Let $V'' = V \setminus V'$. Since V'' is a trap for player 1, it is a subarena. Solve this recursively to get its winning regions W_0'' and W_1''.

Then: $W_0'' \subseteq WR_0$, and the strategy on W_0'' is memoryless by induction (and play stays in W_0'', so is winning in the original game).

Also: $W_1'' \subseteq WR_1$, and the following strategy is memoryless: use the strategy "while in W_1'' play the corresponding ws". Note that if a generated play stays in W_1'' it is won by player 1, but if it ever leaves (to the choice of player 0) it can only leave to V', and thus is also won by player 1.

□

Computing winning region can be done in time $O(2^n)$; just use the recurrence $T(1) = 1$ and $T(n) \leq 2T(n-1) + O(m)$.

More careful analysis gives $O(n^{d+O(1)})$. Recurrence: $T(n,d) \leq n \times T(n, d-1) + O(n^2)$ and $T(n,1) = O(1)$.

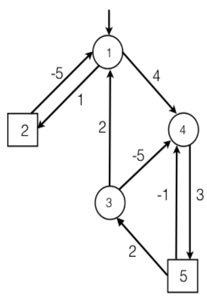
SHOW: solitaire parity games solvable in ptime.

if all nodes belong to player i, find a cycle with largest priority $i mod 2$.

idea: suppose i, d are even. check if there is a path from initial state to a node labeled d and a path from that node to itself. if yes, done. if not, remove all nodes labeled d and $d-1$, and repeat.

5 Games with quantitative objectives

Outline Games with quantitative objectives assign a real number to every play, and one player is trying to maximise this value while the other is trying to minimise it. We will discuss how to solve mean-payoff games.



edge-colored arena $(A, w : E \to [-W, +W])$.

MP-Objective is all $\pi = \pi_0\pi_1\cdots$ such that:

$$(\lim_{n\to\infty} \inf avg_n) \geq 0$$

where $avg_n = \frac{1}{n}\sum_{i=1}^{n}(w(\pi_{i-1}, \pi_i))$.

Theorem 5. *MPG are positionally determined, using uniform strategies, and solving them is in* NP *and co-*NP*.*

Proof is postponed till next lecture.

Meanwhile:

Proposition 1. *Solving Parity games can be reduced to mean-payoff games in polynomial time.*

Proof. WLOG, colors are in $[d-1]$. Define weight of (v, w) to be $(-1)^i n^i$ where $c(v) = i$ and $n = |V|$. In particular, supposing d is even, the weights are from the set $\{-n^{d-1}, n^{d-2}, ..., -n, 1\}$. Claim that for every pair of memoryless strategies, the same strategy wins both games. Indeed, a cycle C is formed which governs the winner. Note that $avg(C) \geq 0$ iff $sum(C) \geq 0$. We show that the max col c on C is even iff $sum(C) \geq 0$. Let x be weight with largest absolute value.

Suppose max col on C is even. Then $x > 0$, and so

$$sum(C) \geq x - (n-1)x/n > 0$$

since there are at most $n-1$ other edges on the cycle, each of whose weight is at most x/n. The other direction is symmetric. □

6 First Cycle Games

We already saw a game that was not positionally determined. Can we characterise those games that are? We introduce FCG. To motivate them notice that if players can win positionally, then fixing positional strategies, the resulting play is a lasso, and who wins (for prefix independent games) depends only on this cycle.

Definition 10. *A* Edge-colored arena *is a tuple* $(A, \lambda : E \to \mathbb{U})$.

A cycle property Y *is a subset of* $\mathbb{U}^*$.

e.g., even-length, parity, average is at least k, average/sum is positive.

Definition 11. *The set of* cycles *of a play is the output of the procedure that reads the edges of the play onto a stack and removes and output the cycles as they appear.*

Let Y be a set of finite sequences. A first-cycle objective *wrt Y consists of all plays whose first cycle is in Y. A* first-cycle game *is a game with a first-cycle objective.*

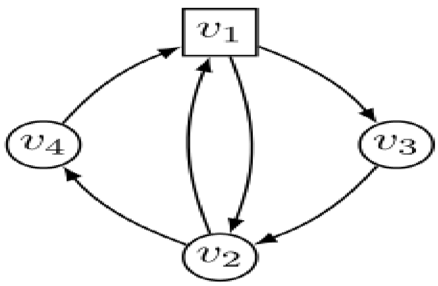
Example 3.

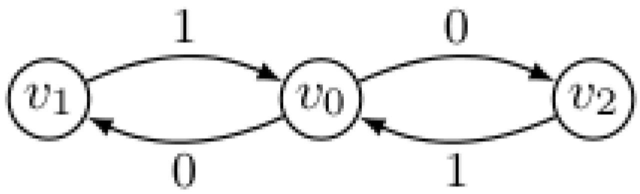
- *the first cycle should contain a vertex from T (i.e., $u \in Y$ iff $u_i \in T$ for some i)*
- *the first cycle should contain a vertex from each of $T_1, T_2, \cdots, T_k$.*
- *the length of the first cycle should be even.*
- *the set of vertices on the first cycle should be the set T.*
- *the set of vertices on the first cycle should be a set from $\{T_1, T_2, \cdots, T_k\}$.*
- *the average of the first cycle should be positive (here we assume that each edge has an integer weight associated with it).*

Fact 1. first-cycle games are determined (why?).

Fact 2. Not necc. positionally determined (use even-length)

Fact 3. Even if positionally determined, not necc. uniform positionally determined.





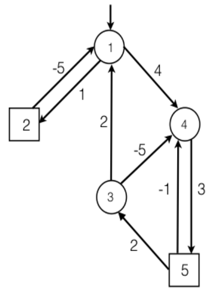
Definition 12. *An* edge-path *is a sequence of edges* $e_1e_2\cdots$ *such that* $last(e_i) = start(e_{i+1})$ *for all* i*. An edge-path is* simple *if* $last(e_j) = start(e_i)$ *implies* $i = j+1$*.*

Algorithm 1 Cycles-Decomposition $CD(s, \pi)$

Require: s is a finite (possibly empty) simple path ▷ initial stack content
Require: π is a finite or infinite path $\pi_1\pi_2\cdots$ ▷ the path to decompose
Require: If s is non-empty then $\mathsf{end}(s) = \mathsf{start}(\pi)$ ▷ $s\pi$ must form a path

```
step = 1
while step ≤ |π| do                              ▷ Start a step
    Append π_step to s                           ▷ Push current edge into stack
    Say s = e_1 e_2 ··· e_m
    if ∃i : e_i e_{i+1} ··· e_m is a cycle then  ▷ If stack has a cycle
        Output e_i e_{i+1} ··· e_m               ▷ output the cycle
        s := e_1 ··· e_{i−1}                     ▷ Pop the cycle from the stack
    end if
    step := step + 1                             ▷ advance to next input edge
end while
```

The code appearing in Algorithm 1 defines an algorithm CD that takes as input a (usually empty) simple path s, which is treated as the initial contents of a stack, and a path π (finite or infinite). At step $j \geq 1$, the edge π_j is pushed onto the stack and if, for some k, the top k edges on the stack form a cycle, this cycle is output, then popped, and the procedure continues to step $j+1$.



Example 4.
What are the sequence of cycles output by the cycles decomposition of

$$(1,2)(2,1)(1,2)(2,1)(1,4)[(4,5)(5,3)(3,4)]^{\omega}$$

We can now define two types of objectives:

Definition 13. *Let Y be a cycle property. The* first-cycle objective based on Y, written $FC(Y)$ *is the set of plays such that the labelling of the first cycle output by CD is in Y. The* all-cycle objective based on Y, written $AC(Y)$ *is the set of plays such that the labelling of every cycle output by CD is in Y.*

Define $N_z(\pi) \in \mathbb{N} \cup \{\infty\}$ to be the index of the first edge that starts with z, if one exists. Formally, $N_z(\pi) := \infty$ if z does not occur on π, and otherwise $N_z(\pi) := \min\{j : \mathsf{start}(\pi_j) = z\}$. Also, define $head_z(\pi) := \pi[1, N_z(\pi) - 1]$ to be the prefix of π before $N_z(\pi)$, and $tail_z(\pi) := \pi[N_z(\pi), |\pi|]$ to be the suffix of π starting at $N_z(\pi)$. By convention, if $N_z(\pi) = \infty$ then $head_z(\pi) = \pi$ and $tail_z(\pi) = \epsilon$.

We now define a game, that is very similar to the first-cycle game, except that one of the nodes of the arena is designated as a "reset" node:

Definition 14. *Fix an arena A, a vertex $z \in V$, and a cycle property Y. Define the objective $FC_z(Y)$ to consist of all plays π satisfying the following property: if $head_z(\pi)$ is not a simple path then $first(\pi) \in Y$, and otherwise $first(tail_z(\pi)) \in Y$.*

Playing the game with objective $FC_z(Y)$ is like playing the first-cycle game over Y, however, if no cycle is formed before reaching z for the first time, the prefix of the play up to that point is ignored. Thus, in a sense, the game is reset. Also note that if play starts from z, then the game reduces to a first-cycle game. It turns out that we may assume that a strategy of $(A, FC_z(Y))$ makes the same move every time it reaches z:

Definition 15 (Forgetful at z from v)**.** *For an arena A, a vertex $v \in V$, a Player $\sigma \in \{0,1\}$, and a vertex $z \in V_\sigma$ belonging to Player σ, we call a strategy T for Player σ* forgetful at z from v *if there exists $z' \in V$ such that $(z, z') \in E$ and for all $\pi \in plays(T, v)$, and all $n \in \mathbb{N}$, if $\mathsf{start}(\pi_n) = z$ then $\mathsf{end}(\pi_n) = z'$.*

Lemma 2 (Forgetful at z from v)**.** *Fix an arena A, a vertex $v \in V$, a Player $\sigma \in \{0,1\}$, and a vertex $z \in V_\sigma$ belonging to Player σ. In the game $(A, FC_z(Y))$, if Player σ has a strategy S that is winning from v, then Player σ has a strategy T that is winning from v and that is forgetful at z from v.*

Sketch. The second time z appears on a play, the winner is already determined, and so the strategy is free to repeat the first move it made at z. On the other hand, when a play visits z the first time, the strategy can make the same move regardless of the history of the play before z, because the winning condition ignores this prefix. $\square$

Definition 16. *Fix Y. An arena is Y-*resettable *if for every $i \in \{0,1\}$, and every node z, we have that $WR^i(A, FC_z(Y)) = WR^i(A, FC(Y))$.*

Theorem 6 (Resettability implies memoryless determinacy)**.** *Suppose that every arena A is Y-resettable. Then every game $(A, FC(Y))$ is memoryless determined.*

Sketch. A node $z \in V$ is a *choice node* of an arena B, if there are at least two distinct vertices $v', v'' \in V$ such that $(z, v') \in E^B$ and $(z, v'') \in E^B$.

Fix arena A. Suppose Player i has a winning strategy in $(A, FC(Y))$ from v (by determinacy, one of the players must). We induct on the number of choice nodes of Player i. Let z be a choice node for Player i (if there are none, the result is immediate). By the resettability assumption applied to A, Player i also wins the game with objective $FC_z(Y)$ from v. By Lemma 2, Player i has a strategy S that is winning from v and that is also forgetful at z from v. Thus we may form a sub-arena B of A by removing all edges from z that are not taken by S. Observe that S is winning from v in $(B, FC_z(Y))$. Applying the resettability assumption to B, Player i also wins $(B, FC(Y))$ from v. But B has less choice nodes for Player i, and thus, by induction, Player i has a memoryless winning strategy from v in $(B, FC(Y))$. This memoryless strategy is also winning from v in A (since we only removed choices of player i, which are not used in this memoryless strategy). $\square$

We also have a a full characterisation in the paper.

Connection with infinite duration games

We now define the connection between first-cycle games and games of infinite duration (such as parity games, etc.), namely the concept of Y-greedy games. We then prove the Strategy Transfer Theorem, which says, roughly, that for every arena, the winning regions of the First-Cycle Game over Y and a Y-greedy game coincide, and that memoryless winning strategies transfer between these two games.

Definition 17 (Greedy)**.** *Say that a* game (A, O) is Y-greedy *if*

$$AC(Y) \subseteq O \text{ and } AC(\neg Y) \subseteq \neg O.$$

Intuitively, a game (A, O) is Y-greedy means that Player 0 can win the game (A, O) if he ensures that every cycle in the cycles-decomposition of the play is in Y, and Player 1 can win if she ensures that every cycle in the cycles-decomposition of the play is not in Y.

Here are some examples.

1. Every all-cycles game $(A, AC(Y))$ is Y-greedy.