Graph expression complexities and simultaneous linear recurrences

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Abstract. The paper investigates relationships between algebraic expressions and graphs. Using the decomposition method we generate special simultaneous systems of linear recurrences for sizes of graph expressions. We propose techniques which provide closed-form solutions for these systems.

1 Introduction

We consider a *labeled* two-terminal directed acyclic graph (*st-dag* in [1]) that has only one source and only one sink in which each edge has a unique label. An algebraic expression is a *graph expression* (a *factoring of a graph* [1]) if it is algebraically equivalent to the sum of edge label products corresponding to all possible paths between the source and the sink of the graph. We define the total number of labels in an algebraic expression as the *complexity of the algebraic expression*. Expressions with a minimum (or, at least, a polynomial) complexity may be considered as a key to generating efficient algorithms on distributed systems.

A *series-parallel graph* is defined recursively so that a single edge is a series-parallel graph and a graph obtained by a *parallel* or a *series composition* of series-parallel graphs is series-parallel [1]. A series-parallel graph expression has a representation in which each label appears only once [1]. Generating an optimum factored form for non-series-parallel graph expressions is a highly complex problem. Interrelations between graphs and expressions are discussed in [1], [4], [5] and other works.

We generate expressions of a number of non-series-parallel graphs using the *decomposition method*. This method is based on revealing subgraphs of approximately equal sizes in the initial graph. The resulting polynomial-size expression is produced by a special composition of subexpressions describing these subgraphs. In many cases, computing of complexity of the obtained expression is reduced to solving a simultaneous system of three linear recurrences. In this paper we propose a method which provides a *closed-form* (explicit) solution for this system.

2 The decomposition method

Consider a graph called a *full square rhomboid* (FSR) [4]. We split every non-trivial graph through two *decomposition vertices* (number 4 in Fig. 1) which are chosen in the middle of the *upper* and the *lower* groups of the graph. Any path from vertex 1 to vertex 7 in Fig. 1 passes either through one of decomposition vertices or through edge b_4. Thus the graph is decomposed into six subgraphs two of which are FSR (connected by b_4 in Fig. 1) and four ones are called *single-leaf full square rhomboids* (FSR_1). Each FSR_1 is decomposed into six new subgraphs in the similar way (Fig. 2(a)). They are one FSR, three FSR_1 and two *dipterous full square rhomboids* (FSR_2). Decomposition of possible varieties of FSR_2 (see the example in Fig. 2(b)) gives two FSR_1 and four FSR_2 subgraphs.

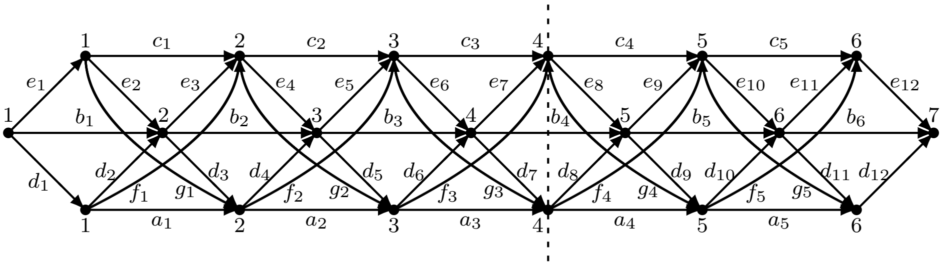

Fig. 1. Decomposition of a full square rhomboid.

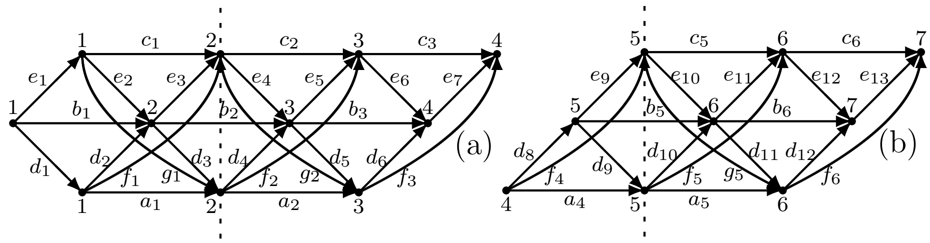

Fig. 2. Decomposition of single-leaf and dipterous full square rhomboids.

The total number of labels $T(m)$ in the expression of a full square rhomboid of size m, i.e., including m vertices of the middle (*basic*) group, for $m = 2^p$ ($p \geq 1$) is defined recursively as follows:

$$\begin{cases} T\left(m\right) = 2T\left(\frac{m}{2}\right) + 4T_1\left(\frac{m}{2}\right) + 1 \\ T_1\left(m\right) = T\left(\frac{m}{2}\right) + 3T_1\left(\frac{m}{2}\right) + 2T_2\left(\frac{m}{2}\right) + 1 \\ T_2\left(m\right) = 2T_1\left(\frac{m}{2}\right) + 4T_2\left(\frac{m}{2}\right) + 1\ , \end{cases} \tag{1}$$

where $T_1(m)$ and $T_2(m)$ are the total numbers of labels in expressions of FSR_1 and FSR_2, respectively, of size m and $T\left(1\right) = 0$, $T_1\left(1\right) = 1$, $T_2\left(1\right) = 3$.

One can see that the sum of the coefficients in each of three simultaneous recurrences (1) equals 6 and $T_1\left(m\right) = \frac{1}{2}T\left(m\right) + \frac{1}{2}T_2\left(m\right)$. Therefore, system (1) may be presented in general terms as a following simultaneous system of three *linear nonhomogeneous recurrences of first order* [6] with constant coefficients:

$$\begin{cases} a_n = \alpha_{11}a_{n-1} + \alpha_{12}b_{n-1} + \alpha_{13}c_{n-1} + \alpha_1 \\ b_n = \alpha_{21}a_{n-1} + \alpha_{22}b_{n-1} + \alpha_{23}c_{n-1} + \alpha_2 \\ c_n = \alpha_{31}a_{n-1} + \alpha_{32}b_{n-1} + \alpha_{33}c_{n-1} + \alpha_3. \end{cases} \tag{2}$$

In this system $\alpha_{11} + \alpha_{12} + \alpha_{13} = \alpha_{21} + \alpha_{22} + \alpha_{23} = \alpha_{31} + \alpha_{32} + \alpha_{33}$ and a sequence b_n is a linear combination of sequences a_n and c_n (a_0, b_0, and c_0 are initial values of a, b, and c, respectively).

Our research indicates that system (2) appears in solving a problem of deriving the explicit form of graph expression complexity for various graphs of regular structure. Some of subgraphs emerged in the result of decomposition are exactly of the same structure as the initial one. Others are supplemented at one end by elements which were in the middle of the split graph. The structure of these *one-sided* subgraphs does not change in the middle and, hence, they are decomposed in the same way. This gives, together with the subgraphs of the initial structure and one-sided subgraphs, *two-sided* subgraphs supplemented at both ends by the elements of the inner structure. The two-sided subgraphs are decomposed likewise and their splitting yields new one-sided and two-sided subgraphs.

The coefficients in (2) equal the numbers of their respective subgraphs. Since subgraphs of all kinds are decomposed into the same numbers of new subgraphs, the sums of coefficients in all recurrences of (2) are equal. It is logical that a characteristic of a graph supplemented by one set of additional elements is a weighted average of characteristics of graphs, one of which has no additional elements and another one has two sets. Specifically, we proved that the weights are equal if the sizes of all revealed subgraphs are equal and the number of subgraphs of a given type revealed from the left of the location of the split is equal to the number of right subgraps of the same type (see system (1)).

Therefore, solving system (2) being a rather special problem for a discrete mathematics as a whole, is a common problem from the perspective of the algorithmic theory.

It is possible to divide (2) into separate recurrences using the *Hamilton-Cayley theorem* [2] and further to attempt to use general methods for linear recurrences solving [6] (*methods* of *characteristic equations* (*roots*), of *generating functions*, etc.). However, using the results obtained in [3], we propose a simpler way that accommodates the restrictions imposed on the coefficients and the unknowns of system (2) and directly gives a closed form for its solution.

3 Results

Lemma 1. *If*

$$\begin{cases} a_n = \alpha_{11}a_{n-1} + \alpha_{12}b_{n-1} + \alpha_1 \\ b_n = \alpha_{21}a_{n-1} + \alpha_{22}b_{n-1} + \alpha_2, \end{cases}$$

$\alpha_{11} + \alpha_{12} = \alpha_{21} + \alpha_{22}$ *and* $\alpha_{12} \neq -\alpha_{21}$, $\alpha_{11} + \alpha_{12} \neq 1$, $\alpha_{11} - \alpha_{21} \neq 1$ *then*

$$\begin{aligned}
a_n = &\left(\alpha_{11}+\alpha_{12}\right)^n a_0 + \alpha_{12}\left(b_0 - a_0\right)\frac{\left(\alpha_{11}+\alpha_{12}\right)^n - \left(\alpha_{11}-\alpha_{21}\right)^n}{\alpha_{12}+\alpha_{21}} + \\
&\alpha_1 \frac{\left(\alpha_{11}+\alpha_{12}\right)^n - 1}{\alpha_{11}+\alpha_{12}-1} + \frac{\alpha_{12}\left(\alpha_2-\alpha_1\right)}{\alpha_{12}+\alpha_{21}} \times \\
&\left(\frac{\left(\alpha_{11}+\alpha_{12}\right)^n - \alpha_{11}-\alpha_{12}}{\alpha_{11}+\alpha_{12}-1} - \frac{\left(\alpha_{11}-\alpha_{21}\right)^n - \alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1}\right) \\
b_n = &\left(\alpha_{11}+\alpha_{12}\right)^n b_0 + \alpha_{21}\left(a_0 - b_0\right)\frac{\left(\alpha_{11}+\alpha_{12}\right)^n - \left(\alpha_{11}-\alpha_{21}\right)^n}{\alpha_{12}+\alpha_{21}} + \\
&\alpha_2 \frac{\left(\alpha_{11}+\alpha_{12}\right)^n - 1}{\alpha_{11}+\alpha_{12}-1} + \frac{\alpha_{21}\left(\alpha_1-\alpha_2\right)}{\alpha_{12}+\alpha_{21}} \times \\
&\left(\frac{\left(\alpha_{11}+\alpha_{12}\right)^n - \alpha_{11}-\alpha_{12}}{\alpha_{11}+\alpha_{12}-1} - \frac{\left(\alpha_{11}-\alpha_{21}\right)^n - \alpha_{11}+\alpha_{21}}{\alpha_{11}-\alpha_{21}-1}\right),
\end{aligned}$$

where a_0 and b_0 are initial values of a and b, respectively.

Lemma 2. *Given system (2) and the following conditions:*
1. $\alpha_{11}+\alpha_{12}+\alpha_{13} = \alpha_{21}+\alpha_{22}+\alpha_{23} = \alpha_{31}+\alpha_{32}+\alpha_{33}$,
2. $\exists$ real constants w_1, w_2, $w_1 + w_2 = 1$, that $\forall$ n, $b_n = w_1 a_n + w_2 c_n$,
three simultaneous recurrences (2) can be presented as the three pairs of the following simultaneous recurrences, respectively:

$$\begin{cases} a_n = \alpha'_{11} a_{n-1} + \alpha'_{12} b_{n-1} + \alpha_1 \\ b_n = \alpha'_{21} a_{n-1} + \alpha'_{22} b_{n-1} + \alpha_2 \end{cases}$$

where $\alpha'_{11} = \alpha_{11} - \frac{w_1}{w_2}\alpha_{13}$, $\alpha'_{12} = \alpha_{12} + \frac{1}{w_2}\alpha_{13}$, $\alpha'_{21} = \alpha_{21} - \frac{w_1}{w_2}\alpha_{23}$, $\alpha'_{22} = \alpha_{22} + \frac{1}{w_2}\alpha_{23}$;

$$\begin{cases} a_n = \alpha'_{11} a_{n-1} + \alpha'_{12} c_{n-1} + \alpha_1 \\ c_n = \alpha'_{21} a_{n-1} + \alpha'_{22} c_{n-1} + \alpha_3 \end{cases}$$

where $\alpha'_{11} = \alpha_{11} + w_1\alpha_{12}$, $\alpha'_{12} = w_2\alpha_{12} + \alpha_{13}$, $\alpha'_{21} = \alpha_{31} + w_1\alpha_{32}$, $\alpha'_{22} = w_2\alpha_{32} + \alpha_{33}$;

$$\begin{cases} b_n = \alpha'_{11} b_{n-1} + \alpha'_{12} c_{n-1} + \alpha_2 \\ c_n = \alpha'_{21} b_{n-1} + \alpha'_{22} c_{n-1} + \alpha_3 \end{cases}$$

where $\alpha'_{11} = \frac{1}{w_1}\alpha_{21} + \alpha_{22}$, $\alpha'_{12} = -\frac{w_2}{w_1}\alpha_{21} + \alpha_{23}$, $\alpha'_{21} = \frac{1}{w_1}\alpha_{31} + \alpha_{32}$, $\alpha'_{22} = -\frac{w_2}{w1}\alpha_{31} + \alpha_{33}$ and for all these pairs of simultaneous recurrences

$$\alpha'_{11} + \alpha'_{12} = \alpha'_{21} + \alpha'_{22}.$$

The following theorem results from Lemmas 1 and 2.

Theorem 1. *Given system (2) and the following conditions:*
1. $\alpha_{11}+\alpha_{12}+\alpha_{13} = \alpha_{21}+\alpha_{22}+\alpha_{23} = \alpha_{31}+\alpha_{32}+\alpha_{33}$,
2. $\exists$ real constants w_1, w_2, $w_1 + w_2 = 1$, that $\forall$ n, $b_n = w_1 a_n + w_2 c_n$,

$$
\begin{aligned}
a_n &= (C_1)^n a_0 + C_2 \frac{(C_1)^n - (C_3)^n}{C_4}(b_0 - a_0) + \alpha_1 \frac{(C_1)^n - 1}{C_1 - 1} + \\
&\quad \frac{(\alpha_2 - \alpha_1) C_2}{C_4}\left(\frac{(C_1)^n - C_1}{C_1 - 1} - \frac{(C_3)^n - C_3}{C_3 - 1}\right) \\
b_n &= (C_1)^n b_0 + C_5 \frac{(C_1)^n - (C_3)^n}{C_4}(a_0 - b_0) + \alpha_2 \frac{(C_1)^n - 1}{C_1 - 1} + \\
&\quad \frac{(\alpha_1 - \alpha_2) C_5}{C_4}\left(\frac{(C_1)^n - C_1}{C_1 - 1} - \frac{(C_3)^n - C_3}{C_3 - 1}\right) \\
c_n &= -\frac{w_1}{w_2} a_n + \frac{1}{w_2} b_n \ ,
\end{aligned}
$$

where $C_1 = \alpha_{11} + \alpha_{12} + \alpha_{13}$ $(C_1 \neq 1)$, $C_2 = \alpha_{12} + \frac{1}{w_2}\alpha_{13}$, $C_3 = \alpha_{11} - \frac{w_1}{w_2}\alpha_{13} - \alpha_{21} + \frac{w_1}{w_2}\alpha_{23}$ $(C_3 \neq 1)$, $C_4 = \alpha_{12} + \frac{1}{w_2}\alpha_{13} + \alpha_{21} - \frac{w_1}{w_2}\alpha_{23}$ $(C_4 \neq 0)$, $C_5 = \alpha_{21} - \frac{w_1}{w_2}\alpha_{23}$, *and* a_0, b_0, c_0 *are initial values of* a, b, c, *respectively.*

4 Conclusions and open problems

We have proposed a method that gives a closed-form solution for a special simultaneous system of three linear recurrences. Sums of coefficients in the recurrences are equal, and each recurrent variable is a linear combination of two other recurrent variables. The solution is applied to deriving explicit forms of graph expression complexities. Specifically, using this method, we have obtained the following formula for the number of labels in expressions of full square rhomboids of size m ($m = 2^p$): $\frac{89}{45}m^{\log_2 6} - \frac{20}{9}m^{\log_2 3} - \frac{1}{5}$. Our intent is to determine the class of graphs for which the complexities of their expressions go hand in hand with these recurrences.

We are going to generalize this method to a system of an arbitrary (or, at least, a larger) number of recurrences and, finally, to develop a method capable to handle simultaneous recurrences with equal sums of coefficients in columns.

References

1. Bein, W.W., Kamburowski, J., Stallmann, M.F.M.: Optimal Reduction of Two-Terminal Directed Acyclic Graphs. SIAM Journal of Computing 21(6), 1112–1129 (1992)
2. Bloom, D. M.: Linear Algebra and Geometry. Cambridge University Press, Cambridge, London (1979)
3. Korenblit, M.: Efficient Computations on Networks, Ph.D. Thesis, Bar-Ilan University, Israel (2004)
4. Korenblit, M.: Decomposition Methods for Generating Algebraic Expressions of Full Square Rhomboids and Other Graphs. Discrete Applied Mathematics 228, 60-72 (2017)
5. Mintz, A., Golumbic, M.C.: Factoring Boolean Functions Using Graph Partitioning. Discrete Applied Mathematics 149(1-3), 131-153 (2005)
6. Rosen, K.H. (ed.): Handbook of Discrete and Combinatorial Mathematics. CRC Press, Chapman & Hall/CRC, Boca Raton, second edition (2010)