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NOTE

UNIQUENESS OF COLORABILITY AND COLORABILITY OF PLANAR 4-REGULAR GRAPHS ARE NP-COMPLETE*

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It is shown that two sorts of problems belong to the NP-complete class. First, it is proven that for a given k-colorable graph and a given k-coloring of that graph, determining whether the graph is or is not uniquely k-colorable is NP-complete. Second, a result by Garey, Johnson, and Stockmeyer is extended with a proof that the coloring of four-regular planar graphs is NP-complete.

The study of the computational complexity of various applied and theoretical problems has become of interest to researchers in a variety of fields outside pure mathematics in recent years. In part, this is due to the widespread recurrence of a particular class of problems known as NP-complete. Such traditional problems as deciding statements of propositional calculus [2] and graph coloring [3] have been shown to belong to this class as well as various applied problems in such diverse areas as marketing, scheduling, circuit design, and traffic control.

Two more problems, both of a graph theoretic nature, are shown to belong to this important class of interreducible problems. The first, that determining whether a given k-colorable graph is or is not uniquely k-colorable is NP-complete suggests that answering the question of uniqueness of colorability is as difficult as determining the chromatic number of an arbitrary graph.

The second result, that the coloring of 4-regular planar graphs is NP-complete extends a previous result [4].

The concept of polynomial reducibility is defined by Karp (3). It is assumed that the reader is familiar with the definitional framework provided there. If this assumption is not completely accurate, any of the references [3], [5], or [6] should provide the necessary background.

Throughout, $L_1 \downarrow L_2$ is used to mean that the language L_1 is polynomially reducible to the language L_2. Also, for two nodes n_1 and n_2 of a graph, n_1 adj n_2 is read "n_1 is adjacent to n_2".

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The following result due to Stockmeyer (7) is stated without proof:

Theorem 1. *k-colorability is NP-complete for $k > 2$. That is, the problem of determining whether a given graph is k-colorable, for a given k, is NP-complete.*

Definition 1. $w_k(G)$ is defined as the number of distinct k-colorings of a graph G. $x(G)$ is defined as the smallest k such that G is k-colorable.

Definition 2. Coloring Uniqueness is the problem of determining for a given graph G and a given k-coloring, X, of G, that G is (or is not) uniquely k-colorable. That is, given a k-colored graph, is $w_k > 1$? Clearly, determining that $w_k(G) > 1$ for a k-colorable graph is equivalent to determining that G is not uniquely k-colorable.

Theorem 2. *Coloring Uniqueness is NP-complete.*

We show that k-colorability $\downarrow$ Coloring Uniqueness. For this, it suffices to show that for any graph G we can construct (in polynomial time) a new graph G' (which is k-colorable) such that G is k-colorable if and only if G' is multiply colorable, that is, $x(G) = k$ iff $w_k(G') > 1$.

Let $G = (N, L)$, $N = \{n_1, n_2, \ldots, n_p\}$. where $p = |N|$. Construct $G' = (N', L')$ as follows:

$$N' = \{n_{1,1}, n_{1,2}, \ldots, n_{1,p}, n_{2,1}, n_{2,2}, \ldots, n_{2,p}, \ldots, n_{k,1}, n_{k,2}, \ldots, n_{k,p}\}.$$

That is, N' consists of k copies, C_i of the nodes of G, where $C_i = \{n_{i,1}, n_{i,2}, \ldots, n_{i,p}\}$. The nodes in $R_i = \{n_{1,i}, n_{2,i}, \ldots, n_{k,i}\}$ are called the k representatives of n_i in G.

The lines of G' are defined for $n_{i,l}$ and $n_{j,m}$ in N' $(i \neq j)$ by

$$n_{i,l} \text{ adj } n_{j,m} \quad \text{if and only if} \quad n_l \text{ adj } n_m \quad (\text{for } n_l, n_m \in N).$$

That is, members of the same copy are not adjacent, and members of different copies are adjacent whenever the nodes which they represent in G are adjacent.

Fig. 1 shows G' when G is a triangle and $k = 3$. Observe that $|N'| = k\,|N|$ and

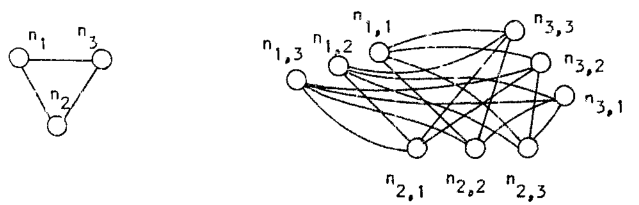

Fig. 1. An illustration of the construction of G' from G' for $k = 3$.

$|L'| = k(k-1)|L|$. Also, G' is clearly k-colorable since we may color each of the nodes in C_j the same color, j, since no two nodes in C_j are connected.

We need to show that G', so constructed, is multiply colorable ($w_k(G') > 1$) if and only if $x(G) \leq k$.

(A). $x(G) \leq k$ implies $w_k(G') > 1$.

The partition $X_1 = \{C_1, C_2, \ldots, C_k\}$ of N' induced by the k copies is one k-coloring of G'. Another coloring which is distinct from this is given by taking the k-coloring of G (which exists since G is k-colorable) and assigning to each representative, $n_{j,i}$, of n_i, the color, color (n_i). Since no two of a node's representatives are mutually adjacent, this may be done. Also, if n_i and n_j in G are colored the same, then n_i is non-adjacent to n_j and the representatives of these nodes in G' are mutually non-adjacent. This partition of the nodes of G' into the color classes of the nodes in G which they represent provides, therefore, a legal k-coloring of G'. This k-coloring is necessarily distinct from the former k-coloring (X_1) since $n_{1,1}$ and $n_{2,1}$ are colored the same in the new coloring whereas they are not in X_1.

(B). $w_k(G') > 1$ implies G is k-colorable ($x(G) \leq k$).

We have the k-coloring $X_1 = \{C_1, C_2, \ldots, C_k\}$ of G' where Copy $i = C_i$ and $C_i = \{n_{i,1}, n_{i,2}, \ldots, n_{i,p}\}$. If $w_k(G') > 1$, we have another k-coloring of G' namely $X_2 = \{S_1, S_2, \ldots, S_k\}$ which is distinct from X_1. Let $n_{i,j}$ and $n_{i,k}$ be two nodes in C_i. Then, if all the representatives of n_i are colored differently, $X_2(n_{i,j}) = X_2(n_{i,k})$, which would impose the coloring X_1. Hence, one may conclude that for each node n_i in G, there are at least two of its representatives in G' which are colored the same by X_2.

We may now define a coloring X_3 of G as follows: for each n_i in G set $X_3(n_i)$ equal to any color which appears on more than one of its representatives under $X_2(G')$; such a color exists by the preceding argument.

Since X_2 used at most k colors, X_3 must then also use at most k colors, so that it remains only to be shown that adjacent nodes in G will be colored differently by X_3. Let n_1 and n_2 be two adjacent nodes in G. If $X_3(n_1) = X_3(n_2)$, then by the definition of X_3, there are representatives $n_{1,i}$ and $n_{1,j}$ of n_1 and $n_{2,k}$ and $n_{2,l}$ of n_2 in G' with $X_3(n_1) = X_2(n_{1,i}) = X_2(n_{1,j})$ and with $X_3(n_2) = X_2(n_{2,k}) = X_2(n_{2,l})$. But, since $n_{1,i}$ and $n_{1,j}$ belong to different copies $n_{2,k}$ must be adjacent to one of them by the construction of G'. Thus, $X_2(n_{1,i}) \neq X_2(n_{2,k})$ and $X_3(n_1) \neq X_3(n_2)$, as required.

Therefore, if G' is not uniquely k-colorable, then G is k-colorable, and the theorem is complete.

Theorem 3 (Garey, Johnson, and Stockmeyer, 4). *3-colorability of planar graphs with node degree at most 4 is NP-complete.*

Theorem 4. *3-colorability of 4-regular planar graphs is NP-complete.*

In accordance with Theorem 3, we start with a planar graph, $G = (N, L)$ in

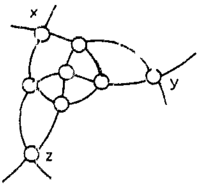

Fig. 2. A node replacement graph for nodes of degree six.

which degree $(n) \leqslant 4$ for each n in N. We construct a planar 4-regular graph G' such that $\chi(G)=3$ if and only if $\chi(G')=3$ as follows:

(1) Delete all nodes of degree less than or equal to 2. This may clearly be done without affecting the coloring of G.

(2) Repeat step (1) until only nodes of degree 3 and 4 remain. Let H refer to the resulting subgraph of G.

(3) Form a multigraph H' from H by "doubling" each edge in H. That is, for each pair of adjacent nodes, n_1 and n_2, in H, the graph H, being a proper graph, has a single edge between n_1 and n_2. H', as defined, has precisely two edges connecting the pair. Thus, the multigraph, H', has the property that every node is either of degree 6 or degree 8.

(4) Form the graph G' by replacing each node of degree six in H' by the "node replacement graph" shown in Fig. 2. Each such node should be replaced so that the edges departing from nodes x, y, and z of Fig. 2 are routed to different nodes of H', thus eliminating multiple edges. Note that this node replacement graph may always be rotated as desired. Also observe that in any 3-coloration of H' the nodes x, y, and z must all be colored the same such that this replacement will not affect the 3-coloration of G'. This node replacement also does not affect the degree of the adjacent nodes, merely of the node being replaced.

(5) Complete the construction of G' by replacing each node of degree eight by the node replacement graph shown in Fig. 3. In this figure, there are eight lines emanating from the nodes w, x, y and z, and each of these four nodes must be colored the same in any 3-coloration of the graph. Again, the figure may be rotated such that lines leading from the same node in the figure lead to different neighbors of the node being replaced.

The graph G' resulting is planar and 4-regular and is 3-colorable if and only if the original graph G is 3-colorable. This completes the proof.

It should be noted that 4 and 5 are the only numbers k such that the coloration

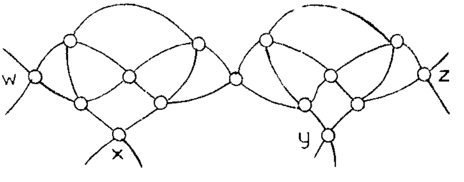

Fig. 3. A node replacement graph for nodes of degree eight.

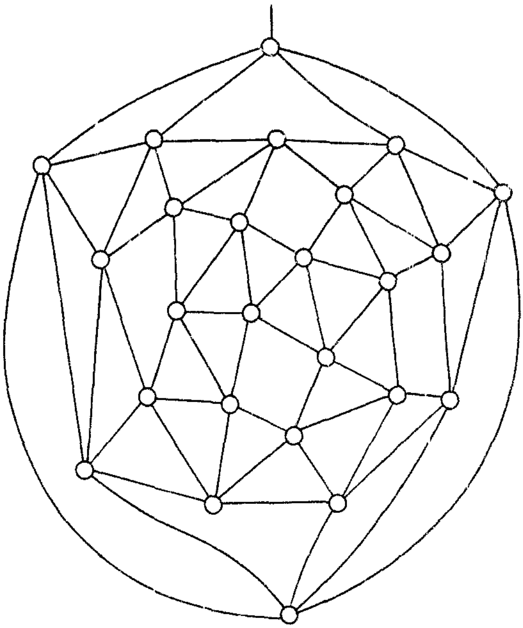

Fig. 4. A 5-regular 3-colorable appendage.

of k-regular planar graphs is NP-complete. When $k = 3$, we have a cubic graph and each such graph is 3-colorable. When $k \geq 6$, the resulting graph is too densely connected to be planar. For $k = 5$, the problem is NP-complete by the following construction: start with a regular graph of degree 4, and merely append one of the graphs shown in Fig. 4 to each of the nodes. The graph shown here has one outlet and is five-regular (with the exception of this outlet) as well as being 3-colorable.

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