Periodic behaviour of generalized threshold functions

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COMMUNICATION

PERIODIC BEHAVIOUR OF GENERALIZED THRESHOLD FUNCTIONS

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It is shown that, for a function Δ from $\{0, 1\}^n$ to $\{0, 1\}^n$ whose components form a symmetric set of threshold functions the repeated application of Δ, leads either to a fixed point or to a cycle of length two.

1. The main theorem

Let $E=\{0, 1\}$ and Δ be a function:

$$\Delta : E^n \to E^n,$$

$$y=(y_1, \ldots, y_n) \to (\phi_1(y), \ldots, \phi_n(y))$$

where each ϕ_i is a threshold function, defined by:

$$\phi_i(y_1, \ldots, y_n)=\begin{cases} 0 & \text{if } \sum_{j=1}^{n} \alpha_{ij} y_j < \theta_i, \\ 1 & \text{otherwise,} \end{cases} \qquad \alpha_{ij}, \theta_i \in \mathbf{R}.$$

We say that Δ is "symmetrically threshold" if for each pair $\{i, j\}$ we have $\alpha_{ij} = \alpha_{ji}$.

This note concerns the periodic behaviour of the iteration:

$$y^{r+1}=\Delta y^r, \quad r=0, 1, 2, \ldots \text{ with } y^0 \in E^n$$

The main theorem is the following:

Theorem. *For any* $y \in E^n$, *there exists* $s \in \mathbf{N}$, *such that*

$$\Delta^{s+2} y = \Delta^s y.$$

This is not true in general if Δ does not satisfy the symmetrical threshold.

2. The tools of proof

Clearly, since E^n is finite, for each $y \in E^n$ there exist $s, t \in \mathbf{N}$, $s \geq 0$, $t > 0$ such that: $\Delta^{s+t}y = \Delta^s y$ and $\Delta^{s+r}y \neq \Delta^s y$ for any r satisfying $0 < r < t$. Then, consider the $n \times t$ matrix:

$$X(y, t) = \begin{pmatrix} x_1(0) & \cdots & x_1(t-1) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ x_n(0) & \cdots & x_n(t-1) \end{pmatrix} = (\Delta^s y, \Delta^{s+1}y, \ldots, \Delta^{s+t-1}y).$$

It is obvious that, for $i \in \{1, \ldots, n\}$:

$$x_i(0) = \phi_i(x_1(t-1), \ldots, x_n(t-1)),$$
$$x_i(l+1) = \phi_i(x_1(l), \ldots, x_n(l)) \quad \text{for } l \in \{0, \ldots, t-2\}.$$

Note. For any integer k, we define $x_i(k)$ as $x_i(r)$ where $r \in \{0, \ldots, t-1\}$ and $k \equiv r \pmod{t}$.

Let γ_i denote the smallest period of the row x_i (γ_i is necessarily a divisor of t). Let us define the mapping L on the set S of rows of $X(y, t)$:

$$L : S \times S \to R,$$
$$(x_i, x_j) \to \alpha_{ij} \sum_{l=0}^{t-1} (x_j(l+1) - x_j(l-1)) x_i(l).$$

Lemma 1. (i) *$L(x_i, x_j) + L(x_j, x_i) = 0$ for $i, j \in \{1, \ldots, n\}$.*
(ii) *If $\gamma_i \leq 2$, then $L(x_i, x_j) = 0$ for $j \in \{1, \ldots, n\}$.*

The proof follows easily from the definition of L.

Lemma 2. *Let $i \in \{1, \ldots, n\}$ such that $\gamma_i \geq 3$, then*

$$\sum_{j=1}^{n} L(x_i, x_j) < 0.$$

The proof, rather technical, can be found in [1].

3. Proof of Theorem

For $y \in E^n$ let $X(y, t)$ be the corresponding matrix as before. If $t \geq 3$, then at least one γ_i is greater than or equal to 3, and by Lemmas 1 (ii) and 2, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} L(x_i, x_j) < 0.$$

But, by Lemma 1 (i):

$$\sum_{i=1}^{n}\sum_{j=1}^{n} L(x_i, x_j) = 0$$

which is a contradiction; therefore $t \leqslant 2$.

If $\alpha_{ij} \neq \alpha_{ji}$, for some pair $\{i, j\}$, this is no longer true in general. The following is a counter example:

$$\Delta : E^3 \to E^3,$$

$$y \to (\phi_1(y), \phi_2(y), \phi_3(y))$$

with $\phi_1(y) = 1[y_2 + 3y_3 - 2]$, $\phi_2(y) = 1[y_1 + y_3 - 1]$, $\phi_3(y) = 1[y_2 - y_1 - 1]$, where $1[v] = 1$ when $v \geqslant 0$, and $1[v] = 0$ when $v < 0$. Let $y^0 = (1, 1, 0)$, then $\Delta^3 y^0 = y^0$ and $\Delta^r y^0 \neq y^0$ for $r = 1, 2$.

4. Comments and remarks

Many authors in various areas: social psychology [2, 4], symmetrical nerve nets [3, 5], uniform cellular automata [6], have already observed, in very special cases, the two-length cycle behaviour of the corresponding iteration. Their proof methods are different from ours, and particular to the problem involved. On the other hand, this theorem can be generalized to the case of $\Delta : \{0, 1, \ldots, q\}^n \to \{0, 1, \ldots, q\}^n$, using an extension of the concept of symmetrical threshold.

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