Automatic structures (draft)

Sasha Rubin

email: `rubin@unina.it`

November 23, 2016

Abstract. In order to compute with infinite mathematical structures one needs, typically at least, a finitary description or presentation of the structure. Because of their close relationship with logic, automata are a natural choice for describing structures. They give rise to the automatic structures whose salient feature is that every definable relation is recognised by an automaton. This chapter focuses on foundational questions: What are equivalent ways of defining automatic structures? Under what operations are automatic structures closed? How does one tell if a given structure is automatic? What does it mean for two presentations of the same structure to be different? As prerequisite one should be familiar with logic and automata on infinite objects such as strings and trees.

Contents

1 Introduction

Decidability. Much elementary algebra and geometry can be expressed in the first-order (FO) language of the structure of *real arithmetic*

$$(\mathbb{R}, +, \times, <, +1, =, 0, 1).$$

Formulas in this FO language allow quantification over elements of $\mathbb{R}$, make use of the operations $+, \times$ and $+1$, the relations $<$ and $=$, the constants 0 and 1, and the Boolean connectives (negation, conjunction, etc). Tarski proved that there is an algorithm that decides the truth or falsity of every FO-sentence in this language (a sentence is a formula with no free variables). One says that the FO-theory (i.e. the set of true FO-sentences) of real arithmetic is decidable. Tarski's proof uses a technique called effective quantifier elimination. The approach looks familiar: the statement that a quadratic has two real roots

$$\exists x \exists y [x \neq y \wedge ax^2 + bx + c = 0 \wedge ay^2 + by + c = 0]$$

(written with abbreviations such as x^2 for $x \times x$) can be replaced by the simpler quantifier-free condition $a \neq 0 \wedge 4ac < b^2$.

This technique does more than just prove decidability. It gives some insight into the nature of the sets and relations definable by formulas of the language. Indeed, Tarski's procedure transforms a given formula into an equivalent formula with no quantifiers and the same free variables (and it also provides a proof from the axioms for a real closed field that the two formulas are equivalent). A model-theoretic consequence is that every FO-definable set in real arithmetic is a finite union of intervals with algebraic endpoints. An algorithmic consequence is that one can effectively determine the number of real roots of a given polynomial. Tarski's paper [73] contains a nice discussion of his result and its uses. For an elementary proof see [59].

There are a number of techniques for proving theories decidable and understanding the definable relations. Notable approaches are providing a finite (or computably enumerable) axiomatisation of a theory[1], and the composition method (associated with the names Feferman, Vaught and Shelah) that leverages a decomposition of a structure into pieces whose theories determine (and can be used to compute) the theory of the original structure.

Of course not all theories are decidable. A good heuristic is that if a long unsolved problem is expressible then the theory is probably undecidable. For instance, the FO-theory of the structure of integer arithmetic $(\mathbb{N}, +, \times, <, =, 0, 1)$ is very expressive and highly undecidable.[2] On the other hand the FO-theory $(\mathbb{N}, +, \mathrm{Pr})$ where $\mathrm{Pr} \subset \mathbb{N}$ is the

---

[1]Here a theory is any set of FO-sentences closed under logical deduction. If, in addition to being finitely axiomatisable, the theory is also complete (every sentence or its negation is in the set), then the theory is decidable. Showing completeness is the bread and butter of model theory.

[2]Since a pairing function is definable one can code finite sequences of integers and so express whether or not a given FO-sentence of arithmetic is provable from a given computable set of FO-axioms; in particular, whether or not a given Turing machine halts on a given input.

set of primes is only known to be undecidable under some number theoretic assumptions related to the twin-primes conjecture [8].

Enter automata. Systems of arithmetics provide a hunting ground for questions of decidability. It is reported in [69] that Tarski had asked: is the (weak) monadic second-order theory of $(\mathbb{N}, +1)$ decidable? In (W)MSO there are additional variables that range over (finite) *subsets* of the domain $\mathbb{N}$. Viewing finite subsets of $\mathbb{N}$ as finite binary strings Büchi [16], Elgot [26] and Trahtenbrot [77] (independently) showed that WMSO-formulas of $(\mathbb{N}, +1)$ define regular languages. In slogan form: formulas can be compiled into automata. This establishes that the WMSO-theory of $(\mathbb{N}, +1)$ (called WS1S) is decidable. Indeed, a sentence of the form $\exists X \phi(X)$ holds in $(\mathbb{N}, +1)$ if and only if the automaton for ϕ has non-empty language (itself a decidable property). Viewing arbitrary subsets of $\mathbb{N}$ as infinite binary strings Büchi [17] then introduced ω-string automata in order to settle that the MSO-theory of $(\mathbb{N}, +1)$ (called S1S) is decidable. There is a natural next step.

Definition 1.1. The structure $\mathfrak{T}_r$ has domain $\{0, 1, \cdots, r-1\}^*$ and binary relations suc_i ($i \in \{0, 1, \cdots, r-1\}$) consisting of the pairs (w, wi), where $w \in \{0, 1, \cdots, r-1\}^*$. The (W)MSO theory of $\mathfrak{T}_r$ is called $(W)SrS$.

Note that $\mathfrak{T}_1$ is isomorphic to $(\mathbb{N}, +1)$ (if we view $+1$ as a binary relation rather than a unary function). Büchi [17] asked whether the MSO-theory of $\mathfrak{T}_2$ is decidable. Subsets of $\{0, 1\}^\star$ are naturally viewed as $\{0, 1\}$-labeled trees. Doner [24] and Thatcher and Wright [74] used tree automata to show that WS2S is decidable. In a milestone paper Rabin [63] introduced automata operating on ω-trees to prove that S2S is decidable.

As expected, more than just decidability was established. We have already seen on the way to decidability that definable relations are, modulo a coding, recognised by automata. The converse also holds. We illustrate for ω-tree automata. For a set $A \subseteq \{0, 1\}^\star$ write $\chi(A)$ for its characteristic $\{0, 1\}$-labelled tree (Definition 2.1). For a tuple of trees $(t_1, \cdots, t_n)$ write $\otimes(\bar{t})$ for the convoluted $\{0, 1\}^n$-labelled tree (Definition 2.2).

Theorem 1.1 ([63]). *A set of tuples $(A_1, \cdots, A_n)$ of sets of strings is MSO definable in $(\{0, 1\}^\star, \mathrm{s}_0, \mathrm{s}_1)$ if and only if the set of $\{0, 1\}^n$-labelled trees $\otimes(\chi(A_1), \cdots, \chi(A_n))$ is recognised by an ω-tree automaton. Moreover, the translation between formulas and automata is effective, and it is decidable whether or not a given ω-tree automaton recognises the empty language.*

An accessible proof of this central theorem can be found in [75]. Similar theorems hold for (W)MSO of $\mathfrak{T}_r$. These theorems can be leveraged to deduce many other decidability and definability results. The method of interpretations is one such lever.

Interpretations We are all familiar with interpretations — eg. rational arithmetic can be defined in terms of integer arithmetic. Formally we would say that the structure $(\mathbb{Q}, +, \times, =)$ is FO-interpretable in $(\mathbb{Z}, +, \times, =, 0)$. That is, every rational is coded by (infinitely many) pairs $(a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and, for instance, two codes $(a, b), (c, d)$ represent equal rationals if and only if $a \times d = b \times c$. Interpretations reduce questions about one structure to questions about another. In our setting those questions concern decidability and definability. Let's illustrate with two standard examples.

Corollary 1.2. *Modulo a certain coding of natural numbers into binary strings, every FO-definable relation of $(\mathbb{N}, +)$ is recognised by an automaton. The translation from formulas to automata is effective. Consequently, the FO-theory of $(\mathbb{N}, +)$ is decidable.*[3]

The following argument establishing this corollary can be found in [69, 16, 26]. There is a *finite-set interpretation* of $(\mathbb{N}, +)$ in $(\mathbb{N}, +1)$. The interpretation codes $n \in \mathbb{N}$ as a *finite subset* of $\mathbb{N}$ (e.g., thirteen is coded first in binary as 1101 and then as the set $\{0, 2, 3\}$). This coding has the property that there is a WMSO formula $\phi_+(X_1, X_2, X_3)$ (written in the signature of $(\mathbb{N}, +1)$) expressing that $X_1' + X_2' = X_3'$, where X_i' is the unique number coded by the set X_i. The formula implements the usual bit-carry procedure for addition: it guesses the existence of the carry set and uses the successor $+1$ to scan the sets one place at a time verifying the addition. This allows one to translate every FO-formula $\phi(y_1, \cdots, y_n)$ of $(\mathbb{N}, +)$ into a WMSO-formula $\Phi(Y_1, \cdots, Y_n)$ of $(\mathbb{N}, +1)$ with the property that the two formulas define the same relation modulo the coding. This relation, viewed as one on strings, is recognised by a finite-string automaton (cf. Theorem 1.1). Finally, every FO-formula over $(\mathbb{N}, +)$ defines a relation that, modulo the coding, is recognised by a finite-automaton. Decidability follows from the fact that the problem of whether or not a given finite-string automaton is empty is decidable.

The second example illustrates that an interpretation in which elements are coded by elements (instead of coding elements by sets, as in the first example) results in more expressive theories being decidable (namely, MSO instead of FO).

Corollary 1.3. *Modulo a certain coding of rational numbers into ω-trees, every MSO-definable relation of $(\mathbb{Q}, <)$ is recognised by an automaton. The translation from formulas to automata is effective. Consequently, the MSO-theory of $(\mathbb{Q}, <)$ is decidable.*

To see this, we follow Rabin [63] who provides a *monadic second-order interpretation* of $(\mathbb{Q}, <)$ in $\mathfrak{T}_2$, which is a restricted type of set-interpretation in which every rational is coded by an *element* of $\{0, 1\}^\star$. First consider the binary relation $\prec$ on $\mathfrak{T}_2$ where $x \prec y$ if $(x \sqcap y)0$ is a prefix of x or $(x \sqcap y)1$ is a prefix of y (here $x \sqcap y$ is the longest common prefix of x and y). Note that $(\{0, 1\}^\star, \prec)$ is a countable dense linear order with no endpoints; so it is isomorphic to $(\mathbb{Q}, <)$. Since the relation $\prec$ is MSO-definable in $\mathfrak{T}_2$ one can translate every MSO-formula $\phi(X_1, \cdots, X_n)$ of $(\mathbb{Q}, <)$ into an MSO-formula $\phi'(X_1, \cdots, X_n)$ of $\mathfrak{T}_2$ with the property that the two formulas define the same relation modulo the isomorphism. Theorem 1.1 ensures that every MSO-definable relation of $(\mathbb{Q}, <)$, viewed as a relation on ω-trees, is recognised by an ω-tree automaton. Decidability follows from the fact that the problem of whether or not a given ω-tree automaton is empty is decidable.

In summary, suppose $\mathcal{A}$ has decidable MSO. If $\mathcal{B}$ is MSO-interpretable in $\mathcal{A}$ then $\mathcal{B}$ has decidable MSO; and if $\mathcal{B}$ is set-interpretable in $\mathcal{A}$ then $\mathcal{B}$ has decidable FO-theory. These examples suggest the following definition.

Automatic Structures   A structure that is set-interpretable in $\mathfrak{T}_2$ is called *Rabin-automatic* (cf. Definition 3.2); for instance $(\mathbb{N}, +)$ is Rabin-automatic, and not only is $(\mathbb{Q}, <)$ Rabin-automatic but also $(\mathcal{P}(\mathbb{Q}), \subset, <)$ (see definition 3.1). The fact that formulas can be com-

---

[3]The FO-theory of $(\mathbb{N}, +)$ is called *Presburger arithmetic* after M. Presburger who showed it decidable using effective quantifier elimination [62].

piled into automata gives the *fundamental theorem of automatic structures* (Theorem 3.3), a fact exploited to prove many results about automatic structures:

- Modulo coding, every FO-definable relation in a Rabin-automatic structure is recognised by an ω-tree automaton.
- The FO-theory of every Rabin-automatic structure is decidable.

Here is an outline of the types of problems discussed in this chapter.

(1) To what extent can we increase the expressive power of the logical language (FO) and still retain decidability of automatic structures?
(2) Which operations on structures preserve being automatic?
(3) What do automatic structures look like? Meaningful answers can be used to show that a given structure is not automatic.
(4) An automatic structure may be set-interpretable in a number of ways. How are these ways different? This is the problem of studying non-equivalent automatic presentations.

Other accounts and omissions. The introduction defines automatic structures in terms of interpretations, following [11, 13, 20], rather than directly in terms of automata.

Automatic structures (over finite and infinite strings) were introduced in [34, 35] for proving decidability. They were independently introduced in [43] where the motivation stresses that automatic structures are part of computable model theory, a subject that looks at the effective content of mathematical statements. Here one asks for the automatic content of standard theorems from mathematics such as Kőnig's tree lemma, Ramsey's partition results and Cantor's results on countable linear orders, see [72, 47, 49, 55, 39]. Besides many foundational contributions, the thesis [11] began the study of automatic structures with automata operating over finite and infinite trees.

The complexity of automatic structures can be measured in a number of ways: the computational complexity of their theories (including query evaluation and model checking problems) can be found in [13, 48, 56, 5, 57]; complexity measures such as Scott rank, Cantor-Bendixson rank and ordinal height of automatic structures can be found in [45, 42, 37, 38]; the complexity of the isomorphism problem for classes of automatic structures can be found in [72, 51, 50, 52]. For model theoretic considerations (quantifier-elimination, VC-dimension, axiomatisations, non-standard models) see [11, 10, 5]. Useful resources regarding the relationship between automatic structures and other finitary formalism (automatic groups, numeration systems, pushdown graphs, etc.) can be found in [13, 5, 40]. The study of whether one can decide if a tree-automatic structure is string-automatic began in [36].

See [31] for a treatment of decision problems for MSO via automata as well as via composition. An equivalent way of identifying MSO and automata is that strings or trees are themselves considered structures (over a certain natural signature) and the regular languages correspond to sets of structures satisfying MSO sentences. This is usually discussed in the context of finite model theory, see for instance [25]. See [64] for FO decision problems via standard techniques (quantifier elimination and completeness considerations).

bringing my attention to Frederic Reinhardt's Diploma Thesis [68]; Wolfgang Thomas and Anil Nerode for clearing up some historical questions; Achim Blumensath for serving as friendly referee; and Jean-Eric Pin for his patient encouragement while writing this chapter.

2 Logic, automata and interpretations

2.1 Logical languages

A *structure* $\mathfrak{A} = (\mathcal{A}, R_1, \cdots, R_N)$ consists of a set $\mathcal{A}$ called the *domain*, and relations[4] $R_i \subseteq \mathcal{A}^{r_i}$. The names of the R_i together with their arities $r_i \in \mathbb{N}$ form the *signature* of the structure. If $\mathfrak{B}$ is the restriction of $\mathfrak{A}$ to some set $\mathcal{B} \subseteq \mathcal{A}$ then write $\mathfrak{B} \subseteq \mathfrak{A}$ and say that $\mathfrak{B}$ is a *substructure* of $\mathfrak{A}$.

Example 2.1. The structure $\mathfrak{T}_r$ has domain $\{0, 1, \cdots, r-1\}^\star$ and for each i from $\{0, 1, \cdots, r-1\}$ a binary relation suc_i consisting of pairs (w, wi). Note that $\mathfrak{T}_1$ is isomorphic to $(\mathbb{N}, +1)$.

We write $(\mathfrak{A}, P)$ for the structure $\mathfrak{A}$ expanded by the predicate P. To express properties of structures we need a logical language. Formulas of monadic second-order logic (MSO) are constructed using logical connectives ('and', 'or', 'not'), individual variables x, y, z (that are intended to range over elements of the domain), set variables $X, Y, Z, \cdots$ (that are intended to range over subsets of the domain), quantification over these variables, and the subset $X \subseteq Y$ and membership $x \in X$ relations, and finally they may use the names of the relations such as $R(x, y)$ from a fixed signature.

Formulas of weak monadic second-order logic (WMSO) are defined as for MSO except that set variables are intended to range over finite subsets of the domain. Formulas of first-order logic (FO) are defined as for MSO but without set variables. A formula written $\phi(X_1, \cdots, X_n, x_1, \cdots, x_m)$ means that ϕ's free variables are included in the set $\{X_1, \cdots, X_n, x_1, \cdots, x_m\}$. A *sentence* is a formula without free variables. Here is a formula of two free individual variables x, y in the signature of the structure $(\mathbb{N}, +1)$:

$$(\forall Z)[x \in Z \wedge (\forall z)(z \in Z \implies z+1 \in Z) \implies y \in Z].$$

We can see that it is satisfied by those pairs of natural numbers (x, y) such that $x \leqslant y$. One can similarly define the prefix relation $\prec_{\texttt{prefix}}$ in $\mathfrak{T}_2$. We often write (W)MSO formulas as $\phi(\overline{X})$ even if some of the free variables are individual variables.

Note that we are appealing to our natural sense of what it means for a sentence to be true of a structure. For a rigourous definition of truth and satisfaction in mathematical logic see, for instance, [14]. We will use the shorthand $\mathfrak{A} \models \Phi$ (read $\mathfrak{A}$ models Φ) to mean that the sentence Φ is true in $\mathfrak{A}$. Two formulas $\phi(\overline{X})$ and $\psi(\overline{Y})$ are *equivalent over* $\mathfrak{A}$ if

$$\mathfrak{A} \models \forall \overline{X} \forall \overline{Y} [\phi(\overline{X}) \iff \psi(\overline{Y})].$$

---

[4]We deal with relational structures. This is no real handicap since we can replace an operation by its graph. For instance, addition is taken as those triples (x, y, z) such that $x + y = z$.

An MSO-formula $\phi(\overline{X}, \overline{x})$, in the signature of $\mathfrak{A}$, *defines* the relation

$$\phi^{\mathfrak{A}} := \{(A_1, \cdots, A_k, a_1, \cdots, a_n) \mid \mathfrak{A} \models \phi(\overline{A}, \overline{a}), A_i \subseteq \mathcal{A}, a_j \in \mathcal{A}\}.$$

A central problem in mathematical logic has been establishing the (non-)decidability of theories. Let $\mathcal{L}$ be one of MSO, WMSO, or FO. The $\mathcal{L}$-*theory* of a structure is the set of $\mathcal{L}$-sentences true in that structure. A set X of sentences is *decidable* if there is an algorithm that correctly decides, given a sentence ϕ in the language of $\mathcal{L}$, whether or not $\phi \in X$. We say that a structure has *decidable $\mathcal{L}$-theory* if its $\mathcal{L}$-theory is decidable.

2.2 Rabin's theorem

The Büchi/Elgot/Trahtenbrot revolution led to increasingly complex structures in which definable relations correspond with some type of automata. The cornerstone is Rabin's theorem: MSO definability in $(\{0,1\}^\star, s_0, s_1)$ coincides with recognisability by ω-tree automata. Tree automata operate on *(binary) A-labeled trees* $T : \{0,1\}^\star \to A$. For a definition of Rabin tree-automaton see [75]. We code tuples of sets as ω-trees in two steps. In the first step code a set as a tree.

Definition 2.1 (characteristic ω-tree). For a set $Y \subseteq \{0,1\}^\star$ define its *characteristic tree* χ_Y as the $\{0,1\}$-labeled ω-tree with a 1 in position $w \in \{0,1\}^\star$ if and only if $w \in Y$.

Second, code a tuple of trees as a single tree by laying the tuples alongside each other.

Definition 2.2 (convoluting ω-trees). Let $\overline{t} = (t_1, \cdots, t_k)$ be a k-tuple of $\{0,1\}$-labelled ω-trees. The *convolution* $\otimes(\overline{t})$ is the $\{0,1\}^k$-labelled ω-tree such that for all positions $w \in \{0,1\}^\star$ the ith component of $\otimes(\overline{t})(w)$ is equal to $t_i(w)$.

Theorem 2.1 (Rabin's theorem [75]). *For each MSO-formula $\phi(\overline{X})$ in the signature of $\mathfrak{T}_2$ there is an ω-tree automaton (and vice-versa) such that the language recognised by the automaton is*

$$\{\otimes(\chi_{X_1}, \cdots, \chi_{X_k}) \mid \mathfrak{T}_2 \models \phi(\overline{X})\}.$$

The translations are effective.

Proposition 2.2. *[75] The emptiness problem for Rabin-automata is decidable and consequently the MSO-theory of $\mathfrak{T}_2$ is decidable. Moreover, there is an effective procedure that given a non-empty automaton M produces (a finite presentation of) a regular ω-tree accepted by M (this fact is called Rabin's basis theorem).*

Remark. Similar results hold with $\mathfrak{T}_1$ replacing $\mathfrak{T}_2$ and are know as Büchi's theorem(s). These can be proven directly or as corollaries by coding $X \subseteq \mathbb{N}$ by the tree T with $T^{-1}(1) = \{0^n \mid n \in X\}$. Similar results also hold for WMSO. The standard reference is [75].

Rabin's theorem with additional set quantifiers We show that we can enrich MSO by certain set quantifiers (such as 'there are finitely many sets X such that . . .') and still get

decidability for $\mathfrak{T}_2$. We do this by showing that formulas with the additional quantifiers are actually equivalent to vanilla MSO formulas.

Lemma 2.3. *The property 'X is finite' is* MSO-*definable in* $\mathfrak{T}_2$.

Proof. The following simple argument is from [63]. The lexicographic (total) ordering $<_{\mathrm{lex}}$ on $\{0,1\}^\star$ is MSO-definable in $\mathfrak{T}_2$. Thus '$X \subseteq \{0,1\}^\star$ is finite' is definable by the formula that says that every $B \subseteq X$ has both a maximum and minimum element with respect to $<_{\mathrm{lex}}$.   □

Remark. So every WMSO-definable relation R of $\mathfrak{T}_2$ is MSO-definable in $\mathfrak{T}_2$ (simply relativise the set quantifiers). In particular the following results also hold with WMSO replacing MSO.

For a cardinal κ let $\exists^{\geqslant \kappa}$ denote the quantifier 'there exists at least κ many sets X such that . . .'. Write MSO($\exists^{\geqslant \kappa}$) for MSO enriched by the quantifier $\exists^{\geqslant \kappa}$.

Proposition 2.4. *[6] For every* MSO($\exists^{\geqslant \aleph_0}$) *formula* $\phi(\overline{X})$ *there is an* MSO *formula* $\phi'(\overline{X})$ *equivalent to* $\phi(\overline{X})$ *over* $\mathfrak{T}_2$.

Proof. The following are equivalent:
  (1) There are only finitely many X satisfying $\phi(X, \overline{Y})$.
  (2) There is a finite set Z such that every pair of different sets X_1, X_2 which both satisfy $\phi(X_i, \overline{Y})$ differ on Z.
The second condition can be expressed in MSO using lemma 2.3.   □

Theorem 2.5. *[6] For every* MSO($\exists^{\geqslant \aleph_1}$) *formula* $\phi(\overline{X})$ *there is an* MSO *formula* $\phi'(\overline{X})$ *equivalent to* $\phi(\overline{X})$ *over* $\mathfrak{T}_2$.

Proof. The proof uses the composition method and basic ideas from descriptive set theory. We sketch a proof of the simpler case of $\mathfrak{T}_1$ following [53]. Say that two subsets of $\mathbb{N}$ have the *same end* if their symmetric difference is finite. There is a constant K (that depends only on ϕ) such that the following are equivalent for all $\overline{Y}$:

  (1) There are uncountably many X satisfying $\phi(X, \overline{Y})$.
  (2) There are K many sets X, each satisfying $\phi(X, \overline{Y})$, and that pairwise have different ends.

The second condition can be expressed in MSO. We argue correctness. The forward direction follows since each end class is countable. For the reverse let K be larger than the number of states of an ω-automaton for ϕ. The idea is that if there are too many sets with different ends we can find two that behave the same and so shuffle these to get uncountably many. For ease of writing assume that $\overline{Y}$ is a singleton and so write Y instead. For a set X_i satisfying $\phi(X_i, Y)$ write ρ_i for some accepting run of the automaton on $\otimes(\chi_{X_i}, \chi_Y)$. There are two sets, say X_1 and X_2, and an infinite set $H \subset \mathbb{N}$ such that $\rho_1[n] = \rho_2[n]$ for all $n \in H$ (otherwise from some point on all pairs of runs disagree contradicting that the automaton has $< K$ states). Without loss we can assume, by passing to an infinite subset if required, that for all $n < m \in H$ with no element of H between them, both $\rho_1[n,m]$ and $\rho_2[n,m]$ mention final states and $\chi_{X_1}[n, m-1] \neq \chi_{X_2}[n, m-1]$.

List H as $h_1 < h_2 < h_3 < \cdots$. By knitting segments of the runs we see that the automaton accepts every string of the form $\otimes(\chi_{X_1}[0, h_1 - 1]z_1 z_2 z_3 \cdots, \chi_Y)$ where $z_n \in \{\chi_{X_1}[h_n, h_{n+1} - 1], \chi_{X_2}[h_n, h_{n+1} - 1]\}$. This gives uncountably many distinct sets X satisfying $\phi(X, Y)$. $\square$

Write $\mathrm{MSO}(\exists^{\mathrm{mod}})$ for MSO enriched by all quantifiers parameterised by $k, m \in \mathbb{N}$ of the form 'exists a set X, whose cardinality is congruent to k modulo m, such that ...'. The proof of the following theorem can be adapted from [44] or [53].

Theorem 2.6. *For every* $\mathrm{MSO}(\exists^{\mathrm{mod}})$ *formula* $\phi(\overline{X})$ *there is an* MSO *formula* $\phi'(\overline{X})$ *equivalent to* $\phi(\overline{X})$ *over* $\mathfrak{T}_2$.

Remark 2.7. Since $\mathfrak{T}_r$ (for $r < \omega$) is MSO-interpretable in $\mathfrak{T}_2$, the results above hold, for instance, with $\mathfrak{T}_1$ replacing $\mathfrak{T}_2$.

2.3 Interpretations

A good reference for interpretations is [33]. Let $\mathcal{L}$ denote FO or MSO or WMSO. Let $\mathcal{I} = (\delta, \Phi_1, \cdots, \Phi_N)$ be $\mathcal{L}$-formulas in the signature of a structure $\mathfrak{B}$ in which the free variables are individual variables. Suppose that δ has 1 free variable and Φ_i has r_i free variables. If $\Phi_i^{\mathfrak{B}}$ are relations over $\delta^{\mathfrak{B}}$ then we can define the structure $\mathcal{I}(\mathfrak{B}) := (\delta^{\mathfrak{B}}, \Phi_1^{\mathfrak{B}}, \cdots, \Phi_N^{\mathfrak{B}})$. This structure is said to be $\mathcal{L}$-*definable in* $\mathfrak{B}$. The tuple $\mathcal{I}$ is called an $\mathcal{L}$-*definition*.

Remark. We have overloaded the phrase '$\mathcal{L}$-definable': here one structure is definable in another; and the earlier meaning is that a relation is definable in a structure.

The following lemma says, in particular, that every relation FO-definable in $\mathcal{I}(\mathfrak{B})$ is FO-definable in $\mathfrak{B}$.

Lemma 2.8. *Fix an* $\mathcal{L}$-*definition* $\mathcal{I}$ *and an* $\mathcal{L}$-*formula* $\Phi(x_1, \cdots, x_k)$ *in the signature of* $\mathcal{I}(\mathfrak{B})$. *There is an* $\mathcal{L}$-*formula* $\Phi_{\mathcal{I}}(x_1, \cdots, x_k)$ *in the signature of* $\mathfrak{B}$ *such that for all elements* b_i *of* $\delta^{\mathfrak{B}}$,

$$\mathcal{I}(\mathfrak{B}) \models \Phi(b_1, \cdots, b_k) \text{ if and only if } \mathfrak{B} \models \Phi_{\mathcal{I}}(b_1, \cdots, b_k).$$

Proof. The idea is to relativise all quantifiers to δ and replace the ith atomic formula R_i in the signature of $\mathcal{I}(\mathfrak{B})$ by Φ_i. Formally, define $\Phi_{\mathcal{I}}$ inductively by $(R_i)_{\mathcal{I}} := \Phi_i$ and for the other cases:

$$
\begin{aligned}
(\Psi \wedge \Xi)_{\mathcal{I}} &:= \Psi_{\mathcal{I}} \wedge \Xi_{\mathcal{I}} & (\neg \Psi)_{\mathcal{I}} &:= \neg \Psi_{\mathcal{I}} \\
(\exists x_i \Psi)_{\mathcal{I}} &:= \exists x_i [\delta(x_i) \wedge \Psi_{\mathcal{I}}] & (\exists X_i \Psi)_{\mathcal{I}} &:= \exists X_i [(\forall x \in X_i \delta(x)) \wedge \Psi_{\mathcal{I}}] \\
(x \in X)_{\mathcal{I}} &:= (x \in X) & (X \subseteq Y)_{\mathcal{I}} &:= (X \subseteq Y).
\end{aligned}
$$

$\square$

Definition 2.3. Let $\mathcal{I}$ be an $\mathcal{L}$-definition. If $\mathfrak{A}$ is isomorphic to $\mathcal{I}(\mathfrak{B})$, say via f, then say that $\mathfrak{A}$ *is* $\mathcal{L}$-*interpretable in* $\mathfrak{B}$ *via co-ordinate map* f.

Proposition 2.9. *Suppose that $\mathfrak{A}$ is $\mathcal{L}$-interpretable in a structure with decidable $\mathcal{L}$-theory. Then $\mathfrak{A}$ has decidable $\mathcal{L}$-theory.*

Proof. For a sentence ϕ of $\mathfrak{A}$, lemma 2.8 produces a sentence $\phi^{\mathcal{I}}$ preserving truth. Apply the given decision procedure to $\phi^{\mathcal{I}}$. $\square$

For example for $r < \omega$ the structure $\mathfrak{T}_r$ is MSO-interpretable in $\mathfrak{T}_2$ and consequently the MSO-theory of $\mathfrak{T}_r$ is decidable.

There are other types of interpretations that could be introduced here, namely (finite)-set interpretations [27] [20]. For the sake of filling in undefined notions in the introduction we define a *(finite)-set interpretation* of $\mathfrak{A}$ in $\mathfrak{B}$ to be like a (W)MSO-interpretation except that the free variables are (finite) set variables. Thus elements of $\mathfrak{A}$ are coded by (finite) subsets of the domain of $\mathfrak{B}$.

3 Automatic structures

3.1 Rabin-automatic structures

MSO-definability can be thought of as FO-definability in a power structure. We use this idea to define the automatic structures. Let $\mathcal{P}(X)$ denote the set of subsets of X.

Definition 3.1. [20] The *power* of a structure $\mathfrak{A} = (\mathcal{A}, R_1, \cdots, R_N)$ is the structure

$$\mathcal{P}[\mathfrak{A}] := (\mathcal{P}(\mathcal{A}), R'_1, \cdots, R'_N, \subseteq)$$

where

$$R'_i := \{(\{x_1\}, \cdots, \{x_{r_i}\}) \mid (x_1, \cdots, x_{r_i}) \in R_i\}.$$

For example, $\mathcal{P}[(\mathbb{N}, +1)]$ is the structure with domain $\mathcal{P}(\mathbb{N})$, the subset relation $\subseteq$, and the binary relation $\{(\{n\}, \{n+1\}) \mid n \in \mathbb{N}\}$. The following says that FO definability in the power of a structure is the same as MSO definability in the structure.

Proposition 3.1. *For every FO-formula $\phi(\overline{x})$ there is an MSO-formula $\Psi(\overline{X})$ (and vice versa) such that for all structures $\mathfrak{A}$ and all $U_i \in \mathcal{P}(\mathcal{A})$*

$$\mathcal{P}[\mathfrak{A}] \models \phi(\overline{U}) \text{ if and only if } \mathfrak{A} \models \Psi(\overline{U}).$$

As a consequence we can transfer MSO-decidability of $\mathfrak{T}_r$ to FO-decidability of the power structure.

Corollary 3.2. *The FO-theory of $\mathcal{P}[\mathfrak{T}_r]$ is decidable.*

Since FO-interpretations preserve decidability we make the following definition.

Definition 3.2. [11] A structure FO-interpretable in $\mathcal{P}[\mathfrak{T}_2]$ is called *Rabin-automatic* or *ω-tree automatic*. The collection of these structures is written $\omega\texttt{T-AutStr}$.

Note that any structure that is FO bi-interpretable with $\mathcal{P}[\mathfrak{T}_2]$ could be used in this definition. For instance we can use the following structure [11]: the domain consists of all finite and infinite trees[5] and the atomic relations are

$$\preceq_{\texttt{ext}}, \equiv_{\text{dom}}, (\text{suc}_a^d)_{a \in \{0,1\}}^{d \in \{0,1\}}, \epsilon_0, \epsilon_1, \mathcal{F}$$

where $T \preceq_{\texttt{ext}} S$ if $\text{dom}(T) \subseteq \text{dom}(S)$ and $S(w) = T(w)$ for $w \in \text{dom}(T)$; $T \equiv_{\text{dom}} S$ if $\text{dom}(T) = \text{dom}(S)$; $\text{suc}_a^d(T) = S$ if the finite tree S is formed from the finite tree T by extending its leaves in direction d and labeling each new such node by a; ϵ_a is the tree with domain $\{\lambda\}$ labelled a; and $\mathcal{F}$ is the set of finite trees.

The elements of the Rabin-automatic structure are naturally viewed as ω-trees.

Definition 3.3. For a Rabin-automatic structure $\mathfrak{A}$ (isomorphic via f to a FO-definable structure $\mathcal{I}(\mathcal{P}[\mathfrak{T}_2])$) and any relation $R \subseteq \mathcal{A}^k$, denote by $\text{code}(R)$ the set of ω-trees

$$\{\otimes(\chi_{f(a_1)}, \cdots, \chi_{f(a_k)}) \mid (a_1, \cdots, a_k) \in R\}.$$

The following is called the *fundamental theorem of automatic structures* and says that FO-definable relations in Rabin-automatic structures are, modulo coding, regular.

Theorem 3.3 (Fundamental theorem, cf. [43, 13]). *Let $\mathfrak{A}$ be Rabin-automatic.*

(1) For every first-order definable relation R in $\mathfrak{A}$ the set of trees $\text{code}(R)$ is recognised by an ω-tree automaton.

(2) The first-order theory of a Rabin-automatic structure is decidable.

Proof. For the first item apply Proposition 3.1 to get that $f(R)$, being FO-definable in $\mathcal{P}[\mathfrak{T}_2]$, is MSO definable in $\mathfrak{T}_2$. Now apply Rabin's theorem. For the second item use the fact that it is decidable whether or not an ω-tree automaton accepts some tree. $\square$

Since formulas are seen as automata, we may view a Rabin-automatic structure as being presented by automata.

Definition 3.4 (ω-tree automatic presentation [13]). Suppose that $f : \mathfrak{A} \simeq (B, S_1, \cdots, S_N)$ and

(1) the elements of B are $\{0, 1\}$-labeled trees;

(2) the set B is recognised by an ω-tree automaton, say M_B;

(3) the set $\{\otimes(\overline{t}) \mid \overline{t} \in S_i\}$ is recognised by an ω-tree automaton, say M_i, for $i \leqslant N$.

Then the data $\langle (M_B, M_1, \cdots, M_N), f \rangle$ is called a ω-*tree automatic presentation* of $\mathfrak{A}$.

The following characterisation is typically taken as a definition of a structure being Rabin-automatic.

Proposition 3.4 (Machine theoretic characterisation [13]). *A structure $\mathfrak{A}$ is FO-interpretable in $\mathcal{P}[\mathfrak{T}_2]$ if and only if $\mathfrak{A}$ has an ω-tree automatic presentation.*

---

[5]Here a tree is a function T from a prefix-closed subset of $\{0, 1\}^\star$ to $\{0, 1\}$.

Proof. The forward direction follows from the fundamental theorem 3.3. For the reverse direction convert regular sets and relations in the presentation into MSO formulas over $\mathfrak{T}_2$ (using Rabin's theorem) and then into FO-formulas using proposition 3.1.   □

Example 3.1. The following structures are Rabin-automatic.

(1) The power structure $\mathcal{P}[\mathfrak{T}_r]$ $(r \geqslant 1)$ as well as its substructure $\mathcal{P}_f[\mathfrak{T}_r]$ whose domain consists of the finite subsets of $\mathfrak{T}_r$.

(2) The power structure of the ordering of the rationals, namely $\mathcal{P}[(\mathbb{Q}, <)]$.

(3) Presburger arithmetic $(\mathbb{N}, +)$.

(4) Skolem arithmetic $(\mathbb{N}, \times)$.

(5) The structure $(\mathcal{P}(\mathbb{N}), \subseteq, =^*)$ where $X =^* Y$ means that X and Y have finite symmetric difference.

(6) Every ordinal $(\alpha, <)$ where $\alpha < \omega^{\omega^{\omega}}$.

Extension of the fundamental theorem   We now rephrase the results on additional quantifiers on MSO into FO. We overload notation so that, for example, $\exists^{\kappa}$ denotes the quantifier 'there exists at least κ many individual elements x such that'. The following theorem is now immediate. Instances of it can be found in [13, 44, 53].

Theorem 3.5 (Extension of fundamental theorem). *Let $\mathfrak{A}$ be Rabin-automatic.*

(1) For every $\mathrm{FO}(\exists^{\geqslant \aleph_0}, \exists^{\geqslant \aleph_1}, \exists^{\mathrm{mod}})$*-definable relation R in $\mathfrak{A}$ the set $\mathrm{code}(R)$ is recognised by an ω-tree automaton.*

(2) The $\mathrm{FO}(\exists^{\geqslant \aleph_0}, \exists^{\geqslant \aleph_1}, \exists^{\mathrm{mod}})$ *theory of $\mathfrak{A}$ is decidable.*

How far can we push this? First we need a rigourous definition of quantifier. This is neatly provided by Lindström's definition of 'generalised quantifier', see [58]. We don't have a clear picture of those generalised quantifiers that can be added to FO and still get the properties as in Theorem 3.5. However here is a special case. Define the *cardinality quantifier parameterised by C*, for C a class of cardinals, as 'there exists exactly α many elements x such that ..., where $\alpha \in C$'. Examples include $\exists, \exists^{\mathrm{mod}}, \exists^{\geqslant \kappa}$, and 'there exist a prime number of elements such that ...'.

It turns out that the only cardinality quantifiers we can add to FO and still get the fundamental theorem are, essentially, the ones mentioned in Theorem 3.5.

Theorem 3.6 (cf. [72]). *Let Q_C be a cardinality quantifier parameterised by C. Suppose for every $\mathfrak{A} \in \omega\mathtt{T\text{-}AutStr}$ and every $\mathrm{FO}(Q)$-definable relation R in $\mathfrak{A}$, the set $\mathrm{code}(R)$ is recognised by a Rabin-automaton. Then every $\mathrm{FO}(Q)$-definable relation is already $\mathrm{FO}(\exists^{\geqslant \aleph_0}, \exists^{\geqslant \aleph_1}, \exists^{\mathrm{mod}})$-definable in $\mathfrak{A}$.*

Proof. We illustrate the proof for a set $C \subset \mathbb{N}$ of finite cardinals. Consider the Rabin-automatic presentation of $\mathfrak{A} := (\mathbb{N}, \leqslant)$ in which $n \in \mathbb{N}$ is coded by the set $\{0^n\} \subset \{0, 1\}^*$. Since the set $C \subset \mathbb{N}$ is $\mathrm{FO}(Q_C)$-definable in $\mathfrak{A}$, $\mathrm{code}(C)$ is recognised by a tree-automaton. But the trees in $\mathrm{code}(C)$ are essentially unary strings and so the language $\mathrm{code}(C)$ is ultimately periodic. So C is already $\mathrm{FO}(\exists^{\mathrm{mod}})$-definable in $\mathfrak{A}$.   □

What about extensions of FO by set quantification? Unfortunately WMSO is too much to hope for. Since the configuration graph $\mathfrak{G}$ of a Turing machine (with the single-transition edge relation) is automatic, and reachability is expressible in WMSO, the halting problem reduces to the WMSO-theory of a certain $\mathfrak{G}$. It is a research programme to understand which quantifiers can be added to automatic structures and retain decidability, see [54, 71].

3.2 Other classes of automatic structures

We introduce subclasses of $\omega\texttt{T-AutStr}$ related to automata on finite strings/trees and infinite strings. Each has a machine theoretic characterisation as in Definition 3.4. A member of any of the four classes is said to be *automatic*.

Finite-string automatic structures   Recall $\mathcal{P}_f(\mathfrak{A})$ is the substructure of $\mathcal{P}(\mathfrak{A})$ restricted to the finite subsets of $\mathcal{A}$.

Definition 3.5. A structure is called *finite-string automatic* if it is FO-interpretable in $\mathcal{P}_f[\mathfrak{T}_1]$. This collection of structures is written $\texttt{S-AutStr}$.

We have seen in Corollary 1.2 that $(\mathbb{N}, +)$ is *finite-string automatic*. Note that such structures have countable domain.

Example 3.2. The following structures are bi-interpretable with $\mathcal{P}_f[\mathfrak{T}_1]$:
  (1) For $|\Sigma| \geqslant 2$, $(\Sigma^*, \{\sigma_a\}_{a\in\Sigma}, \prec_{\texttt{prefix}}, \text{el})$ where σ_a holds on pairs (w, wa), el holds on pairs (u, v) such that $|u| = |v|$, and $\prec_{\texttt{prefix}}$ is the prefix relation.
  (2) For $k \geqslant 2$, $(\mathbb{N}, +, |_k)$ where $|_k$ is the binary relation on $\mathbb{N}$ with $x|_k y$ if x is a power of k and x divides y.

The following definition turns a finite set Y into a finite string χ_Y as in $\chi_{\{1,3\}} = 0101$.

Definition 3.6 (characteristic finite-string). For a finite set $Y \subset \mathbb{N}$ define its *characteristic string* χ_Y as the $\{0,1\}$-labeled string of length $\max_{y\in Y} y + 1$ with a 1 in position n if and only if $n \in Y$.

Definition 3.7 (convoluting finite strings). Let $\overline{w} = (w_1, \cdots, w_k)$ be a k-tuple of $\{0,1\}$-labelled finite strings. Let $l := \max_i |w_i|$. The *convolution* $\otimes(\overline{w})$ is the $\{0, 1, \square\}^k$-labelled string of length l such that for all positions $n \leqslant l$ the ith component of $\otimes(\overline{w})[n]$ is equal to $w_i[n]$ if $n \leqslant |w_i|$ and the blank symbol $\square$ otherwise.

Definition 3.8. Suppose f is an isomorphism witnessing $\mathfrak{A} \in \texttt{S-AutStr}$. For a relation $R \subseteq \mathcal{A}^k$ denote by $\text{code}(R)$ the set of finite-strings $\{\otimes(\chi_{f(a_1)}, \cdots, \chi_{f(a_k)}) \mid \overline{a} \in R\}$.

Just as for Rabin-automatic structures, there is a fundamental theorem for finite-string automatic structures. We do not state it in full; simply replace Rabin-automatic by finite-string automatic and ω-tree automata by finite-string automata. However, we do slightly generalise the analogous definition of automatic presentation to cover an arbitrary alphabet Σ.

Definition 3.9 (finite-string automatic presentation [43]). Fix a finite alphabet Σ. Suppose that $f : \mathfrak{A} \simeq (\mathcal{B}, S_1, \cdots, S_N)$ and

(1) the elements of $\mathcal{B}$ are finite strings from Σ^*;
(2) the set $\mathcal{B}$ is recognised by a finite-string automaton, say $M_\mathcal{B}$;
(3) the set $\{\otimes(\bar{t}) \mid \bar{t} \in S_i\}$ is recognised by a finite-string automaton, say M_i, for $i \leqslant N$.

The data $\langle (M_\mathcal{B}, M_1, \cdots, M_N), f \rangle$ is called a *finite-string automatic presentation* of $\mathfrak{A}$.

Proposition 3.7 (Machine theoretic characterisation [13]). *A structure is FO-interpretable in $\mathcal{P}_f[\mathfrak{T}_1]$ if and only if it has a finite-string automatic presentation over an alphabet with $|\Sigma| \geqslant 2$.*

Proof. If $|\Sigma| > 2$ recode a string w by replacing individual symbols with binary blocks of size $\log_2 |\Sigma|$. $\square$

Relationships amongst the classes of automatic structures There are two more standard classes of automatic structures, see [11]. A *Büchi-automatic structure* is one FO-interpretable in $\mathcal{P}[\mathfrak{T}_1]$. Collectively these are denoted ωS-AutStr. An ω-*string* is a function from $\mathbb{N}$ to $\{0, 1\}$ and automata operating on (convolutions of tuples of) these are called ω-*string automaton*. One can similarly characterise ω-string automatic structures as those with ω-*string automatic presentations*. For example, $([0, 1), +, <) \in \omega$S-AutStr where $+$ is taken modulo 1 (the usual binary coding works). A *finite-tree automatic structure* is one FO-interpretable in $\mathcal{P}_f[\mathfrak{T}_2]$. Collectively these are denoted T-AutStr. A *finite tree* is a function from a finite prefix-closed subset of $\{0, 1\}^\star$ to $\{0, 1\}$ and automata operating on (convolutions of tuples of) these are called *finite-tree automata*. These automatic structures are those with *finite-tree automatic presentations* (see [11, 9]). For example $(\mathbb{N}, \times) \in$ T-AutStr (decompose n into $\prod_i p_i^{e_i}$ where p_i is the ith prime and so code n as a tree with e_i written in binary on branch $0^i 1^*$).

Proposition 3.8. S-AutStr *is a proper subset of* ωS-AutStr *and of* T-AutStr, *each of which is a proper subset of* ωT-AutStr.

Proof. The inclusions follow since finite-strings are special cases of finite-trees, etc. The structure of Skolem arithmetic $(\mathbb{N}, \times)$ separates T-AutStr from S-AutStr (see [11]). We will see (Theorem 6.9) that a structure separating ωT-AutStr from ωS-AutStr is $(\mathcal{P}(\{l, r\}^\star), \subset, V)$ where V is the unary relation consisting of those sets X such that the characteristic tree of X (a function $\{l, r\}^\star \to \{0, 1\}$) has the property that every infinite path is labelled with only finitely many 1s. $\square$

Proposition 3.9. *If* $\mathfrak{A} \in \omega$S-AutStr *is countable then* $\mathfrak{A} \in$ S-AutStr. *If* $\mathfrak{A} \in \omega$T-AutStr *is countable then* $\mathfrak{A} \in$ T-AutStr.

Proof. The reason is that every ω-string automaton whose language is countable accepts only ultimately periodic strings with a uniform bound on the length of the periods. A similar conditions holds for ω-tree automata that accept countable languages. $\square$

3.3 Operations on automatic structures

This section asks and gives some basic answers to the question:

> Which operations on automatic structures preserve automaticity?

See [12] for a survey of operations that preserve decidability.

Closure under interpretations Let $\circ$ stand for 'finite-string', 'ω-string', 'finite-tree', or 'ω-tree'. Since FO-definitions compose we have that:

Proposition 3.10. *[11] The $\circ$-automatic structures are closed under FO-interpretations.*

Example 3.3. $\circ$-automatic structures are closed under FO-definable expansions: if $\mathfrak{A}$ is $\circ$-automatic and ϕ is a FO-formula over the signature of $\mathfrak{A}$ then $(\mathfrak{A}, \phi^{\mathfrak{A}})$ is $\circ$-automatic.

There is a more general notion of interpretation called a *FO-interpretation of dimension d*. Here δ has $d \in \mathbb{N}$ free variables and each Φ_i has $d \times r_i$ free variables.

Proposition 3.11. *[13] The $\circ$-automatic structures are closed under FO-interpretations of dimension d.*

Proof. We illustrate the idea for `S-AutStr`. Suppose $\mathfrak{B} \in$ `S-AutStr`. It is enough to show that if $\mathfrak{A} = \mathcal{I}(\mathfrak{B})$ then we can find a finite-string automatic presentation of $\mathfrak{A}$. An element a of $\mathfrak{A}$ is a d-tuple of elements $(b_1, \cdots, b_d)$ of $\mathfrak{B}$ each of which is coded by a finite string $\mathrm{code}(b_i)$. Coding the element a by the string $\otimes(\mathrm{code}(b_1), \cdots, \mathrm{code}(b_d))$ we get a finite-string automatic presentation of $\mathfrak{A}$ over alphabet $\{0, 1, \square\}^d$. By proposition 3.7 $\mathfrak{A}$ is in `S-AutStr`. $\qquad\square$

Say $\mathfrak{A}$ and $\mathfrak{B}$ are FO-interpretable in $\mathfrak{U}$. Then the disjoint union of $\mathfrak{A}$ and $\mathfrak{B}$ is 2-dimensionally interpretable in $\mathfrak{U}$. Similarly for their direct product. Thus $\circ$-automatic structures are closed under disjoint union and direct product.

The *(weak) direct power* of $\mathfrak{A}$ is a structure with the same signature as $\mathfrak{A}$, its domain consists of (finite) sequences of $\mathcal{A}$, and the interpretation of a relation symbol R is the set of sequences σ such that $R^{\mathfrak{A}}(\sigma(n))$ holds for all n. For example the weak direct power of $(\mathbb{N}, +)$ is isomorphic to $(\mathbb{N}, \times)$; the isomorphism sends n to the finite sequence $(e_i)_i$ where $\prod p_i^{e_i}$ is the prime power decomposition of n. Since $(\mathbb{N}, \times)$ is neither in `S-AutStr` nor ω`S-AutStr`, these string classes are not closed under weak direct power (or direct power).

Proposition 3.12. *[11] Each of* `T-AutStr` *and* ω`T-AutStr` *is closed under weak direct power. The class* ω`T-AutStr` *is closed under direct power.*

Proof. We illustrate the second statement. Let $\mathfrak{A}$ be a Rabin-automatic structure with relation symbol R. Let $\sigma = (a_n)_n$ be an element of the direct power of $\mathfrak{A}$. Code the sequence σ by the tree t_σ whose subtree at $0^n 1$ is the tree $\mathrm{code}(a_n)$. The interpretation of R in the direct power is recognised by a tree automaton: it processes t_σ by checking that the convolution of the subtrees rooted at $0^n 1$ is recognised by the automaton for $R^{\mathfrak{A}}$. $\qquad\square$

Closure under quotients Let $\mathfrak{A} = (\mathcal{A}, R_1, \cdots, R_N)$ be a structure. An equivalence relation ϵ on the domain $\mathcal{A}$ is called a *congruence for* $\mathfrak{A}$ if each relation R_i satisfies the following property: for every pair of r_i-tuples $\overline{a}, \overline{b}$ of elements of $\mathcal{A}$, if $(a_j, b_j) \in \epsilon$ for $1 \leqslant j \leqslant r_i$ then $R_i(\overline{a})$ if and only if $R_i(\overline{b})$. The *quotient of* $\mathfrak{A}$ *by* ϵ, written $\mathfrak{A}/\epsilon$ is the structure whose domain is the set of equivalence classes of ϵ and whose ith relation is the image of R_i by the map sending $u \in \mathcal{A}$ to the equivalence class of u. Consider the question: If $(\mathfrak{A}, \epsilon)$ is $\circ$-automatic, is $\mathfrak{A}/\epsilon$ $\circ$-automatic?[6]

S-AutStr**:** Yes [13]. There is a regular well-ordering of the set of finite strings, for instance the length-lexicographic ordering $<_{\text{llex}}$. Use this order to define a regular set D of unique ϵ-representatives. Then restrict the presentation of $\mathfrak{A}$ to D to get a presentation of $\mathfrak{A}/\epsilon$.

T-AutStr**:** Yes [20]. Except in the finite string case, there is no regular well ordering of the set of all finite trees. However one can still convert a finite-tree automatic presentation of $(\mathfrak{A}, \epsilon)$ into one for $\mathfrak{A}/\epsilon$. The idea is to associate with each tree t a new tree $\hat{t}$ of the following form: the domain is the intersection of the prefix-closures of the domains of all trees that are ϵ-equivalent to t; a node is labelled σ if t had label σ in that position; a leaf x is additionally labelled by those states q from which the automaton for ϵ accepts the pair consisting of the subtree of t rooted at x and the tree with empty domain. Using transitivity and symmetry of ϵ, if $\hat{t} = \hat{s}$ then t is ϵ-equivalent to s. Moreover each equivalence class is associated with finitely many new trees, and so a representative may be chosen using any fixed regular linear ordering of the set of all finite trees.[7]

ωS-AutStr**:** It depends. Kuske and Lohrey [53] observed that there is no unique set of representatives of the equal almost-everywhere relation $\sim_{\text{ae}}$ that is regular. Thus we can't quotient using the trick that worked for S-AutStr. In fact, there is a structure in ωS-AutStr whose quotient is not ω-string automatic [32]. The proof actually shows that the structure has no Borel presentation, see Theorem 6.10. However, every regular ω-string equivalence relation with countable index has an ω-regular set of unique representatives. This follows from the following more general result [7]: If ϵ has countable index on A then there exist finitely many ω-strings $x_1, \cdots, x_c$ so that every x in A is ϵ-equivalent to some y which is $\sim_{\text{ae}}$-equivalent to some x_i. Thus if $\mathfrak{A}/\epsilon$ is countable and $(\mathfrak{A}, \epsilon) \in \omega$S-AutStr then $\mathfrak{A}/\epsilon \in \omega$S-AutStr.

ωT-AutStr**:** A finer analysis shows that the quotient of the Rabin-automatic structure from the previous paragraph is not in ωT-AutStr [32]. Thus ωT-AutStr is not closed under quotients. However it is not yet known what happens in the case that congruence-relation has countable index.

---

[6]This question is sometimes phrased as asking if every structure with a non-injective automatic presentation has an injective automatic presentation.

[7]The construction given in [20] is slightly more general and allows one to effectively factor finite-subset interpretations in any tree.

Nonetheless, quotients still have decidable theory since to decide the truth in the quotient structure, for a given sentence replace $=$ by ϵ and apply the decision procedure for $(\mathfrak{A}, \epsilon)$.

Proposition 3.13. *[20] If $(\mathfrak{A}, \epsilon)$ is ∘-automatic then the quotient $\mathfrak{A}/\epsilon$ has decidable FO-theory.*

Countable elementary substructures In this section we show a way of producing, from an uncountable Büchi- or Rabin-automatic structure $\mathfrak{B}$, a countable substructure $\mathfrak{A}$ with the same theory. Thus although $\mathfrak{A}$ may not itself be automatic, it has decidable theory.

Example 3.4. This example is taken from [21] (pg. 106). Code reals in base 2 and so get an ω-string automatic presentation of $\mathfrak{A} := (\mathbb{R}, +, <)$. The substructure of $\mathfrak{A}$ consisting of those reals coded by ultimately periodic ω-strings is isomorphic to $\mathfrak{B} := (\mathbb{Q}, +, <)$. We will see that the FO-theory of these two structures are identical and so conclude that $(\mathbb{Q}, +, <)$ has decidable FO-theory. A breakthrough paper establishes that $(\mathbb{Q}, +)$ is not in `S-AutStr` [78]. It is not known whether or not $(\mathbb{Q}, +, <) \in \omega\texttt{T-AutStr}$.

Two structures with the same FO-theory are called *elementary equivalent*. Let $\mathfrak{A}, \mathfrak{B}$ have the same signature. Say that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and for all formulas $\phi(\overline{x})$ and all $\overline{a}$ from $\mathcal{A}$,

$$\mathfrak{A} \models \phi(\overline{a}) \text{ if and only if } \mathfrak{B} \models \phi(\overline{a}) \ \ (\dagger)$$

Then in particular: $\mathfrak{A}$ and $\mathfrak{B}$ are elementary equivalent (take ϕ to be a sentence) and $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ (they agree on the atomic relations of $\mathfrak{A}$). There is a simple characterisation of being an elementary substructure.

Lemma 3.14 (Tarski-Vaught [33]). *Let $\mathfrak{A}$ be a substructure of $\mathfrak{B}$. Then $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ if and only if for every FO-formula $\phi(x, \overline{y})$ and all $\overline{a}$ from $\mathcal{A}$*

$$\mathfrak{B} \models \exists x \phi(x, \overline{a}) \implies \mathfrak{A} \models \exists x \phi(x, \overline{a}).$$

Say $f : \mathfrak{A} \simeq (\mathcal{B}, S_1, \cdots, S_N)$ is an ω-string automatic presentation of $\mathfrak{A}$. Write $\mathfrak{A}_{\mathrm{up}}$ for the substructure of $\mathfrak{A}$ isomorphic via f to the substructure whose domain consists of the ultimately periodic strings from $\mathcal{B}$. Similarly if $\mathfrak{A} \in \omega\texttt{T-AutStr}$ define $\mathfrak{A}_{\mathrm{reg}}$ as the substructure of $\mathfrak{A}$ isomorphic via f to the substructure consisting of regular trees from $\mathcal{B}$.

Proposition 3.15. *[7, 32]*
  (1) Let $\mathfrak{A} \in \omega\texttt{S-AutStr}$. The structure $\mathfrak{A}_{\mathrm{up}}$ is an elementary substructure of $\mathfrak{A}$.
  (2) Let $\mathfrak{A} \in \omega\texttt{T-AutStr}$. The structure $\mathfrak{A}_{\mathrm{reg}}$ is an elementary substructure of $\mathfrak{A}$.

Proof. Use the fact that an automaton — possibly instantiated with ultimately periodic strings $\overline{a}$ — is non-empty only if it contains an ultimately periodic string. Similarly for the tree case with 'regular tree' replacing 'ultimately periodic string'. □

Similar reasoning shows that $\mathfrak{A}_{\mathrm{reg}}$ and $\mathfrak{A}_{\mathrm{up}}$ have decidable FO($\exists^{\geqslant \aleph_0}, \exists^{\mathrm{mod}}$)-theory.

4 Anatomy of structures

What do structures in `S-AutStr` or `T-AutStr` look like?

Here is a useful pumping observation:

Proposition 4.1. *[43] Suppose that the partial function $F : A^n \to A$ is finite-string/tree regular, and let p be the number of states of the automaton. If $\overline{x}$ is in the domain of F then the length/height of the string/tree $F(\overline{x})$ is at most p more than the length/height of the largest string/tree in $\overline{x}$.*

Proof. Otherwise, take a counterexample $\overline{x}$. After all of $\overline{x}$ has been read, and while still reading $F(\overline{x})$, some path in the run must have a repeated state. So the automaton also accepts infinitely many tuples of the form $(\overline{x}, \cdot)$ contradicting the functionality of F.   □

Growth of generation

Definition 4.1. [43] Let $\mathfrak{A}$ be a structure with functions $f_1, \cdots, f_k$ of arities $r_1, \cdots, r_k$ respectively. Let $D \subset \mathcal{A}$ be a finite set. Define the *nth growth level*, written $G_n(D)$, inductively by $G_0(D) = D$ and $G_{n+1}(D)$ is the union of $G_n(D)$ and

$$\bigcup_{i \leqslant k} \{f_i(x_1, \cdots, x_{r_i}) \mid x_j \in G_n(D) \text{ for } 1 \leqslant j \leqslant r_i\}.$$

How fast does $|G_n(D)|$ grow as a function of n? For example, consider the free group with generating set $D = \{d_1, \cdots, d_m\}$. For $m \geqslant 2$ the set $G_n(D)$ includes all strings over D of length (in the generators) at most 2^n; so $|G_n(D)|$ is at least m^{2^n}.

Proposition 4.2. *[43] Let $\mathfrak{A} \in$ `T-AutStr` and $D \subset \mathcal{A}$ be a finite set. Then there is a linear function $t : \mathbb{N} \to \mathbb{N}$ so that for all $e \in G_n(D)$ the tree $\mathrm{code}(e)$ has height at most $t(n)$.*

Proof. Iterate proposition 4.1.   □

Corollary 4.3. *If $\mathfrak{A} \in$ `T-AutStr` then $|G_n(D)| \leqslant 2^{2^{O(n)}}$. If $\mathfrak{A} \in$ `S-AutStr` then $|G_n(D)| \leqslant 2^{O(n)}$.*

Proof. Count the number of $\{0, 1\}$-labelled trees (strings) of height at most k.   □

Thus the free group on more than one generator is not in `S-AutStr`.

Growth of projections We will be considering structures $(\mathfrak{A}, R)$ where R is a relation of arity > 1. For a tuple $\overline{u}$ of elements from $\mathcal{A}$ define $R(\cdot, \overline{u}) := \{a \in \mathcal{A} \mid (a, \overline{u}) \in R\}$.

Definition 4.2. For finite $E \subset \mathcal{A}$ the *shadow cast by $\overline{u}$ on E via R* is the set $R(\cdot, \overline{u}) \cap E$ and the *shadow count of E via R* is the number distinct shadows cast on E via R as $\overline{u}$ varies over tuples of elements of $\mathcal{A}$. We may suppress mention of R.

For example the random graph $(\mathcal{A}, R)$ has the property that for every pair of disjoint finite sets $E, F \subset \mathcal{A}$ there is a point $x \in \mathcal{A}$ that has an edge to every element in E and to no element of F. So for a given finite $E \subset \mathcal{A}$, the shadow count of E via R is the largest possible, namely $2^{|E|}$. The following propositions are due to Christian Delhommé [22] (the first one independently due to Frank Stephan) and limit the possible shadow counts in automatic structures.

Proposition 4.4. *Suppose $(\mathfrak{A}, R) \in \mathtt{S-AutStr}$. Then there is a constant k, that depends on the automata for domain $\mathcal{A}$ and R, and arbitrarily large finite subsets $E \subset \mathcal{A}$ such that the shadow count of E via R is at most $k|E|$.*

Proof. To simplify readability we suppose R is binary. Let $\mathcal{A}_n$ be the set of strings in $\mathcal{A}$ of length at most n. Let Q be the state set of the automaton for R. First, there is a constant $c := |Q|^{|Q|}$ such that for all n and all $x \in \mathcal{A}$ there is a $y \in \mathcal{A}_{n+c}$ such that x and y cast the same shadow on $\mathcal{A}_n$. Indeed, consider the sequence of functions $f_i : Q \to Q$ with $f_i(q)$ defined to be the state reached when the automaton for R starts in q and reads the string $\otimes(x[n+1, n+i], \lambda)$. If $|x| > n + c$ then there are two positions $k < l$ such that $f_k = f_l$. If we remove the segment $x[k, l-1]$ from x we get a shorter string x' that casts the same shadow on $\mathcal{A}_n$ as x does. Repeat until the string is short enough. Second, consider the sequence of sets $\mathcal{A}_{b+nc}$ where b is fixed so that $\mathcal{A}_b \neq \emptyset$. Write X_n for the cardinality of $\mathcal{A}_{b+nc}$ and S_n for the shadow count of $\mathcal{A}_{b+nc}$. We know that $S_n \leqslant X_{n+1}$. Suppose t were such that for almost all n, $S_n > tX_n$. Then $2^{b+(n+1)c} \geqslant X_{n+1} \geqslant S_n > t^n X_0$ for almost all n. So t is smaller than a constant that depends on b, c and X_0. So take k larger than this constant and conclude that $S_n \leqslant kX_n$ for infinitely many n, as required. $\qquad\square$

The proof of the following proposition is similar.

Proposition 4.5. *Suppose $(\mathfrak{A}, R) \in \mathtt{T-AutStr}$. Then there is a constant k, that depends on the automata for $\mathcal{A}$ and R, and arbitrarily large finite subsets $E \subset \mathcal{A}$ such that the shadow count of E via R is at most $|E|^k$.*

These are used to show that certain structures are not automatic. An immediate application is that the random graph is not in $\mathtt{T-AutStr}$.

Sum- and product-decompositions   All definitions and results in this section are due to Delhommé [22]. In these definitions all structures are over the same signature.

Definition 4.3. Say that a structure $\mathfrak{B}$ is *sum-decomposable* using a set of structures $\mathbf{C}$ if there is a finite partition of $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_n$ such that for each i the substructure $\mathfrak{B} \restriction \mathcal{B}_i$ is isomorphic to some structure in $\mathbf{C}$.

Theorem 4.6. *Suppose $(\mathfrak{A}, R) \in \mathtt{S-AutStr}$. There is a finite set of structures $\mathbf{C}$ so that for every tuple of elements $\overline{u}$ from $\mathcal{A}$, the substructure $\mathfrak{A} \restriction R(\cdot, \overline{u})$ is sum-decomposable using $\mathbf{C}$.*

Proof. To simplify readability we suppose that $\mathfrak{A} = \langle \mathcal{A}, \prec \rangle$ and for each $T \in \{\prec, R\}$ fix a deterministic automaton $(Q_T, \iota_T, \Delta_T, F_T)$ recognising $\mathrm{code}(T^{\mathfrak{A}})$. Naturally extend

Δ_T to all strings and so write $\Delta_T(q, w)$. From now on we supress writing $\mathrm{code}()$. Given a tuple of strings $\overline{u}$ write $|\overline{u}| := \max\{|u_i|\}$. Observe that we can partition the set $R(\cdot, \overline{u})$ into the finitely many sets: the singletons $\{c\}$ such that $R(c, \overline{u})$ and $|c| < |\overline{u}|$; as well as the sets

$$R^a(\cdot, \overline{u}) := \{aw \in \mathcal{A} \mid (aw, \overline{u}) \in R, w \in \{0, 1\}^\star\}$$

where $|a| = |\overline{u}|$. There are finitely many isomorphism types amongst substructures of the form $\mathcal{A} \restriction \{c\}$, for $c \in \mathcal{A}$. So, it is sufficient to show that as we vary the tuple $(a, \overline{u})$ subject to $|a| = |\overline{u}|$, there are finitely many isomorphism types amongst substructures of the form $\mathfrak{A} \restriction R^a(\cdot, \overline{u})$.

We do this by bounding the number of isomorphism types in terms of the number of states of the automata. To this end, define a function f as follows. Its domain consists of tuples $(a, \overline{u})$ satisfying $|a| = |\overline{u}|$; and f sends $(a, \overline{u})$ to the pair of states

$$\langle \Delta_R(\iota_R, \otimes(a, \overline{u})), \Delta_\prec(\iota_\prec, \otimes(a, a)) \rangle .$$

The range of f is bounded by $|Q_R| \times |Q_S|$; in particular, the range is finite.

To finish the proof, we argue that the isomorphism type of the substructure $\mathfrak{A} \restriction R^a(\cdot, \overline{u})$ depends only on the value $f(a, \overline{u})$. This follows from the fact that if $f(a, \overline{u}) = f(a', \overline{u'})$, then the corresponding substructures are isomorphic via the mapping $aw \mapsto a'w$ ($w \in \{0, 1\}^\star$). For instance $(aw, \overline{u}) \in R$ if and only if the automaton for R starting in state $\Delta_R(\iota_R, \otimes(a, \overline{u}))$ and reading $\otimes(w, \lambda, \cdots, \lambda)$ reaches a final state if and only if starting in $\Delta_R(\iota_R, \otimes(a', \overline{u'}))$ and reading $\otimes(w, \lambda, \cdots, \lambda)$ it reaches a final state if and only if $(a'w, \overline{u'}) \in R$. This established that f is a bijection between the domains $R^a(\cdot, \overline{u})$ and $R^{a'}(\cdot, \overline{u'})$. A similar argument shows that $aw_1 \prec aw_2$ if and only if $a'w_1 \prec a'w_2$. □

Corollary 4.7. *The ordinal $(\omega^\omega, <)$ is not in* `S-AutStr`.

Proof. Suppose for a contradiction that $(\omega^\omega, <)$ has an automatic presentation and let **C** be the finite set of structures guaranteed by the theorem using $<$ for R. For $n \in \mathbb{N}$ consider the substructure restricted to $R(\cdot, \omega^n)$. It has order type ω^n. It can be shown that if ordinal ω^α is sum-decomposable using a finite set of ordinals **C** then at least one ordinal in **C** has order type ω^α (see [23]). Thus **C** must contain a structure with order type ω^n, for each $n \in \mathbb{N}$, contradicting the finiteness of **C**. □

We now state the analogous results for `T-AutStr`. In these definitions all structures are over the same signature and all sequences of structures are finite.

Definition 4.4. Let $(\mathfrak{C}_i)_{i \leqslant k}$ be a sequence of structures. Their *synchronous product* is the structure with domain $\prod_{i \leqslant k} \mathcal{B}_i$ and relations defined as follows. Write π_j for the projection $\prod_{i \leqslant k} \mathcal{B}_i \to \mathcal{B}_j$. The interpretation of an r-ary relation symbol R consists of those tuples $(\overline{x_1}, \cdots, \overline{x_r})$ such that for all $i \leqslant k$, $\mathfrak{B}_i \models (\pi_i(\overline{x_1}), \cdots, \pi_i(\overline{x_r})) \in R$. If **C** is a set of structures write $\mathrm{SYNCH}(\mathbf{C})$ for the set of all synchronous products of sequences (of arbitrary finite length) of structures from **C**.

Let $(\mathfrak{C}_i)_{i \leqslant k}$ be a finite family of structures each with domain A. Their *superposition* is the structure with domain A and relation symbol R interpreted as $\cup_{i \leqslant k} R^{\mathfrak{C}_i}$. Write $\mathrm{SUPER}(\mathbf{C})$ for the set of all superpositions formed from families of structures from **C**.

Theorem 4.8. *Suppose $(\mathfrak{A}, R) \in$ T-AutStr. There is a finite set of structures $\mathbf{C}$ so that for every tuple of elements $\overline{u}$ from A, the substructure $\mathfrak{A} \upharpoonright R^{\mathfrak{A}}(\cdot, \overline{u})$ is sum-decompable using the set* SUPER(SYNCH($\boldsymbol{C}$)).

Proof. We adapt the proof of Theorem 4.6. There $\mathfrak{A} = (\mathcal{A}, \prec)$ and we do not distinguish between a tuple of elements of $\mathcal{A}$ and its code, or between a relation and the automaton defining it. We may assume all our automata M are deterministic leaf to root. If tree t has k leaves, say at positions $p_1, \cdots, p_k$ ordered lexicographically write $\delta_M(q_1, \cdots, q_k, t)$ for the label of the root of the run on input t that for all $i \leqslant k$ labels position p_i by state q_i. Write $\Delta_M(t)$ for $\delta_M(\iota_M, \cdots, \iota_M, t)$; that is, for the state labelling the root of the run on input t in which leaves are labelled with the initial state ι_M.

For a fixed tuple of trees $\overline{u}$ partition $R^{\mathfrak{A}}(\cdot, \overline{u})$ into finitely many sets $R^a(\cdot, \overline{u})$. Here a is a tree whose domain is a subset of the domain of the tree $\otimes(\overline{u})$ and $R^a(t, \overline{u})$ means that t extends a and that $\mathfrak{A} \models R(t, \overline{u})$. Fix an a and suppose for simplicity that a has the same domain as $\otimes(\overline{u})$ and that this domain has k leaves at positions $p_1, \cdots, p_k$. For a sequence $\overline{q} := q_1, \cdots, q_k$ of states of Q_R with $\delta_R(\overline{q}, \otimes(a, \overline{u})) \in F_R$ write $R^a_{\overline{q}}(\cdot, \overline{u})$ for those trees t such that $R^a(t, \overline{u})$ and the run (of the automaton for R on input $\otimes(t, \overline{u})$) at position p_i has label q_i (for all $i \leqslant k$). Thus $R^a(\cdot, \overline{u})$ is itself partitioned into finitely many pieces each of the form $R^a_{\overline{q}}(\cdot, \overline{u})$. Fix one of these sequences $\overline{q}$. Recall our goal is to express $R^a_{\overline{q}}(\cdot, \overline{u})$ as a superposition of synchronous products. For $q \in Q_R$ define the set $\mathcal{C}_q$ of all trees t (with any number of leaves) such that $\Delta_R(\otimes(t, \lambda, \cdots, \lambda)) = q$. For $q \in Q_R, s \in Q_\prec$ define a structure $\mathfrak{C}_{q,s}$ with domain C_q and relation $t \prec_s t'$ if $\Delta_\prec(\otimes(t, t')) = s$. The required set $\mathbf{C}$ consists of the structures $\mathfrak{C}_{q,s}$; and up to isomorphism there are at most $|Q_R| \times |Q_\prec|$. It is straightforward to check that $R^a_{\overline{q}}(\cdot, \overline{u})$ is in the set SUPER(SYNCH($\mathbf{C}$)). Indeed, the domain of $R^a_{\overline{q}}(\cdot, \overline{u})$ is $C_{q_1} \times \cdots \times C_{q_k}$ and $(R^a_{\overline{q}}(\cdot, \overline{u}), \prec)$ is the superposition of synchronous products $\mathfrak{C}_{q_1, s_1} \times \cdots \times \mathfrak{C}_{q_k, s_k}$ for $\overline{s} = (s_1, \cdots, s_k)$ satisfying $\delta_\prec(\overline{s}, \otimes(a, a)) \in F_\prec$. $\square$

Corollary 4.9. *The ordinal $(\omega^{\omega^\omega}, <)$ is not in* T-AutStr.

Proof. If ω^{ω^ω} were tree-automatic then take R to be $<$ and consider the substructure restricted to $R(\cdot, \omega^{\omega^n})$ (for $n \in \omega$). It has order type ω^{ω^n}. Since this is a power of ω, at least one substructure in SUPER(SYNC($\mathbf{C}$)) has order type ω^{ω^n}. It can be shown that if ω^{ω^α} is in SUPER($\mathbf{E}$) then $\mathbf{E}$ must contain ω^{ω^α} (see [23, Remark 8]). It is straightforward to check that if SYNCH($\mathbf{C}$) contains ω^{ω^n} then already $\mathbf{C}$ contains ω^{ω^n}. This contradicts finiteness of $\mathbf{C}$. $\square$

5 Equivalent presentations

In this section we focus on finite-string automatic presentations of a fixed structure. The main result in this section is due to Bárány [3]. We illustrate with base k presentations of $(\mathbb{N}, +)$ where $2 \leqslant k \in \mathbb{N}$. Write μ_k for the map sending $n \in \mathbb{N}$ to the base-k

representation of n. The translation between bases p and q is the map $\mu_q \circ \mu_p^{-1}$. It sends a string in base-p to that string in base-q that represents the same natural number.

Call two bases p and q *multiplicatively dependent* if for some positive integers k, l

$$p^k = q^l.$$

Proposition 5.1. *If p and q are multiplicatively dependent, then every relation $R \subseteq \mathbb{N}^r$ is regular when coded in base p if and only if it is regular when coded in base q.*

To see this we may use semi-synchronous rational relations: these can be thought of as being recognised by a multi-tape automaton where each read-head advances at a different, but still constant, speed. In the following definition the ith head moves m_i symbols at a time.

Definition 5.1. Fix a finite alphabet Σ and a vector of positive integers $\underline{m} = (m_1, \cdots, m_r)$. Let $\square$ be a symbol not in Σ and write $\Sigma_\square$ for $\Sigma \cup \{\square\}$. For each component m_i introduce the alphabet $(\Sigma_\square)^{m_i}$. The $\underline{m}$-*convolution of a tuple* $(w_1, \cdots, w_r) \in (\Sigma^\star)^r$ is formed as follows. First, consider the intermediate string $(w_1 \square^{a_1}, \cdots, w_r \square^{a_r})$ where the a_i are minimal such that there is some $k \in \mathbb{N}$ so that for all i, $|w_i| + a_i = km_i$. Second, partition each component $w_i \square^{a_i}$ into k many blocks of size m_i, and view each block as an element of $(\Sigma_\square)^{m_i}$. Thus the string $\otimes_{\underline{m}}(w_1, \cdots, w_r)$ is formed over alphabet $(\Sigma_\square)^{m_1} \times \cdots \times (\Sigma_\square)^{m_r}$. The $\underline{m}$-*convolution of a relation* $R \subseteq (\Sigma^\star)^r$ is the set $\otimes_{\underline{m}}R$ defined as

$$\{\otimes_{\underline{m}}\overline{w} \mid \overline{w} \in R\}.$$

A relation R is $\underline{m}$-*synchronous rational* if there is a finite automaton recognising $\otimes_{\underline{m}}R$. Call R *semi-synchronous* if it is $\underline{m}$-synchronous rational for some $\underline{m}$.

For example, if $\underline{m} = (1, \cdots, 1)$ then $\otimes_{\underline{m}}$ is the same as $\otimes$. For another example, the base-changing translation from base p to base q assuming $p^k = q^l$ is (k, l)-synchronous. Proposition 5.1 now follows from the fact that the image of a regular relation under a semi-synchronous transduction is regular. The converse of Proposition 5.1 is also true and follows from the Cobham-Semenov theorem, see [15]. For instance, if p and q are multiplicatively independent then the set of powers of p is regular in base-p but not regular in base-q. This discussion is the inspiration for the following generalisation.

For a given finite-string automatic presentation $\mu : \mathfrak{A} \simeq (\mathcal{B}, S_1, \cdots, S_N)$ write μ_{Reg} for the collection of relations

$$\{\mu^{-1}(R) \mid R \subseteq \mathcal{B}^k \text{ is a regular relation}, k \in \mathbb{N}\}.$$

Let $\nu : \mathfrak{A} \simeq (\mathcal{C}, R_1, \cdots, R_N)$ be another finite-string automatic presentation of $\mathfrak{A}$.

Definition 5.2. [3] The presentations μ and ν of $\mathfrak{A}$ are *equivalent* if $\mu_{\mathrm{Reg}} = \nu_{\mathrm{Reg}}$.

For instance if p and q are multiplicatively dependent then Proposition 5.1 says that the presentations μ_p and μ_q are equivalent.

Theorem 5.2. *[3] The presentations μ and ν are equivalent if and only if the map $\nu\mu^{-1} : \mathcal{B} \to \mathcal{C}$, namely*

$$\{(\mu(x), \nu(x)) \in \mathcal{B} \times \mathcal{C} \mid x \in \mathcal{A}\},$$

is semi-synchronous.

Proof. The interesting case is the forward direction. Let f denote the translation $\nu\mu^{-1}$: $\mathcal{B} \to \mathcal{C}$. Here is an outline: starting with $x \in \mathcal{B}$, we apply f to get $f(x) \in \mathcal{C}$, then pad to get $f'(x) \in \mathcal{C}'$, then cut into blocks to get $f''(x) \in \mathcal{C}''$. Write π for the padding $\mathcal{C} \to \mathcal{C}'$ and β for the blocking $\mathcal{C}' \to \mathcal{C}''$. Then f can be decomposed into semi-synchronous maps

$$\mathcal{B} \xrightarrow{f''} \mathcal{C}'' \xrightarrow{\beta^{-1}} \mathcal{C}' \xrightarrow{\pi^{-1}} \mathcal{C}.$$

We need some definitions. For a set X of strings, write $\mathbb{L}_X$ for the regular relation of pairs $(x, y) \in X \times X$ such that $|x| \geqslant |y|$. The *growth* of a function g between regular sets is the function $G : n \mapsto \max_{|a| \leqslant n} |g(a)|$. A bijection g is *length preserving* if $|g(x)| = |x|$. It is *length-monotonic* if $|x_1| \leqslant |x_2|$ implies $|g(x_1)| \leqslant |g(x_2)|$. It has δ-*delay* if $|x_2| > |x_1| + \delta$ implies $|g(x_2)| > |g(x_1)|$.

Claim 1. There is a constant δ such that f has δ-delay.

Since $f^{-1}(\mathbb{L}_\mathcal{C}) := \{(a, b) \mid |f(a)| \geqslant |f(b)|\} \subseteq \mathcal{B} \times \mathcal{B}$ is regular (by assumption of equivalence) and locally finite (every a is related to finitely many bs) there is a δ (by a pumping argument as in proposition 4.1) such that $(a, b) \in f^{-1}(\mathbb{L}_\mathcal{C})$ implies $|b| \leqslant |a| + \delta$.

The next claim says that the strings shorter than x are not translated into strings that are more than a constant longer than the string x is translated into.

Claim 2. There is a constant K with $F(|x|) - |f(x)| \leqslant K$, where F is the growth of f.

Since $f(\mathbb{L}_\mathcal{B}) := \{(f(a), f(b)) \mid |a| \leqslant |b|\} \subseteq \mathcal{C} \times \mathcal{C}$ is regular and locally finite, $|a| \leqslant |x|$ implies $|f(a)| \leqslant |f(x)| + K$. Thus $F(|x|) := \max_{|a| \leqslant |x|} |f(a)|$ is at most $|f(x)| + K$.

Let $\natural$ be a new symbol. Define $f' : x \mapsto f(x)\natural^{F(|x|)-|f(x)|}$, for $x \in \mathcal{B}$. Thus f' pads $f(x)$ by $\natural$ symbols so that the length of the translation by f' of x is no smaller than the length of the translation of any shorter string than x. Since the property $\{x \mid F(|x|) - |f(x)| = i\}$ is regular for fixed i the graph of f' is regular (by the previous claim). Write $\mathcal{C}'$ for $f'(\mathcal{C})$. Write F' for the growth of f'.

Claim 3. The translation $f' : \mathcal{B} \to \mathcal{C}'$ is length-monotonic and has δ-delay.

Use the fact that $|f'(x)| = F(|x|) = F'(|x|)$.

Claim 4. There exists $p, s \in \mathbb{N}$ such that the sequence $F'(n + p) - F'(n) = s$ for almost all n.

Let $l_0 < l_1 < \cdots$ be the sequence of integers l for which there is a string in $\mathcal{B}$ of length l. Let $u_n \in \mathcal{C}'$ denote the length-lexicographically smallest element amongst $\{f'(x) \mid |x| = l_n, x \in \mathcal{B}\}$. The set L of all such u_n is regular. Note that $|u_n| = F'(l_n)$ and so $|u_i| \leqslant |u_{i+1}|$ (length-monotonic) and $|u_i| < |u_{i+\delta}|$ (δ-delay). Thus L has at most δ many strings of any given length. So partition L into regular sets L_k for $k \leqslant \delta$: $x \in L_k$ if there are exactly k strings of length $|x|$ in L. The length-preserving projection of these onto 0^* results in unary presentations of L_k. These are ultimately periodic.

For simplicity assume the previous claim holds for all n. Now for $x \in \mathcal{B}$ of length n write $f'(x)$ as $v_1 v_2 \cdots v_n$ where $|v_i| = s$ (if $|f'(x)|$ is not a multiple of s, append a (new) blank symbol until it is). For a string w of length s write $\widehat{w}$ for a new alphabet symbol. Define $f'' : x \mapsto \widehat{v_1} \cdots \widehat{v_n}$. Write $\mathcal{C}''$ for $f''(\mathcal{B})$. Clearly the translation $f'' : \mathcal{B} \to \mathcal{C}''$ is length-preserving.

Claim 5. Since f'' is length-preserving and preserves all regular relations, the graph of f'' is regular.

The idea is that we can use lengths of elements of $\mathcal{B}$ (and $\mathcal{C}''$) as pointers to simultaneously identify the symbols in x and $f''(x)$. For simplicity, suppose that for every $n \in \mathbb{N}$ there is an element in $\mathcal{B}$ of length n (in general the gap between lengths is bounded). For a symbol σ define the regular relation $S_\sigma(p, b) \subset \mathcal{B} \times \mathcal{B}$ saying that σ occurs in b at position $|p|$. Write $R_\sigma \subset \mathcal{C}'' \times \mathcal{C}''$ for the image of S_σ under f''. It is also regular. Then $f''(x) = y$ if and only if $|x| = |y|$ and for all $p \in \mathcal{B}$ and $q \in \mathcal{C}''$ with $|p| = |q|$ and each symbol σ we have $S_\sigma(p, x) \iff R_\sigma(q, y)$. This latter condition is regular.

Finally, write $\pi : \mathcal{C} \to \mathcal{C}'$ for the map sending $f(x) \mapsto f'(x)$, and $\beta : \mathcal{C}' \to \mathcal{C}''$ for the map sending $f'(x) \mapsto f''(x)$. Since π^{-1} is a projection its graph is regular. Finally, β^{-1} is semi-synchronous sending blocks of size 1 to blocks of size s. $\qquad\square$

6 Relatives of automatic structures

6.1 Expansions by predicates and automatic with advice

Elgot and Rabin [27] use automata theoretic arguments to show that certain expansion of $\mathfrak{T}_1$ by unary predicates have decidable MSO. For instance they showed that $(\mathbb{N}, +1, \text{Fact})$ with $\text{Fact} := \{n! \mid n \in \mathbb{N}\}$ has decidable MSO-theory. For a predicate $P \subseteq \mathbb{N}$, the *P-membership problem* is to decide, given a Büchi-automaton M, whether or not M accepts χ_P. Recall that we write $(\mathfrak{A}, P)$ for the structure $\mathfrak{A}$ expanded by the predicate P.

Lemma 6.1. *[27] For every predicate $P \subseteq \mathbb{N}$, the structure $(\mathfrak{T}_1, P)$ has decidable MSO-theory if and only if the P-membership problem is decidable.*

Proof. Let Φ be a sentence of $(\mathfrak{T}_1, P)$. Let X be a variable not used in Φ. Build a formula $\Psi(X)$ from Φ in which every occurence of P has been replaced by the variable X. By construction $(\mathfrak{T}_1, P) \models \Phi$ if and only if $\mathfrak{T}_1 \models \Psi(P)$. The latter condition is equivalent to the problem of whether the automaton corresponding to Ψ accepts P or not. $\qquad\square$

We now briefly discuss how to find explicit predicates whose P-membership problem is decidable.

The contraction method for $P \subseteq \mathbb{N}$. Almost trivially, ultimately periodic P have decidable P-membership problem. For more general predicates, like the factorials Fact, we

use the *contraction method* of [27] and its generalisation by Carton and Thomas [18] that we now explain.

Call P *residually ultimately-constant* if there is an infinite sequence $x_0 < x_1 < \cdots$ of numbers such that for every semigroup morphism $h : \{0,1\}^\star \to S$ with S finite, the sequence $(h(u_i))_{i \geqslant 0}$ is ultimately constant, where $u_i = \chi_P[x_i, x_{i+1})$. Call P *effectively residually ultimately-constant* if the function $i \mapsto x_i$ is computable and given $h(0), h(1)$ and S one can compute an integer l, *a lag*, such that for all $m \geqslant l$, $h(u_l) = h(u_m)$. For example, it can be shown that the set of factorials Fact is effectively residually ultimately-constant.

Proposition 6.2. *[18] If P is effectively residually ultimately-constant then the P-membership problem is decidable.*

Proof. There is a standard effective way to associate with a given automaton M a morphism and finite semigroup $h : \{0,1\}^\star \to (S_M, \star)$ with the following property[8]: if a string $\alpha_0 \alpha_1 \alpha_2 \cdots$ ($\alpha_i \in \{0,1\}^\star$) is accepted by M then every string β that can be decomposed into $\beta_0 \beta_1 \beta_2 \cdots$ ($\beta_i \in \{0,1\}^\star$) with $h(\alpha_i) = h(\beta_i)$ (for all i) is also accepted by M. Then χ_P is accepted by M if and only if the ultimately periodic string uv^ω is accepted by M, where $u = \chi_P[0, x_l)$ and $v = \chi_P[x_l, x_{l+1})$. The latter property is decidable since we can compute a lag l and the elements $h(u), h(v)$. $\qquad\square$

It turns out that P being effectively residually ultimately-constant is also a necessary condition for $(\mathfrak{T}_1, P)$ having decidable MSO. See [67, 65] for a proof of this and other characterisations of $(\mathfrak{T}_1, P)$ having decidable MSO. See [18, 4] for explicit effectively residually ultimately-constant predicates, including the morphic predicates.

Why restrict to expansions by unary predicates? The reason is that expansions by non-trivial binary relations result in undecidability.

Theorem 6.3. *[27] Let $g : \mathbb{N} \to \mathbb{N}$ be a function such that $x < y$ implies $1 + g(x) < g(y)$. The expansion of $\mathfrak{T}_1$ by the relation $G_g := \{(n, g(n)) \mid n \in \mathbb{N}\}$ has undecidable* WMSO-*theory.*

In fact, one shows that one can quantify over finite relations which in turn gives the power of recursion to define addition and then multiplication. An example is taking $g(n) := 2n$.

What about expansions of $\mathfrak{T}_2$? For predicate $P \subseteq \{0,1\}^\star$, define the P-membership problem as above but with Rabin automata instead. Then identical arguments show that $(\mathfrak{T}_2, P)$ has decidable MSO if and only if the P-membership problem is decidable. The pushdown/Caucal hierarchy is a well studied collection of trees (and graphs) with decidable MSO [19] [76]. We illustrate an approach for decidability due to Fratani [29] (that can yield decidability of all trees in the hierarchy). To a semigroup $(\mathcal{M}, +)$ with finitely many generators $g_1, \cdots, g_k$, associate the Cayley structure $(\mathcal{M}, S_1, \cdots, S_k)$

---

[8] The idea appeared in Büchi's complementation proof: define $h : \{0,1\}^\star \to 2^{Q \times Q \times \{=, \neq\}}$ so that $(q, q', \oplus) \in h(u)$ if and only if there is a path in M from state q to state q' labelled u such that if C are the states occurring on this path then $C \cap F \oplus \emptyset$. Note that if $h(u_i) = h(v_i)$ for $i = 1, 2$ then $h(u_1 u_2) = h(v_1 v_2)$. Thus define the associative operation $\star$ on S by $s_1 \star s_2 = h(u_1 u_2)$ where u_i is any element such that $h(u_i) = s_i$.

where $S_i(m) = m + g_i$. For instance, the semigroup of strings under concatenation $(\{0,1\}^\star, \cdot)$ viewed as a Cayley structure is $\mathfrak{T}_2$.

Theorem 6.4. *[29] Take a surjective semigroup morphism* $\mu : \{0,1\}^\star \to \mathcal{M}$ *and a set* $R \subset \mathcal{M}$. *If the MSO-theory of* $(\mathcal{M}, S_1, \cdots, S_k, R)$ *is decidable then the MSO-theory of* $(\mathfrak{T}_2, \mu^{-1}(R))$ *is decidable.*

Proof. Use the relationship between Rabin automata and parity games (see [75]) to show that a given tree automaton (with state set Q, initial state ι, transition set Δ) accepts $\mu^{-1}(R)$ if and only if the first player has a memoryless winning strategy in the parity game defined as follows: the arena is $\mathcal{M} \times (Q \cup \Delta)$; the priority of (m, q) and $(m, (q, \sigma, q_0, q_1))$ is the priority of q, the starting node is $(\mu(\lambda), \iota)$, and for every transition $\delta = (q, \sigma, q_0, q_1) \in \Delta$ and $m \in \mathcal{M}$ such that $m \in R \iff \sigma = 1$ the first player's moves are of the form (m, q) to (m, δ) and the second player's moves are of the form (m, δ) to $(m + \mu(i), q_i)$ for $i \in \{0, 1\}$. Having a memoryless winning strategy is expressible in MSO over $(\mathcal{M}, S_1, \cdots, S_k, R)$. $\qquad\square$

Example 6.1. Consider the semigroup morphism $\mu : \{0,1\}^\star \to \mathbb{N}$ that sends u to the number of 1s in u (the operation on $\mathbb{N}$ is addition). We have seen that $(\mathfrak{T}_1, \text{Fact})$ has decidable MSO and so conclude that $(\mathfrak{T}_2, \mu^{-1}(\text{Fact}))$ does too.

Automatic with advice If $(\mathfrak{T}_2, P)$ has decidable MSO-theory then every structure FO-interpretable in $\mathcal{P}[(\mathfrak{T}_2, P)]$ has decidable FO-theory. This justifies the following definitions (see [20]).

Definition 6.1. A structure is *Rabin-automatic with advice* $P \subseteq \{0,1\}^\star$ if it is FO-interpretable in $\mathcal{P}[(\mathfrak{T}_2, P)]$. A structure is *Büchi-automatic with advice* $P \subseteq \mathbb{N}$ if it is FO-interpretable in $\mathcal{P}[(\mathfrak{T}_1, P)]$.

A machine theoretic characterisation holds. A *Rabin-automaton with advice* $P \subseteq \{0,1\}^\star$ is one that, while in position $u \in \{0,1\}^\star$, can decide on its next state using the additional information of whether or not $u \in P$.[9] In other words, the advice P is simply read as part of the input. Thus a structure has a presentation by Rabin-automata with advice P if and only if it is FO-interpretable in $\mathcal{P}[(\mathfrak{T}_2, P)]$. An analogous statement holds for Büchi-automata with advice $P \subseteq \mathbb{N}$.

The theory of Rabin-automatic structures with advice is yet to be developed. These generalise Rabin-automatic structures which themselves still hold some mystery, e.g. are the countable quotients already finite-tree automatic? It is known that the extension of the fundamental theorem (Theorem 3.5) holds: if $\mathfrak{A}$ is Rabin-automatic with advice P then the code of every $\text{FO}(\exists^{\geqslant \aleph_0}, \exists^{\geqslant \aleph_1}, \exists^{\text{mod}})$-definable relation is recognised by a Rabin-automaton with advice P, see [6].

On the other hand, structures that are Büchi-automatic with advice have received some attention. For instance, the ordinal ω^ω is not Büchi-automatic with advice [66].

---

[9] The word 'advice' is meant to connote that we can ask for a bit of information based on the current state and the current symbol being read. The other term found in the literature is 'oracle' which I choose not to use because in computability theory it means that the machine can ask if the whole content written on a tape is in the oracle language.

Structures in which elements are coded by *finite* strings/trees have received some consideration.

Definition 6.2. A structure is *finite-tree automatic with advice* $P \subseteq \{0,1\}^\star$ if it is FO-interpretable in $\mathcal{P}_f[(\mathfrak{T}_2, P)]$. A structure is *finite-string automatic with advice* $P \subseteq \mathbb{N}$ if it is FO-interpretable in $\mathcal{P}_f[(\mathfrak{T}_1, P)]$.

Again, if $(\mathfrak{T}_i, P)$ has decidable WMSO then $\mathcal{P}_f[(\mathfrak{T}_i, P)]$ has decidable FO-theory. Machine models, i.e., finite-string/tree automata with advice, are studied (sometimes implicitly) in [27, 18, 67, 2, 20, 46, 30].

Example 6.2. The structure $(\mathbb{Q}, +)$, although not finite-string automatic [78], is finite-string automatic with advice.[10] To simplify the exposition we give a presentation $([0,1) \cap \mathbb{Q}, +)$ by finite strings over the alphabet $\{0, 1, \#\}$ where the automata have access to the advice string

$$10\#11\#100\#101\#110\#111\#1000\# \cdots$$

which is a version of the Champernowne-Smarandache string and known to have decidable MSO [4]. To every rational in $[0,1)$ there is a unique *finite* sequence of integers $a_1 \cdots a_n$ such that $0 \leqslant a_i < i$ and $\sum_{i=2}^{n} \frac{a_i}{i!}$ and n minimal. The presentation codes this rational as $f(a_2)\#f(a_3)\#f(a_4)\cdots\#f(a_n)$ where f sends a_i to the binary string of length $\lceil \log_2 i \rceil + 1$ representing a_i. Addition $a + b$ is performed least significant digit first (right to left) based on the fact that

$$\frac{a_i + b_i + c}{i!} = \frac{1}{(i-1)!} + \frac{a_i + b_i + c - i}{i!}$$

where $c \in \{0,1\}$ is the carry in. In other words, if $a_i + b_i + c \geqslant i$ then write $a_i + b_i + c - i$ in the ith segment and carry a 1 into the $(i-1)$st segment; and if $a_i + b_i + c < i$ then write this under the ith segment and carry a 0 into the $(i-1)$st segment. These comparisons and additions can be performed since the advice tape is storing i in the same segment as a_i and b_i. Of course since the automaton reads the input and advice from left to right it should non-deterministically guess the carry bits and verify the addition.

Structures that are finite-tree automatic with advice were first studied in [20].

Theorem 6.5. *[20] If $(\mathfrak{A}, \equiv)$ is finite-tree automatic with advice P and $\equiv$ is a congruence on $\mathfrak{A}$ then $\mathfrak{A}/_{\equiv}$ is also finite-tree automatic with advice P.*

Theorem 6.6. *[20] If $\mathcal{P}_f[\mathfrak{S}]$ is finite-tree automatic with advice P then $\mathfrak{S}$ is WMSO-interpretable in $(\mathfrak{T}_2, P)$.*

Consequently the following structures are not finite-tree automatic with any advice: the free monoid on two or more generators; the random graph; the structure $\mathcal{P}_f[(\mathbb{N}, +)]$. For instance, the last item follows from the fact that $(\mathbb{N}, +)$ is not WMSO-interpretable in any tree. It is not known what happens in these examples and theorems if we replace WMSO by MSO and $\mathcal{P}_f$ by $\mathcal{P}$.

---

[10] This was communicated independently by Frank Stephan and Joe Miller and reported in [61].

6.2 Descriptive set theory and Borel presentations

A standard reference for classical descriptive set theory is [41]. See [60] for a short survey about Borel presentable structures. There is a natural topology, called the Cantor topology, on $\{0,1\}^\omega$, namely the one whose basic open sets are of the form $\{\alpha \mid \tau \preceq_{\texttt{prefix}} \alpha\}$ for $\tau \in \{0,1\}^\star$. A subset $X \subseteq \{0,1\}^\omega$ is called *Borel (over $\{0,1\}$)* if it is in the smallest class of subsets of $\{0,1\}^\omega$ containing the basic open sets and closed under countable unions and complementation.

Example 6.3.   (1) A set X is a countable union of basic open sets if and only if there exists $W \subseteq \{0,1\}^\star$ such that

$$\alpha \in X \iff (\exists i)\alpha[i] \in W.$$

These are the *open* sets.

(2) Complements of open sets are called *closed*. Every singeleton is closed and thus every countable subset of $\{0,1\}^\omega$ is Borel.

(3) A set X is a countable intersection of open sets if and only if there exists $W \subseteq \{0,1\}^\star$ such that
$$\alpha \in X \iff (\forall j)(\exists i > j)\alpha[i] \in W$$

(4) A set X is a countable union of closed sets if and only if there exists $W \subseteq \{0,1\}^\star$ such that
$$\alpha \in X \iff (\exists j)(\forall i > j)\alpha[i] \in W$$

(5) A language X recognised by a *deterministic* Muller automaton is a boolean combination of sets of the previous two forms. Thus every ω-regular language is Borel.

In a similar way we can form Borel subsets of A^ω where A is a finite set, not just $\{0,1\}$. Thus we can define Borel relations: we call $S \subseteq (\{0,1\}^\omega)^r$ Borel if $\otimes S$ is a Borel subset of $(\{0,1\}^r)^\omega$.

Lemma 6.7. *Borel relations are closed under Boolean combinations and instantiation, i.e. if R is Borel over X and $x \in X^\omega$ is fixed, then*

$$\{(x_1, \cdots , x_{i-1}, x_{i+1}, \cdots , x_r) \mid (x_1, \cdots , x_{i-1}, x, x_{i+1}, \cdots , x_r) \in R\}$$

is Borel for every i.

Definition 6.3 (Borel presentation). A *Borel structure* $(\mathcal{B}, S_1, \cdots , S_N)$ is one for which $\mathcal{B} \subseteq \{0,1\}^\omega$ is Borel and each of the relations $S_1, \cdots , S_N$ are Borel. Any structure isomorphic to it is called *Borel presentable*.

Example 6.4. The following structures are Borel presentable.

(1) Every structure with a countable domain.

(2) The field $(\mathbb{C}, +, \times)$.

(3) The order $(\mathcal{P}(\{0,1\}^\star), \subseteq)$.

(4) The power structure of $(\mathbb{N}, +, \times)$.

(5) Every Büchi-automatic structure.

Although Borel presentable structures do not neccessarily have decidable first-order theories, methods of descriptive set theory can be used to answer questions about automatic presentations. To illustrate we will make use of expansions and extensions of the Borel presentable structure $(\mathcal{P}(\{0,1\}^\star), \subseteq)$.

Lemma 6.8 ([32]). *Let C be a countable set. If Φ is an isomorphism between two Borel presentations of $(\mathcal{P}(C), \subseteq)$, say (D, L) and (D', L'), then the graph of Φ is Borel.*

Proof. Every Borel presentation of $(\mathcal{P}(C), \subseteq)$ can be expanded to include the countable unary predicate Sing $:= \{\{c\} \mid c \in C\}$. So consider expansions (D, L, S) and (D', L', S') that are Borel. Note that $\Phi(x) = y$ if and only if for every u in the countable set S, $(u, x) \in L \iff (\phi(u), y) \in L'$. For fixed $(u, \Phi(u))$ the relation $(u, x) \in L \iff (\Phi(u), y) \in L'$ is Borel by Lemma 6.7. Thus the graph of Φ is the intersection over a countable set of Borel relations, and so is itself Borel. $\qquad\square$

We will use the following facts.

Fact 1. The graph of a function is Borel if and only if the preimage of every Borel set is Borel.

Fact 2. Suppose a function $F : \{0,1\}^\omega \to \{0,1\}^\omega$ satisifies that for all X, Y the strings X and Y are eventually equal if and only if $F(X) = F(Y)$. Then the graph of F is not Borel.

The following theorem separates ωT-AutStr from ωS-AutStr.

Theorem 6.9 ([32]). *There is a structure in ωT-AutStr that has no Borel presentation; in particular this structure is not in ωS-AutStr.*

Proof. Consider the structure $\mathfrak{V} = (\mathcal{P}(\{0,1\}^\star), \subseteq, V)$ where V is the unary relation consisting of those sets X such that the characteristic tree of X has the property that every infinite path is labelled with only finitely many 1s. The structure $\mathfrak{V}$ has a natural presentation placing it in ωT-AutStr. A bijection between $\{0,1\}^\star$ and $\mathbb{N}$ allows us to identify $\mathcal{P}(\{0,1\}^\star)$ with $\{0,1\}^\omega$. Write $\mathfrak{V}' = (\{0,1\}^\omega, \subseteq, V')$ for the corresponding structure. Now if $\mathfrak{V}'$ were Borel presentable, say via isomorphism Ψ, then by Lemma 6.8 the graph of Ψ would be Borel and so would V' be Borel. But this contradicts the fact that V' is Π^1_1-complete. $\qquad\square$

The following theorem says that ωS-AutStr is not closed under quotients. A more sophisticated argument using the same counter-example structure shows that ωT-AutStr is not closed under quotients [32].

Theorem 6.10 ([32]). *There is a structure $(\mathfrak{A}, \approx)$ in ωS-AutStr whose quotient $\mathfrak{A}/_\approx$ has no Borel presentation, and is thus not in ωS-AutStr.*

Proof. Let $\mathfrak{B}_0$ and $\mathfrak{B}_1$ be structures with disjoint domain, each isomorphic to $(\mathcal{P}(\mathbb{N}), \subseteq)$. Define $\mathfrak{A}$ to be the structure with domain $B_0 \cup B_1$ and relations $\leqslant, U, f$ where $\leqslant$ is $\subseteq$ restricted to each B_i, U holds on the elements of B_0 and $f : B_0 \to B_1$ is the identity. Let $=^*$ be an equivalence relation on the $B_0 \cup B_1$ which is the identity on B_0 and 'the symmetric difference is finite' on B_1. Clearly $(\mathfrak{A}, =^*) \in \omega$S-AutStr. We now sketch

the proof that $\mathfrak{A}/_{=_*}$ is not Borel presentable. Assume that $\mathfrak{A}/_{=_*}$ is isomorphic to a Borel structure $(B', \leqslant', U', f')$ via Φ and let Φ_0 be the restriction of Φ to $[B_0]_{=_*}$ (which is B_0). By Lemma 6.8 Φ_0 is Borel. Then by Fact 1 the composition $F := f' \circ \Phi_0$ is Borel, contradicting Fact 2. $\qquad\square$

Methods from (descriptive) set theory have also been used to study the isomorphism problem for automatic structures. Specifically, writing $\mathcal{M} = (M_B, M_1, \cdots, M_N)$ for a tuple of automata presenting a structure, the *isomorphism problem* for a collection **C** is the set of pairs $(\mathcal{M}, \mathcal{M}')$ such that both $\mathcal{M}$ and $\mathcal{M}'$ present the same structure from **C**. See [71, 51, 52] in which various natural isomorphism problems can be placed within the arithmetic and analytic hierarchy (see [70] for definitions). For instance, the complexity of the isomorphism problem for S-AutStr is Σ_1^1-complete.

We end with a stunning result:

Theorem 6.11 ([28]). *The isomorphism problem for the collection*

$$\{\mathfrak{A}/_{\approx} \mid (\mathfrak{A}, \approx) \in \omega\text{T-AutStr}\}$$

is not determined by the axiomatic system ZFC.

Proof. Here is the barest of sketches. Write FIN for the ideal of finite subsets of $\mathbb{N}$, and ANC for the ideal of subsets of $\{l, r\}^\star$ with no infinite antichain with respect to the prefix-order. Both $(\mathcal{P}(\mathbb{N}), \cap, \cup, \neg, \text{FIN})$ and $(\mathcal{P}(\{l, r\}^\star), \cap, \cup, \neg, \text{ANC})$ are Rabin-automatic. Write $\mathfrak{B}_1$ for the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{FIN}$ and $\mathfrak{B}_2$ for $\mathcal{P}(\{l, r\}^\star)/\text{ANC}$. Using results from descriptive set theory it can be proved that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are isomorphic under ZFC+CH, but not isomorphic under ZFC+OCA (open colouring axiom). $\qquad\square$

7 Questions

This chapter focused on what I consider foundational problems of automatic structures. Here are some directions and questions raised along the way.

(1) Theorem 3.5 describes extensions of the fundamental theorem by certain unary generalised quantifiers. Which other generalised quantifiers can be added? What if one restricts to a specific automatic structure?

(2) Section 3.3 states that Rabin-automatic structures are not closed under quotient. The counterexample mentioned has size continuum. Is every counterexample this large? That is, if $(\mathfrak{A}, \epsilon)$ is Rabin-automatic and the quotient $\mathfrak{A}/\epsilon$ is countable, then is the quotient Rabin-automatic (and hence finite-tree automatic)? If not, then which countable ordinals are regular quotients of Rabin-automatic structures?

(3) Can automata be used to explain why the countable random graph $\mathcal{G}$, which is not finite-tree automatic with any advice, has a decidable FO-theory? Although this question seems hard to formalise we may ask: Is $\mathcal{G}$, or an elementarily equivalent structure, the regular quotient of a Rabin-automatic structure with advice? The same question can be asked of other important structures such as real arithmetic $(\mathbb{R}, +, \times, <)$ which is decidable in double exponential time. In [1] it is proved that real arithmetic is not the regular quotient of a Büchi-automatic structure.

(4) Theorem 6.6 allows one to prove that certain structures of the form $\mathcal{P}_f[\mathfrak{S}]$ are not finite-tree automatic with advice P. Is it the case that if the power structure $\mathcal{P}[\mathfrak{S}]$ is Rabin-automatic with advice Q then $\mathfrak{S}$ is MSO-interpretable in $(\mathfrak{T}_2, Q)$?

References

[1] F. Abu Zaid, E. Grädel, Ł. Kaiser, and W. Pakusa. Model-theoretic properties of *omega-automatic structures*. *Theory of Computing Systems*, pages 1–25, 2013.

[2] V. Bárány. A hierarchy of automatic omega-words having a decidable mso theory. In *On-line proceedings of the 11th Journes Montoises, Rennes 2006*, 2006.

[3] V. Bárány. Invariants of automatic presentations and semi-synchronous transductions. In *Proceedings of the 23rd Annual Symposioum on Theoretical Aspects of Computer Science, STACS 2006*, pages 289–300, 2006.

[4] V. Bárány. *Automatic Presentations of Infinite Structures*. Phd thesis, RWTH Aachen, 2007.

[5] V. Bárány, E. Grädel, and S. Rubin. Automata-based presentations of infinite structures. In *Finite and Algorithmic Model Theory*, London Mathematical Society Lecture Note Series (No. 379). Cambridge University Press, 2011.

[6] V. Bárány, L. Kaiser, and A. Rabinovitch. Cardinality quantifiers in MLO over trees. In *Computer Science Logic*, pages 117–131, 2009.

[7] V. Bárány, L. Kaiser, and S. Rubin. Cardinality and counting quantifiers on omega-automatic structures. In *STACS '08: Proceedings of the 25th Annual Symposium on Theoretical Aspects of Computer Science*, 2008.

[8] P. T. Bateman, C. G. J. Jr., and A. R. Woods. Decidability and undecidability of theories with a predicate for the primes. *J. Symb. Log.*, 58(2):672–687, 1993.

[9] M. Benedikt, L. Libkin, and F. Neven. Logical definability and query languages over ranked and unranked trees. *ACM Transactions on Computational Logic*, 8(2), 2007.

[10] M. Benedikt, L. Libkin, T. Schwentick, and L. Segoufin. Definable relations and first-order query languages over strings. *J. ACM*, 50(5), 2003.

[11] A. Blumensath. *Automatic Structures*. Diploma thesis, RWTH Aachen, 1999.

[12] A. Blumensath, T. Colcombet, and C. Löding. Logical theories and compatible operations. In J. Flum, E. Grädel, and T. Wilke, editors, *Logic and Automata: History and Perspectives*, Texts in Logic and Games, pages 73–106. Amsterdam University Press, 2007.

[13] A. Blumensath and E. Grädel. Automatic structures. In *15th Symposium on Logic in Computer Science* (LICS), pages 51–62, 2000.

[14] G. Boolos, Burgess, and Jeffrey. *Computability and Logic*. Cambridge University Press, 2007.

[15] V. Bruyère, G. Hansel, C. M. Christian, and R. Villemaire. Logic and p-recognizable sets of integers. *Bulletin of the Belgian Mathematical Society. Simon Stevin.*, 1(2):191–238, 1994. Journées Montoises (Mons, 1992).

[16] J. Büchi. Weak second-order arithmetic and finite automata. *Z. Math. Logik Grundlagen Math.*, 6:66–92, 1960.

[17] J. Büchi. On a decision method in restricted second order arithmetic. *Logic, Methodology and Philosophy of Science (Proc.* 1960 *Internat. Congr)*, pages 1–11, 1962.

[18] O. Carton and W. Thomas. The monadic theory of morphic infinite words and generalizations. *Inf. Comput.*, 176(1):51–65, 2002.

[19] D. Caucal. On infinite terms having a decidable monadic theory. In *Mathematical foundations of computer science 2002*, volume 2420 of *Lecture Notes in Comput. Sci.*, pages 165–176, Berlin, 2002. Springer.

[20] T. Colcombet and C. Löding. Transforming structures by set interpretations. *Logical Methods in Computer Science*, 3(2), 2007.

[21] M. Dauchet. Rewriting and tree automata. In *Term Rewriting*, pages 95–113, 1993.

[22] C. Delhommé. Automaticité des ordinaux et des graphes homogènes. *Comptes Rendus Mathematique*, 339(1):5–10, 2004.

[23] C. Delhommé. Height of a superposition. *Order*, 23:221–233, 2006.

[24] J. Doner. Tree acceptors and some of their applications. *Journal of Computer and System Sciences*, 4:406–451, 1970.

[25] H. Ebbinghaus and J. Flum. *Finite Model Theory*. Perspectives in Mathematical Logic. Springer, 1995.

[26] C. Elgot. Decision problems of finite automata design and related arithmetics. *Transactions of the American Mathematical Society*, 98:21–51, 1961.

[27] C. Elgot and M. Rabin. Decidability of extensions of theory of successor. *Journal of Symbolic Logic*, 31(2):169–181, 1966.

[28] O. Finkel and S. Todorčević. The isomorphism relation between tree-automatic structures. *Central European Journal of Mathematics*, 8:299–313, 2010.

[29] S. Fratani. *Automates piles de piles ... de piles*. PhD thesis, Bordeaux, 2005.

[30] S. Fratani. Regular sets over extended tree structures. *Theoretical Computer Science*, 418(0):48 – 70, 2012.

[31] Y. Gurevich. Monadic second-order theories. In J. Barwise and S. Feferman, editors, *Model-Theoretic Logics*, pages 479–506. Springer-Verlag, Perspective in Mathematical Logic, 1985.

[32] G. Hjorth, B. Khoussainov, A. Montalbán, and A. Nies. From automatic structures to Borel structures. In *23rd Symposium on Logic in Computer Science (LICS)*, 2008.

[33] W. A. Hodges. *Model Theory*. Cambridge University Press, 1993.

[34] B. Hodgson. *Théories décidables par automate fini*. Phd thesis, University of Montréal, 1976.

[35] B. Hodgson. Decidabilite par automate fini. *Annales de Sciences Math/'ematiques du Qu/'ebec*, 7:39–57, 1983.

[36] M. Huschenbett. Word automaticity of tree automatic scattered linear orderings is decidable. In *CiE*, pages 313–322, 2012.

[37] M. Huschenbett. The rank of tree-automatic linear orderings. In *STACS*, pages 586–597, 2013.

[38] M. Huschenbett, A. Kartzow, J. Liu, and M. Lohrey. Tree-automatic well-founded trees. *Logical Methods in Computer Science*, 9(2), 2013.

[39] M. Huschenbett and J. Liu. A polychromatic ramsey theory for ordinals. In *MFCS*, pages 559–570, 2013.

[40] A. Kartzow. Collapsible pushdown graphs of level 2 are tree-automatic. *Logical Methods in Computer Science*, 9(1), 2013.

[41] A. Kechris. *Classical Descriptive Set Theory*. Springer-Verlag, 1995.

[42] B. Khoussainov and M. Minnes. Model-theoretic complexity of automatic structures. *Ann. Pure Appl. Logic*, 161(3):416–426, 2009.

[43] B. Khoussainov and A. Nerode. Automatic presentations of structures. *Lecture Notes in Computer Science*, 960:367–392, 1995.

[44] B. Khoussainov, S. Rubin, and F. Stephan. Definability and regularity in automatic structures. In *STACS 2004*, volume 2996 of *LNCS*, pages 440–451. Springer, Berlin, 2004.

[45] B. Khoussainov, S. Rubin, and F. Stephan. Automatic linear orders and trees. *ACM Transactions on Computational Logic (TOCL)*, 6(4):675–700, 2005.

[46] A. Kruckman, S. Rubin, J. Sheridan, and B. Zax. A myhill-nerode theorem for automata with advice. In *GandALF*, pages 238–246, 2012.

[47] D. Kuske. Is Cantor's theorem automatic ? *Proceedings of the 10th International Conference on Logic for Programming, Artificial Intelligence, and Reasoning (LPAR)*, 2850:332–345, 2003.

[48] D. Kuske. Theories of automatic structures and their complexity. In *CAI*, pages 81–98, 2009.

[49] D. Kuske. Is Ramsey's theorem omega-automatic? *27th International Symposium on Theoretical Aspects of Computer Science (STACS 2010)*, 5:537–548, 2010.

[50] D. Kuske. Isomorphisms of scattered automatic linear orders. In *CSL*, pages 455–469, 2012.

[51] D. Kuske, J. Liu, and M. Lohrey. The isomorphism problem on classes of automatic structures. In *LICS*, pages 160–169, 2010.

[52] D. Kuske, J. Liu, and M. Lohrey. The isomorphism problem for *omega*-automatic trees. *Ann. Pure Appl. Logic*, 164(1):30–48, 2013.

[53] D. Kuske and M. Lohrey. First-order and counting theories of omega-automatic structures. *J. Symb. Log.*, 73(1):129–150, 2008.

[54] D. Kuske and M. Lohrey. Hamiltonicity of automatic graphs. *FIP TCS 2008*, 2008.

[55] D. Kuske and M. Lohrey. Some natural decision problems in automatic graphs. *J. Symb. Log.*, 75(2):678–710, 2010.

[56] D. Kuske and M. Lohrey. Automatic structures of bounded degree revisited. *J. Symb. Log.*, 76(4):1352–1380, 2011.

[57] D. Kuske and T. Weidner. Size and computation of injective tree automatic presentations. In *MFCS*, pages 424–435, 2011.

[58] P. Lindström. First order predicate logic with generalized quantifiers. *Theoria*, 32:186–195, 1966.

[59] C. Michaux and A. Ozturk. Quantifier elimination following muchnik. Technical Report 10, Universit de Mons-Hainaut, 2002.

[60] A. Montalbán and A. Nies. Borel structures: a brief survey. In N. Greenberg, J. D. Hamkins, D. R. Hirschfeldt, and R. G. Miller, editors, *Effective Mathemematics of the Uncountable*, number 41 in ASL Lecture Notes in Logic, pages 124–134. Cambridge University Press, 2013.

[61] A. Nies. Describing groups. *Bulletin of Symbolic Logic*, 13(3):305–339, 2007.

[62] M. Presburger. Über die vollständigkeit eines gewissen systems der arithmetik ganzer zahlen, in welchem die addition als einzige operation hervortritt. *Comptes Rendus du I congrès de Mathématiciens des Pays Slaves. Warszawa: 92101*, 1929.

[63] M. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*, 141:1–35, 1969.

[64] M. O. Rabin. Decidable theories. In J. Barwise, editor, *HANDBOOK OF MATHEMATICAL LOGIC*, volume 90 of *Studies in Logic and the Foundations of Mathematics*, pages 595 – 629. Elsevier, 1977.

[65] A. Rabinovich. On decidability of monadic logic of order over the naturals extended by monadic predicates, 2007.

[66] A. Rabinovich and S. Rubin. Interpretations in trees with countably many branches. In *LICS*, pages 551–560, 2012.

[67] A. M. Rabinovich and W. Thomas. Decidable theories of the ordering of natural numbers with unary predicates. In *CSL*, 2006.

[68] F. Reinhardt. Automatic Structures with Parameters. Diploma thesis, RWTH-Aachen, 2013.

[69] R. M. Robinson. Restricted set-theoretical definitions in arithmetic. *Proc. Amer. Math. Soc.*, 9:238–242, 1958.

[70] H. Rogers. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, 1967.

[71] S. Rubin. *Automatic Structures*. Phd thesis, University of Auckland, 2004.

[72] S. Rubin. Automata presenting structures: a survey of the finite string case. *Bull. Symbolic Logic*, 14(2):169–209, 2008.

[73] A. Tarski. A decision method for elementary algebra and geometry, 1951.

[74] J. W. Thatcher and J. B. Wright. Generalized finite automata theory with an application to a decision problem of second-order logic. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 2:57–81, 1968.

[75] W. Thomas. *Automata on infinite objects*, pages 133–191. Elsevier, 1990.

[76] W. Thomas. Constructing infinite graphs with a decidable MSO-theory. In *Mathematical foundations of computer science 2003*, volume 2747 of *Lecture Notes in Comput. Sci.*, pages 113–124, Berlin, 2003. Springer.

[77] B. Trahtenbrot. Finite automata and the logic of one-place predicates. Russian. *Siberian Mathematical Journal*, 3:103–131, 1962. English translation: American Mathematical Society Translations, Series 2, 59 (1966), 23-55.

[78] T. Tsankov. The additive group of the rationals does not have an automatic presentation. *Journal of Symbolic Logic*, 76(4):13411351, 2011.