Automatic Linear Orders and Trees

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We investigate partial orders that are computable, in a precise sense, by finite automata. Our emphasis is on trees and linear orders. We study the relationship between automatic linear orders and trees in terms of rank functions that are related to Cantor–Bendixson rank. We prove that automatic linear orders and automatic trees have finite rank. As an application we provide a procedure for deciding the isomorphism problem for automatic ordinals. We also investigate the complexity and definability of infinite paths in automatic trees. In particular, we show that every infinite path in an automatic tree with countably many infinite paths is a regular language.

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1. INTRODUCTION

Consider a class of infinite structures, such as the class of graphs, partial orders, trees, groups, or lattices, etc. A given structure in this class may or may not be computable. If it is, one then naturally asks whether or not the structure,

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or algorithmic problems of the structure, are feasibly computable. In case that 'feasible structure' means computable by finite automata, one has an *automatic structure* (Definition 2.4). The automata in this article operate synchronously on finite words. Using the closure of these automata under Boolean operations and projection, one has that the first-order theory of an automatic structure is decidable, see, for instance, Khoussainov and Nerode [1994]. From a computer science point of view, this result suggests that automatic structures may be suitable objects that can be effectively queried. This is illustrated in the related concept of automatic groups from computational group theory [Epstein et al. 1992]. There it is proven that a finitely generated automatic group is finitely presentable and that its word problem is solvable in quadratic time. The general notion of structures presentable by automata has been recently studied in Blumensath [1999], Blumensath and Grädel [2000], Delhommé [2004], Delhommé et al. [2002], Ishihara et al. [2002], Khoussainov and Nerode [1994], Khoussainov and Rubin [2001, 2003], Khoussainov et al. [2004a, 2004b], Kuske [2003] and Lohrey [2003]. Throughout this article, we will use the following more general theorem proved in Blumensath and Grädel [2000] without explicit mention.

THEOREM 1.1. *Given an automatic structure $\mathcal{A}$ and a relation R, which is first-order definable in $\mathcal{A}$ (with the quantifier $\exists^\infty$ which stands for 'there exist infinitely many'), one can effectively construct an automaton recognizing R.*

Our work is motivated by the following general problem:

Problem 1.2. Given a class of structures $\mathcal{C}$, characterize the isomorphism types of the automatic structures in $\mathcal{C}$.

The isomorphism type of a structure $\mathcal{A}$ is defined as the set of structures that are isomorphic to $\mathcal{A}$. A satisfactory answer to this problem would give a non-automata-theoretic description of those isomorphism types that contain an automatic structure. We note that the isomorphism problem for automatic graphs is Σ_1^1–complete [Khoussainov et al. 2004a] and that the isomorphism problem for the class of all automatic structures has the same complexity. Consequently, we restrict the class of structures under consideration for better results; for instance, we show that the isomorphism problem of automatic ordinals (represented as automatic well-orderings) is recursive. This article is concerned with the class of partially ordered structures, with an emphasis on trees and linear orderings. A *partial order* (*partial ordering*) is a structure $(A, \preceq)$ such that $\preceq$ is a reflexive, transitive and anti-symmetric binary relation on the domain A. A *linear order* $\mathcal{L}$ is a partial order $(L, \leq)$ in which $\leq$ is total, that is $(\forall x)(\forall y)[x \leq y \vee y \leq x]$.

Classically linear orderings are characterized in terms of scattered and dense linear orderings as follows: One says that $\mathcal{L}$ is *dense* if for all distinct a and b in L with $a < b$ there exists an $x \in L$ with $a < x < b$. There are only five types of countable dense linear orderings up to isomorphism: the order of rational numbers with or without least or greatest elements, and the order type of the trivial linear order with exactly one element. One says that $\mathcal{L}$ is *scattered* if it does not contain a nontrivial dense subordering. Examples of scattered linear orders are finite sums (see Definition 3.1) of Cartesian products of ω (the order type of the natural numbers) and ζ (the order type of the integers). The following theorem is the classical representation of countable linear orderings–it is proved below.

THEOREM 3.2. *Every countable linear ordering $\mathcal{L}$ can be represented as a dense sum of countable scattered linear orderings.*

The scattered linear orderings can be characterized inductively whereby to each scattered linear order $\mathcal{L}$ one associates a countable ordinal–called the VD–rank of $\mathcal{L}$ (Definition 3.3), a version of Cantor–Bendixson rank for topological spaces. One of the results in this paper (Proposition 4.6) says that the VD–rank of every automatic scattered linear order is finite.

Definition 3.8 introduces the FC-rank of a (not necessarily scattered) linear order, with the property that FC-rank and VD-rank agree on scattered linear orders. Combining Proposition 4.6 and Theorem 3.2 gives the following necessary condition for automatic linear orders.

THEOREM 4.7. *If $\mathcal{L}$ is an automatic linear order, then its FC-rank is finite.*

This condition is not sufficient for a linear order to be automatic; indeed, there is a scattered linear order of FC-rank 2 that is not isomorphic to any automatic linear order (Remark 4.8).

The proof of Theorem 4.7 generalizes a novel technique of Delhommé. In a manuscript circulated in 2001, he gives a full characterization of automatic ordinals.

COROLLARY [DELHOMMÉ 2004]. *An ordinal α is isomorphic to an automatic ordinal if and only if $\alpha < \omega^\omega$.*

There is an effective procedure that given an automatic presentation of an ordinal, produces the ordinal's Cantor-normal-form.

THEOREM 5.3. *The isomorphism problem for automatic ordinals is decidable.*

Recently, Delhommé [2004] independently generalized his technique and notes that Theorem 4.7 can be obtained as a corollary. He also generalizes the technique to tree-automatic structures and in particular characterizes the tree-automatic ordinals.

The second topic of this paper concerns the class of partial orders known as trees. A *tree* $\mathcal{T} = (T, \preceq)$ is a partial order that has a minimum element and in which every set of the form $\{y \in T \mid y \preceq x\}$ forms a finite linear order. Elements of trees are called nodes. A *path* of a tree $(T, \preceq)$ is a subset $P \subseteq T$, which is linearly ordered and maximal (with respect to set theoretic inclusion) with this property. An *infinite path* is a path P consisting of infinitely many nodes. We are interested in understanding algebraic, model-theoretic as well as computational properties of automatic trees.

We deal with trees by associating to each tree its Kleene–Brouwer ordering. This transformation preserves automaticity, and associates the Cantor–Bendixson rank (CB-rank for short) of trees with the FC-ranks of the associated linear orders. Informally the CB-rank of the tree tells us how complicated its infinite paths are in terms of ordinals (see e.g., Kechris [1995]). This relationship between trees and linear orders yields the next result.

THEOREM 7.10. *The CB-rank of an automatic tree is finite.*

It is known that every infinite finitely branching tree has an infinite path– usually referred to as König's Lemma. The proof of this fact does not produce an infinite path constructively. In fact, there are even examples of computable finitely branching trees with *exactly* one infinite path, and that path is *not* computable. Moreover, if one omits the assumption that the tree is finitely branching, then there are examples of computable trees in which every infinite path is not even arithmetical (see Rogers [1967]). This negative phenomenon fails dramatically when one considers automatic trees, and not only finitely branching ones. To start with, here is an automatic version of König's Lemma.

THEOREM 8.2. *Every infinite automatic finitely branching tree* $(T, \preceq)$ *has a regular infinite path. That is, there exists a regular set* $P \subseteq T$ *so that* P *is an infinite path of the tree.*

This is because the length-lexicographically leftmost path is definable using the quantifier $\exists^\infty$ in $(T, \preceq)$. We can significantly strengthen this theorem under the assumption that the tree has at most countably many paths. Indeed from Theorem 7.10, we derive that if an automatic finitely branching tree $\mathcal{T}$ has countably many infinite paths then every path of $\mathcal{T}$ is regular (Theorem 8.3). This is because the set of paths in such trees is definable. Moreover, one may even omit the assumption that the tree be finitely branching.

THEOREM 8.7. *Every infinite path in an automatic tree with countably many infinite paths is regular.*

2. PRELIMINARIES

All classical definitions and results on linear orderings can be found in Rosenstein [1982]. Countable means finite or countably infinite. All structures are assumed to be countable. Definable means first-order definable with the additional quantifier $\exists^\infty$.

A thorough introduction to automatic structures can be found in Blumensath [1999] and Khoussainov and Nerode [1994]. A survey paper of Khoussainov and Rubin [2003] discusses the basic results and possible directions for future work in the area. Familiarity with the basics of finite automata theory is assumed though for completeness and to fix notation the necessary definitions are included here.

A *finite automaton* $\mathcal{A}$ over an alphabet Σ is a tuple (S, ι, Δ, F), where S is a finite set of *states*, $\iota \in S$ is the *initial state*, $\Delta \subseteq S \times \Sigma \times S$ is the *transition table* and $F \subseteq S$ is the set of *final states*. A *computation* of $\mathcal{A}$ on a word $\sigma_1\sigma_2 \cdots \sigma_n$ ($\sigma_i \in \Sigma$) is a sequence of states, say $q_0, q_1, \ldots, q_n$, such that $q_0 = \iota$ and $(q_i, \sigma_{i+1}, q_{i+1}) \in \Delta$ for all $i \in \{0, 1, \ldots, n-1\}$. If $q_n \in F$, then the computation is *successful*. If a word has a successful computation, then we say that automaton $\mathcal{A}$ *accepts* the word. The *language* accepted by the automaton $\mathcal{A}$ is the set of all words accepted by $\mathcal{A}$. In general, $D \subseteq \Sigma^\star$ is *finite automaton recognizable*, or *regular*, if D is equal to the language accepted by $\mathcal{A}$ for some finite automaton $\mathcal{A}$. An automaton $\mathcal{A}$ is *deterministic* if, for every $q \in S$ and $\sigma \in \Sigma$, there is a unique $q' \in S$ such that $(q, \sigma, q') \in \Delta$.

We briefly mention Büchi automata as they will be used later in Lemma 8.6. A (nondeterministic) Büchi automaton (S, ι, Δ, F) over Σ accepts an infinite string $\alpha \in \Sigma^\omega$ if it has a run $(q_i)_{i\in\mathbb{N}}$ such that there is some state $f \in F$ with $f = q_j$ for infinitely many $j \in \mathbb{N}$. The language accepted by such an automaton is called *Büchi recognizable*. See, for instance, Khoussainov and Nerode [2001] for a modern treatment.

Classically finite automata recognize sets of words. The following definition extends recognizability to relations of arity n, by *synchronous n–tape automata*. Informally, a synchronous n–tape automaton can be thought of as a one-way Turing machine with n input tapes [Eilenberg et al. 1969]. Each tape is regarded as semi-infinite, having written on it a word in the alphabet Σ followed by an infinite succession of blanks, $\diamond$ symbols. The automaton starts in the initial state, reads simultaneously the first symbol of each tape, changes state, reads simultaneously the second symbol of each tape, changes state, etc., until it reads a blank on each tape. The automaton then stops and accepts the n-tuple of words if it is in a final state. The set of all n-tuples accepted by the automaton is the relation recognized by the automaton. Here is a formalization:

Definition 2.1. Write $\Sigma_\diamond$ for $\Sigma \cup \{\diamond\}$ where $\diamond$ is a symbol not in Σ. The *convolution of a tuple* $(w_1, \dots, w_n) \in \Sigma^{\star n}$ is the string $\otimes(w_1, \dots, w_n)$ of length $\max_i |w_i|$ over alphabet $(\Sigma_\diamond)^n$ defined as follows. Its kth symbol is $(\sigma_1, \dots, \sigma_n)$ where σ_i is the kth symbol of w_i if $k \le |w_i|$ and $\diamond$ otherwise.

The *convolution of a relation* $R \subseteq (\Sigma^\star)^n$ is the relation $\otimes R \subseteq ((\Sigma_\diamond)^n)^\star$ formed as the set of convolutions of all the tuples in R.

Definition 2.2. *An n-tape automaton* on Σ is a finite automaton over the alphabet $(\Sigma_\diamond)^n$. An n–ary relation $R \subseteq \Sigma^{\star n}$ is *finite automaton recognizable* or *regular* if its convolution $\otimes R$ is recognizable by an n-tape automaton.

For instance, let $\preceq_p$ be the *prefix* relation. That is, for $x, y \in \Sigma^\star$ define $x \preceq_p y$ if there exists $z \in \Sigma^\star$ such that $xz = y$. If z is not the empty string ϵ, then x is a *proper prefix* of y, written $x \prec_p y$. So $\preceq_p$ and $\prec_p$ are regular binary relations over Σ. For example, if $\Sigma = \{0, 1\}$, then $\otimes(\preceq_p) = \{\binom{1}{1}, \binom{0}{0}\}^\star\{\binom{\diamond}{1}, \binom{\diamond}{0}\}^\star$.

PROPOSITION 2.3 (KHOUSSAINOV AND NERODE [1994]). *n-tape automata can be effectively determinised, and are effectively closed under Boolean operations and projection.*

We now relate n-tape automata to structures. A *structure* $\mathcal{A}$ consists of a set A called the *domain* and some constants, relations and operations on A. We may assume that $\mathcal{A}$ only contains relational predicates as the operations can be replaced with their graphs and constants can be thought of as operations of arity 0. We write $\mathcal{A} = (A, R_1^A, \dots, R_k^A)$ where R_i^A is an n_i-ary relation on $\mathcal{A}$. The *signature* of $\mathcal{A}$ is $(R_1, \dots, R_k)$. A structure is finite (countably infinite) if its domain has a finite (countably infinite) number of elements. An *isomorphism* between structures $\mathcal{A}$ and $\mathcal{B}$ of the same signature is a bijective mapping $\nu : A \to B$ such that for every relational symbol from the signature, say R of arity i, $(a_1, \ldots, a_i) \in R^A$ if and only if $(\nu(a_1), \ldots, \nu(a_i)) \in R^B$ for every tuple $(a_1, \ldots, a_i)$.

Definition 2.4. A structure $\mathcal{A}$ is *automatic over* Σ if its domain $A \subseteq \Sigma^\star$ and the relations $R_i^A \subseteq (\Sigma^\star)^{n_i}$ are finite automaton recognizable.

An isomorphism from a structure $\mathcal{B}$ to a structure $\mathcal{A}$ that is automatic over Σ is an *automatic presentation* of $\mathcal{B}$ in which case $\mathcal{B}$ is called *automatically presentable* over Σ. A structure is called *automatic* (*automatically presentable*) if it is automatic (automatically presentable) over some alphabet.

For instance, the structure $(\Sigma^\star, \preceq_p)$ is automatic. Other examples of automatically presentable structures are Presburger arithmetic $(\mathbb{N}, S, +, 0)$, the group of integers $(\mathbb{Z}, +)$, and the Boolean algebra of finite or co-finite subsets of $\mathbb{N}$.

We now mention some important examples of automatic linear orders. Fix an ordering on Σ, say $\sigma_1 < \sigma_2 < \cdots < \sigma_k$. Define x *lexicographically less* than y, written $x <_{lex} y$, if either x is a proper prefix of y, or else in the first place where they differ the symbol in x is $<$ the symbol in y. Then, $(\Sigma^\star, \leq_{lex})$ is an automatic linear order. Also define x *length-lexicographically less* than y, written $x <_{llex} y$, if $|x| < |y|$ or else $|x| = |y|$ and $x <_{lex} y$. Then $(\Sigma^\star, \leq_{llex})$ is an automatic linear order of type ω.

If $\mathcal{A} = (A, (R_i)_i)$ is an automatic structure over Σ, then $(A, (R_i)_i, \leq_{lex}, \leq_{llex})$ is also automatic over Σ. Consequently, every automatic presentation of $\mathcal{A}$ can be expanded to include the regular relations $\leq_{lex}$ and $\leq_{llex}$ (restricted to the domain A). This fact will be used repeatedly.

Examples of automatically presentable linear orders are $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$ and the order on the rationals $(\mathbb{Q}, \leq)$. Moreover, if $\mathcal{L}_1 = (L_1, \leq_1)$ and $\mathcal{L}_2 = (L_2, \leq_2)$ are automatic linear orders, then so is their sum and their product. Hence (the ordering given by) the ordinal ω^n is automatically presentable for every $n \in \mathbb{N}$. An example of such a presentation is the lexicographical ordering $\leq_{lex}$ restricted to the domain $(1^*0)^n$.

Below, we present two generic examples of automatic trees.

Example 2.5. Let R be a nonempty regular language and let *Pref* (R) be the set of prefixes of strings in R. Recall the prefix relation $\preceq_p$. Then, the partial orders $(\mathit{Pref}(R), \preceq_p)$ and $(R \cup \{\epsilon\}, \preceq_p)$ are automatic trees.

Example 2.6. Let R be a regular language containing ϵ. Consider the partial order $\mathcal{T} = (R, \leq)$, where $x \leq y$ iff $x = y$ or $|x| < |y|$ and x is lexicographically smallest among all $x' \in R$ such that $|x| = |x'|$. Note that by Proposition 2.3, the relation $\leq$ is regular since it is first-order definable from regular predicates. Hence, $\mathcal{T}$ is an automatic tree.

3. LINEAR ORDER PRELIMINARIES

If $\mathcal{L}$ is a linear ordering, then, unless specified, we denote its domain by L and ordering by $\leq_L$ or simply $\leq$. Write $<$ for the corresponding strict order; that is, $x < y$ is defined by $x \leq y \wedge x \neq y$. If $S \subseteq L$, then we write $\mathcal{S} = (S, \leq_S)$ for the ordering with domain S and ordering $\leq$ restricted to S. In this case, we say that $\mathcal{S}$ is a *subordering* of $\mathcal{L}$.

Write ω for the (order) type of the positive integers, $\omega^\star$ for the negative integers, ζ for the integers, η for the rationals and $\mathbf{n}$ for the finite order on n elements. The empty ordering is written $\mathbf{0}$ and the ordering with exactly one element is written $\mathbf{1}$. A subordering $\mathcal{S}$ of $\mathcal{L}$ is an *interval* if for every $x, y \in S$ with $x <_L y$ it is the case that $z \in S$ for every $z \in L$ satisfying $x <_L z <_L y$. An interval is *closed* if it is of the form $\{z \in L \mid x \leq z \leq y\}$ if $x \leq y$ and $\{z \in L \mid y \leq z \leq x\}$ otherwise; either way the interval is written $[x, y]$.

Definition 3.1. Consider a linear order $\mathcal{I}$ as an index set for a set of linear orderings $\{\mathcal{A}_i\}_{i\in I}$. The $\mathcal{I}$-*sum*

$$\mathcal{L} = \Sigma\{\mathcal{A}_i \mid i \in I\}$$

is the linear order with domain $\cup_i A_i$ (we may assume that the domains A_i are pairwise disjoint). For $x \in A_i$, $y \in A_j$, define $x \leq_L y$ if $(i <_I j) \vee (i = j \wedge x \leq_{A_i} y)$.

We refer to the case when I is dense as a *dense sum*. Classically linear orderings are characterized in terms of scattered and dense linear orderings. Recall that $\mathcal{L}$ is *scattered* if it does not contain a nontrivial dense subordering. If every $\mathcal{A}_i$ is scattered, and $\mathcal{I}$ is scattered, then the sum is scattered. If $\mathcal{A}_i = \mathcal{B}$ for every $i \in I$, then the sum is written as a product $\mathcal{B}\mathcal{I}$. For instance, $\omega 2$ is $\omega + \omega$. The following classical characterization, whose proof is given below, is due to Hausdorff [1908].

THEOREM 3.2. *Every countable linear ordering $\mathcal{L}$ can be represented as a dense sum of countable scattered linear orderings.*

In turn, the scattered linear orders can be characterized inductively, where to each linear order one associates an ordinal ranking, called the VD-rank. VD stands for very discrete.

Definition 3.3. For each countable ordinal α, define the set VD_α of linear orders inductively as

(1) $\mathrm{VD}_0 := \{\mathbf{0}, \mathbf{1}\}$.
(2) $\mathrm{VD}_\alpha :=$ all linear orderings formed as $\mathcal{I}$-sums where $\mathcal{I}$ is of the type ω, $\omega^\star$, ζ or $\mathbf{n}$ for some $n < \omega$ and every $\mathcal{A}_i$ is a linear ordering from $\bigcup\{\mathrm{VD}_\beta \mid \beta < \alpha\}$.

Define the class VD as the union of the VD_α. The VD-*rank* of a linear ordering $\mathcal{L} \in \mathrm{VD}$, written $\mathrm{VD}(\mathcal{L})$, is the least ordinal α such that $\mathcal{L} \in \mathrm{VD}_\alpha$.

So the only linear orders of VD-rank 0 are $\mathbf{0}$ and $\mathbf{1}$. The linear orders of VD-rank 1 are exactly those of type ω, $\omega^\star$, ζ and $\mathbf{n}$ for $1 < n < \omega$.

Example 3.4. Let $\mathcal{L}_1 = \Sigma\{\zeta + \mathbf{n} \mid n \in \omega\}$, $\mathcal{L}_2 = (\zeta \cdot \zeta) \cdot \zeta$. Then $\mathrm{VD}(\mathcal{L}_1) \leq 2$ since $\mathcal{L}_1$ can be re-expressed as an ω-sum of orders of VD-rank 1. And in fact $\mathrm{VD}(\mathcal{L}_1) = 2$ since $\mathcal{L}_1$ does not have rank 1. Similarly, $\mathrm{VD}(\mathcal{L}_2) = 3$ and $\mathrm{VD}(\mathcal{L}_1 + \mathcal{L}_2) = 4$.

More generally, if $\alpha = \max(\mathrm{VD}(\mathcal{L}_1), \mathrm{VD}(\mathcal{L}_2))$, then $\alpha \leq \mathrm{VD}(\mathcal{L}_1 + \mathcal{L}_2) \leq \alpha + 1$ (see Rosenstein [1982, Lemma 5.15]).

Example 3.5. Let α, β be countable ordinals. Then $\mathrm{VD}(\beta) \leq \alpha$ iff $\beta \leq \omega^\alpha$. In particular, $\mathrm{VD}(\omega^\alpha) = \alpha$.

THEOREM 3.6 [HAUSDORFF 1908]. *A countable linear ordering $\mathcal{L}$ is scattered if and only if $\mathcal{L}$ is in VD.*

There is an alternative definition of ranking that generalizes VD-rank and assigns an ordinal rank to nonscattered linear orders as well. We proceed with the definitions.

Definition 3.7. A *condensation (map)* of $\mathcal{L}$ is a mapping c from L to nonempty intervals of L such that $c(y) = c(x)$ whenever $y \in c(x)$. The *condensation* of $\mathcal{L}$ is the linear order $c[\mathcal{L}]$ whose domain consists of the collection of nonempty intervals $c(x)$ for $x \in L$ ordered by $c(x) \trianglelefteq c(y)$ if $c(x) = c(y)$ or $(\forall x' \in c(x))(\forall y' \in c(y))[x' <_L y']$.

The relation x *is condensed (by c) to* y defined as $x \in c(y)$ is an equivalence relation.

As an illustration of the definition, we prove that every countable linear ordering can be represented as a dense sum of scattered linear orderings (Theorem 3.2).

PROOF. The mapping $c_S : x \mapsto \{y \in L \mid [x, y] \text{ is scattered}\}$ is a condensation since if $y \in c_S(x)$ then for all a, $[y, a]$ does not contain a dense subordering if and only if $[x, a]$ does not contain a dense subordering. Now $\mathcal{L} = \sum\{a \mid a \in c[\mathcal{L}]\}$ and each $a = c_S(x) \in c[\mathcal{L}]$ is scattered. Finally, $c_S[\mathcal{L}]$ is dense since for $c_S(x) \triangleleft c_S(y)$, if there is no z with $c_S(x) \triangleleft c_S(z) \triangleleft c_S(y)$ then $[x, y]$ is scattered. □

Definition 3.8. Define $c_{FC}(x)$ as $\{y \in L \mid [x, y] \text{ is a finite interval of } \mathcal{L}\}$.
For every ordinal α, define a condensation map c^α_{FC} of $\mathcal{L}$ inductively:

$$c^\alpha_{FC}(x) = \left\{ y \in L : y = x \vee (\exists \beta < \alpha)\left[c_{FC}\big(c^\beta_{FC}(y)\big) = c_{FC}\big(c^\beta_{FC}(x)\big)\right] \right\}.$$

In the expression $c_{FC}(c^\beta_{FC}(y))$, the term c_{FC} is a condensation map of the linear order $c^\beta[\mathcal{L}]$; hence, $c_{FC}(c^\beta_{FC}(y))$ is the set of elements of $c^\beta[\mathcal{L}]$ that are condensed to the element $c^\beta_{FC}(y)$. Note that this definition implicitly gives $c^0_{FC}(x) = \{x\}$ and $c^1_{FC}(x) = c_{FC}(x)$.

Here FC stands for finite condensation and indeed c^α_{FC} is a condensation map of $\mathcal{L}$. The idea is that $c^1_{FC}(x)$ is the set of elements of $\mathcal{L}$ that are only finitely far away from x; $c^2_{FC}(x)$ is the set of elements of $\mathcal{L}$ that are in intervals of $c_{FC}[\mathcal{L}]$ which themselves are only finitely far away in $c_{FC}[\mathcal{L}]$ from the interval $c^1_{FC}(x)$, etc.

Definition 3.9. The least ordinal α such that $c^\beta_{FC}(x) = c^\alpha_{FC}(x)$ for all $x \in L$ and $\beta \geq \alpha$ is called the FC-*rank* of $\mathcal{L}$, written $\mathrm{FC}(\mathcal{L})$.

For instance, a nonempty linear order $\mathcal{L}$ is dense if and only if its FC-rank is 0. A dense sum of orders of FC-rank α has FC-rank α. **From now on, we write c for $\mathbf{c_{FC}}$.**

Example 3.10. The FC-rank of $\mathcal{L}$ is the least ordinal α such that $c^\alpha[\mathcal{L}]$ is dense. So $\mathcal{L}$ is scattered if and only if $c^\alpha[\mathcal{L}] \simeq \mathbf{1}$ for some ordinal α.

The following theorem connects FC-ranks and VD-ranks of scattered linear orderings.

THEOREM 3.11 [HAUSDORFF 1908]. *If $\mathcal{L}$ is scattered, then its VD-rank equals its FC-rank.*

For instance, the ordinal ω^n is scattered and has VD-rank and FC-rank n.

Given linear order $\mathcal{L}$ and $A \subseteq L$, we will use c to denote the condensation of $\mathcal{L}$ and c_A to denote the condensation of the linear order $\mathcal{A}$. For instance, if $a \in A$, then $c_A(a) = \{y \in A \mid [a, y] \cap A \text{ is finite}\}$. Here are some useful properties.

LEMMA 3.12

(1) [Rosenstein 1982, Lemma 5.14]. If $\mathcal{L}$ is scattered and $M \subseteq L$, then $\mathrm{FC}(\mathcal{M}) \leq \mathrm{FC}(\mathcal{L})$.
(2) [Rosenstein 1982, Lemma 5.13 (2)]. $\mathrm{FC}(c^\alpha(x)) \leq \alpha$ and $c^\alpha(x)$ is a scattered interval of $\mathcal{L}$ for every ordinal α and $x \in L$.
(3) [Rosenstein 1982, Exercise 5.12 (1)]. If I is an interval of $\mathcal{L}$, then $c_I^\alpha(x) = c^\alpha(x) \cap I$ for every ordinal α and $x \in I$.
(4) For every $x, y \in L$, if $[x, y]$ is scattered, then $c^\alpha_{[x,y]}(x) = c^\alpha_{[x,y]}(y)$ if and only if $\mathrm{FC}([x, y]) \leq \alpha$.

PROOF. We prove the last item. Let $x, y \in L$ and α be an ordinal. Then, by definition, $\mathrm{FC}([x, y]) \leq \alpha$ means that, (†) for every $z \in [x, y]$, $c^\alpha_{[x,y]}(z) = c^{\alpha+1}_{[x,y]}(z)$, which necessarily equals $[x, y]$ since $[x, y]$ is scattered. Denote the condition $c^\alpha_{[x,y]}(x) = c^\alpha_{[x,y]}(y)$ by (††).

Then, (†) clearly implies (††) by considering $z \in \{x, y\}$. For the converse, suppose (††). We first claim that $c^\alpha_{[x,y]}(x) = [x, y]$. Indeed, (††) implies that $y \in c^\alpha_{[x,y]}(x)$ since $c^\alpha_{[x,y]}$ is a condensation, which means that $[x, y]$ is a subset of the interval $c^\alpha_{[x,y]}(x)$. But also $c^\alpha_{[x,y]}(x) \subseteq [x, y]$ by item (3). Hence, $c^\alpha_{[x,y]}(x) = [x, y]$ as claimed. So if $z \in [x, y] = c^\alpha_{[x,y]}(x)$, then $c^\alpha_{[x,y]}(x) = c^\alpha_{[x,y]}(z)$ by the property of being a condensation. Hence, $z \in [x, y]$ implies that $c^\alpha_{[x,y]}(z) = [x, y]$. In particular, then also $c^{\alpha+1}_{[x,y]}(z) = [x, y]$, which implies (†) as required. □

4. RANKS OF AUTOMATIC LINEAR ORDERS

We now prove the central technical result, Theorem 4.7, via three propositions. As a matter of convenience, we introduce the following variation of VD-rank.

Definition 4.1. If $\mathcal{L}$ is scattered, define its VD_**-rank*, written $\mathrm{VD}_*(\mathcal{L})$, as the least ordinal α such that $\mathcal{L}$ can be written as a finite sum of orderings of VD-rank $\leq \alpha$.

For example, it is not hard to check that $\mathrm{VD}(\omega) = \mathrm{VD}_*(\omega) = 1$ and that $\omega 2 + 1$ has VD-rank 2 but VD_*-rank 1. We list some basic properties.

PROPERTY 4.2. *Suppose $\mathcal{L}$ is scattered.*

(1) *$c^\alpha[\mathcal{L}]$ is a finite linear order if and only if $\mathrm{VD}_*(\mathcal{L}) \leq \alpha$. So $\mathrm{VD}_*(\mathcal{L})$ is the least ordinal such that $c^\alpha[\mathcal{L}]$ is a finite linear order.*

(2) *If* $M \subseteq L$*, then* $\mathrm{VD}_*(\mathcal{M}) \leq \mathrm{VD}_*(\mathcal{L})$.

(3) $\mathrm{VD}_*(\mathcal{L}) \leq \mathrm{VD}(\mathcal{L}) \leq \mathrm{VD}_*(\mathcal{L}) + 1$.

(4) $\mathrm{VD}_*(\mathcal{L}) = \alpha$ *implies that* $\mathcal{L}$ *contains an interval, say* M*, with* $\mathrm{VD}(\mathcal{M}) = \alpha$ *and* $\mathrm{VD}_*(\mathcal{M}) = \alpha$.

Proof. For the first item observe that for every α, $\mathcal{L} = \Sigma\{a \mid a \in c^\alpha[\mathcal{L}]\}$. Each a is an interval of the form $c^\alpha(x)$ for some $x \in L$. So, by Lemma 3.12, every a has VD-rank at most α. So, if $c^\alpha[\mathcal{L}]$ is finite, then $\mathrm{VD}_*(\mathcal{L}) \leq \alpha$. Conversely, let $\mathcal{L} = \mathcal{L}_1 + \cdots + \mathcal{L}_k$ and $\mathrm{VD}(\mathcal{L}_i) \leq \alpha$. Then, $L_i \subseteq c^\alpha(x)$ if $x \in L_i$. Since, for $x \in L$, the $c^\alpha(x)$ are pairwise disjoint, $c^\alpha[\mathcal{L}]$ is finite.

For the second item, suppose $\mathcal{L}$ can be expressed as a finite sum $\mathcal{L}_1 + \cdots + \mathcal{L}_K$ where $\mathrm{VD}(\mathcal{L}_i) \leq \alpha$. Define $M_i = M \cap L_i$. By Lemma 3.12(1), the VD-rank of M_i is at most α. But $\mathcal{M} = \mathcal{M}_1 + \cdots + \mathcal{M}_i$ so $\mathrm{VD}_*(\mathcal{M}) \leq \alpha$.

The third follows from item (1) above and the property that $\mathrm{VD}(\mathcal{L})$ is the least ordinal β such that $c^\beta[\mathcal{L}]$ is isomorphic to $\mathbf{1}$.

For the last item suppose $\mathrm{VD}_*(\mathcal{L}) = \alpha$. Then $\mathcal{L}$ can be expressed as a finite sum of orders of VD-rank (and by item (2) also VD_*-rank) at most α. There is a summand with VD_*-rank α for otherwise every summand can be written as a finite sum of linear orders of VD-rank $< \alpha$, and hence $\mathcal{L}$ could be written as a finite sum of linear orders of VD-rank $< \alpha$, contrary to assumption. Finally by item (3) if a summand has VD_*-rank α then it has VD-rank α. □

Lemma 4.3. *Suppose* $\mathcal{L}$ *is a scattered linear order containing* $\Sigma_{i \in I} A_i$ *as a subordering and* $\mathrm{VD}_*(\mathcal{A}_i) = \beta$*, where* $\mathcal{I}$ *has order type* ω *or* $\omega^\star$ *and each* A_i *is nonempty. Then* $\mathrm{VD}_*(\mathcal{L}) > \beta$.

Proof. By Property 4.2(4) we can assume without loss of generality that $\mathrm{VD}_*(\mathcal{A}_i) = \mathrm{VD}(\mathcal{A}_i) = \beta$. This means that A_i can not be written as a finite sum of orders of VD-rank $< \beta$. Suppose that $\mathcal{I}$ has order type ω, the other case being similar. Let $\mathcal{A} = \Sigma_i \mathcal{A}_i$. For every i choose some $x_i \in A_i$.

Suppose that $c^\beta_A(x_i) = c^\beta_A(x_{i+2})$ for some i. In other words, x_i is condensed to x_{i+2} in at most β steps. Then, $\beta > 0$ since $c^0_A(x) = \{x\}$ for every x. Moreover, there is some $\gamma < \beta$ so that there are finitely many elements in $c^\gamma_A[\mathcal{A}]$ between $c^\gamma_A(x_i)$ and $c^\gamma_A(x_{i+2})$. These finitely many elements are of the form $c^\gamma_A(x)$ for $x \in A$ and so have VD-rank at most γ. In particular, $\mathcal{A}_{i+1}$ can be written as a finite sum of orders of VD-rank at most γ, contrary to assumption. We conclude that $c^\beta_A(x_i) \neq c^\beta_A(x_{i+2})$ for every i.

Hence, $c^\beta_A[\mathcal{A}]$ is infinite and so $\mathrm{VD}_*(\mathcal{A}) > \beta$ by Property 4.2(1). So $\mathrm{VD}_*(\mathcal{L}) > \beta$ by Property 4.2(2). □

Proposition 4.4. *Suppose* $\mathcal{L}$ *is a scattered linear ordering and consider a finite partition of the domain* $L = A_1 \cup A_2 \cup \cdots \cup A_k$*. Then there exists some* $\delta \in \{1, \ldots, k\}$ *with* $\mathrm{VD}_*(\mathcal{A}_\delta) = \mathrm{VD}_*(\mathcal{L})$.

Proof. The proof is done by induction. So assume that $\mathrm{VD}_*(\mathcal{L}) = \alpha$ is the ordinal to be addressed and that the proposition holds for all $\beta < \alpha$. Let $L, k, A_1, \ldots, A_k$ be as in the statement of the proposition.

If $\alpha = 0$, then $\mathrm{VD}_*(\mathcal{L}) = \mathrm{VD}_*(\mathcal{A}_\epsilon) = 0$ for every nonempty subset A_ϵ of L.

Otherwise, by Property 4.2 item (4), there is some interval of $\mathcal{L}$, say $\mathcal{M}$, with $\mathrm{VD}(\mathcal{M}) = \alpha$ and $\mathrm{VD}_*(\mathcal{M}) = \alpha$. Then, $\mathcal{M}$ is an $\mathcal{I}$–sum of linear orders $\{\mathcal{M}_i\}$ of VD-rank $< \alpha$, where $\mathcal{I}$ is an infinite linear order of the type $\omega, \omega^\star$ or ζ. So suppose that $\mathcal{I}$ is of type ω (the other two order types are similar).

Suppose α is a successor ordinal, say $\alpha = \beta + 1$. There are infinitely many i such that $\mathrm{VD}_*(\mathcal{M}_i) = \beta$, for otherwise we could write $\mathcal{M}$ as a finite sum of orders of VD-rank β, and conclude that $\mathrm{VD}_*(\mathcal{M}) \leq \beta$. For each such i let $A_{\delta,i} = M_i \cap A_\delta$, where $\delta \in \{1, \ldots, k\}$. Applying the induction hypothesis to every $\mathcal{M}_i$ we see that there is an $\epsilon \in \{1, \ldots, k\}$ and infinitely many j such that $\mathrm{VD}_*(\mathcal{A}_{\epsilon,j}) = \mathrm{VD}_*(\mathcal{M}_j) = \beta$. Hence A_ϵ contains an ω-sum of linear orders of VD_*-rank β. By Lemma 4.3, $\mathrm{VD}_*(\mathcal{A}_\epsilon) = \alpha$.

Suppose that α is a limit ordinal. The supremum of the VD-ranks of the $\mathcal{M}_i$ is α. Using the notation of the case above, and applying induction, we see that there is an $\epsilon \in \{1, \ldots, k\}$ and infinitely may j such that $\mathrm{VD}_*(\mathcal{A}_{\epsilon,j}) = \mathrm{VD}_*(\mathcal{M}_j)$, and the supremum of the VD_*-ranks of these $\mathcal{A}_{\epsilon,j}$ is α. Then, $\mathrm{VD}_*(\mathcal{A}_\epsilon) = \alpha$ as required. □

PROPOSITION 4.5. *Let $\mathcal{L}$ be a scattered order with VD-rank at least α. Then, for every $\beta < \alpha$, there exists a closed interval of $\mathcal{L}$ of VD-rank $\beta + 1$.*

PROOF. Recall that VD-ranks and FC-ranks coincide on scattered linear orders, Theorem 3.11. Fix $\beta < \alpha$. Since $\mathcal{L}$ has FC-rank $> \beta$, by definition, there is some $x \in L$ such that $c^\beta(x) \neq c^{\beta+1}(x)$. Pick $y \in c^{\beta+1}(x) \setminus c^\beta(x)$. Then, $c^\beta(x) \neq c^\beta(y)$ and $c^{\beta+1}(x) = c^{\beta+1}(y)$. Recall that $c^\beta_{[x,y]}$ is the condensation mapping c^β within the interval $[x, y]$. Hence, by Lemma 3.12(3), $c^\beta_{[x,y]}(x) \neq c^\beta_{[x,y]}(y)$ and $c^{\beta+1}_{[x,y]}(x) = c^{\beta+1}_{[x,y]}(y)$. By Lemma 3.12(4) the first fact implies that $\mathrm{FC}([x, y]) > \beta$ and the second fact implies that $\mathrm{FC}([x, y]) \leq \beta + 1$. So the FC-rank of $[x, y]$ is exactly $\beta + 1$. □

PROPOSITION 4.6. *The VD-rank of every automatic scattered linear ordering is finite.*

PROOF. Given an automatic scattered linear order $\mathcal{L}$ over $\Sigma^\star$, let $(Q_\leq, \iota_\leq, \Delta_\leq, F_\leq)$ be a deterministic 2–tape automaton recognizing the ordering of $\mathcal{L}$. Similarly, let $(Q_A, \iota_A, \Delta_A, F_A)$ be a deterministic 3-tape automaton recognizing the definable relation $\{(x, z, y) \mid x \leq z \leq y\}$. We assume the state sets Q_A and $Q_\leq$ are disjoint.

For $x, y \in L$ and $v \in \Sigma^\star$, define $[x, y]_v$ as the set of all $z \in L$ such that $x \leq z \leq y$ and z has prefix v. For $|v| \geq |x|, |y|$ define $I(x, v, y) \in Q_A$ and $J(x) \in Q_\leq$ as follows. $I(x, v, y)$ is the state in Q_A that results from the initial state ι_A after reading the convolution of (x, v, y), namely $(x\diamond^n, v, y\diamond^m)$ where $n, m \geq 0$ are chosen so that the length of each component is exactly $|v|$. That is, define $I(x, v, y) := \Delta_A(\iota_A, \otimes(x, v, y))$. Similarly, define $J(v) := \Delta_\leq(\iota_\leq, \otimes(v, v))$. Write $K(x, v, y)$ for the ordered pair $(I(x, v, y), J(v))$.

Now, if $K(x, v, y) = K(x', v', y')$, then the subordering with domain $[x, y]_v$ is isomorphic to the subordering with domain $[x', y']_{v'}$ via the map $vw \mapsto v'w$ for

$w \in \Sigma^\star$. Indeed the domains are isomorphic since for every $w \in \Sigma^\star$,

$$vw \in [x, y]_v$$

if and only if

$$\Delta_A(\Delta_A(\iota_A, \otimes(x, v, y)), \otimes(\epsilon, w, \epsilon)) \in F_A$$

if and only if

$$\Delta_A(\Delta_A(\iota_A, \otimes(x', v', y')), \otimes(\epsilon, w, \epsilon)) \in F_A$$

if and only if

$$v'w \in [x', y']_{v'}.$$

The map preserves the ordering since for $w_1, w_2 \in \Sigma^\star$ such that $vw_1, vw_2 \in [x, y]_v$ and $v'w_1, v'w_2 \in [x', y']_{v'}$ we have

$$vw_1 \leq vw_2$$

if and only if

$$\Delta_\leq(\Delta_\leq(\iota_\leq, \otimes(v, v)), \otimes(w_1, w_2)) \in F_\leq$$

if and only if

$$\Delta_\leq(\Delta_\leq(\iota_\leq, \otimes(v', v')), \otimes(w_1, w_2)) \in F_\leq$$

if and only if

$$v'w_1 \leq v'w_2.$$

Hence, the number of isomorphism types of suborderings with domain $[x, y]_v$ for $|v| \geq |x|, |y|$ is bounded by the number of distinct pairs $K(x, v, y)$ which is at most $|Q_A| \times |Q_\leq|$, denoted by d. In particular (†), there are at most d many VD$_*$-ranks among suborderings with domain of the form $[x, y]_v$ for $|v| \geq |x|, |y|$.

Now suppose there exists a closed interval $[x, y]$ of $\mathcal{L}$ with VD-rank at least $2(d + 2)$. Using Proposition 4.5, for every $1 \leq i \leq 2(d + 2)$, the interval $[x, y]$ contains a closed interval, say $[x_i, y_i]$, of VD-rank i. So by Property 4.2(3), at least $d + 2$ many of these intervals have different VD$_*$-ranks; and at least $d + 1$ many of these intervals have nonzero VD$_*$–rank. Say $[x_j, y_j]$ is one of these $d + 1$ many intervals. Set $n = \max\{|x_j|, |y_j|\}$ and partition $[x_j, y_j]$ into the set $[x_j, y_j] \cap \Sigma^{<n}$ and the finitely many sets of the form $[x_j, y_j]_v$ where $|v| = n$. Since the finite set $[x_j, y_j] \cap \Sigma^{<n}$ has VD$_*$-rank 0, by Proposition 4.4 there is some v_j with $|v_j| = n$ so that the subordering on domain $[x_j, y_j]_{v_j}$ has the same VD$_*$-rank as $[x_j, y_j]$. Hence there are at least $d + 1$ many intervals of the form $[x_j, y_j]_{v_j}$ all with different VD$_*$-ranks. This contradicts (†) and so we conclude that the VD-rank of every closed interval $[x, y]$ of $\mathcal{L}$ is at most $e = 2(d + 2)$. So, for every $x, y \in L$, $c^e(x) = c^e(y)$ and so $\mathrm{VD}(\mathcal{L}) \leq e$ as required. □

As a corollary of the proposition just proved we derive the following result for all automatic linear orderings:

THEOREM 4.7. *The FC–rank of every automatic linear order is finite.*

PROOF. Let $\mathcal{L}$ be a linear order and write it as $\sum\{\mathcal{L}_i \mid i \in D\}$ where $\mathcal{D}$ is dense and each $\mathcal{L}_i$ is scattered. We will show that for every $i \in D$ and every $a, b \in L_i$, the VD–rank of $[a, b]$ is uniformly bounded. Let $(Q_{\leq}, \iota_{\leq}, \Delta_{\leq}, F_{\leq})$ be a deterministic 2–tape automaton recognizing the ordering of $\mathcal{L}$. Let $(Q_A, \iota_A, \Delta_A, F_A)$ be a deterministic 3–tape automaton recognizing the definable relation $\{(x, z, y) \mid x \leq z \leq y\}$. Now consider an interval $[a, b]$ of $\mathcal{L}_i$ for some $i \in D$. The proof of the previous theorem ensures that the VD-rank of the scattered interval $[a, b]$ is at most e, where the constant e does not depend on $[a, b]$ or i but only on $|Q_A|$ and $|Q_{\leq}|$. Therefore, the VD-rank of interval $[a, b]$ is at most e. Hence, $\text{VD}(\mathcal{L}_i) \leq e$ for every $i \in D$ and so $\text{FC}(\mathcal{L}) \leq e$. □

Remark 4.8. This result is a necessary though not sufficient condition for a linear order to be automatically presentable. Indeed there are linear orders of rank 2 that are not automatically presentable. For instance if $R \subset \mathbb{N}$ is a noncomputable set, then the linear order $\Sigma_{n \in R}(\zeta + \mathbf{n})$ does not have decidable first order theory and so is not automatically presentable.

COROLLARY 4.9 [DELHOMMÉ 2004]. *An ordinal α is automatically presentable if and only if $\alpha < \omega^\omega$.*

PROOF. Suppose α is an automatically presentable ordinal. Then, by Theorem 4.7, it has finite FC-rank and so, by Example 3.5, $\alpha < \omega^\omega$ as required.

Conversely, given $\alpha < \omega^\omega$, there exists $n < \omega$ such that $\alpha < \omega^n$. But ω^n is automatically presentable. Say $(W, \leq)$ is an automatic presentation. Let $p \in W$ be the string corresponding to α. So the suborder of $\mathcal{W}$ on the definable domain $\{x \in W \mid x < p\}$ is isomorphic to α. Hence, α is automatically presentable. □

PROPOSITION 4.10. *It is decidable whether or not an automatic linear order $\mathcal{L}$ is scattered. If it is not scattered, then a regular dense subordering is effectively computable from a presentation for $\mathcal{L}$.*

PROOF. Let $\mathcal{L}$ be an automatic order. The proof of Theorem 4.7 says that a bound e on the FC-rank of $\mathcal{L}$ is computable given automata for the order and the interval relation. The condensation c_{FC}, viewed as the equivalence relation x related to y if $x \in c_{FC}(y)$, is definable in $\mathcal{L}$ since $c_{FC}(x) = c_{FC}(y)$ if and only if $[x, y]$ is finite. Since the ordering on $c_{FC}[\mathcal{L}]$ is also definable from $\mathcal{L}$ (see Definition 3.7) the linear orders $c^i_{FC}[\mathcal{L}]$ are definable for every $i \in \mathbb{N}$, and hence automatic. So consider $c^e_{FC}[\mathcal{L}]$. By Example 3.10, it is isomorphic to $\mathbf{1}$ if and only if $\mathcal{L}$ is scattered. So using the decidability of the theory of $c^e_{FC}[\mathcal{L}]$, check this with the sentence $(\exists x)(\exists y)[x < y]$. In case $c^e_{FC}[\mathcal{L}]$ is not the singleton it must be an infinite dense ordering. One may view c^e_{FC} as an automatic equivalence relation on $\mathcal{L}$ (the $c^e_{FC}(x)$ partition $\mathcal{L}$), and so the $<_{llex}$-smallest representatives from every equivalence class forms a dense subordering of $\mathcal{L}$ that is a regular subset of L. □

5. DECIDABILITY RESULTS FOR AUTOMATIC ORDINALS

Theorem 4.7 can now be applied to prove decidability results for automatic ordinals. Contrast this with the fact that the set of computable structures that are well orderings is Π^1_1-complete (see Rogers [1967]).

Proposition 5.1. *Let $\mathcal{L} = (L, \leq)$ be an automatic structure. It is decidable whether $\mathcal{L}$ is isomorphic to an ordinal.*

Proof. First, check that $\leq$ linearly orders L, by testing whether $\mathcal{L}$ is reflexive, transitive and antisymmetric–all first-order axioms and hence computable properties. Although being a well-order is not first-order expressible (see, e.g., Theorem 13.13 in Rosenstein [1982]), the following algorithm can be used:

(1) **Input** the presentation $(L, \leq)$ of $\mathcal{L}$.
(2) Let $D = L$.
(3) **While** $(D, \leq)$ is not dense and $(\forall x \in D)[\omega^*$ does not embed in the interval $c(x)]$
Do Replace $(D, \leq)$ by a presentation for $c[\mathcal{D}]$.
(4) **End While**
(5) If $\mathcal{D}$ is isomorphic to **1** then **Output** $\mathcal{L}$ **is an ordinal**,
else **Output** $\mathcal{L}$ **is not an ordinal**.

Every step in the algorithm is computable. Indeed, the equivalence relation on pairs (x, y) satisfying $c(x) = c(y)$ is definable as $(\neg\exists^\infty z)[x < z < y]$. So a presentation for $c[\mathcal{D}]$ is computed by factoring $\mathcal{D}$ by c. The while test is expressible as

$$\neg(\forall x \neq y)(\exists z)[x < z < y]$$

and

$$(\forall x)(\neg\exists^\infty y)(c(x) = c(y) \wedge y < x).$$

The final test is expressible by $(\exists x)(\forall y)[x = y]$.

Since the FC-rank of $\mathcal{L}$ is finite, say k, the algorithm terminates after at most $k + 1$ many while-loop tests. If $\mathcal{L}$ is an ordinal, then $c[\mathcal{L}]$ is an ordinal and for every $x \in L$, $c(x)$ is either finite or isomorphic to ω. By induction on k, for every $0 \leq i \leq k$, $c^i[\mathcal{L}]$ passes the $(i + 1)$'th while-test. The resulting order $\mathcal{D} = c^k[\mathcal{L}]$ is isomorphic to **1** as required.

If $\mathcal{L}$ is not an ordinal, then there exists an infinite decreasing sequence of elements. Suppose there exists such a sequence $x_1 > x_2 > x_3 > \cdots$ and an $n_0 \in \mathbb{N}$ such that for all $i \geq n_0$ $c(x_i) = c(x_{n_0})$. Then, the while-test fails the first time it is executed and the resulting order $\mathcal{D} = \mathcal{L}$ is not isomorphic to the ordinal **1**. If there is no such sequence (x_i) and n_0, then there exists a sequence, say $y_1 > y_2 > y_3 > \cdots$ such that $c(y_{i+1}) \lhd c(y_i)$ for all $i \in \mathbb{N}$; this is an infinite decreasing sequence of elements in $c[\mathcal{L}]$. Continue inductively in this way with $c[\mathcal{L}]$ in place of $\mathcal{L}$. Suppose the while-test fails the mth time for some $1 \leq m \leq k$. If it fails because there is some $x \in c^{m-1}[\mathcal{L}]$ for which $\omega^\star$ embeds in $c(x)$ then $\mathcal{D} = c^{m-1}[\mathcal{L}]$ is infinite and so not isomorphic to **1**. If there is no such m, then the while-test must fail the $(k + 1)$'st time. In this case $\mathcal{D} = c^k[\mathcal{L}]$ is dense but as before there is a sequence $y_1 > y_2 > y_3 > \ldots$ with $c^k(y_{i+1}) \lhd c^k(y_i)$ for every $i \in \mathbb{N}$. In this case, $\mathcal{D}$ is not isomorphic to **1**. $\square$

We now show that the isomorphism problem for automatic ordinals is decidable. Recall that by Cantor's Normal Form Theorem if α is an ordinal then it can be uniquely decomposed as $\omega^{\alpha_1}n_1 + \omega^{\alpha_2}n_2 + \cdots + \omega^{\alpha_k}n_k$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are ordinals satisfying $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ and $k, n_1, n_2, \ldots, n_k$ are natural numbers. The proof of deciding the isomorphism problem for automatic ordinals is based on the fact that Cantor's normal form can be extracted from automatic presentations of ordinals.

THEOREM 5.2. *If α is an automatic ordinal, then its normal form is computable from an automatic presentation of α.*

PROOF. Let $(R, \leq_{ord})$ be an automatic presentation over Σ of α. Recall that the unknown ordinal is of the form $\alpha = \omega^m n_m + \omega^{m-1} n_{m-1} + \cdots + \omega^2 n_2 + \omega n_1 + n_0$ where $m, n_m, n_{m-1}, \ldots, n_1, n_0$ are natural numbers. Now one can compute the values $m, n_0, n_1, \ldots$ by the following algorithm:

(1) **Input** the presentation $(R, \leq_{ord})$.
(2) Let $D = R$, $m = 0$, $n_m = 0$.
(3) **While** $D \neq \emptyset$ **Do**
(4) **If** D has a maximum u
Then Let $n_m = n_m + 1$, let $D = D - \{u\}$.
Else Let $L \subseteq D$ be the set of limit ordinals in D; that is L is the set of all $x \in D$ with no immediate predecessor in D. Replace D by L, let $m = m + 1$, let $n_m = 0$.
(5) **End While**
(6) **Output** the formula

$$\omega^m n_m + \omega^{m-1} n_{m-1} + \ldots + \omega^2 n_2 + \omega n_1 + n_0$$

using the current values of $m, n_0, \ldots, n_m$.

Since the first-order theory of an automatic structure is decidable, each step in the algorithm is computable. Removing the maximal element from D reduces the ordinal represented by D by 1 while the corresponding n_m is increased by 1. Replacing D by the set of its limit ordinals is like dividing the ordinal represented by D by ω; the set of limit ordinals (including 0) strictly below $\omega^m a_m + \cdots + \omega^1 a_1$ has order type $\omega^{m-1} a_m + \cdots + \omega^1 a_2 + a_1$. So the next coefficient can start to be computed. Based on this it is easy to verify that the algorithm computes the coefficients $n_0, n_1, \ldots$ in this order. The algorithm eventually terminates since m is bounded by the finite bound on the VD-rank of the ordinal. □

The following is an immediate corollary.

THEOREM 5.3. *The isomorphism problem for automatic ordinals is decidable.*

Compare this with the fact that the isomorphism problem for automatic structures and even permutation structures [Blumensath 1999, compare Ishihara et al. 2002] is not decidable.

Problem 5.4. Is the isomorphism problem for automatic linear orders decidable ?

6. AUTOMATIC TREE PRELIMINARIES

The remaining sections deal with trees viewed as partial orders. Theorems 7.7 and 7.10 give a necessary condition for certain trees to be automatic. The condition is similar to that for linear orders and says that the Cantor–Bendixson rank (Definition 7.1) of the tree be finite.

A *tree* $\mathcal{T} = (T, \preceq)$ is a partial order that has a least element r, called the root, and in which $\{y \in T \mid y \preceq x\}$ is a finite linear order for each $x \in T$. So we think of trees as growing upwards. Write $x \| y$ if $x \not\preceq y$ and $y \not\preceq x$. A partial order $(T, \preceq)$ is a *forest* if there is a partition of the domain $T = \cup T_i$ such that every $(T_i, \preceq)$ is a tree. The subtree rooted at x, written $\mathcal{T}(x)$, has domain $T(x) = \{y \in T \mid x \preceq y\}$ with order $\prec$ restricted to this domain. The set $S(x)$ of immediate successors of x is defined as

$$S(x) = \{y \in \mathcal{T} \mid x \prec y \wedge (\forall z)[x \preceq z \preceq y \rightarrow (z = x \vee z = y)]\}.$$

A tree $\mathcal{T}$ is *finitely branching* if $S(x)$ is finite for each $x \in T$. A *path* of a tree $(T, \preceq)$ is a subset $P \subseteq T$ which is linearly ordered by $\preceq$ and maximal (under set-theoretic inclusion) with this property. Note that a path contains the root. A path with finitely many nodes is called a *finite path*; otherwise it is called an *infinite path*.

Recall that $<_{llex}$ is the length lexicographic order on $\Sigma^\star$ defined as $x <_{llex} y$ if either $|x| < |y|$ or $|x| = |y|$ but x lexicographic before y. For example, $\epsilon <_{llex} 0 <_{llex} 1 <_{llex} 00 <_{llex} 01 <_{llex} \cdots$ in the case that $\Sigma = \{0, 1\}$. Thus, if $\mathcal{T}$ is an automatic tree with $T \subseteq \Sigma^\star$, then the length-lexicographic order on $\Sigma^\star$ is inherited by each set $S(x)$. This permits one to talk about the first, second, third, ... successor of x.

7. RANKS OF AUTOMATIC TREES

Our approach to proving facts about trees is to associate a linear order with a tree, in such a way that the tree is automatic if and only if the linear order is automatic. Then, by Theorem 4.7, the linear order has finite rank which it turns out implies that the rank of the tree is finite. More precisely, in this section, it is shown that every automatic tree has finite Cantor–Bendixson Rank.

Given a tree $\mathcal{T}$, define a subset of T as consisting of those nodes $x \in T$ with the property that there exist at least two distinct infinite paths in the subtree of $\mathcal{T}$ rooted at x. It follows from downward closure that this subpartial order, $d(\mathcal{T})$, is a subtree of $\mathcal{T}$ with the same root.

For each ordinal α, define the iterated operation $d^\alpha(\mathcal{T})$ inductively as follows:

(1) $d^0(\mathcal{T}) = \mathcal{T}$.
(2) $d^{\alpha+1}(\mathcal{T})$ is $d(d^\alpha(\mathcal{T}))$.
(3) If α is a limit ordinal, then $d^\alpha(\mathcal{T})$ is $\cap_{\beta<\alpha} d^\beta(\mathcal{T})$.

Definition 7.1. The *Cantor–Bendixson Rank* of a tree $\mathcal{T}$, written $\mathrm{CB}(\mathcal{T})$, is the least ordinal α such that $d^\alpha(\mathcal{T}) = d^{\alpha+1}(\mathcal{T})$.

Remark 7.2. The Cantor–Bendixson Rank of an arbitrary topological space X is defined as above, using D given as $D\,X = \{P \in X \mid p \text{ is not isolated}\}$ instead of d. Recall that P is isolated if $\{P\}$ is an open set. So given a tree $\mathcal{T} = (T, \preceq)$, consider the following topological space. The set of elements are the infinite paths in $\mathcal{T}$, written $[\mathcal{T}]$. For $P \in [\mathcal{T}]$ and $x \in T$ write $x \prec P$ if $x \in P$ and say that x is on P. The basic open sets are of the form $\{P \in [\mathcal{T}] \mid x \prec P\}$ for every $x \in T$. Then, the Cantor–Bendixson Rank of this topological space, $\mathrm{CB}[\mathcal{T}]$, is just the least ordinal α such that $D^{\alpha+1}[\mathcal{T}] = D^{\alpha}[\mathcal{T}]$. Given an infinite path P of $\mathcal{T}$, the following statements are equivalent:

—There is a node $x \prec P$ such that P is the only infinite path of $\mathcal{T}$ going through x;

—$P \notin D(\mathcal{T})$;

—There is a node $x \prec P$ with $x \notin d(\mathcal{T})$.

It follows that $D[\mathcal{T}]$ consists of exactly the infinite paths of $d(\mathcal{T})$. It can be proven by transfinite induction that also

$$D^{\alpha}[\mathcal{T}] = [d^{\alpha}(\mathcal{T})].$$

Assume now that $\alpha = \mathrm{CB}[\mathcal{T}]$. Then $d^{\alpha}(\mathcal{T})$ and $d^{\beta}(\mathcal{T})$ contain the same infinite paths for all $\beta > \alpha$, but $d^{\alpha}(\mathcal{T})$ might contain some nodes which are not on any infinite paths and therefore not contained in $d^{\alpha+1}(\mathcal{T})$. Thus, the two CB-ranks might differ, but they differ at most by 1:

$$\mathrm{CB}[\mathcal{T}] \leq \mathrm{CB}(\mathcal{T}) \leq \mathrm{CB}[\mathcal{T}] + 1.$$

A witness $\mathcal{T}$ with $\mathrm{CB}[\mathcal{T}] \neq \mathrm{CB}(\mathcal{T})$ is the tree where the domain consists of the root 0 and, for every $n > 0$, the strings $01^{a_1}01^{a_2}0 \cdots 1^{a_n}0$ with $a_1 \geq a_2 \geq \cdots \geq a_n$; the ordering is the prefix-relation $\preceq$ restricted to this domain. One has for every node $01^{a_1}01^{a_2}0 \cdots 1^{a_n}0 \in T$ that $01^{a_1}01^{a_2}0 \ldots 1^{a_n}0 \in d^m(\mathcal{T}) \Leftrightarrow a_n \geq m$. So $d^{\omega} = \{0\}$. It follows that $\mathrm{CB}[\mathcal{T}] = \omega$ by $D^{\omega}(\mathcal{T}) = \emptyset$ while $\mathrm{CB}(\mathcal{T}) = \omega + 1$ by $d^{\omega+1}(\mathcal{T}) = \emptyset \neq d^{\omega}(\mathcal{T})$. This witness is also robust to small changes in the definition of d. If one, for example, takes $d(\mathcal{T})$ to contain exactly those nodes which are on infinitely many infinite paths of $\mathcal{T}$, then the resulting trees $d^{\alpha}(\mathcal{T})$ and derived CB-ranks are the same.

Here are some basic properties of CB-rank.

Property 7.3. *If $\mathcal{T}$ is a countable tree with* $\mathrm{CB}(\mathcal{T}) = \alpha$ *then*

(1) *α is a countable ordinal.*

(2) *If $d^{\alpha}(\mathcal{T}) \neq \emptyset$, then $d^{\alpha}(\mathcal{T})$ and $\mathcal{T}$ contain uncountably many infinite paths.*

(3) *If $d^{\alpha}(\mathcal{T}) = \emptyset$, then $\mathcal{T}$ contains only countably many infinite paths. Furthermore, α is either* 0 *or a successor ordinal.*

Proof. For each β, let $x_{\beta} \in d^{\beta}(\mathcal{T}) \setminus d^{\beta+1}(\mathcal{T})$. Since T is countable, and $\alpha \neq \beta$ implies that $x_{\alpha} \neq x_{\beta}$, the set of ordinals β such that $d^{\beta}(\mathcal{T}) \setminus d^{\beta+1}(\mathcal{T}) \neq \emptyset$ is also countable. Hence, its least upper bound, a countable ordinal, say α, is $\mathrm{CB}(\mathcal{T})$. This proves (1).

If $d^{\alpha}(\mathcal{T})$ is not the empty tree, then, for every $x \in d^{\alpha}(\mathcal{T})$ there exist $y, z \in d^{\alpha}(\mathcal{T})$ with $x \prec y, z$ and $y \| z$. In particular, the full binary tree $(\{0,1\}^{\star}, \preceq_p)$ embeds in $d^{\alpha}(\mathcal{T})$. Since $d^{\alpha}(\mathcal{T})$ is a subset of $\mathcal{T}$, the full binary tree also embeds in $\mathcal{T}$. This proves (2).

If $d^{\alpha}(\mathcal{T})$ is the empty tree, then one shows that $\mathcal{T}$ has only countably many infinite paths as follows: For every infinite path P of $\mathcal{T}$, there is a minimum ordinal $\beta_p \leq \alpha$ such that $P \not\subseteq d^{\beta_P}(\mathcal{T})$. Furthermore, there is a node x_P in P such that $x_P \notin d^{\beta_P}(\mathcal{T})$. Since $x_P \in d^{\gamma}(\mathcal{T})$ for all $\gamma < \beta_P$, it follows that β_P

is a successor ordinal $\delta + 1$. Furthermore, P is the only infinite path of $d^{\delta}(\mathcal{T})$ which contains x_P. Thus, the mapping $P \to (x_P, \beta_P)$ of the infinite paths of $\mathcal{T}$ to pairs of nodes and successor ordinals up to α is one-one. Since the range of this mapping is countable, so is its domain. Now for every $\gamma < \alpha$ the root of $\mathcal{T}$ is in $d^{\gamma}(\mathcal{T})$. So if $\alpha > 0$, then it is a successor ordinal. This proves (3). $\square$

As a matter of convenience we introduce a variation of CB-rank.

Definition 7.4. Suppose that $\mathcal{T}$ has countably many infinite paths. Define the CB$_*$-*rank* of $\mathcal{T}$, written CB$_*(\mathcal{T})$, as the least ordinal α so that $d^{\alpha}(\mathcal{T})$ has finitely many nodes.

This is well defined since $d^{\alpha}(\mathcal{T}) = \emptyset$ for some α. Note that CB$_*$-rank is non-increasing in the sense that, if $x \preceq y$ then $\mathrm{CB}_*(\mathcal{T}(y)) \leq \mathrm{CB}_*(\mathcal{T}(x))$. Also since finite trees have no infinite paths, $\mathrm{CB}_*(\mathcal{T}) \leq \mathrm{CB}(\mathcal{T}) \leq \mathrm{CB}_*(\mathcal{T}) + 1$.

LEMMA 7.5. *Suppose $\mathcal{T}$ has countably many infinite paths and that $\mathcal{T}$ is finitely branching.*

(1) $\mathrm{CB}_*(\mathcal{T})$ *is* 0 *or a successor ordinal.*
(2) *If* $\mathrm{CB}_*(\mathcal{T}) \geq \beta + 1$*, then there is some* $x \in T$ *with* $\mathrm{CB}_*(\mathcal{T}(x)) = \beta + 1$.

PROOF. Say $\mathrm{CB}_*(\mathcal{T}) = \alpha$ and $\alpha > 0$ is a limit ordinal. Then $d^{\alpha}(\mathcal{T})$ contains an infinite path as follows. The root of $\mathcal{T}$, call it x_0, is in $d^{\alpha}(\mathcal{T})$ for otherwise $d^{\gamma}(\mathcal{T})$ is empty for some $\gamma < \alpha$. Since x_0 has finitely many immediate successors in $\mathcal{T}$, there must be one, call it x_1, with the property that $x_1 \in d^{\alpha}(\mathcal{T})$ for otherwise the maximum of the CB$_*$-ranks of the immediate successors of x_0 is $< \alpha$ and so $\mathrm{CB}_*(\mathcal{T}) < \alpha$. Proceed in this way to build an infinite path $x_0, x_1, x_2, \cdots$ of $d^{\alpha}(\mathcal{T})$. In particular then the CB$_*$-rank of $\mathcal{T}$ is not α. This proves (1).

Let $\mathrm{CB}_*(\mathcal{T}) \geq \beta + 1$. Then $d^{\beta}(\mathcal{T})$ is an infinite finitely branching tree and so contains some infinite path P. Moreover there must be some infinite path of $P \subseteq d^{\beta}(\mathcal{T})$ so that $P \not\subseteq d^{\beta+1}(\mathcal{T})$; for otherwise we could embed a copy of the infinite binary tree in $d^{\beta}(\mathcal{T})$ and so conclude that $\mathcal{T}$ has uncountably many infinite paths. Hence, pick $x \in P$ with $x \notin d^{\beta+1}(\mathcal{T})$. Then $\mathrm{CB}_*(\mathcal{T}(x)) = \beta + 1$. This proves (2). $\square$

For the first result, one associates the Kleene–Brouwer ordering with a tree.

Definition 7.6 (*see Rogers* [1967]). Let $(T, \preceq)$ be a tree and $\leq_{llex}$ be the length lexicographic order induced by the presentation of T as a subset of Σ^*. Let x, y be nodes on T. Define $x \leq_{kb} y$ to mean either $y \preceq x$ or there are u, v, w such that $v, w \in S(u)$, $v \preceq x$, $w \preceq y$ and $v <_{llex} w$. Write $\mathcal{KB}_T$ for the structure $(T, \leq_{kb})$.

In words, $x \leq_{kb} y$, if and only if either x is above y in the tree or x is to the left of y (with respect to $<_{llex}$ restricted to immediate successors). Note that $\leq_{kb}$ linearly orders T and $(T, \leq_{kb})$ is first-order definable from $(T, \preceq, \leq_{llex})$. For example, if $y_1 <_{llex} y_2 <_{llex} \cdots <_{llex} y_l$ are the immediate successors of the root r of $\mathcal{T}$, then $\mathcal{KB}_T = \mathcal{KB}_{T(y_1)} + \cdots + \mathcal{KB}_{T(y_l)} + \mathbf{1}$. Recall that $\mathcal{T}(x)$ denotes the subtree of $\mathcal{T}$ with root x and that its domain is written $T(x)$.

THEOREM 7.7. *The CB-rank of an automatic finitely branching tree with countably many infinite paths is finite.*

PROOF. Suppose $\mathcal{T}$ is finitely branching with countably many infinite paths. We now prove (†) that $\mathcal{KB}_T$ is scattered and $\mathrm{CB}_*(\mathcal{T}) = \mathrm{VD}_*(\mathcal{KB}_T)$. Consequently, if $\mathcal{T}$ is automatic, then so is $\mathcal{KB}_T$, which by Theorem 4.7 has finite VD-rank and hence finite VD_*-rank. Then, the CB_*-rank of $\mathcal{T}$ must be finite as required.

To prove (†) proceed by induction on $\mathrm{CB}_*(\mathcal{T})$. If $\mathcal{T}$ has CB_*-rank 0, then $\mathcal{KB}_T$ is finite and so has VD_*-rank 0. Before proceeding to the general case, we make some observations. Suppose $\mathcal{T}$ contains an infinite path $x_1, x_2, x_3, \ldots$. For a given i list $S(x_i)$ as follows: $y_1 <_{llex} \cdots <_{llex} y_k <_{llex} x_{i+1} <_{llex} z_1 <_{llex} \cdots <_{llex} z_l$. Define the set $L_i \subseteq T$ as $\cup T(y_j)$ and define $R_i \subset T$ as $\cup T(z_j)$. So $\mathcal{L}_i$ and $\mathcal{R}_i$ are forests of disjoint subtrees of $\mathcal{T}$. Abuse notation and define $\mathcal{KB}_{L_i}$ as the linear order $\mathcal{KB}_{T(y_1)} + \mathcal{KB}_{T(y_2)} + \cdots + \mathcal{KB}_{T(y_k)}$. Similarly define $\mathcal{KB}_{R_i}$ as the linear order $\mathcal{KB}_{T(z_1)} + \mathcal{KB}_{T(z_2)} + \cdots + \mathcal{KB}_{T(z_l)}$. Then, by definition of $<_{kb}$,

$$\mathcal{KB}_T = (\mathcal{KB}_{L_1} + \mathcal{KB}_{L_2} + \mathcal{KB}_{L_3} + \cdots) + (\cdots + \mathcal{KB}_{R_3} + \mathbf{1}_3 + \mathcal{KB}_{R_2} + \mathbf{1}_2 + \mathcal{KB}_{R_1} + \mathbf{1}_1), \quad (1)$$

where $\mathbf{1}_i$ has order type $\mathbf{1}$ and represents the element x_i. In particular, suppose $\mathcal{T}$ contains exactly one infinite path. Then every $\mathcal{KB}_{L_i}$ and $\mathcal{KB}_{R_i}$ is a finite linear order. So depending on whether there are infinitely many i such that $\mathcal{KB}_{L_i}$ (or $\mathcal{KB}_{R_i}$) is the empty linear order, $\mathcal{KB}_T$ has one of the following scattered order types: $\omega^\star$, $\mathbf{n} + \omega^\star$ for some $n \in \mathbb{N}$, or $\omega + \omega^\star$. Note that these orders have VD_*-rank 1.

For the general case, suppose the CB_*-rank of $\mathcal{T}$ is not 0. Then by Lemma 7.5(1) it is $\beta+1$ for some ordinal β. Let $X = \{x \in T \mid \mathrm{CB}_*(\mathcal{T}(x)) = \beta+1\}$. Then X is a downward closed subset of $\mathcal{T}$, and so $\mathcal{X}$ is a tree.

The tree $\mathcal{X}$ has infinitely many nodes. Indeed for every $x \in X$, the finitely branching tree $d^\beta(\mathcal{T}(x))$ is infinite and so contains an infinite path (w_i). For every i the tree $\mathcal{T}(w_i)$ has CB_*-rank $\beta + 1$ and so w_i is in X. So $\mathcal{X}$, also being finitely branching, has at least one infinite path. Now if $\mathcal{X}$ has infinitely many infinite paths, then, since $\mathcal{X}$ is finitely branching, we can construct an infinite path (z_i) of $\mathcal{X}$ such that for every i there are infinitely many infinite paths in $\mathcal{X}(z_i)$ (the subtree of $\mathcal{X}$ with root z_i). For infinitely many i, there is a $y \in S(z_i) \setminus \{z_{i+1}\}$ with $y \in X$. So $\mathcal{T}$ contains the infinite path (z_i) with $\mathrm{CB}_*(\mathcal{T}(z_i)) = \beta + 1$ and for infinitely many i there is $y \in T$ that is in $S(z_i) \setminus \{z_{i+1}\}$ with $\mathrm{CB}_*(\mathcal{T}(y)) = \beta + 1$. So $d^{\beta+1}(\mathcal{T})$ contains the infinite path (z_i) contradicting that $\mathrm{CB}_*(\mathcal{T}) = \beta + 1$. We conclude that $\mathcal{X}$ contains a nonzero finite number of infinite paths.

Let (x_i) be some infinite path of $\mathcal{X}$ and define $L_i \subset T$ and $R_i \subset T$ as above. The forest $\mathcal{L}_i$ (or $\mathcal{R}_i$) consists of finitely many disjoint subtrees of $\mathcal{T}$; list these as $\mathcal{T}(w_1), \ldots, \mathcal{T}(w_k)$, where $w_j \in S(x_i)$. Then $\mathrm{CB}_*(\mathcal{T}(w_j)) \leq \beta + 1$. Moreover, since $\mathcal{X}$ has only finitely many infinite paths, there exists $c \in \mathbb{N}$ such that for every $i \geq c$, every tree $\mathcal{T}(w_j)$ of $\mathcal{L}_i$ and $\mathcal{R}_i$ has CB_*-rank $\leq \beta$. By induction, $\mathcal{KB}_{T(w_j)}$ is scattered and $\mathrm{CB}_*(\mathcal{T}(w_j)) = \mathrm{VD}_*(\mathcal{KB}_{T(w_j)})$. So for every $i \geq c$, the linear order $\mathcal{L}_i$ (and $\mathcal{R}_i$) being a finite sum of such $\mathcal{T}(w_j)$ is scattered and has VD_*-rank equal to the supremum of $\mathrm{VD}_*(\mathcal{T}(w_1)), \ldots, \mathrm{VD}_*(\mathcal{T}(w_k))$ which is at most β. Moreover, by Lemma 7.5(2), there are infinitely many $m \geq c$ for which there exists a tree $\mathcal{T}(w_j)$ in $\mathcal{L}_m$ (or $\mathcal{R}_m$) that has CB_*-rank β. We conclude that there are infinitely many $\mathcal{KB}_{L_m}$ (or infinitely many $\mathcal{KB}_{R_m}$) with VD$_*$-rank exactly β. Hence, using Eq. (1) and Lemma 4.3, the linear order $\mathcal{KB}_{T(x_c)}$ has VD$_*$-rank $\beta + 1$.

Pick $n < c$ so that $\mathcal{L}_n$ (or $\mathcal{R}_n$) contains a tree $\mathcal{T}(w_j)$ of CB$_*$-rank $\beta + 1$. Argue as before with $\mathcal{T}(w_j)$ in place of $\mathcal{T}$. To this end, define X' as $\{x \in T(w_j) \mid \mathrm{CB}_*(\mathcal{T}(x)) = \beta + 1\}$. Then, as before, $\mathcal{X}'$ is a subtree of $\mathcal{T}(w_j)$ with finitely many infinite paths. However, since $\mathcal{X}'$ does not contain the previous infinite path (x_i), the tree $\mathcal{X}'$ has fewer infinite paths than $\mathcal{X}$. This guarantees that after a finite number of iterations $c = 0$.

Since $\mathcal{X}$ has a finite (nonzero) number of infinite paths, we can write $\mathcal{KB}_T$ as a finite (nonzero) sum of linear orders of VD$_*$-rank $\beta + 1$. This completes the induction. □

THEOREM 7.8. *The CB-rank of every finitely branching automatic tree is finite.*

PROOF. Let $\mathcal{T}$ be a finitely branching automatic tree and $\mathcal{KB}_T$ the Kleene–Brouwer ordering of $\mathcal{T}$. Call a node $a \in T$ *scattered* if $\mathcal{T}(a)$ contains countably many infinite paths. By the proof Theorem 7.7, $\mathcal{KB}_{T(a)}$ is a scattered linear ordering and $\mathrm{CB}_*(\mathcal{T}(a)) = \mathrm{VD}_*(\mathcal{KB}_{T(a)})$. But applying the proof of Theorem 4.7 to the automatic linear order $\mathcal{KB}_T$, there exists $e \in \mathbb{N}$ such that $\mathrm{VD}([x, y]) \le e$ for every scattered closed interval $[x, y]$ of $\mathcal{KB}_T$. In particular, for every scattered $a \in T$, $\mathrm{VD}_*(\mathcal{KB}_{T(a)}) \le \mathrm{VD}(\mathcal{KB}_{T(a)}) \le e$. So $\mathrm{CB}(\mathcal{T}(a)) \le e$ and $d^e(\mathcal{T})$ contains no scattered nodes.

We claim that $d^e(\mathcal{T}) = d^{e+1}(\mathcal{T})$. It is always the case that $d^{e+1}(\mathcal{T}) \subseteq d^e(\mathcal{T})$. If $\mathcal{T}$ has countably many infinite paths, then $d^e(\mathcal{T})$ is empty. Otherwise, suppose $x \in d^e(\mathcal{T})$. Then, x is not a scattered node since all the scattered nodes have been removed, and so $\mathcal{T}(x)$ contains uncountably many infinite paths. In particular, $x \in d^{e+1}(\mathcal{T})$. So $d^e(\mathcal{T}) \subseteq d^{e+1}(\mathcal{T})$, as required. □

Next, we remove the condition that the tree be finitely branching.

Definition 7.9. Given a tree $(T, \preceq)$, define a partial order $x \preceq' y$ on T by

$$x \preceq y \vee (\exists v, w \in T)[x, w \in S(v) \wedge x \le_{llex} w \wedge w \preceq y];$$

where $\le_{llex}$ is the length lexicographic order and $S(v)$ the set of immediate successors of v with respect to $\preceq$.

Recall the set $S(x)$ is the set of $\prec$-immediate successors of $x \in T$. Then, since $\le_{llex}$ restricted to $S(x)$ has order type ω if $S(x)$ is infinite, $(T, \preceq')$ is indeed a tree which we denote by $\mathcal{T}'$. Let $s(x)$ be the length-lexicographically least element of $S(x)$ for the case $S(x) \ne \emptyset$ and let $s(x) = u$ for a default value $u \notin T$ if $S(x) = \emptyset$.

Note that $\preceq'$ extends $\preceq$. For $x \in T$ let $S'(x)$ be the set of successors with respect to $\preceq'$. Then, $S'(x)$ contains $s(x)$ whenever $s(x) \ne u$ and the length-lexicographically next sibling y of x with respect to $\preceq$ whenever this y exists. Recall that y is a sibling of x with respect to $\preceq$ if there is a node z with $x, y \in S(z)$. Hence, $\mathcal{T}' = (T, \preceq')$ is a finitely branching tree that is automatic if $\mathcal{T}$ is automatic.

THEOREM 7.10. *The CB-rank of an automatic tree $\mathcal{T} = (T, \preceq)$ is finite.*

PROOF. Let U and U' be the sets of infinite paths of $\mathcal{T} = (T, \preceq)$ and $\mathcal{T}' = (T, \preceq')$, respectively. Since every infinite path of $\mathcal{T}$ generates an infinite path of $\mathcal{T}'$, there is a one-one continuous mapping q from U to U'. This mapping satisfies for all $P \in U$ and all $x \in T$: $x \in P$ iff $s(x) \in q(P)$. Furthermore, U' contains besides the paths of the form $q(P)$ for some $P \in U$ also the paths generated by those sets $S(x)$ where $S(x)$ is infinite. Since there are countably many of these additional paths one has the following equivalence for all x: $\{P \in U : x \in P\}$ is uncountable iff $\{P' \in U' : s(x) \in P'\}$ is uncountable.

Now one shows by induction over n that the following implication holds for all $x \in T$ with $s(x) \neq u$ and $n \in \mathbb{N}$: $x \in d^n(\mathcal{T}) \Rightarrow s(x) \in d^n(\mathcal{T}')$. The property clearly holds for $n = 0$. Now assume the inductive hypothesis for n and consider any $x \in d^{n+1}(\mathcal{T})$. There are two distinct infinite paths $P, Q \in U$ such that $x \in P \cap Q$ and $P \cup Q \subset d^n(\mathcal{T})$. It follows that $s(x) \in q(P) \cap q(Q)$. By induction hypothesis and by q being one-one, $s(x)$ is a member of the two distinct infinite paths $q(P), q(Q)$ of $d^n(\mathcal{T}')$ and thus $s(x) \in d^{n+1}(\mathcal{T}')$. This completes the proof of this property.

By Theorem 7.8, there is a natural number n such that $d^n(\mathcal{T}')$ contains exactly those nodes of the form $s(x)$ which are in uncountably many members of U'. Then all $x \in d^n(\mathcal{T})$ satisfy that x is in uncountably many members of U. On the other hand, every x being in uncountably many members of U is in $d^n(\mathcal{T})$. So $d^n(\mathcal{T})$ contains exactly the nodes x that are in uncountably many members of U and $d^{n+1}(\mathcal{T}) = d^n(\mathcal{T})$. The CB-rank of $\mathcal{T}$ is at most n. $\square$

8. AUTOMATIC VERSIONS OF KÖNIG'S LEMMA

König's Lemma says that every infinite finitely branching tree has at least one infinite path. This section consists of automatic versions of this result. If one considers Turing machines instead of finite automata there are trees that have infinite paths, but no hyperarithmetic one, and in particular no computable infinite path. Furthermore, even finitely branching trees might have infinite paths but none of them is computable. In contrast to this, the following results state that every automatic tree, not necessarily finitely branching, either has a regular infinite path or does not have an infinite path at all.

PROPOSITION 8.1. *It is decidable whether an automatic tree has an infinite path.*

PROOF. Let $(T, \preceq)$ be an automatic tree and recall that $(T, <_{kb})$ is an automatic linear order. By Proposition 5.1, it is decidable whether this order is isomorphic to an ordinal. And this is the case if and only if $(T, \preceq)$ has no infinite path. To prove this last statement recall that a linear order is isomorphic to an ordinal if and only if it has no infinite decreasing chain. So suppose $(T, \preceq)$ has an infinite path $x_1 \prec x_2 \prec x_3 \cdots$. Then $x_1 >_{kb} x_2 >_{kb} x_3 \cdots$ is an infinite decreasing chain in $(T, \leq_{kb})$, and so $(T, \leq_k b)$ is not isomorphic to an ordinal. Conversely, suppose $(T, <_{kb})$ is not isomorphic to an ordinal and let $x_1 >_{kb} x_2 >_{kb} x_3 \cdots$ be an infinite decreasing chain. We define an infinite path (p_i) of $(T, \prec)$ as follows.

(1) Let $i = 1$ and $j = 1$.

(2) **Repeat**

(a) Define $p_i = x_j$.
(b) Replace j with the smallest $k > j$ for which there is a $u \in S(p_i)$ with $u \preceq x_l$ for every $l \geq k$.
(c) Replace i with $i + 1$.

(3) **End Repeat**

If such a k exists in Step (2)(b) of every stage of the repeat loop, then the resulting sequence (p_i) is an infinite path in $(T, \preceq)$. So suppose that the algorithm has computed $p_1, p_2, \ldots, p_n$ with $p_1 \prec p_2 \prec \cdots \prec p_n$. So $i = n$ and $j \in \mathbb{N}$. For every $m > j$ define $u(x_m)$ as the immediate successor of p_i that is $\preceq x_m$. Then this sequence satisfies $u(x_m) \geq_{llex} u(x_{m+1}) \geq_{llex} u(x_{m+2}) \geq_{llex} \cdots$ since $x_m >_{kb} x_{m+1} >_{kb} x_{m+2} >_{kb} \cdots$. But since $\leq_{llex}$ is isomorphic to an ordinal (of type ω) it can not have an infinite decreasing sequence. Thus, the sequence is eventually constant; that is, there is a (smallest) $k > j$ such that for every $l \geq k$ one has $u(x_k) = u(x_l) \preceq x_l$ as required. □

8.1 Finitely Branching Automatic Trees

An infinite tree is *pruned* if every element is on some infinite path. Note that for a tree $\mathcal{T}$ the set of elements $E(\mathcal{T})$ above which there are infinitely many elements is definable as $\{x \in T \mid (\exists^\infty y) x \preceq y\}$. So if $\mathcal{T}$ is finitely branching, then $E(\mathcal{T})$ consists of those nodes of $\mathcal{T}$, that are on some infinite path. Indeed, if $x \in E(\mathcal{T})$, then by König's Lemma it is on an infinite path. Conversely, if $x \notin E(\mathcal{T})$ then there are only finitely many elements above it (in $\mathcal{T}$) and so it is not on an infinite path. Hence, the subtree $(E(\mathcal{T}), \preceq)$ is pruned and contains every infinite path of $\mathcal{T}$. Further, if $\mathcal{T}$ is automatic, then so is $E(\mathcal{T})$.

Theorem 8.2 (Automatic König's Lemma version 1). *If $\mathcal{T} = (T, \preceq)$ is an infinite finitely branching automatic tree then it has a regular infinite path. That is, there exists a regular set $P \subseteq T$ so that P is an infinite path of $\mathcal{T}$.*

Proof. By the previous remark, replace $\mathcal{T}$ with the automatic pruned tree $(E(\mathcal{T}), \preceq)$, and call the resulting tree $\mathcal{T}$. Recall that the length-lexicographic order $<_{llex}$ on Σ^* is automatic and therefore one can extend the presentation of $\mathcal{T}$ to include $<_{llex}$, namely $(T, \preceq, <_{llex})$ is an automatic structure. Now define the leftmost infinite path P with respect to the length-lexicographic order of the successors of any node. P contains those nodes x for which every $y \prec x$ satisfies that $\forall z, z' \in S(y)[z \preceq x \Rightarrow z <_{llex} z']$, and so by Proposition 2.3 P is regular. This means, that the unique node $z \in S(y)$, which is below x, is just the length-lexicographically least element of $S(y)$. Since the length-lexicographic ordering of Σ^* is a well-ordering (of type ω), this minimum always exists.

We briefly check that P is an infinite path. First P is closed downward. Indeed, given $x \in P$, let $a \preceq x$. Then for every $y \prec a$, if $z, z' \in S(y)$ and $z \preceq y \preceq x$ so by hypothesis then $z \leq_{llex} z'$, as required. Second P is linearly ordered. For otherwise, if $x, a \in P$ with $x \| a$, then let z be their $\prec$-maximal common ancestor. Consider two successors of z say v and w with $v \prec x$ and $w \prec a$. Without loss of generality, suppose that $v <_{llex} w$. Then, z, v and w form a counterexample to a's membership in P. Finally, P is infinite (and hence maximal with these properties). Indeed, if $x \in P$, then the $<_{llex}$-smallest element in $S(x)$ is also in P. Hence, P is an infinite regular path in $\mathcal{T}$, as required. □

If in the hypothesis above $\mathcal{T}$ contains finitely many infinite paths, then every infinite path is regular since after defining P, one considers the tree on domain $T \setminus P$ to find the next infinite path. The next theorem generalizes this to the case when $\mathcal{T}$ contains countably many infinite paths.

THEOREM 8.3 (AUTOMATIC KÖNIG'S LEMMA VERSION 2). *If $\mathcal{T}$ is an automatic tree that is finitely branching and has countably many infinite paths, then every infinite path in it is regular.*

PROOF. As before replace $\mathcal{T}$ with the automatic pruned tree $(E(\mathcal{T}), \preceq)$. Then, the derivate $d(\mathcal{T})$ is definable and so the elements of the tree $\mathcal{T} \setminus d(\mathcal{T})$ form a regular subset of T, call it R. Then, R consists of countably many disjoint infinite paths, each definable as follows. For every $\prec$-minimal $a \in R$, define the infinite path P_a as $\{x \in T \mid x \preceq a \vee (a \prec x \wedge x \in R)\}$.

Now replace $\mathcal{T}$ by $d(\mathcal{T})$ and repeat the steps in the previous paragraph. Since $\mathrm{CB}(\mathcal{T})$ is finite, these steps can be iterated at most $\mathrm{CB}(\mathcal{T})$ times; after which time the resulting tree will be empty and every infinite path in the original $\mathcal{T}$ will have been generated at some stage. □

Remark 8.4. The assumption that $\mathcal{T}$ have countably many infinite paths can not be dropped, since otherwise $\mathcal{T}$ necessarily has non-regular (indeed, uncountably many noncomputable) infinite paths.

8.2 The General Case

It turns out that automaticity allows one to remove the condition that $\mathcal{T}$ be finitely branching, under the assumption of course that $\mathcal{T}$ has at least one infinite path. This can be done if given an automatic tree $\mathcal{T}$, one can effectively construct an automatic copy of the pruned tree $E(\mathcal{T})$, the set of elements of $\mathcal{T}$ that are on an infinite path in $\mathcal{T}$. Then, as in the finitely branching case, Theorem 8.2, the $<_{llex}$-least path is definable and hence regular.

THEOREM 8.5 (AUTOMATIC KÖNIG'S LEMMA VERSION 3). *If $\mathcal{T}$ is an automatic tree with an infinite path, then it has a regular infinite path.*

This follows immediately from the following construction.

LEMMA 8.6. *If $\mathcal{T}$ is an automatic tree, then $E(\mathcal{T}) \subseteq T$ is a regular language.*

PROOF. Let $\mathcal{T} = (T, \preceq)$ be an automatic tree. Writing T' for $E(\mathcal{T})$, it is required that the set $T' \subseteq \Sigma^\star$ of all nodes in T that are on an infinite path is a regular language.

The idea is to construct a Büchi recognizable language $\mathcal{B}$ over the alphabet $\Delta = \Sigma_\diamond \times \Sigma$ so that its projection (on the first co-ordinate) is of the form $T' \cdot \{\diamond\} \cdot W^\omega$ for some regular $W \subseteq \Sigma_\diamond^\star$. Then, T' is regular since Büchi automata are closed under projection and an automaton for T' can be extracted from one for B.

Say that a word x *is on* $c_0 c_1 \ldots$, where each c_i is $(a_i, b_i) \in \Sigma_\diamond \times \Sigma$, iff there exist $m, n \in \mathbb{N}$ such that

— either $m = 0$, $x = a_0 a_1 \cdots a_n$ and $a_{n+1} = \diamond$
— or $n \geq m > 0$, $x = b_0 b_1 \cdots b_{m-1} a_m a_{m+1} \cdots a_n$, $a_{m-1} = \diamond$ and $a_{n+1} = \diamond$.

In the first case, we say that x is the *first word on* $c_0c_1\cdots$. Consider the set of all sequences $(a_0, b_0)(a_1, b_1)\ldots \in \Delta$ such that there are infinitely many words on the sequence and the words on the sequence generate an infinite path of T. More formally,

—$\exists^\infty n(a_n = \diamond)$;
—if y, z are on $(a_0, b_0)(a_1, b_1)\cdots$ and $|y| \leq |z|$ then $y \preceq z$ and $y, z \in T$.

There is a Büchi automaton $\mathcal{B}$ accepting such sequences because the orderings $\preceq$ and length-comparison are automatic and T is regular. Further, using that $\mathcal{T}$ is transitive, one need only check that adjacent words y, z on the sequence satisfy $y \preceq z$.

To complete the proof, we prove that $x \in T'$ if and only if x is the first word on some sequence $c_0c_1\cdots$ satisfying the two conditions. The reverse implication is clear. For the forward implication, let $x \in T$ be given and P be an infinite path witnessing that $x \in T'$. Define the sequences $a_0a_1\cdots$ and $b_0b_1\cdots$ described below.

(1) **Choose** $n, a_0, a_1, \ldots, a_n$ such that $x = a_0a_1\ldots a_n$. **Let** $a_{n+1} = \diamond$.
(2) **Let** $m = 0$. **Let** $y = x$.
(3) **Find** $b_mb_{m+1}\ldots b_{n+1}$ such that infinitely many nodes in P extend $b_0b_1\ldots b_{m-1}$ as strings.
(4) **Update** $m = n + 2$.
(5) **Find** a new value for n and $a_ma_{m+1}\ldots a_n$ such that $n \geq m$, the path P contains the node $z = b_0b_1\ldots b_{m-1}a_ma_{m+1}\ldots a_n$ and $y \preceq z$. **Let** $a_{n+1} = \diamond$.
(6) **Let** $y = z$. **Go to** 3.

Note that it is an invariant of the construction that whenever the algorithm comes to Step (3), either $m = 0$ or infinitely many nodes in P extend the string $b_0b_1\cdots b_{m-1}$. As there are only finitely many choices for the new part $b_mb_{m+1}\cdots b_{n+1}$, one can choose this part such that still infinitely many nodes in P extend $b_0b_1\cdots b_{n+1}$ as a string. In Step(4), m is chosen such that the precondition of Step (3) holds again and b_m is the first of the b-symbols not yet defined. For every $y \in P$, it holds that all but finitely many nodes z in P satisfy $y \preceq z$. Furthermore, for every finite length l, almost all nodes in P are represented by strings longer than l. Thus, one can find a node z as specified in Step (5) and the algorithm runs forever defining the infinite sequence $(a_0, b_0)(a_1, b_1)\cdots$ in the limit. In particular, such a sequence exists. It is not required that the sequence can be constructed effectively since the path P might not even be computable. □

From Theorem 8.5, we see that if an automatic tree has *finitely* many infinite paths, then each is regular. The next theorem generalizes this to trees with *countably* many infinite paths.

THEOREM 8.7 (AUTOMATIC KÖNIG'S LEMMA VERSION 4). *Every infinite path in an automatic tree with countably many infinite paths is regular.*

PROOF. Let $\mathcal{T} = (T, \preceq)$ be an automatic tree with countably many infinite paths. Then, the extendible part of $\mathcal{T}$, $E(\mathcal{T}) \subseteq T$, is regular by Lemma 8.6. So the derivative $d(\mathcal{T})$ is automatic. Write $E^i(\mathcal{T}) \subseteq T$ for the extendible part of the domain of $d^i(\mathcal{T})$. Then, since $\mathcal{T}$ is automatic CB($\mathcal{T}$) is finite, say n. And since $\mathcal{T}$ has countably many infinite paths, $d^n(\mathcal{T})$ is the empty tree. So the structure $(T, E^0(\mathcal{T}), E^1(\mathcal{T}), \dots, E^n(\mathcal{T}), \preceq)$ is automatic.

Now, for every $x \in \mathcal{T}$, there exists an $m < n$ such that x is in the domain the tree $d^m(\mathcal{T})$ and not in the domain of the tree $d^{m+1}(\mathcal{T})$. In particular, if P is an infinite path of $\mathcal{T}$, then there is a largest $m < n$ such that $P \subseteq E^m(\mathcal{T})$. The path P is isolated on $(E^m(\mathcal{T}), \preceq)$ since otherwise P would also be an infinite path of $d^{m+1}(\mathcal{T})$ and a subset of $E^{m+1}(\mathcal{T})$. Define $x_P \in T$ to be the least, with respect to $\preceq$, element of P, which is not in $E^{m+1}(\mathcal{T})$. Then P is the only infinite path of $E^m(\mathcal{T})$ containing x_P. So P is the set of all $y \in E^m(\mathcal{T})$, which are comparable to x_P with respect to $\preceq$. Hence, P is regular. $\square$

Remark 8.8. If $\mathcal{T} = (T, \preceq)$ is an automatic tree with countably many infinite paths, then there is a formula specifying all these paths that is built from the parameters $n, E^0(\mathcal{T}), E^1(\mathcal{T}), \dots, E^n(\mathcal{T})$ defined in the previous proof. The formula is the following one:

$$\begin{aligned}\Phi(a,b) = \bigvee_{i=0}^{n-1} & [a \in E^i(\mathcal{T}) \wedge a \notin E^{i+1}(\mathcal{T}) \wedge b \in E^i(\mathcal{T})] \\ & \wedge [(b \preceq a) \vee (a \prec b \wedge (\forall c, d, e \in E^i(\mathcal{T})) \\ & \times [a \preceq c \wedge d, e \in S(c) \wedge d \preceq b \Rightarrow d \leq_{llex} e])].\end{aligned}$$

The formula Φ and each set P_a defined as $\{b \in T \mid \Phi(a, b)\}$ satisfy the following conditions:

—If $a \in E^0$, then P_a is an infinite path of $\mathcal{T}$;
—If $a \notin E^0$, then P_a is empty;
—For every infinite path P of $\mathcal{T}$, there is an a with $P = P_a$.

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REFERENCES

BLUMENSATH, A. 1999. Automatic structures. Diploma Thesis, RWTH Aachen.

BLUMENSATH, A. AND GRÄDEL, E. 2000. Automatic structures. In *Proceedings of the 15th Symposium on Logic in Computer Science (LICS2000)*. IEEE Computer Society Press, Los Alamitos, CA, 51–62.

DELHOMMÉ, C. 2004. Automaticité des ordinaux et des graphes homogénes. *Comput. Rend. Math. 339*, 1, 5–10.

DELHOMMÉ, C. 2004. Automaticité des ordinaux et des graphes homogènes. *C.R. Acad. Sci. Paris Ser. I 339*, 5–10.

EILENBERG, S., ELGOT, C., AND SHEPHERDSON, J. 1969. Sets recognized by n-tape automata. *J. Alg. 13*, 4, 447–464.

EPSTEIN, D. B. H., CANNON, J. W., HOLT, D. F., LEVY, S. V. F., PATERSON, M. S., AND THURSTON, W. P. 1992. *Word processing in groups*. Jones and Bartlett.

HAUSDORFF, F. 1908. Grundzüge einer theorie der geordnete mengen. *Math. Ann 65*, 435–505.

ISHIHARA, H., KHOUSSAINOV, B., AND RUBIN, S. 2002. Some results on automatic structures. In *Proceedings of the 17th Symposium on Logic in Computer Science* (*LICS2002*). IEEE Computer Society Press, Los Alamitos, CA, 235–242.

KECHRIS, A. 1995. *Classical Descriptive Set Theory*. Springer-Verlag, New York.

KHOUSSAINOV, B. AND NERODE, A. 1994. Automatic presentations of structures. In *Logic and Computational Complexity,* International Workshop LCC 1994 (Indianapolis, IN), D. Leivant, Ed. Lecture Notes in Computer Science, vol. 960. Springer-Verlag, New York, 367–392.

KHOUSSAINOV, B. AND NERODE, A. 2001. *Automata theory and its applications*. Progress in Computer Science and Applied Logic, vol. 21. Birkhäuser Boston Inc., Boston, MA.

KHOUSSAINOV, B., NIES, A., RUBIN, S., AND STEPHAN, F. 2004a. Automatic structures: Richness and limitations. In *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science (LICS)*. IEEE Computer Society Press, Los Alamitos, CA, 44–53.

KHOUSSAINOV, B. AND RUBIN, S. 2001. Graphs with automatic presentations over a unary alphabet. *J. Automata, Lang. Combinat. 6*, 4, 467–480.

KHOUSSAINOV, B. AND RUBIN, S. 2003. Automatic structures—Overview and future directions. *J. Automata, Lang. Combinat. 8*, 2, 287–301.

KHOUSSAINOV, B., RUBIN, S., AND STEPHAN, F. 2004b. Definability and regularity in automatic structures. In *Proceedings of the 21st Symposium on Theoretical Aspects of Computer Science* (*STACS 2004*). 440–451.

KUSKE, D. 2003. Is Cantor's theorem automatic ? In *Proceedings of the 10th International Conference on Logic for Programming Artificial Intelligence and Reasoning* (*LPAR 2003*). 330–343.

LOHREY, M. 2003. Automatic structures of bounded degree. In *Proceedings of the 10th International Conference on Logic for Programming Artificial Intelligence and Reasoning* (*LPAR 2003*). 344–358.

ROGERS, H. 1967. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, New York.

ROSENSTEIN, J. 1982. *Linear Orderings*. Academic Press, Orlando, FL.

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