\mathsf{end}(\pi_n) = z'$.*

Lemma 2 (Forgetful at z from v)**.** *Fix an arena A, a vertex $v \in V$, a Player $i \in \{0, 1\}$, and a vertex $z \in V_i$ belonging to Player i. In the game $(A, FC_z(Y))$, if Player i has a strategy S that is winning from v, then Player i has a strategy T that is winning from v and that is forgetful at z from v.*

Sketch. The second time z appears on a play, the winner is already determined, and so the strategy is free to repeat the first move it made at z. On the other hand, when a play visits z the first time, the strategy can make the same move regardless of the history of the play before z, because the winning condition ignores this prefix. □

Definition 15. *Fix Y. An arena is Y-*resettable *if for every $i \in \{0, 1\}$, and every node z, we have that $WR^i(A, FC_z(Y)) = WR^i(A, FC(Y))$.*

Theorem 6 (Resettability implies memoryless determinacy)**.** *Suppose that every arena A is Y-resettable. Then every game $(A, FC(Y))$ is memoryless determined.*

Sketch. A node $z \in V$ is a *choice node* of an arena B, if there are at least two distinct vertices $v', v'' \in V$ such that $(z, v') \in E^B$ and $(z, v'') \in E^B$.

Fix arena A. Suppose Player i has a winning strategy in $(A, FC(Y))$ from v (by determinacy, one of the players must). We induct on the number of choice nodes of Player i. Let z be a choice node for Player i (if there are none, the result is immediate). By the resettability assumption applied to A, Player i also wins the game with objective $FC_z(Y)$ from v. By Lemma 2, Player i has a strategy S that is winning from v and that is also forgetful at z from v. Thus we may form a sub-arena B of A by removing all edges from z that are not taken by S. Observe that S is winning from v in $(B, FC_z(Y))$. Applying the resettability assumption to B, Player i also wins $(B, FC(Y))$ from v. But B has less choice nodes for Player i, and thus, by induction, Player i has a memoryless winning strategy from v in $(B, FC(Y))$. This memoryless strategy is also winning from v in A (since we only removed choices of player i, which are not used in this memoryless strategy). □

We also have a full characterisation in the paper.

1. Say that Y is *closed under cyclic permutations* if $ab \in Y$ implies $ba \in Y$, for all $a \in \mathbb{U}, b \in \mathbb{U}^*$.

2. Say that Y is *closed under concatenation* if $a \in Y$ and $b \in Y$ imply that $ab \in Y$, for all $a, b \in \mathbb{U}^*$.

Theorem 7 (Memoryless Determinacy Characterisation of FCGs)**.** *The following are equivalent for every cycle property Y:*

1. *Y is closed under cyclic permutations, and every arena A is Y-resettable.*
2. *Every game with objective $FC(Y)$ is uniform memoryless determined.*

Connection with infinite duration games

We now define the connection between first-cycle games and games of infinite duration (such as parity games, etc.), namely the concept of Y-greedy games. We then prove the Strategy Transfer Theorem, which says, roughly, that for every arena, the winning regions of the First-Cycle Game over Y and a Y-greedy game coincide, and that memoryless winning strategies transfer between these two games.

Definition 16 (Greedy)**.** *The* all-cycle objective based on Y, written $AC(Y)$ *is the set of plays such that the labelling of every cycle output by CD is in Y.*

Say that a game (A, O) is Y-greedy *if*

$$AC(Y) \subseteq O \text{ and } AC(\neg Y) \subseteq \neg O.$$

Intuitively, a game (A, O) is Y-greedy means that Player 0 can win the game (A, O) if he ensures that every cycle in the cycles-decomposition of the play is in Y, and Player 1 can win if she ensures that every cycle in the cycles-decomposition of the play is not in Y.

Here are some examples.

1. Every all-cycles game $(A, AC(Y))$ is Y-greedy.
2. Every parity game is Y-greedy where Y says "the largest occuring color is even".
3. Every game with mean-payoff winning condition is Y-greedy where Y says "the average of the cycle is non-negative".

We state the Strategy Transfer Theorem:

Theorem 8 (Strategy Transfer)**.** *Let (A, O) be a Y-greedy game, and let $i \in \{0, 1\}$.*

1. *The winning regions for Player i in the games (A, O) and $(A, FC(Y))$ coincide.*
2. *For every memoryless strategy S for Player i, and vertex $v \in V$ in arena A: S is winning from v in the game (A, O) if and only if S is winning from v in the game $(A, FC(Y))$.*

To prove the Strategy Transfer Theorem we need a lemma that states that one can pump a strategy S that is winning for the first-cycle game to get a strategy $S^{\circlearrowleft}$ that is winning for the all-cycles game by *following S until a cycle is formed, removing that cycle from the history, and continuing.* The fact that every winning strategy in the first-cycle game of Y can be pumped to obtain a winning strategy in a Y-greedy game, is why we call such games "greedy".

Recall from the Definitions that for a finite path $\pi \in E^*$, the stack content at the end of $CD(\epsilon, \pi)$ (Algorithm 1) is denoted $stack(\pi)$.

Definition 17 (Pumping Strategy)**.** *Fix an arena A, a Player $i \in \{0,1\}$, and a strategy S for Player i. Let the* pumping strategy *of S be the strategy $S^{\circlearrowleft}$ for Player i, defined, on any history $u = v_1 \dots v_k$ ending in a node of player i, as follows: Let $\pi \in E^*$ be the path corresponding to u, i.e., $\pi = (v_1, v_2)(v_2, v_3) \cdots (v_{k-1}, v_k)$.*

1. *$S^{\circlearrowleft}(u) := S(v_k)$ if $k = 1$ or $stack(\pi) = \epsilon$, and otherwise*
2. *$S^{\circlearrowleft}(u) = S(stack(\pi))$,*

Note that $S^{\circlearrowleft}$ is well-defined since if $stack(\pi) \neq \epsilon$ then $stack(\pi)$ ends with $v_k \in V_i$ and so $stack(\pi)$ is in the domain of S.

NB. if S is memoryless then the pumping strategy $S^{\circlearrowleft} = S$.

Lemma 3. *Fix Player $i \in \{0,1\}$ and let (A, O) be a Y-greedy game. If S is a strategy for Player i that is winning from v in $(A, FC(Y))$ then $S^{\circlearrowleft}$ is winning from v in (A, O).*

Proof. The strategy $S^{\circlearrowleft}$ says to follow S, and when a cycle is popped by CD, remove that cycle from the history and continue. Thus, for every cycle C that is popped, let l be the time at which the first edge of C is being pushed, and note that the stack up to time l followed by C is a path consistent with S whose first cycle is C.

Thus if S is a strategy of player 0 that is winning from v in the game $(A, FC(Y))$ then for every play $\pi \in plays(S^{\circlearrowleft}, v)$, every cycle in $cycles(\pi)$ is in Y. By definition of Y-greedy, this means that $S^{\circlearrowleft}$ is winning from v in the game (A, O). The case of player 1 is symmetric. $\square$

Proof of Strategy Transfer Theorem. Let Y be a cycle property and A an arena. Suppose that (A, O) is Y-greedy. We begin by proving the first item. Use Lemma 3 to get that for $i \in \{0,1\}$,

$$\mathrm{WR}^i(A, FC(Y)) \subseteq \mathrm{WR}^i(A, O).$$

Since first-cycle games are determined, the winning regions $\mathrm{WR}^0(A, FC(Y))$ and $\mathrm{WR}^1(A, FC(Y))$ partition V. Thus, since $\mathrm{WR}^0(A, O)$ and $\mathrm{WR}^1(A, O)$ are disjoint, the containments above are equalities, as required for item 1.

We prove the second item. Since $S = S^{\circlearrowleft}$ if S is memoryless, conclude by Lemma 3: if S is winning from v in the game $(A, FC(Y))$ then it is winning from v in the game (A, O). For the other direction, assume by contraposition that S is not winning from v in the game $(A, FC(Y))$. Since S is memoryless, plays of A consistent with S are exactly infinite paths in the induced sub-arena $A^{||S}$. Hence, there is a path π in the induced solitaire arena $A^{||S}$ for which the first cycle, say $\pi[i, j]$, satisfies Y^{1-i}. Define the infinite path $\pi' := \pi[1, i-1] \cdot (\pi[i, j])^{\omega}$ and note that, being a path in $A^{||S}$, it is a play of A consistent with S. Moreover, π' has the property that every cycle in its cycles-decomposition (i.e., $\pi[i, j]$) satisfies Y^{1-i}. Since (A, O) is Y-greedy, S is not winning from v in the game (A, O). $\square$

Putting together we get:

Theorem 9. *If every arena is Y-resettable and (A, O) is Y-greedy then (A, O) is memoryless determined.*

Recipe for positional determinacy

Question 14. *How to check if every arena is Y-resettable?*

Definition 18. *Fix an arena A. Let $TAC(Y)$ consist of all plays π of A such that some suffix of π is in $AC(Y)$. An arena A is Y-unambiguous if there is no play of A that is in $TAC(Y) \cap TAC(\neg Y)$.*

Theorem 10 (Strategy Transfer 2)**.** *Every arena that is Y-unambiguous is also Y-resettable.*

Proof. First, by Theorem 8 the winning regions of $(A, FC(Y))$ and $(A, TAC(Y))$ coincide.

Second, we now show that the winning regions of $(A, FC_z(Y))$ and $(A, TAC(Y))$ coincide. As usual, it is sufficient to show containment. So, suppose player i has a winning strategy S from v in $(A, FC_z(Y))$. We define a strategy T...

There are two cases:

- every path consistent with σ has a cycle before it visits z (if it sees z at all). Thus $T = S^{\circlearrowleft}$ is winning for $AC(Y)$ and thus $TAC(Y)$.
- some path consistent with σ sees z along a simple path h. Define

  $$T(u) := \begin{cases} S^{\circlearrowleft}(u) & \text{if } z \text{ does not appear on } u, \\ (S_z)^{\circlearrowleft}(tail_z(u)) & \text{otherwise.} \end{cases}$$

  where $S_z(u) = S(hu)$ if u starts in z (and, otherwise arbitrarily).

  In words, T behaves like the pumping strategy $S^{\circlearrowleft}$. Once (and if) z is reached, T erases all its memory and switches to $(S_z)^{\circlearrowleft}$ (which itself is winning from z in the FCG).

Form the strategy σ^z as follows: "follow σ and remove the cycles that are formed; the first time we reach z (if at all), remove the □

Question 15. *Ok, so how to check if A is Y-unambigous?*

Lemma 4. *If (A, Obj) is Y-greedy and Obj and $\neg Obj$ are prefix-independent, then A is Y-unambiguous.*

Proof. Suppose not. Take a play π of some arena such that some suffix $\pi' \in AC(Y) \subseteq Obj$ and some suffix $\pi'' \in AC(\neg Y) \subseteq \neg Obj$. But one of these is a suffix of the other, so $\pi' \in Obj$ iff $\pi'' \in Obj$. Contradiction. □

But prefix-independence is typically easy to check!

> To conclude that every game with objective Obj is memless determined:
>
> 1. Check that Obj and $\neg Obj$ are prefix independent.
> 2. Finitise the winning condition Obj to get a cycle property Y.
> 3. Show that every game with objective Obj is Y-greedy.

Note. If also Y is closed under cyclic permutations, then can conclude uniform memless determined (not shown here).

Example 5. *Apply the recipe to Büchi objective, parity-objective, mean-payoff objective.*

What about uniformity?

Theorem 11. *If Y is closed under cyclic permutations then every Y-greedy game that is memoryless determined is also uniform memoryless determined.*

Proof. Enough to show that if $(A, FC(Y))$ memoryless determined implies it is uniform memoryless determined (then use strategy transfer theorem). □

What if the objective is not prefix independent?

Theorem 12. *Let Y be a cycle property. If Y is closed under cyclic permutations, and both Y and $\neg Y$ are closed under concatenation, then every arena is Y-resettable.*

To conclude that (A, Obj) is uniform memless determined:

1. Finitise the winning condition Obj to get a cycle property Y.
2. Check that Y is closed under cyclic permutations and both Y and $\neg Y$ are closed under concatenation.
3. Check that (A, Obj) is Y-greedy.

Exercise. Energy games: Use this to show that either there is an initial credit with which Player 0 (the "energy" player) wins (i.e., the sum of the credit and weights seen so far is never negative), or that for every initial credit Player 1 wins.

7 Games with multiple players

Outline Games with multiple players require different types of solutions. We will define Nash equilibria on games on graphs, and show how to decide their existence (and compute an equilibrium, if it exists) for certain objectives that we have already encountered.

8 Different approaches to solving games

1. PDL satisfiability (Racer, FaCT, Pellet)
2. simulation based techniques
3. safety games, GR(1), ATL
4. Directed search (Stroeder and Pagnucco 09)
5. Planning