$MSO + \exists^{\infty} + \exists^{>\omega} + \exists^{mod}$

on

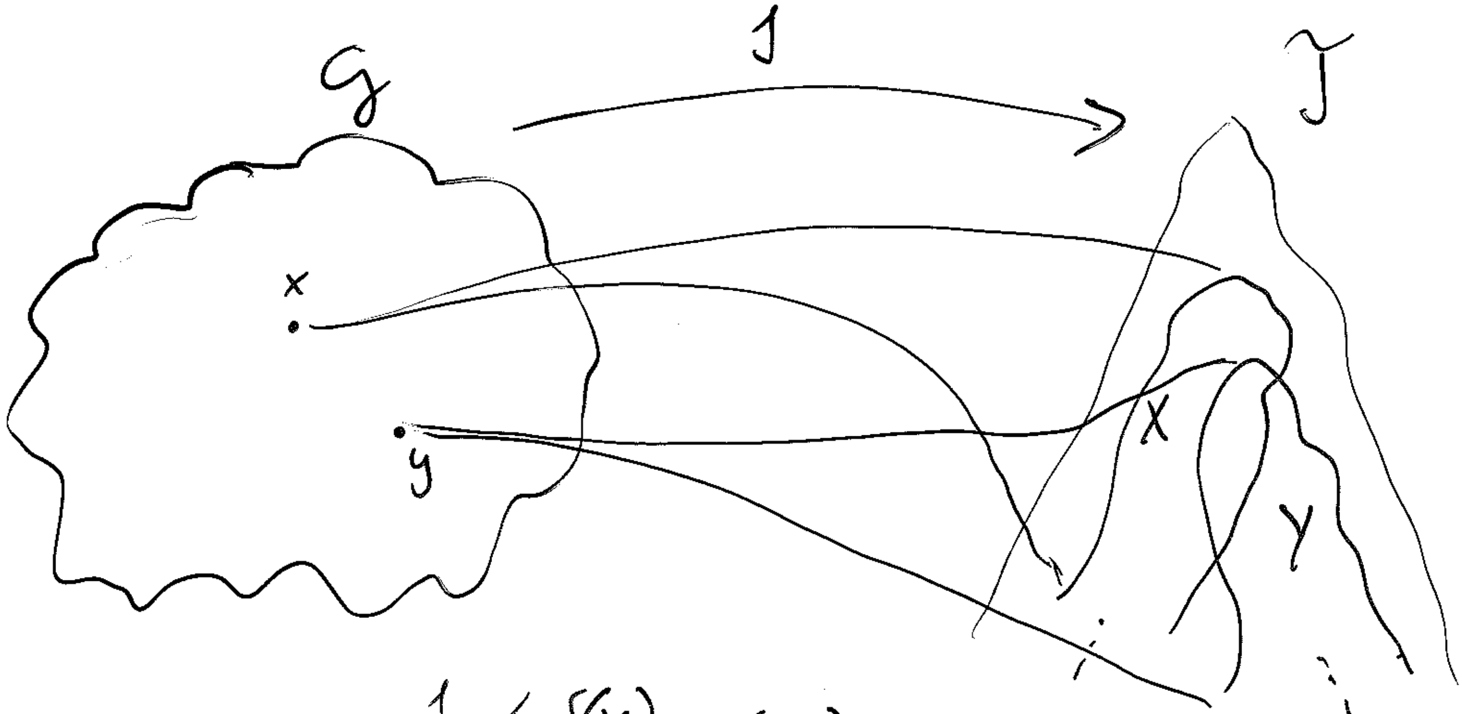
trees and countable linear orders

Vince Bárány

joint work with

Ł. Kaiser, A. Rabinovich and Sasha Rubin

Subset Interpretations



$$\mathcal{I} = \langle \delta(X),\ \varepsilon(X,Y),\ \varphi(X,Y), \dots \rangle$$

$x \in V(\mathcal{G})$	$\longleftrightarrow$	$\mathcal{J} \models \delta(X)$
$x = y$		$\mathcal{J} \models \varepsilon(X,Y)$
$(x,y) \in E(\mathcal{G})$		$\mathcal{J} \models \varphi(X,Y)$

First-Order Logic (FO) — Monadic Second-Order Logic (MSO)

decidable FO-theory $\Longleftarrow$ decidable MSO-theory

Examples:

$(\mathbb{N}, +) \leq_{\text{subset-int.}} (\mathbb{N}, succ)$ / in S1S

$(\mathbb{Q}, <) \leq_{\cancel{\text{subset}}\text{-int}} (\{0,1\}^*, succ_0, succ_1)$ / in S2S

$(\mathbb{N}, \times)$ — " —

$(\mathcal{P}(\mathbb{N}), \cap, \cup, -)/_{\approx_{\text{finite diff}}} \leq_{\text{subset-int}} (\mathbb{N}, succ)$

Classical Cases of Interest

finite-subset interpretations: $\delta(X) \Rightarrow X$ is finite

injective interpr.: $\varepsilon(X,Y) \Rightarrow X=Y$

finite-subset interpr. in $(\mathbb{N}, succ)$: automatic structures

subset interpr. in $(\mathbb{N}, succ)$: ω-automatic / Büchi-aut. str.

finite-subset interpr. in $(\{0,1\}^*, succ_0, succ_1)$: tree-automatic str.

subset interpr. in — " — : Rabin-automatic str.

Generalisations:

- adding parameters (unary predicates) to $(\mathbb{N}, succ)$ or to the full binary tree
- interpretations in "higher-order trees"

  in countable ordinals, linear orders

Facts, Questions, Results

* $\underline{\text{finite}}$-subset interpr. in $(\mathbb{N}, \text{succ})$ and in the full binary tree can be turned into injective ones.

* not known for non-finite-subset interpretations
  [Kuske-Lohrey FoSSACS '06] $\approx_{\text{finite diff}}$ on $\mathcal{P}(\mathbb{N})$
  has no regular/definable set of representatives
  breaking news: Hjörth, Nies, et al. have a counter-example on $(\mathbb{N}, \text{succ})$

? Is every countable ω-aut. str. also automatic? [Blumensath '99]

* assuming $\mathcal{I}$ is <u>injective</u>
  the $FO + \exists^{\infty} + \exists^{>\omega} + \exists^{\text{mod}}$ - theory of $\mathcal{G}$
  can be reduced to the MSO-theory of $(\mathbb{N}, \text{succ})$, resp., the binary tree
  [BG '00, KRS '04, KL '06, Colcombet '04, + here + ?]

+ over $(\mathbb{N}, \text{succ})$ this holds for non-injective $\mathcal{I}$ as well
  $\Rightarrow$ YES to Blumensath's question

* [Niwinski '91] gave a decidable characterisation of <u>countable</u> tree-regular (S2S-def.) languages

+ assuming $\mathcal{I}$ is <u>injective</u>
the $FO + \exists^\infty + \exists^{>\omega}$-theory of $\mathcal{G}$
reduces to the MSO-theory of $\mathcal{T}$
for $\mathcal{T}$ any <u>finitely branching tree</u> or
any <u>countable linear order</u>

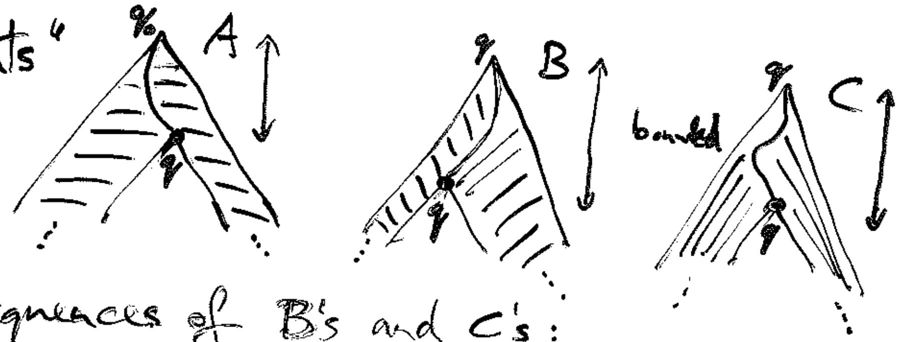
$\equiv$ $MSO + \exists^\infty + \exists^{>\omega}$ effectively reduces to MSO on $\begin{cases} \text{fin. br. trees} \\ \text{countable lin. ord.} \end{cases}$
(and uniformly!)
to MSO + "X is finite"
on infinitely branching trees

characterisation and proof technique (composition, shuffle) are similar to those of Niwinski and of Kuske-Lohrey only more elaborate

Intuition of Niwiński '91

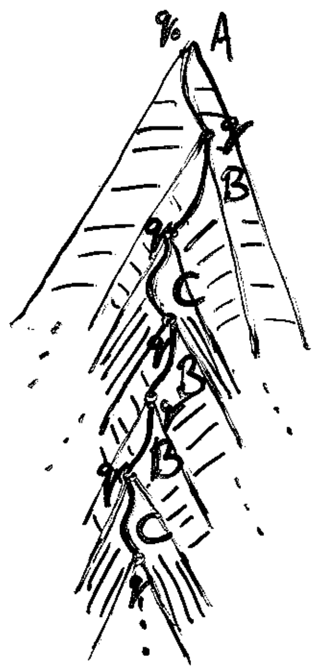
regular lang. L of infinite trees is <u>uncountable</u>

$\Leftrightarrow$ ex. "tree segments"



such that for all sequences of B's and C's:

$$\boxed{\Delta_2 \models \exists^{>\omega} X.\ \varphi(X) \text{ is decidable}}$$



$\in L$

via composition!

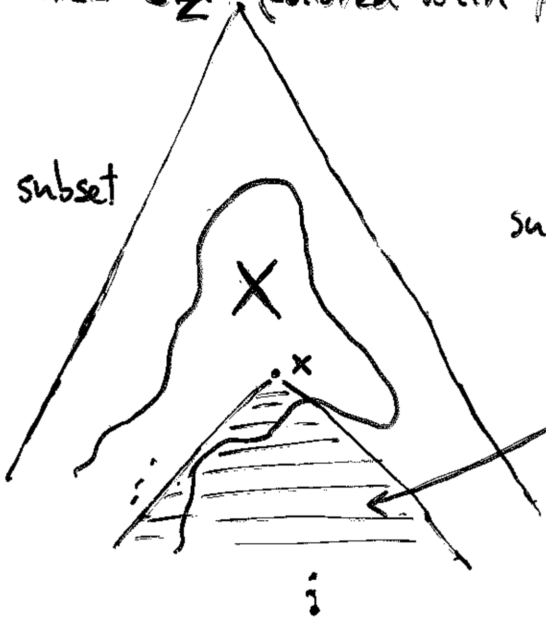
<u>Note</u>: this way of shuffling is only conceivable because the underlying tree (Δ_2) is <u>regular</u>

will <u>not</u> work in the presence of <u>parameters</u>, e.g. <u>for</u> $\exists^{>\omega} X.\ \varphi(X, \vec{Z})$

Intuition of general technique and condition for trees

tree $T_{\vec{Z}}$ (colored with parameters $\vec{Z}$)

fix subset



such that $T \models \varphi(X, \vec{Z})$

ϑ_x = n-type of  (colored with $X, \vec{Z}$)

x is a U-node / D-node wrt. X

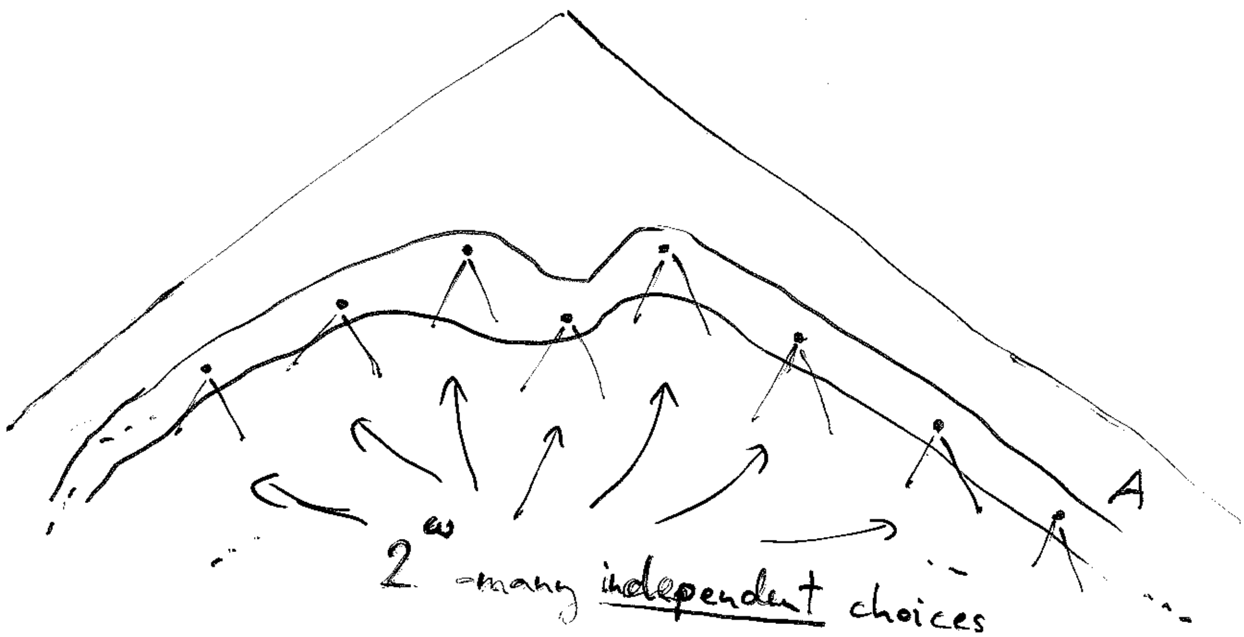
if ϑ_x <u>uniquely determines</u> X on the subtree / if it does not

* The set of D-nodes for X is <u>prefix-closed</u>

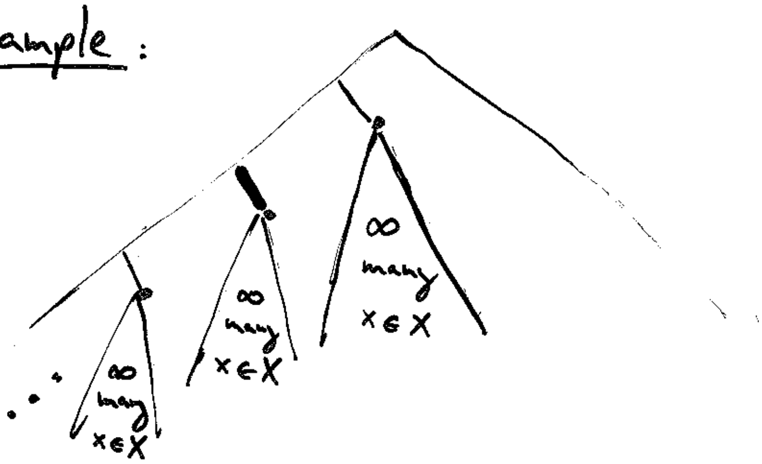
case A (antichain) :

if there is an X satisfying $\varphi(X, \vec{z})$ s.t.
there is an infinite antichain A of D-nodes for X

$\Rightarrow$ there are 2^{ω}-many X satisfying $\varphi(X, \vec{z})$



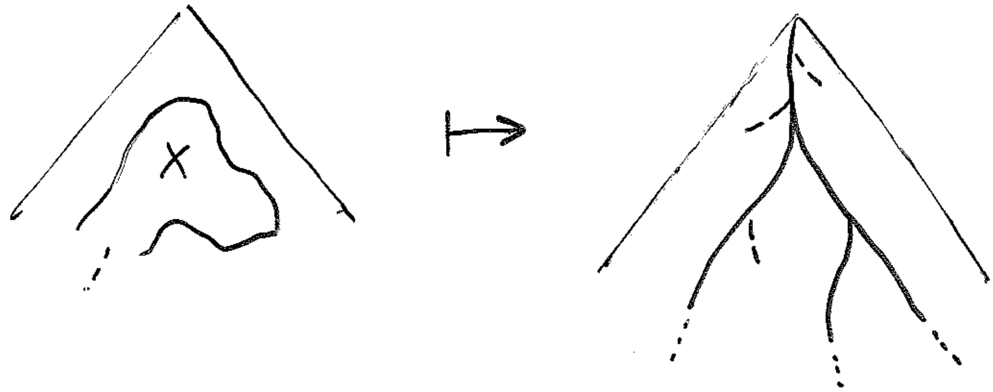
Example :



<u>assuming case A does not hold</u>:

We can associate (definably) to every X

<u>finitely many</u> infinite D-paths (paths of D-nodes)



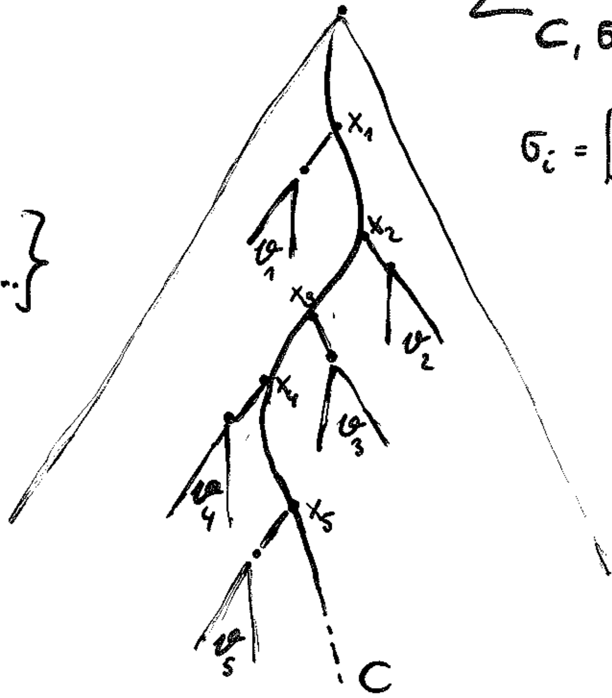
case C (chain):

if there is an X sat. $\varphi(X, \vec{Z})$ and
an infinite D-path C assoc. to X

composition

$$\vartheta = \sum_{C,\sigma} \vartheta_i$$

$$\sigma_i = [\![x_i \in X]\!]$$



such that the ω-language

$$L = \{ (\vartheta_i, \sigma_i)_i \mid \vartheta = \sum_{C,\sigma} \vartheta_i \ldots \}$$

is uncountable (then 2^ω)

$\Rightarrow$ there are 2^ω-many
X satisfying $\varphi(X, \vec{Z})$

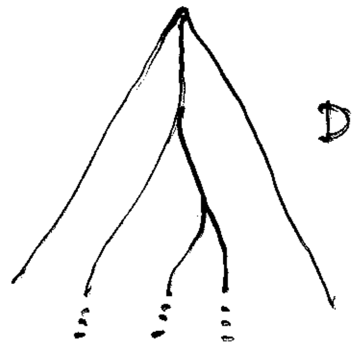
* this case corresponds to [Kuske-Lohrey '06]:
$\exists^{>\omega}$-quantifier over ω-words

$\Rightarrow$ condition C is also MSO-expressible

Example: $\varphi(X)$ = every subtree along the left-most path is either contained in X or is disjoint from X

assume neither case A nor case C hold

$\Rightarrow$ the mapping

$X \longmapsto$  D-paths of X

is <u>countable-to-one</u>

$$\Rightarrow \quad \exists^{>\omega} X.\ \varphi(X, \vec{Z}) \iff \exists^{>\omega} C.\ \text{“}C \text{ an infinite path”} \wedge \exists X.\ \text{“}C \text{ is a D-path for } X\text{”} \wedge \varphi(X, \vec{Z})$$

<u>claim</u>: $\{ C \mid C \text{ an infinite path and } \varphi(C, \vec{Z}) \}$

seen as a subset of $\{0,1\}^{\mathbb{N}}$ (or of $\mathbb{N}^{\mathbb{N}}$)

is analytic, i.e. Σ^1_1

(Suslin 1916 $\Rightarrow$ is in fact Borel)

Suslin 1916 $\Rightarrow$ it is uncountable $\iff$ it <u>contains a perfect subset</u>

($\Rightarrow$ has cardinality 2^{ω})

this we can express in MSO

QED

Remarks

* same idea works for countable linear orders

<u>Thm</u>: there is an effective reduction
$\mathrm{MSO} + \exists^{>\omega} + \exists^{\infty} \ni \varphi \mapsto \varphi^* \in \mathrm{MSO}$
such that uniformly over any countable lin. ord. $\mathcal{L}$:
$\mathcal{L} \models \varphi \iff \mathcal{L} \models \varphi^*$

*

* $\mathrm{MSO} + \exists^{\infty} \rightsquigarrow \mathrm{MSO}$ + "X is finite" via König
$\rightsquigarrow \mathrm{MSO}$ if and how finiteness is definable

* on infinitely branching trees
our technique gives a reduction
$\mathrm{MSO} + \exists^{>\omega} + \exists^{\infty} \rightsquigarrow \mathrm{MSO}$ + "X is finite"

* this is best possible, since finiteness
is expressible using either $\exists^{\infty}$ or $\exists^{>\omega}$

this is joint work with Ł. Kaiser and A. Rabinovich

with thanks to S. Rubin

$\exists^{\infty}, \exists^{>\omega}, \exists^{\text{mod}}$ over $(\mathbb{N}, \text{succ})$ modulo an MSO-definable equivalence

jointly with Ł. Kaiser and S. Rubin

Main Technical Lemma:

given $\mathcal{E}(X,Y)$ defining an equiv., $\varphi(X,\vec{Z})$ in MSO
there is a computable C s.t.
for all $\vec{Z}$ subsets of $\mathbb{N}$ the following are equivalent:

1) there are countably many X mod $\mathcal{E}$ satisfying $\varphi(X,\vec{Z})$

2) $\exists X_1 \dots X_C . \bigwedge_i \varphi(X_i,\vec{Z}) \wedge \forall X. \varphi(X,\vec{Z}) \rightarrow \exists Y. \mathcal{E}(X,Y) \wedge \bigvee_i |Y \triangle X_i|$ fin

Proof uses ω-semigroups, is simple but elusive :-)

Consequences:

* $\text{MSO} + \exists^{\infty} + \exists^{>\omega} + \exists^{\text{mod}}$ modulo $\mathcal{E}$ $\rightarrow$ MSO on $(\mathbb{N}, \text{succ})$

* if $\mathcal{E}$ has only countably many equiv. classes then every class contains an ult. per. (semilinear) set with bounded period

* every countable ω-automatic structure is automatic:

$\mathfrak{A} \leq_{\text{subset-int}} (\mathbb{N}, \text{succ})$, $\mathfrak{A}$ countable

$\Rightarrow \mathfrak{A} \leq_{\text{finite-subset int}} (\mathbb{N}, \text{succ})$