Choosing products in social networks

Krzysztof Apt and Sunil Simon

CWI, Amsterdam

Social networks

- Facebook
- LinkedIn
- Google+

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- Facebook
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- Google+
- Tupperware party 1960s (Source: Wikipedia)

Social networks

Essential components of the model

- Finite set of agents
- Influence of “friends”
- Finite product set for each agent
- Resistance level in adopting a product

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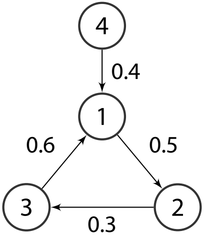

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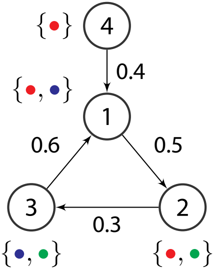

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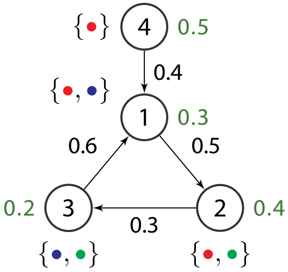

The model

Social network [Apt, Markakis 2011]

- Weighted directed graph: $G = (V, \to)$ consisting of a finite set of agents $V = \{1, \dots, n\}$ and a weight function $w_{ij} \in [0, 1]$: weight of the edge $i \to j$
- Products: A finite set of products $\mathcal{P}$
- Product assignment: A map $P : V \to 2^{\mathcal{P}} \setminus \{\varnothing\}$ which assigns to each agent a non-empty set of products
- Threshold function: For each agent i the threshold value $0 < \theta(i) \leq 1$

- Neighbour of node i: $\{j \in V \mid j \to i\}$
- Source nodes: Agents with no neighbours

The associated strategic game

Interaction between agents: Each agent i can adopt a product from the set $P(i)$ or choose not to adopt any product (t_0)

Social network games

- Players: Agents in the network
- Strategies: Set of strategies for player i is $P(i) \cup \{t_0\}$

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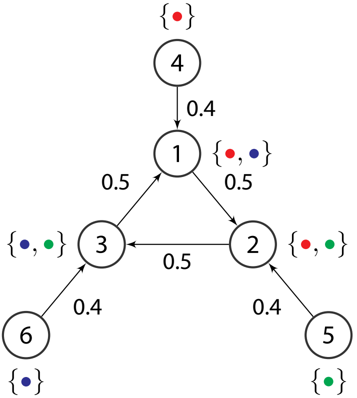
  - if $i \notin source(\mathcal{S})$, $p_i(s) = \begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}$

Notation: $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted in s the product t.

Example



Threshold is 0.3 for all the players

- $\mathcal{P} = \{{\color{red}\bullet}, {\color{green}\bullet}, {\color{blue}\bullet}\}$

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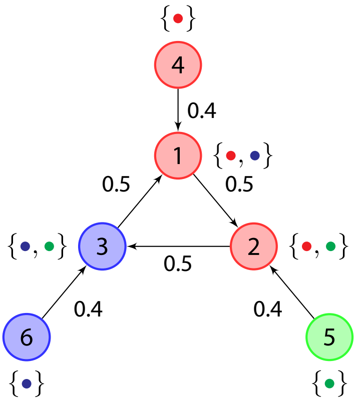

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Payoff:

- $p_4(s) = p_5(s) = p_6(s) = c$

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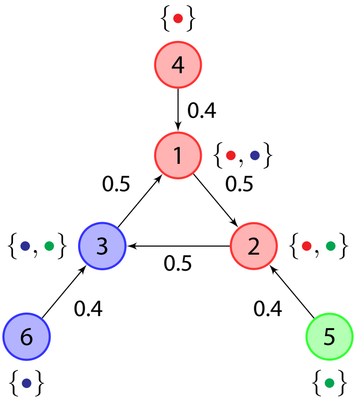

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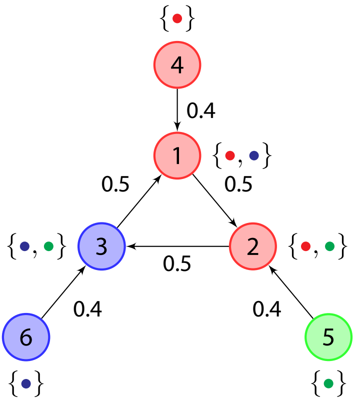

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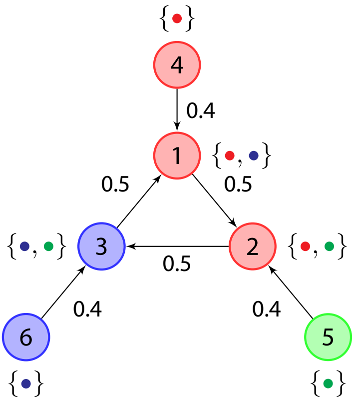

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Social network games

Properties

- Graphical game: The payoff for each player depends only on the choice made by his neighbours
- Join the crowd property: The payoff of each player weakly increases if more players choose the same strategy

Solution concept

Best response

A strategy s_i of player i is a best response to a joint strategy s_{-i} if for all s_i', $p_i(s_i', s_{-i}) \leq p_i(s_i, s_{-i})$.

Nash equilibrium

A strategy profile s is a Nash equilibrium if for all players i, s_i is the best response to s_{-i}.

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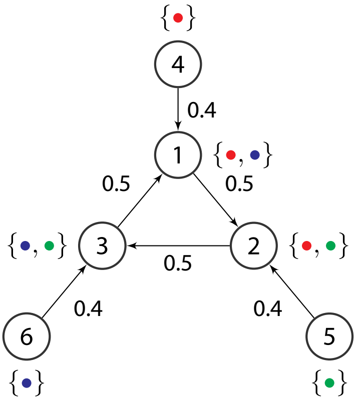
Answer: No

Nash equilibrium



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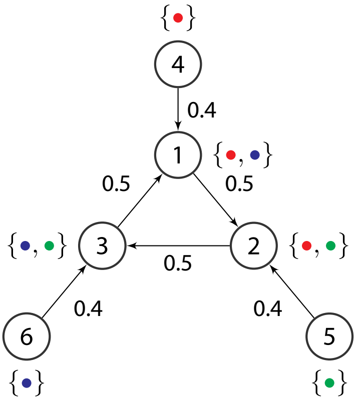

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- Source nodes can ensure a payoff of $c > 0$
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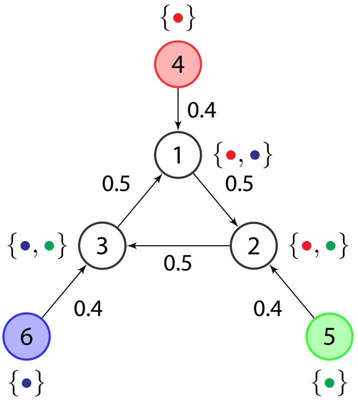

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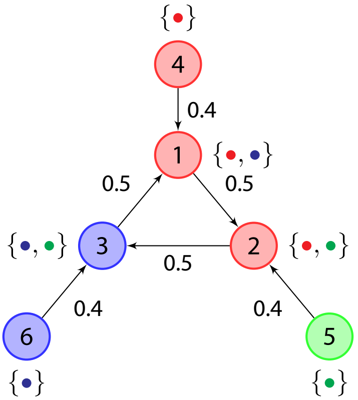

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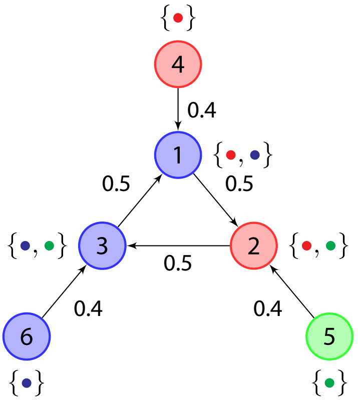

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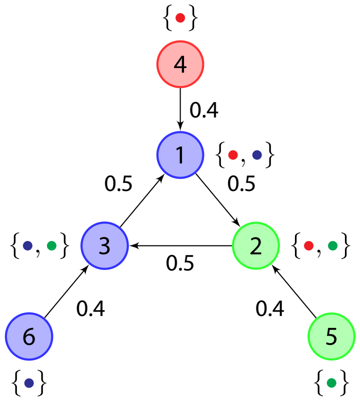

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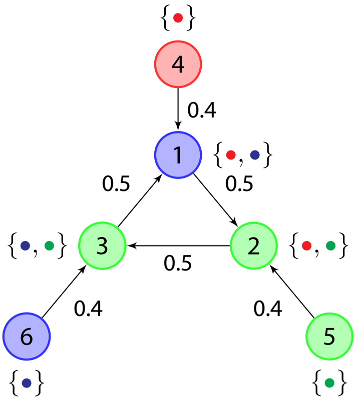

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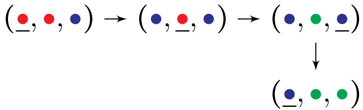

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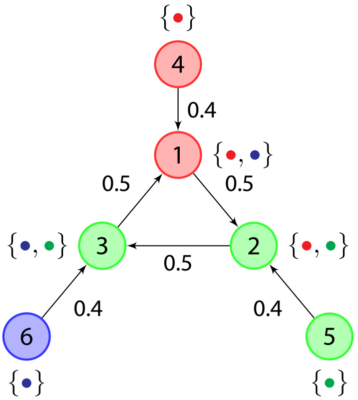

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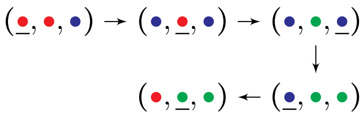

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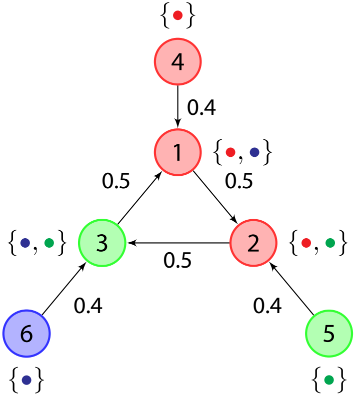

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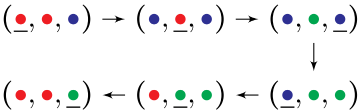

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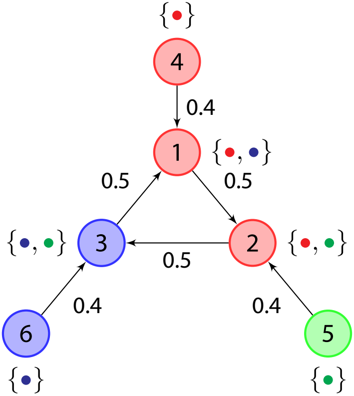

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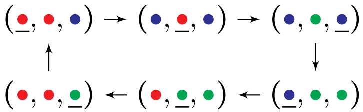
Best response dynamics



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Question: Given a social network S, what is the complexity of deciding if $G(S)$ has a Nash equilibrium?

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The PARTITION problem

Input: n positive rational numbers $(a_1, \ldots, a_n)$ such that $\sum_i a_i = 1$.

Question: Is there a set $S \subseteq \{1, 2, \ldots, n\}$ such that

$$\sum_{i \in S} a_i = \sum_{i \notin S} a_i = 1/2.$$

Hardness

Reduction: Given an instance of the PARTITION problem $P = (a_1, \ldots, a_n)$, construct a network $\mathcal{S}(P)$ such that there is a solution to P iff there is a Nash equilibrium in $\mathcal{S}(P)$.

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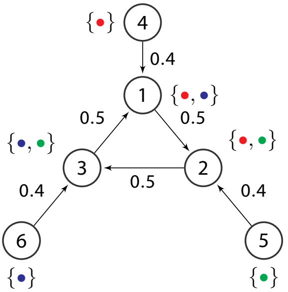

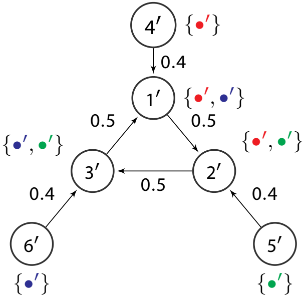

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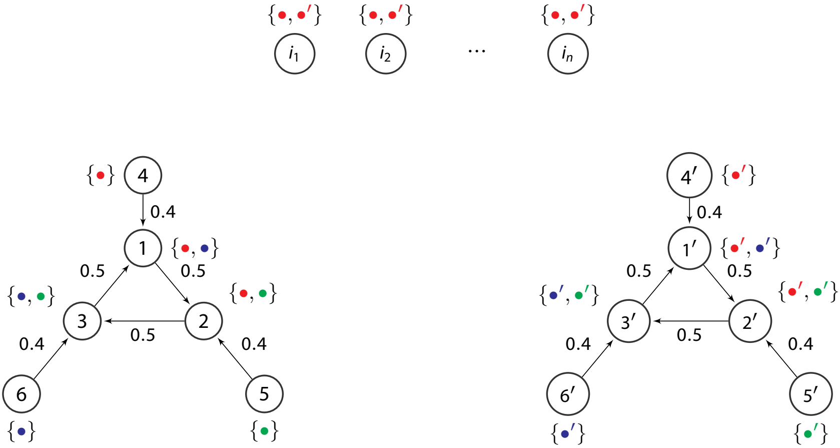

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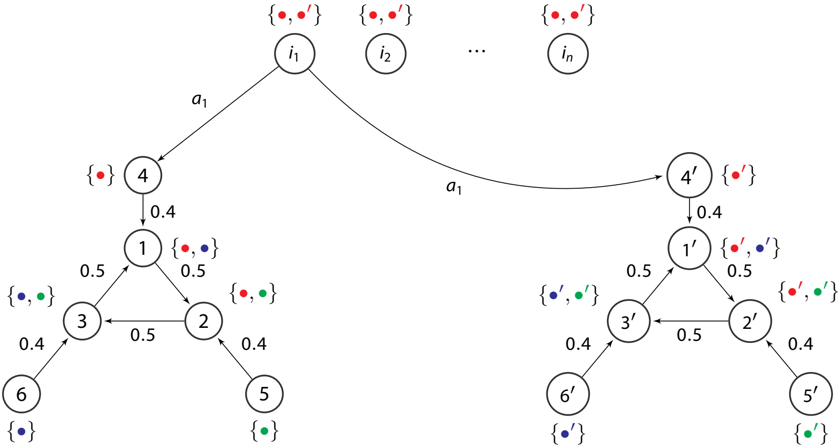

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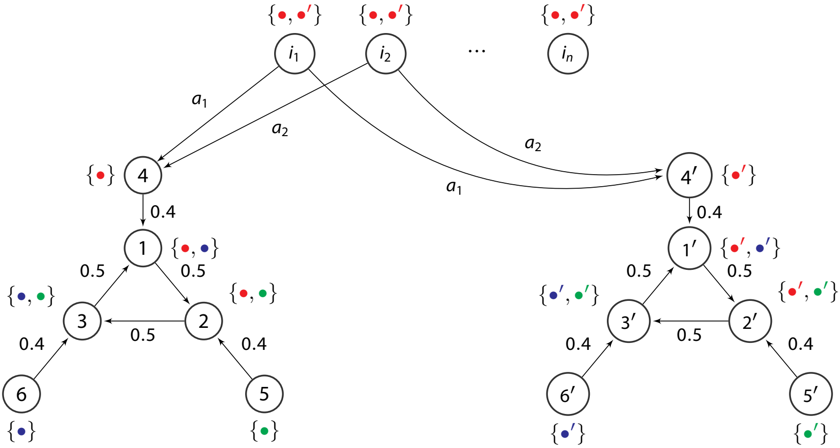

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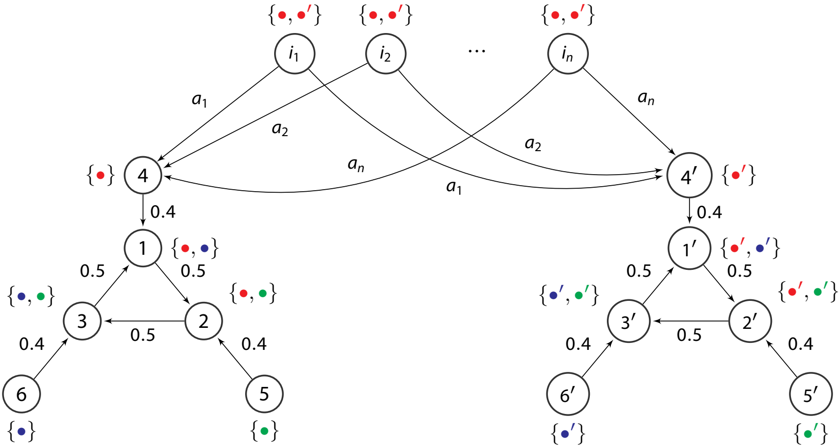
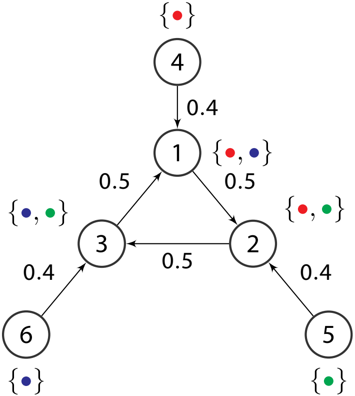
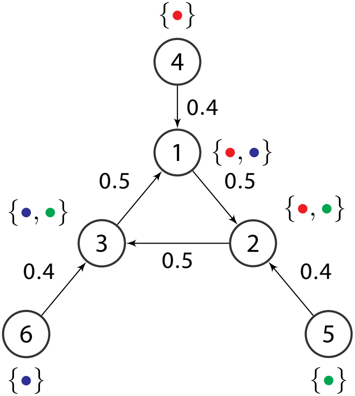


Nash equilibrium



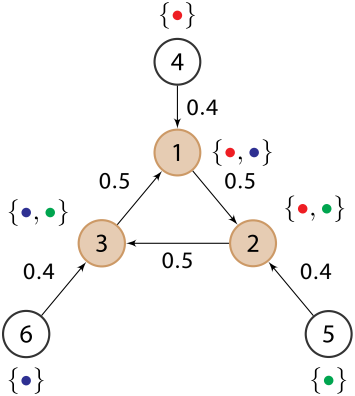
Theorem. If there exists $X \subseteq \mathcal{P}$ where $|X| \leq 2$ such that for all source nodes i, $P(i) \cap X \neq \varnothing$ then $\mathcal{S}$ has a Nash equilibrium and it can be computed in polynomial time.

Nash equilibrium



Properties of the underlying graph:

Nash equilibrium



Properties of the underlying graph:

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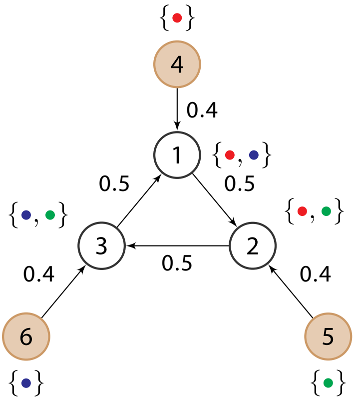

Properties of the underlying graph:

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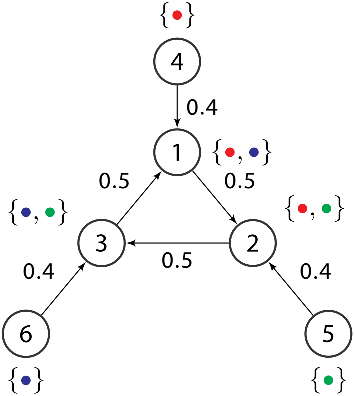


Properties of the underlying graph:

- Contains a cycle
- Contains source nodes

Question: Does Nash equilibrium always exist in social networks when the underlying graph

- is acyclic?
- is free of source nodes?

Directed acyclic graphs

Theorem. In a DAG, a non-trivial Nash equilibrium always exist.

Procedure to generate a non-trivial Nash equilibrium

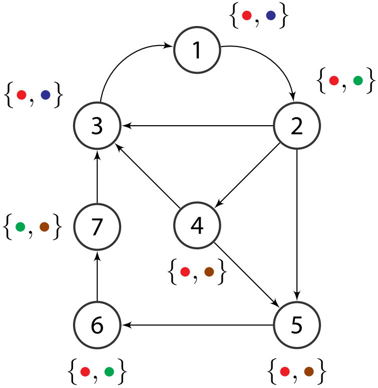
Initialise: Assigns a product for each source node

Repeat until all nodes are labelled:

- Pick a node which is not labelled and for which all neighbours are labelled
- Assign the product which maximises the payoff

Theorem. A strategy profile s is a Nash equilibrium iff there is a run of the labelling procedure such that s is the profile defined by the labelling function.

Graphs with no source nodes



"Circle of friends": everyone has a neighbour

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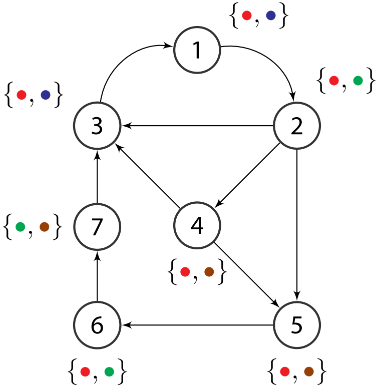
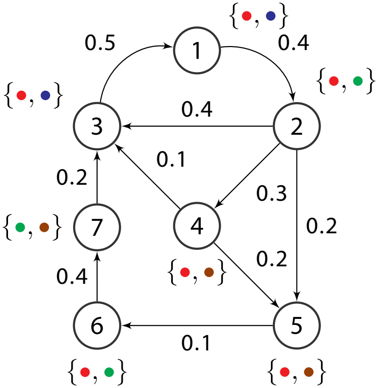


“Circle of friends”: everyone has a neighbour

Observation: $\overline{t_0}$ is always a Nash equilibrium

Question: When does a non-trivial Nash equilibrium exist?

Graphs with no source nodes



Threshold=0.3

Self sustaining subgraph

A subgraph C_t is self sustaining for product t if it is strongly connected and for all i in C_t,

- $t \in P(i)$
- $\sum_{j \in \mathcal{N}(i) \cap C_t} w_{ji} \geq \theta(i)$

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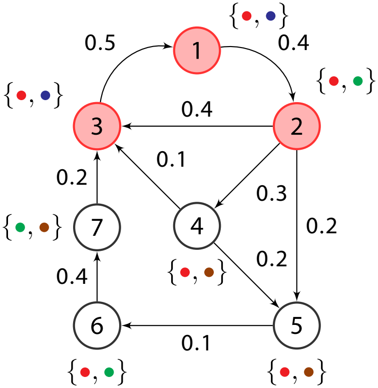

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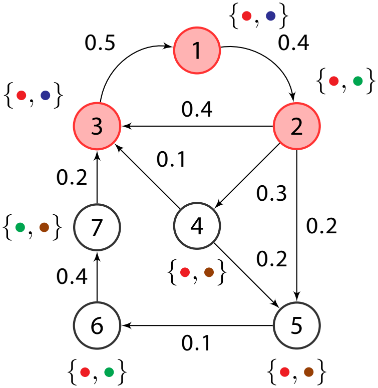

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Theorem. There is a non-trivial Nash equilibrium iff there exists a product t and a self sustaining subgraph C_t for t.

Graphs with no source nodes

An efficient procedure

For a product t,

- $X_t^0 := \{i \in V \mid t \in P(i)\}$
- $X_t^{m+1} := \{i \in V \mid \sum_{j \in \mathcal{N}(i) \cap X_j^m} w_{ji} \geq \theta(i)\}$
- $X_t := \bigcap_{m \in \mathbb{N}} X_t^m$

Theorem. There is a non-trivial equilibrium iff there exists a product t such that $X_t \neq \emptyset$.

Complexity

- For a fixed product t, the set X_t can be computed in $\mathcal{O}(n^3)$.
- Running time: $\mathcal{O}(|\mathcal{P}| \cdot n^3)$

Nash equilibrium

Networks where everyone is forced to adopt a product

Theorem. Nash equilibrium may not exist even for a simple cycle.

Theorem. Checking if Nash equilibrium exists in a graph with no source nodes is NP-complete.

Network dynamics (with K. Apt & E. Markakis)

Consequence of adding new products

Observation. Starting at a Nash equilibrium, suppose an additional product t become available to a single player i. The best response path can lead to a new Nash equilibrium where everyone is worse off (including player i).

Addition of links

The same observation holds for addition of new links in a network.

Thank You