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ALFRED TARSKI'S ELIMINATION THEORY FOR REAL CLOSED FIELDS

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Introduction. Tarski made a fundamental contribution to our understanding of **R**, perhaps mathematics' most basic structure. His theorem is the following.

(1) *To any formula $\phi(X_1, \ldots, X_m)$ in the vocabulary $\{0, 1, +, \cdot, <\}$ one can effectively associate two objects*: (i) *a **quantifier free formula** $\bar{\phi}(X_1, \ldots, X_m)$ in the same vocabulary, and* (ii) *a **proof** of the equivalence $\phi \leftrightarrow \bar{\phi}$ that uses only the axioms for real closed fields.* (*Reminder*: *real closed fields are ordered fields with the intermediate value property for polynomials.*)

Everything in (1) has turned out to be crucial: that arbitrary formulas are considered rather than just sentences, that the equivalence $\phi \leftrightarrow \bar{\phi}$ holds in all real closed fields rather than only in **R**; even the *effectiveness* of the passage from ϕ to $\bar{\phi}$ has found good theoretical uses besides firing the imagination.

We begin this survey with some history in §1. In §2 we discuss three other influential proofs of Tarski's theorem, and in §3 we consider some of the remarkable and totally unforeseen ways in which Tarski's theorem functions nowadays in mathematics, logic and computer science.

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§1. History. The first sign in print that Tarski possessed the powerful theorem (1) occurs in the abstract [30[a]], from which I quote:

(2) "In order that a set of numbers A be arithmetically definable it is necessary and sufficient that A be a union of finitely many (open or closed) intervals with algebraic endpoints." (My translation.)

This follows of course directly from the case $m = 1$ of the fundamental theorem (1). Tarski gave no hint in [30[a]] how he arrived at (2) except to say that it can be established "by purely mathematical means". (In those days there was "metamathematics" and "mathematics", and notions of definability were supposed to belong to "metamathematics".) It seems unlikely that he could have obtained (2) without actually proving (1) for general m.

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A precise formulation of (1) and a clear outline of its proof are given in ***The completeness of elementary algebra and geometry*** [67^{m}a], a document from 1940 whose publication was stopped by the war. As Tarski shows, the key point is a generalized form of Sturm's theorem, namely a criterion for the solvability of a system $f(X) = 0, g_1(X) > 0, \ldots, g_k(X) > 0$ which is *rational* in the coefficients of the polynomials $f, g_1, \ldots, g_k$.

The title of the monograph suggests a change of emphasis from definability questions to *completeness*, the fact that the truth of any *sentence* in $\mathbf{R}$ can be established purely on the basis of the axioms for real closed fields. (This corresponds to the case $m = 0$ of the fundamental theorem.) Tarski's axioms here are just the familiar ones for ordered fields plus an axiom schema saying that sign-changing polynomials have a zero. Remarkably, he does refer to van der Waerden's section [1937, p. 229ff.] on real closed fields, but does not mention these fields, and only indicates one other model of his axioms besides $\mathbf{R}$, namely the field of real algebraic numbers.

In the light of many later developments the following quote [67^{m}a, p. 10] sounds rather curious:

> "A relatively restricted scope appears to be left to future investigations in this field and in particular to efforts to strengthen the results of the present work."

In this connection Tarski mentions Gödel's incompleteness theorem concerning addition and multiplication of integers.

A detailed proof of (1) finally appeared in ***A decision method for elementary algebra and geometry*** [48^{m}], prepared for publication by J.C.C. McKinsey and published by the RAND corporation in 1948. The title indicates a second change of emphasis, now from completeness to decidability. In Tarski's words in the Preface to the 1951 edition of this monograph:

> "As was to be expected it reflected the specific interests which the RAND Corporation found in the results. The decision method ... was presented in a systematic and detailed way, thus bringing to the fore the possibility of constructing an actual decision machine. Other, more theoretical aspects of the problems discussed were treated less thoroughly, and only in notes."

Indeed, it is surprising that the long introduction to [48^{m}] (and 2nd ed., 1951) mentions the fundamental theorem only in passing, at the end:

> "We are often concerned, not with a *sentence* of elementary algebra, but with a *condition* involving parameters $a, b, c, \ldots$ and formulated in terms of elementary algebra; ... we are interested in reducing it to a standard form, in which it appears as a combination of algebraic equations and inequalities in a, b, c, ... The decision method developed below will give the assurance that such a reduction is always possible." (My italics.)

Nicely contrasting with Tarski's decidability of $\mathbf{R}$—and mentioned several times by Tarski in this connection—is Julia Robinson's undecidability of $\mathbf{Q}$. (Her beautiful proof [1949] that $\mathbf{Z}$ is definable in the field $\mathbf{Q}$ has not yet been superseded.)

More than thirty years later the many asides, notes and supplementary notes of the 1951 edition of this monograph are still worth reading, and the questions raised there had a large influence on later research by logicians; see for example A. Robinson [1956a], [1959b], J. Ax [1968], A. Macintyre [1976], [1986], M. Ziegler [1982], B. Dahn [1984], L. van den Dries [1986]. (For extensions of Ax's work, see the recent volume ***Field arithmetic*** by M. Fried and M. Jarden [1986].)

We refer to the article by Vaught [1986] for remarks on the origin of *quantifier elimination* in work of Löwenheim and Skolem, and its systematic development as a logical tool by Tarski. Here we should say a few words on the (lack of) connection to the much older tradition of elimination theory in classical algebra. In this subject one "eliminates unknowns" from a system of equations to end up with necessary (and sometimes sufficient) conditions for solvability: "vanishing of resultants". Logically speaking one is eliminating quantifiers from certain kinds of existential formulas. However, to view it this way and to see the point of eliminating quantifiers from more general formulas one needs familiarity with the logical concepts involved in this description, like "quantifier", "Boolean combination" and "atomic formula". Those notions and the simple laws governing them are obvious, once pointed out, but only emerged into consciousness late in the last century. The clear distinction between sentences and formulas, and the interpretation of the latter as descriptions of sets and of quantifiers as operating on these 'definable' sets came even later; see e.g. Tarski's abstract [30[a]]. Löwenheim, Skolem and Langford eliminated quantifiers to settle logical questions about newfangled structures like Boolean algebras and other ordered sets, where a parallel to older concerns of classical algebra was not at all evident. This origin may explain why Tarski refers only once briefly to the connection between quantifier elimination and algebraic elimination theory; cf. [31, p. 133 of the English translation]. Also, Tarski never seems to have explicitly mentioned in print that the theory of algebraically closed fields admits quantifier elimination. He obviously knew this result; cf. [48[m]], p. 54, Note 16] (and p. 373 of Seidenberg [1954]), but only drew attention to its zero-dimensional consequences: decidability, and completeness in each characteristic. Now, eliminating quantifiers for algebraically closed fields is, modulo familiar logic, just a simple exercise in the Euclidean algorithm for multivariable polynomials over **Z**, much easier than doing the same thing for real closed fields. Still, this exercise would have been worthwhile in 1948; a well-known and widely used later result, Chevalley's constructibility theorem, is a special case, and even Grothendieck's ambitious version [1964, p. 239] of the constructibility theorem where the characteristic is "variable" is immediate from it.

In the writings of A. Seidenberg [1954], [1956] and A. Robinson [1956a], [1958], [1959a] the kinship of logical quantifier elimination to algebraic elimination theory, specializing parameters, Nullstellensätze and similar ideas becomes more explicit and very fruitful.

Tarski himself seems to have been more interested in pursuing implications of his work for questions on axiomatization and interpretability of Euclidean and related geometries. My guess is that his fascinating question about adding "exponentiation" as an extra operation to the field operations came about in this way: plane *hyperbolic* geometry with the distance function among the primitives is interpretable in the *exponential* field of reals, but apparently not in the field of reals.

§2. Other proofs of Tarski's theorem, and their uses.

A. *Seidenberg's proof and Hörmander's inequality.* It seems that initially only a few specialists in logic regarded the study of Tarski's RAND monograph as profitable. Fortunately a colleague of Tarski in Berkeley, A. Seidenberg, published in 1954 another proof of Tarski's result in a much read mathematical journal; cf. [1954]. A new feature is that in eliminating an entire block of existential quantifiers in front of a system of equations and inequalities Seidenberg uses Sturm's method only once, and he describes the elimination process in a very concise geometric and algebraic style without using the logical formalism of predicate calculus. This makes some formulations a bit unwieldy, and ultimately it makes no sense to avoid the logical formalism if one wants to appreciate the full scope of Tarski's theorem, but the practical effect was to make Tarski's result much more accessible to many mathematicians. In analysis for instance, Tarski's theorem, restricted to the field $\mathbf{R}$, is generally known as Seidenberg's theorem (or as the Seidenberg-Tarski theorem), and has been applied in several questions concerning differential operators and distributions. Hörmander [1955] initiated these applications and he used the theorem in particular as the basic ingredient in his proof of the following inequality:

> *For each polynomial* $f(X_1,\ldots,X_m) \in \mathbf{R}[X_1,\ldots,X_m]$ *there are positive constants* c *and* r *such that*
>
> $$|f(x)| \geq c \cdot d(x, Z(f))^r \quad \textit{for all } x \in \mathbf{R}^m, |x| \leq 1,$$
>
> *where* $Z(f) \subset \mathbf{R}^m$ *is the zero-set of* f *and* $d(x,y) = |x - y|$ *is the (Euclidean) distance.*

For an early survey of results of this nature, see Gorin [1961]. Here is one sentence from this article (p. 94): "Some of the quoted results may appear 'obvious', but if one attempts to give a direct proof for them not using the Seidenberg-Tarski theorem one meets with serious difficulty."

It is appropriate to mention here also Seidenberg's construction [1956] of an elimination theory for algebraic differential equations in Ritt's sense. Seidenberg gives conditions for "solvability in a differential field extension of the differential field of coefficients". Later A. Robinson [1959a] used this work to *define* differentially closed fields and to rephrase Seidenberg's result in terms of "quantifier elimination for the elementary theory of differentially closed field". Further progress along model-theoretic lines in this area is due to L. Blum [1977], Poizat [1983] and others.

B. *Robinson's model-theoretic proof and applications of elementary logic to local fields.* Henceforth we shall write QE for "quantifier elimination". Clearly, an L-theory T admitting QE is *model-complete*; that is, if $\mathcal{M}$ and $\mathcal{N}$ are T-models and $\mathcal{M} \subset \mathcal{N}$, then an $L_{\mathcal{M}}$-sentence σ is true in $\mathcal{M}$ if and only if σ is true in $\mathcal{N}$. (Equivalently: every L-formula is T-equivalent to an *existential* L-formula.)

This model-completeness property drew Abraham Robinson's attention, and in ***Complete theories*** he showed that model-completeness of RCF, the elementary theory of real closed fields, follows on *general grounds* from two facts known since the classical Artin-Schreier paper [1926]:

(i) *Each ordered integral domain has a real closure.*

(ii) *If R is a real closed field then a simple ordered field extension $R(x)$, with x distinguished, is up to R-isomorphism uniquely determined by the set $\{r \in R: r < x\}$.*

Besides these two algebraic facts only the simplest model-theoretic techniques (compactness, diagrams) are used in Robinson's argument, which runs to no more than two pages. Robinson went on to treat in a similar way other important structures: algebraically closed *valued* fields [1956a] and certain kinds of ordered abelian groups (Robinson and Zakon [1960]). In [1959b] he proved by these methods the completeness of the theory of real closed fields with a predicate for a dense real closed proper subfield, solving one of Tarski's problems from [48$^{\mathrm{m}}$].

The very title of Robinson's monograph, ***Complete theories***, suggests that he, like Tarski, saw this type of work mainly in traditional terms of "completeness" and "decidability". These are prestigious and attractive logical notions, inherited from Hilbert, but experience has shown that *in the absence of QE* their mathematical content is disappointing. (Old habits die hard!)

In [1958] Robinson showed that under certain conditions a model-complete theory admits QE. Perhaps he overlooked here that it is usually easier to test *directly* for QE than first for the weaker property of model-completeness. The oversight (which is repeated in Sacks [1972] and Chang and Keisler [1973]) was corrected by Shoenfield [1971], [1977]. A careful analysis of many cases shows that the following variant of the *Robinson-Shoenfield* QE-*test* is the most useful:

An L-theory T admits QE if it enjoys the following two properties:

(i) *Existence of T-closures*: if $\mathscr{A}$ is a substructure of a T-model then $\mathscr{A}$ has a T-closure $\mathscr{A}^-$; that is, $\mathscr{A} \subset \mathscr{A}^- \models T$ and $\mathscr{A}^-$ can be embedded over $\mathscr{A}$ into any T-model extending $\mathscr{A}$.

(ii) *Specializability of selected elements*: if $\mathscr{M}, \mathscr{N} \models T$, $\mathscr{M} \subsetneq \mathscr{N}$, then there is $x \in \mathscr{N} \backslash \mathscr{M}$ and a set of quantifier free $L_{\mathscr{M}}$-formulas $\Phi = \{\phi_i(v): i \in I\}$, realized by x and determining its isomorphism type over $\mathscr{M}$, such that each finite subset of Φ can be realized by an element of $\mathscr{M}$.

(The analogy with Robinson's treatment of real closed fields is clear.) The freedom to choose x in (ii) is almost never stated but helpful in some situations. It also suffices to check (i) for finitely generated $\mathscr{A}$ and (ii) for $\mathscr{M}$ a T-closure of a finitely generated substructure. This simply reflects the Kroneckerian nature of "elimination" theory.

For the applicability of this test it is of course crucial that "closure" notions are abundantly available: injective hulls, algebraic and real closures, Henselizations, and all sorts of combinations of these. In fact, some new "closures" were discovered in connection with QE: differential closure (Blum [1977], Shelah [1973]) and p-adic closure (van den Dries [1978, p. 50]). Algebraic properties of these closures may sometimes reveal unsuspected definability aspects, e.g. the "rigidity" of p-adic closures implies the existence of definable Skolem functions for the p-adic field $\mathbf{Q}_p$ (van den Dries [1984]).

Robinson made Tarski's theorem not only an immediate consequence of Artin-Schreier theory, but also its culmination: Artin's solution [1927] of Hilbert's 17th problem by means of a delicate specialization becomes in Robinson's hands [1955] a memorable one-liner that can be applied over and over again: *what holds at a generic point (with coordinates in a real closed extension) holds by model-completeness for some point with coordinates in the real closed ground field.* Generalizations of Artin's theorem to, say, nonrational function fields are now a trivial matter.

Robinson [1957] also obtained recursive bounds in Hilbert's 17th problem, and this inspired the following piecewise uniform version by Henkin [1960] who gave a very simple argument for it using Tarski's theorem.

Given a "general" polynomial $f(C,X) \in \mathbf{Z}[C,X]$, $C = (C_1,\ldots,C_m)$, $X = (X_1,\ldots,X_n)$, *one can partition the parameter space* $\mathbf{R}^m$ *into finitely many definable subsets* $D_1,\ldots,D_k$ *such that if* $f(c,X) \in \mathbf{R}[X]$ *is positive semi-definite,* $c \in D_i$, $1 \leq i \leq k$, *then* $f(c,X)\cdot g_i(c,X)^2 = \Sigma_j\, \alpha_{ij}(c)\cdot h_{ij}(c,X)^2$, *with* $\alpha_{ij}(c) \geq 0$, $g_i(c,X) \neq 0$, *for certain* $\alpha_{ij} \in \mathbf{Z}[C]$ *and* g_i, $h_{ij} \in \mathbf{Z}[C,X]$ *which only depend on* f *(and not on* c*).*

For further literature and constructive improvements, along somewhat different lines originating with Kreisel [1960], see Delzell [1984].

Robinson's simple methods efficiently organize and create new algebraic knowledge, but more important is that these methods suggest fruitful analogies. The theory of real closed fields assumes here the role of paradigm. It is no accident that a basic corner of p-adic theory was entirely developed by logically trained mathematicians. We refer here to the breakthrough papers by Ax and Kochen [1965a], [1965b], [1966] and Eršov [1965], [1966], [1967] (who were of course also inspired by other ideas from logic, in particular by the then popular ultraproducts), and the subsequent articles by P. J. Cohen [1969] (a direct frontal attack), A. Macintyre [1976] (who combined Ax-Kochen with Shoenfield's test to eliminate quantifiers for $\mathbf{Q}_p$ in a natural language, thus gaining access to the definable sets) and J. Denef [1984] (who combined Macintyre's theorem with ideas hidden in Cohen [op. cit.] to develop a new method for computing hitherto uncomputable p-adic integrals).

I recommend Prestel and Roquette [1984] for a leisurely treatment of some of these p-adic topics, and the recent survey by Macintyre [1986] for further background and references. In both these accounts the suggestive analogies with the real case are a leading theme, and all I have to add here is that there is actually a two-way traffic: p-adic arguments often stimulate a better understanding of the real situation. See for example Denef and van den Dries [1988].

C. *Łojasiewicz's proof and the structure of definable sets.* Topological and geometric aspects of definable sets emerged early in Tarski's work but received only in the sixties a convincing treatment, by Łojasiewicz in [1964] and [1965]. Łojasiewicz's beautiful proof of Tarski's theorem in [1965, pp. 105–110] deserves to be better known. It goes as follows.

Let X be a topological space and R a ring of $\mathbf{R}$-valued continuous functions on X. Define an *R-set* to be a finite union of sets of the form $\{x \in X : f(x) = 0, g_1(x) > 0,\ldots,g_k(x) > 0\}$, where $f, g_1,\ldots,g_k \in R$. (So the R-sets form a Boolean algebra of subsets of X.)

We say that the pair (X,R) has the *Ł-property* if each R-set has only finitely many connected components, and each component is also an R-set. (If X is a one-point space, then (X,R) trivially has the Ł-property.) Note that the polynomial ring $R[T]$

can be considered as a ring of functions on the product space $X \times \mathbf{R}$. Now Łojasiewicz proves:

THEOREM. *If* (X, R) *has the Ł-property then* $(X \times \mathbf{R}, R[T])$ *also has the Ł-property, and moreover the image of each* $R[T]$*-set under the projection* $X \times \mathbf{R} \to X$ *is an* R*-set.*

The proof gives important extra information in the form of a *cylindrical decomposition* of each $R[T]$-set over X. To explain this roughly, consider a polynomial $f_d T^d + \cdots + f_0 \in R[T]$. Then there is a partition $X = X_1 \cup \cdots \cup X_r$ into finitely many R-sets and there are $d_1, \ldots, d_r \in \{0, 1, \ldots, d, \infty\}$ such that for each $x \in X_i$ the polynomial $f_d(x)T^d + \cdots + f_0(x)$ has exactly d_i *complex* zeros, not counting multiplicities. (This requires only a special case of QE for $\mathbf{C}$!) Now let Y be a connected component of X_i and suppose $d_i \neq \infty$. Then Łojasiewicz shows there are continuous $\mathbf{R}$-valued functions $r_1, \ldots, r_k$ on Y, $k \leq d_i$, such that for each $y \in Y$ the *real* zeros of $f_d(y)T^d + \cdots + f_0(y)$ are exactly $r_1(y), \ldots, r_k(y)$, and $r_1(y) < \cdots < r_k(y)$. This gives a cylindrical decomposition of the zero-set of our polynomial.

This is all we shall say about Łojasiewicz's proof. For more details, see Łojasiewicz [1965, pp. 105–110]. Here we only remark that Łojasiewicz's proof does not use Sturm's theorem, and that a nice elementary lemma due to Thom (see §3, A(ii) below) plays an important role in the proof of the topological part of the theorem.

Starting with the one-point space and using induction on m, we obtain from Łojasiewicz's theorem that the pair $(\mathbf{R}^m, \mathbf{Z}[T_1, \ldots, T_m])$ has the Łojasiewicz property. This says essentially that the ordered field $\mathbf{R}$ admits QE and that each (quantifier free) definable subset of $\mathbf{R}^m$ has only finitely many connected components, each of these also quantifier free definable. All this remains true when we replace $\mathbf{Z}[T_1, \ldots, T_m]$ by $\mathbf{R}[T_1, \ldots, T_m]$ and "definable" by "definable using constants from $\mathbf{R}$".

Subsets of $\mathbf{R}^m$ that are quantifier free definable using constants from $\mathbf{R}$ are called *semialgebraic* subsets of $\mathbf{R}^m$ by Thom [1962, p. 29], who formulates (an important case of) Tarski's theorem as follows: the image of a semialgebraic subset of $\mathbf{R}^m$ under a polynomial map $\mathbf{R}^m \to \mathbf{R}^n$ is a semialgebraic subset of $\mathbf{R}^n$. This formulation of Tarski's theorem is taken over by Łojasiewicz and many others. It bypasses logic and has a corresponding drawback. To illustrate this, if the function $f\colon \mathbf{R}^{m+1} \to \mathbf{R}$ has semialgebraic graph then the set $\{x \in \mathbf{R}^m : \lim_{y \to \infty} f(x, y) \in \mathbf{R}\}$ is semialgebraic. This is just one of many little facts that are obvious using a bit of logical symbolism: simply express the condition on $x \in \mathbf{R}^m$ by

$$\exists a \, \forall \varepsilon > 0 \, \exists r \, \forall y > r \, \exists z \, (f(x, y) = z \wedge |z - a| < \varepsilon).$$

If instead of this formula one describes a series of Boolean and projection operations, then the result is a lengthy and unintuitive argument.

In [1964] Łojasiewicz proved the very basic fact that every semialgebraic set has a finite semialgebraic triangulation, so that there are only countably many (semialgebraic) homeomorphism types among semialgebraic sets. Later Hardt [1980] used this triangulability result and Tarski's theorem to prove that among the sets in a semialgebraic *family* there are only finitely many (semialgebraic) homeomorphism types. (Hardt's theorem is in fact much more precise.)

Łojasiewicz also extended many "semialgebraic" results to the much larger class of *semianalytic* sets. (A subset of $\mathbf{R}^m$ is called semianalytic if it is defined locally around each point of $\mathbf{R}^m$ by a finite Boolean combination of analytic equations and inequalities.) His theorem on the Ł-property quoted above is very useful in this regard: by the Weierstrass preparation theorem (and a suitable linear transformation) one can replace a system of analytic equations and inequalities around, say, the origin by an equivalent system that is polynomial in the last variable. In this way Łojasiewicz obtained far-reaching extensions of Hörmander's inequality, among many other things. (Actually Łojasiewicz had already proved in 1958, independently of Hörmander, an analytic generalization of this inequality, but that proof was quite complicated.)

For many analytic purposes the class of semianalytic sets is not yet large enough. It is in particular not closed under projecting, even if one restricts to bounded semianalytic sets. To alleviate this inconvenience Gabriėlov [1968] enlarged the class of semianalytic sets to the class of *subanalytic* sets and extended several of Łojasiewicz's results to this class. (See van den Dries [1986] for some "definability" consequences of Gabriėlov's main theorem.) The term "subanalytic" is due to Hironaka, who made a very thorough study of these sets in [1973a] and [1973b] using his powerful resolution of singularities. Recently the hitherto rather complicated theory of subanalytic sets was greatly simplified in a way that highlights once more the crucial role of Tarski's theorem, in conjunction with the Weierstrass preparation theorem; cf. Denef and van den Dries [1988]. This paper also contains the first treatment of the p-adic case, with applications to p-adic varieties.

Needless to say, this will not be the end of the story: for example, the graph of the function $x \mapsto x^{\sqrt{2}}\,(x > 0)$ is not subanalytic despite its simple definition. There is every reason to expect that the class of subanalytic sets can be usefully enlarged further to contain also this set.

§3. Topics related to Tarski's theorem.

A. *Some loose ends.*

(i) I have not even mentioned yet the nowadays most cited proof of Tarski's theorem, the one given by P. J. Cohen [1969]. See also Hörmander [1983, pp. 362–371] for a presentation of this proof and for an intriguing *analytic continuation* property of semialgebraic functions observed by Cohen.

(ii) To study various finer properties of semialgebraic sets Tarski's theorem is often combined with *Thom's lemma*, a simple case of which says that *if* $f(X) \in \mathbf{R}[X]$ *is a polynomial of degree* $d \geq 1$, *then each set of the form*

$$A = \bigcap_{i=0}^{d} \{x \in \mathbf{R} : f^{(i)}(x) \square_i 0\}$$

(where each $\square_i$ is one of the signs $<, >, =$) *is connected, and if* $A \neq \emptyset$ *its closure is obtained by relaxing the strict inequalities.* (See Coste [1982] for more details.)

A result of Tarski-Ax-Kochen type that I found of considerable help in the study of *limit functions* of a semialgebraic family of functions is the following theorem by Cherlin and Dickmann [1983]: *the theory of real closed fields with a distinguished proper convex subring admits QE in the vocabulary* $\{0, 1, +, \cdot, <, |\}$ *where* $a | b$ *is interpreted as* ($\exists x \in$ *distinguished subring*) $ax = b$.

(iii) In a totally different direction, here are two facts indicating "logical optimality" of Tarski's theorem.

(a) Each finitely axiomatized theory in the vocabulary $\{0, 1, +, \cdot, <\}$ that has **R** as model is hereditarily undecidable (Ziegler [1982]).

(b) A linearly ordered ring with $1 \neq 0$ whose elementary theory admits QE is necessarily a real closed field (Macintyre, McKenna and van den Dries [1983]).

B. *Use of effectivity.*

(i) The majority of applications of Tarski's theorem only depend on the *existence* of a quantifier elimination for RCF. The one case I know where the effectiveness of the elimination is exploited is Kreisel [1960]: let $f(C,X) \in \mathbf{Z}[C,X]$, $C = (C_1, \ldots, C_m)$, $X = (X_1, \ldots, X_n)$, be a "general" polynomial in X. By elimination one converts the formula $\forall X \; f(C,X) \geq 0$ into quantifier free form. Kreisel found a connection between the "complexity" of this conversion and the "complexity" of the weighted sums of squares for f appearing in Henkin's result mentioned in §2B.

(ii) A notable use of *decidability* of Th(**R**) occurs in articles by Grunewald and Segal [1980]. Among many other things they solve the decision problem for isomorphism of two finitely presented nilpotent groups. Very roughly speaking this reduces to deciding if two explicitly given points on an explicitly given algebraic variety are conjugate under a certain explicitly given arithmetic group action. The explicit construction of suitable *semialgebraic* "fundamental domains" for these arithmetic groups then comes to play an important role.

C. *Semialgebraic geometry and topology.* A distinctive feature of this new and active area is that Artin-Schreier theory, including Tarski's theorem, is built into its foundations. For motivation, see the Introduction of Brumfiel [1979]; for representative papers, cf. Delfs and Knebusch [1981] and Coste and Coste-Roy [1982], and for useful surveys, cf. Lam [1984], Dickmann [1985], and Becker [1986].

My impression is that this is an area where elementary logic (à la A. Robinson) could be exploited much more.

An intriguing prospect seems to me to give a purely (semi) algebraic proof of Riemann's existence theorem for compact Riemann surfaces.

D. *The real numbers as an exponential field.* The exponential map is not semialgebraic, but relates the two basic semialgebraic operations on **R**, addition and multiplication. Already in 1940 [67$^{\mathrm{m}}$a] Tarski asked logical questions about the exponential field of reals, but only during the last ten years have there been sustained efforts, by many people, to unravel the model theory of this structure. There are many partial results, obtained by a great diversity of methods. Because of lack of space I refrain from discussing this fascinating topic further. Let me only mention the articles by Hovanskiĭ [1980], [1985] and Dahn [1984]. One goal that seems now much more plausible than two years ago (at least to me) is to show model-completeness of $(\mathbf{R}, <, +, \cdot, \exp)$.

E. *Computational aspects of Tarski's theorem.* In [1975] Collins does something very much like Łojasiewicz [1965, pp. 105–110], calls it *cylindrical decomposition*, introduces the notion of *sample point* and bases a decision procedure for the elementary theory of **R** on this idea. It runs much faster than Tarski's [48$^{\mathrm{m}}$], but is, to my knowledge, still too slow for any practical purpose. (At about the same time

Monk and Solovay did similar work. See also Macintyre [1986, p. 149] for an informative account and some analogous questions on $\mathbf{Q}_p$ that are still open. Collins [1982] lists other related papers.)

A lot of activity in this area is recent, and takes place in computer science fields such as symbolic computation, probabilistic algorithms, automated "theorem proving", and robotics. It would be desirable to have some critical reflection on what is going on here, as in Hao Wang [1984, pp. 59–69] on artificial "intelligence".

There is, I guess, a general feeling that a single algorithm for the full elementary theory of **R** can hardly be practical, except in very rare situations that still have to be discovered. On the other hand there are time-honoured algorithms (Lagrange multipliers, simplex method), only intended for specific classes of elementary problems on **R**, that perform very well. An apparently new algorithm of this sort that Hao Wang [op. cit.] draws attention to is the work on elementary geometry by Wu Wen-Tsün [1984]; cf. also Chou Shang-Ching [1984]. Roughly speaking, Wu Wen-Tsün's algorithm treats geometry problems that, in terms of Hilbert's ***Grundlagen*** [1899], can be formulated using only the notions of incidence, congruence and parallelism, and do not involve order. I do not know if Tarski ever heard of this work, which first appeared in Chinese in the late 1970s, but it would perhaps have pleased him: Hilbert's classic was a major influence on his own work, as it was on Wu Wen-Tsün's.

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