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Local and global price of anarchy of graphical games☆

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ABSTRACT

This paper initiates a study of connections between local and global properties of graphical games. Specifically, we introduce a concept of local price of anarchy that quantifies how well subsets of agents respond to their environments. We then show several methods of bounding the global price of anarchy of a game in terms of the local price of anarchy. All our bounds are essentially tight.

1. Introduction

The model of graphical games [14], is a representation method of games in which the dependencies among the agents are represented by a graph. In a graphical game, each agent is identified by a vertex, and its utility is determined solely by its own strategy and the strategies of its graph neighbors. Note that every game can be represented by a graphical game with a complete graph. Yet, often, a much more succinct representation is possible. While the original motivation of defining graphical games was computational, we believe that an important property of the model is that it enables an investigation of many natural structural properties of games.

In this work we investigate connections between local and global properties of graphical games. Specifically, we study the Price of Anarchy (PoA) which is the ratio between the welfare of a worst Nash equilibrium and the optimal possible welfare [15]. We introduce a novel notion of a *local* price of anarchy which quantifies how well subsets of agents respond to their environments. We then study the relations between this local measure and the global price of anarchy of the game. We provide several methods of bounding the global price of anarchy in terms of the local price of anarchy, and demonstrate the tightness of these bounds.

One possible interpretation of our results is as follows: if a decentralized system is comprised of smaller, well behaved units, with small overlap between them, then the whole system behaves well. This holds independently of the size of the small units, and even when the small units only behave well on average. From a computational perspective, the price of anarchy of large games is likely to be extremely hard to compute. However, computing the local price of anarchy of small units is relatively easy since they correspond to much smaller games. Once these are computed, our methods can be invoked to bound the price of anarchy of the overall game. We believe that our approach can assist in studying games that may be too complex to be analyzed directly.

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The model of graphical games was introduced in [14]. The original motivation for the model was computational as it permitted a succinct representation of many games of interest. Moreover, for certain graph families, there are properties that can be computed efficiently. For example, although computing a Nash equilibrium is usually a hard task [8,7], it can be computed efficiently for graphical games on graphs with maximum degree 2 [10]. Rather surprisingly, the proofs of the hardness of computing Nash equilibria of normal form games are conducted via reductions to graphical games [8].

Several works have studied the connections between combinatorial structure and game theoretic properties. For example, Galeotti et al. [12] investigate the structure of equilibria of graphical games under some symmetry assumptions on the utility of the agents. It shows that in these games, there always exists a pure strategy equilibrium. For such games of incomplete information, [12] shows that there is a monotone relationship between the degree of a vertex and its payoff, and investigates further the connections between the level of information the game possesses and the monotonicity of the players' degree in equilibria. In addition, a few works coauthored by Michael Kearns also explore economic and game theoretic properties which are related to structure (e.g. [13]). The questions addressed in these works are somewhat different from the ones we address here.

The price of anarchy [15] is a natural measure of games. After the discovery of fundamental results regarding the price of anarchy of congestion games [20,2], the price of anarchy and the price of stability[1] [1] have become almost standard methods for evaluating games. We use the price of anarchy as the sole criterion throughout this work.

Another work that presents bounds on the price of anarchy is [9], where a special graphical game was built by imposing the same two player game on each edge of a graph, and letting the utility of a player be its aggregate utility over all its neighbors. For a game taken from a class called *coordination game*, upper and lower bounds were given on the price of anarchy of the graphical game in terms of the original two player game.

Bilò et al. [4] analyze the impact of a social knowledge among the players on congestion games with linear latency functions. On games where the payoff of each player is affected only by the strategies of the neighbors in a social knowledge graph, they give a characterization of the games which have a pure Nash equilibrium. They also give bounds on the price of anarchy and price of stability in terms of the global maximum degree of the graph. In [6,5] Bilò et al. considered the price of anarchy and the price of stability of graphical multicast cost sharing games, and proved that if a central authority can enforce a certain graph it can lower the price of anarchy to a large extent.

Throughout the work we derive global bounds by only testing local properties. In the same sense Linial et al. [16] investigate deductions that can be made on global properties of graphs after examining only local neighborhoods. They show that for any graph G, where $V[G] = n$, and a function $f : V \to \Re^+ \cup \{0\}$, if the local average of f over every ball of radius which is positive and no more than r—and is not less than α, then the global average of f is at least $\frac{\alpha}{n^{O(1/\log r)}}$. The tightness of this bound was also established in [16].

In this work we make an extensive use of graph covers, but we do not introduce a method for finding them. Algorithms that find good covers can be found, for example, in [3,17]. Due to the game theoretic nature of our setup, these algorithms cannot be applied to it directly, and the question of how to design an algorithm that finds a good cover remains open for future investigation.

In general, the field of property testing in computer science examines the connections between local and global properties of combinatorial objects (see for example [11] for a survey). For many properties it is known that if an object satisfies a property in a local sense, then it is "not too far" from satisfying it globally. As we shall see, the additivity of the welfare function, enables even stronger connections between the local and the global perspectives in our setup.

2. Preliminaries

Throughout we deal with an *n-player game*. Every player in the game is associated with a set of possible strategies, and it needs to choose a distribution over this set. When the support of this distribution is greater than one it is called a *mixed strategy*, otherwise the strategy is *pure*. Every player in the game is also associated with one *game matrix*, through which its *utility* is determined for every pure vector of strategies. For a mixed strategy the utility is determined by the expectation of utilities over the distribution of pure strategies. We denote the utility of a player $i \in [n] = \{1, 2, \dots, n\}$ by u_i. In this work we focus on games where the utility of each player is non-negative, i.e. $u_i \geq 0$. Every player wishes to maximize its own utility by choosing the best strategy. The following was first defined by [14]. We use a definition similar to the one on [19].

Definition 2.1 (*[19] Chapter 7 Graphical Game*). An n player *graphical game* is a pair $(\mathcal{G}, \mathcal{M})$, where $\mathcal{G}$ is an undirected graph over the vertices $[n]$ and $\mathcal{M}$ is a set of n *local game matrices*. For every strategy vector $\vec{a}$, the local game matrix $M_i \in \mathcal{M}$ specifies the utility $M_i(\vec{a}^i)$ for player i, which is determined only by its strategy and by the strategies of its neighbors.

In other words, each vertex of the graph corresponds to a player, and the utility of each player is determined solely by its own strategy and the strategies of its neighbors. A few examples of such games are given at Section 4. Since we only discuss graphical representations of games in this paper, the terms game and graphical game are treated as synonyms. We note again that every game can be represented as a graphical game.

[1] The price of stability is the ratio between the *best* Nash equilibrium and the optimum of the game.

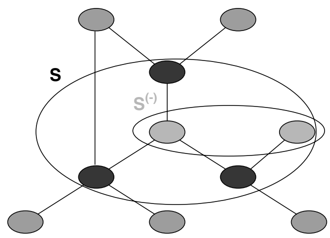

Fig. 1. A set S and its' interior $S^{(-)}$.

Definition 2.2 (*Welfare*)**.** Let G be a game and let $\vec{a}$ be a vector of agent strategies. The *welfare* is the sum of the agents' utilities resulting from $\vec{a}$, i.e. $\sum_i u_i$, when players adopt $\vec{a}$.

The welfare of a game is a common measure of the aggregation of the agents' utilities. It is by no means the only aggregation method. In this paper we focus on maximizing the welfare as the sole criterion of how good a game is. Our results can immediately be generalized to any measure of the form $\sum_i \psi_i(u_i)$ where $\psi_i : \Re^+ \to \Re^+$ are non decreasing functions.

Definition 2.3 (*[18] Nash Equilibrium*)**.** An strategy vector where no player can unilaterally deviate from and increase its utility is called a *Nash equilibrium*.

A Nash equilibrium always exists, however it does not have to be unique [18]. In this work we are only interested in *global worst Nash equilibria*, i.e. in equilibria that obtain the minimal welfare of the whole game. From compactness and continuity considerations, a global worst Nash equilibrium also always exists, and it is also not necessarily unique; but its value is fixed for a game so we can just pick one such arbitrary strategy vector to work with. Let $U_{\mathrm{WN}}(G)$ be the utility vector of one such global worst Nash strategy vector and $|U_{\mathrm{WN}}(G)|$ the worst Nash welfare. We denote by $U_{\mathrm{OPT}}(G)$ a utility vector for which there exists an strategy vector that achieves the optimal welfare of the game, and by $|U_{\mathrm{OPT}}(G)|$ the optimal welfare. We use U_{WN} and U_{OPT} i.e., suppress the G from the notation, when the context is clear.

Definition 2.4 (*[15] Price of Anarchy*)**.** For a game G, the ratio between the welfare of a worst Nash equilibrium and the optimal welfare is called the *price of anarchy* (PoA). That is: PoA$=\frac{|U_{\mathrm{WN}}|}{|U_{\mathrm{OPT}}|}$.

Note that the price of anarchy is always between 0 and 1. A price of anarchy of 1 means that all Nash equilibria are optimal. It is natural to define the price of anarchy of sub-games as well. We thus denote the PoA of the whole game by GPoA (global price of anarchy). The following combinatorial definitions are standard.

Definition 2.5 (*Cover, Partition*)**.** A *cover* $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ of a graph $\mathcal{G}$ is a collection of subsets of $V[\mathcal{G}]$ such that for every vertex $v_j \in V[\mathcal{G}]$, there exists a set $S_i \in \mathcal{S}$, where $v_j \in S_i$. We say that $\mathcal{S}$ is a *partition* (aka *disjoint cover*) if for every vertex $v_j \in V[\mathcal{G}]$, there exists a unique set $S_i \in \mathcal{S}$, such that $v_j \in S_i$.

Definition 2.6 (*Width*)**.** Let $\mathcal{S}$ be a cover of $V[\mathcal{G}]$. The *width* of $v \in V[\mathcal{G}]$ is the number of sets that contain it. The *width* of a cover $\mathcal{S}$ is β, if β is the maximum width of a vertex in $V[\mathcal{G}]$. That is: $\beta = \max_{v \in V[\mathcal{G}]} |\{S_i | \, v \in S_i\}|$.

Note that a partition has width $\beta = 1$. For a set S, we let $S^{(-)}$ denote the set S minus its internal boundary (i.e. $S^{(-)}$ contains only nodes that do not have neighbors that are not in S (see Fig. 1 for an example)). We let $S^{(+)}$ denote the set S plus its external boundary (its neighbors). For a collection of sets $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$, we let $\mathcal{S}^{(-)}$ denote the collection $\{S_1^{(-)}, S_2^{(-)}, \ldots, S_l^{(-)}\}$, and $\mathcal{S}^{(+)} = \{S_1^{(+)}, S_2^{(+)}, \ldots, S_l^{(+)}\}$.

Observation 1. *Let $\mathcal{S}$ be a partition of a graph $\mathcal{G}$ of max degree d, then $\mathcal{S}^{(+)}$ is a cover of width $d + 1$ at the most.*

Next we introduce a basic definition of the local price of anarchy. Note that if $S \subseteq V[G]$ is a set of players in a graphical game G, the utility of S depends only on the strategies of the players in S and the strategies of S's neighbors. Therefore every strategy vector to the neighbors of S induces a *sub-game* on S. The local price of anarchy is defined then in terms of subsets of vertices and their neighborhoods. Subsets of vertices (without their boundary) do not form a sub-game, therefore they cannot come straight as a basis for a local game theoretic parameter definition.

Definition 2.7 (*Local Price of Anarchy*)**.** Let G be a graphical game. The *local price of anarchy* of a set of players S is α_S, if for every set of strategies of its neighbors, the PoA of the induced sub-game is at least α_S. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover of $V[G]$. We say that the *local PoA* of G with respect to $\mathcal{S}$ ($\mathrm{LPoA}_{\mathcal{S}}(G)$) is α, if α_{S_i} the local price of anarchy of every subset $S_i \in \mathcal{S}$ is at least α.

Intuitively, a high local price of anarchy means that every set S_i in the cover responds well to its neighbors' strategies. The local price of anarchy of the set of all players, equals the global PoA of the game. Note that we could focus only on neighbors' strategies which are part of a global Nash equilibrium and still obtain all the results in this paper.

We wish in this work to bound the global price of anarchy in terms of the local. A good local price of anarchy alone does not suffice. Consider, for instance, a graphical game G and a cover by singletons $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ where $\forall i, S_i = \{v_i\}$. It is clear that $\text{LPoA}_{\mathcal{S}}(G) = 1$ since if a single player cannot increase its utility (Nash) then its utility is maximized (optimum). Yet the global price of anarchy can be as bad as we want. We will see at the next section that the usage of interiors facilitates such bounds.

We denote by $U_{\text{MAX}}(S) = \max_s \sum_{i \in S} u_i$ the maximum welfare that a set S can achieve (over all the possible vectors of strategies of $S^{(+)}$). Note that $U_{\text{MAX}}(S) \geq U_{\text{OPT}}(G)|_S$ where $U_{\text{OPT}}(G)|_S$ is the aggregated utility of players in S when all players in the graph $\mathcal{G}$ play some optimal strategy. We let $U_{\text{WN}}(S)$ denote the sum of utilities of S when the game is in a specific *global* worst Nash equilibrium. Similarly, we let $U_{\text{WN}}(i)$ denote the utility of player i (u_i) in this equilibrium. Note that if the game is in a global Nash equilibrium then *all* the subsets are also in local Nash equilibria (i.e. all the induced sub-games are in equilibrium).

3. A basic bound on the price of anarchy

In this section we introduce a basic lower bound of the global price of anarchy in terms of the local price of anarchy.

Definition 3.1 ((α, β)-*cover*)**.** Let G be a graphical game. A cover $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ of $V[G]$ is called an (α, β)-*cover* if the following hold:

1. $\text{LPoA}_{\mathcal{S}}(G) \geq \alpha$
2. $\mathcal{S}$ is of width at most β
3. The collection of interiors $\mathcal{S}^{(-)}$ is also a cover.

Intuitively, an (α, β)-cover is good when $0 \leq \alpha \leq 1$ is high, and $\beta \geq 1$ is low. It is possible to view an (α, β)-cover in the following manner: every set S_i is well-behaved, that is, reacts well to its external conditions; and the interaction (overlap) between sets is limited. The former is established by the local price of anarchy bound and the latter by the combinatorial width. The requirement that $\mathcal{S}^{(-)}$ is also a cover, is crucial as explained above.

Theorem 2. *Let G be a graphical game and $\mathcal{S}$ an (α, β)-cover, then GPoA$(G) \geq \alpha/\beta$.*

Proof. We will see that the fact that the local price of anarchy is α helps us in bounding by an α factor the ratio between every Nash equilibrium of S_i and the maximum welfare of $S_i^{(-)}$ (Claim 3.2). The requirement for $\mathcal{S}^{(-)}$ to be a cover will thus be used to bound the global optimum (Claim 3.3), and the width of $\mathcal{S}$ will generate the $1/\beta$ factor (Claim 3.4).

Claim 3.2. *Let $S_i \in \mathcal{S}$ be a set in the cover. Then, $U_{\text{WN}}(S_i) \geq \alpha U_{\text{MAX}}(S_i^{(-)})$.*

Proof. The welfare of $S_i^{(-)}$ only depends on the strategies of $S_i^{(-)}$ and the neighbors of $S_i^{(-)}$, that is, the welfare only depends on S_i. Thus, the best utility for the set S_i (for all strategy vectors of the neighbors of S_i) is not less than $U_{\text{MAX}}(S_i^{(-)})$ (recall that the utilities are always non-negative). The claim now follows from the definition of the local price of anarchy. $\square$

Summing the former over all the sets in the cover $\mathcal{S}$ yields:

$$\sum_{S_i \in \mathcal{S}} U_{\text{WN}}(S_i) \geq \alpha \sum_{S_i \in \mathcal{S}} U_{\text{MAX}}(S_i^{(-)}).$$

Claim 3.3. $\sum_{S_i \in \mathcal{S}} U_{\text{MAX}}(S_i^{(-)}) \geq |U_{\text{OPT}}|$.

Proof. Since $\mathcal{S}^{(-)}$ is a cover, the subsets $H_i^{(-)} = S_i^{(-)} - \bigcup_{j<i} S_j^{(-)}$ compose a disjoint cover $H^{(-)}$ of the graph. Since the utilities are non-negative, $|U_{\text{OPT}}|(H_i^{(-)}) \leq |U_{\text{OPT}}|(S_i^{(-)})$ for all i. (Note that OPT refers to the same strategy vector for both covers.) Since $H^{(-)}$ is a cover we get that

$$|U_{\text{OPT}}| = \sum_i |U_{\text{OPT}}|(H_i^{(-)}) \leq \sum_i |U_{\text{OPT}}|(S_i^{(-)}) \leq \sum_{S_i \in \mathcal{S}} U_{\text{MAX}}(S_i^{(-)})$$

where the last inequality is due to the optimality of $U_{\text{MAX}}(S_i^{(-)})$. $\square$

Claim 3.4. $\sum_{i \in V} U_{\text{WN}}(i) \geq \frac{1}{\beta} \sum_{S_i \in \mathcal{S}} U_{\text{WN}}(S_i)$.

Proof. $\mathcal{S}$ is of width β at the most. Therefore every element on the left-hand side appears at most β times in the sum on the right hand side. $\square$

Putting it all together we get:

$$|U_{\text{WN}}| = \sum_{i \in V} U_{\text{WN}}(i) \geq \frac{1}{\beta} \sum_{S_i \in \mathcal{S}} U_{\text{WN}}(S_i) \geq \frac{\alpha}{\beta} \sum_{S_i \in \mathcal{S}} U_{\text{MAX}}(S_i^{(-)}) \geq \frac{\alpha}{\beta} |U_{\text{OPT}}|.$$

This completes the proof of Theorem 2. $\square$

Remarks. While the local and global price of anarchies refer to Nash equilibria, it is possible to obtain a similar bound for many solution concepts (e.g. correlated or strong equilibria). The width parameter is purely combinatorial and can be interpreted as a measure of interaction between the sub-games (subsets). The α parameter is a measure of how well the small subsets behave. Later, we will average these parameters and also study the effects of other local parameters on the global price of anarchy. An interesting algorithmic issue is how to decompose a large game into small units such that the resulting bound on the global price of anarchy is as tight as possible, i.e. how to find a good cover.

3.1. Covers by balls

A natural way of obtaining a cover is by taking all balls of a certain radius.

Definition 3.5 (*r-LPoA*). Let G be a graphical game. We say that the *r-LPoA* of G is at least α, if every ball B of radius r has a local price of anarchy of at least α.

Corollary 3.6. *Let G be a graphical game with maximum graph degree d. If the 1-LPoA of G is at least α, then the GPoA$(G) \geq \frac{\alpha}{d+1}$.*

Proof. Consider the cover $\mathcal{S}$ of all balls of radius 1. Since this cover is of width $\beta \leq d+1$ and $\mathcal{S}^{(-)}$ is also a cover, it is an $(\alpha, d+1)$-cover. Now we can apply Theorem 2, and the corollary follows. □

Open problem. Interestingly, an r-LPoA $\geq \alpha$ only guarantees a bound of $O(\frac{\alpha}{r^{d+1}})$ on the GPoA. On the other hand it is natural to conjecture that the right bound is $\Theta(\frac{\alpha}{d})$. We leave this as an interesting open problem. When the game is relatively balanced, we can show that indeed $\Theta(\frac{\alpha}{d})$ is the right bound (see Section 5.3).

4. Examples

Before refining Theorem 2, let us consider a few simple examples. The example of covering a torus by grids (or covering a big grid by small ones), apart from being a natural example, demonstrates the need for refining the basic bound. The star-of-cliques game demonstrates how to use the theorem. The biased consensus game shows that the theorem is tight, i.e. that in the general case, it is not possible to improve the α/β bound.

4.1. Covering a torus by grids

Many games of interest are 'embedded' in some planar graph, for example the *grid* graph. In the grid graph, the vertices are labeled by two indices (i, j), where $i, j \in [m]$. Two vertices (i, j) and (i', j') are connected if and only if $i' = i+1, j' = j$ or $i' = i, j' = j+1$. The *torus* graph is just the grid graph when this calculation is made $\mod m$.

Example 3. Consider the $m \times m$ torus and let G be a graphical game played on it. Let k be a divisor of m and let $\mathcal{S}^{(-)}$ be a partition of the torus into $k \times k$ grids.

Consider the case of $k > 2$. Here, $\beta = 3, \forall i,\ |S_i| = k^2 + 4k$, and $|S_i^{(-)}| = k^2$. When k is large, almost all the vertices have a width of 1. An immediate conclusion of Theorem 2 is as follows.

Corollary 4.1. *(of Theorem 2) For a game played on the torus example, if LPoA$_{\mathcal{S}}(G) = \alpha$, then GPoA $\geq \alpha/3$.*

In other words, an LPoA of α implies a GPoA of about $\alpha/3$. The whole game might be very complex but if every local group responds well to its environment we know that the game is efficient. The example however demonstrates the need for refining the basic theorem: While the width of the cover is 3, almost all the vertices have a width of 1 and are therefore counted only once in $\sum_i U_{\mathrm{WN}}(S_i)$. Thus, typically, one should expect a GPoA of around α and not $\alpha/3$. This can be addressed by the various refinements of the basic theorem shown in Section 5.

4.2. Star-of-cliques

The following toy example demonstrates how to use the basic theorem. The game takes place at some school of witchcraft and wizardry. There is one player, the *head principle*, who only interacts with k *houses' heads*. Each house head interacts with the head principle and with her $l-1$ *house students*. All house students of a certain house are interconnected amongst themselves and in a relationship with their house head. Combinatorially, the graph we encounter is a k-star of l-cliques.

Definition 4.2 (*(k, l) Star-of-Cliques Graph*). A (k, l) *star-of-cliques* is a graph $\mathcal{G}G = (V, E)$, where

$$V = \{w, v_i, x_i^j | \forall i \in [k], j \in [l-1]\}$$

$$E = \{\{\{w, v_i\} | \forall i \in [k]\} \cup \{\{v_i, x_i^j\} | \forall i \in [k], j \in [l-1]\} \cup \{\{x_i^{j_1}, x_i^{j_2}\} | \forall i \in [k], j_1, j_2 \in [l-1]\}\}.$$

That is, the vertices $w, v_1, \ldots, v_k$ form a star with w in the center, and for each i, the l vertices $v_i, x_i^1, x_i^2, \ldots, x_i^{l-1}$ form a clique. See Fig. 2 as an example of a star-of-cliques graph.

Now assume there are two big concerts that take place on the same time at the school of witchcraft and wizardry. The players can only participate in one concert, i.e. they can choose one strategy out of two. The different players want to be in a concert where other players 'of their kind' are present, i.e. students with students, adults with adults. The utility of the players is as follows. If the player is a student of house h, then she gets one point for every other student of house h that had the same decision as hers (going to one concert or another), and in addition, she gets $l-2$ points more when she does not take the same strategy as h's house head. The h's house head gets one point for every h's student playing the opposite of her and another $l-1$ points if she plays the same as the head principle. The head principle only gets 1 if she plays as the majority

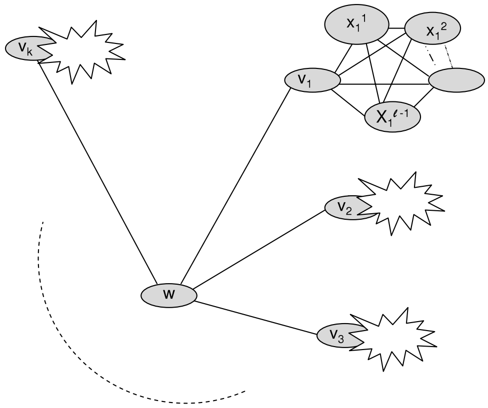

Fig. 2. A (k, l) star-of-cliques graph. Here $l \geq 5$ and $k \geq 4$. Note that the star clouds represent cliques of size at-least 5.

of house heads and 0 otherwise. More formally, The game has three types of players w, v, and x for the head principle, house head and student respectively. On each type the set of strategies is $\{0, 1\}$. For each vertex y we denote by $N_a(y)$ the number of y's neighbors of type $a \in \{w, v, x\}$ that play the same strategy as y, and by $\overline{N_a(y)}$ the number of y's neighbors of type $a \in \{w, v, x\}$ that play the opposite of y. We denote by $Act_{MAJ}(y)$ the strategy of the majority of y's neighbors.

Example 4 (*(k, l) Star-of-Cliques Game*)**.** A (k, l) *star-of-cliques game*, is an n-player game, where the graph of the game is a (k, l)-star-of-cliques, and the utilities of the players are given by:

$$u_y = \begin{cases} N_x(y) + (l-2)\overline{N_v(y)} & \text{if } y \text{ is of type } x \\ \overline{N_x(y)} + (l-1)N_w(y) & \text{if } y \text{ is of type } v \\ 1 & \text{if } y \text{ is of type } w \text{ and its strategy is } Act_{\text{MAJ}}(y) \\ 0 & \text{if } y \text{ is of type } w \text{ and its strategy is not } Act_{\text{MAJ}}(y). \end{cases}$$

The following will be improved later on by Theorem 12.

Proposition 4.3. *Let G be a (k, l)-star-of-cliques game. Then, $GPoA(G) \geq \frac{1}{2(k+1)}$.*

Proof (sketch). Consider the cover $\mathcal{S} = \{S_1, S_2, \dots, S_k, S_{k+1}\}$, where $S_i = \{w, v_i, x_i^1, x_i^2, \dots, x_i^{l-1}\}$ for $i \in [k]$ and $S_{k+1} = \{w, v_1, v_2, \dots, v_k\}$, i.e. each S_i ($i \leq k$) is a ball of radius 1 around v_i, and S_{k+1} is a ball of radius 1 around w. Note that $\mathcal{S}^{(-)}$ is a partition containing the singleton $\{w\}$ and all the surrounding cliques. In order to apply Theorem 2 to the star-of-cliques game we need a few simple observations.

Observation 5. *The width of w is $k + 1$ and since all other vertices are of lower width this is the width of $\mathcal{S}$.*

Observation 6. *For every set S_i where $i \leq k$, the following hold:*

- *The neighbors of the set only influence w. Since w's utility is negligible if we take large k or l, their strategies hardly effect the welfare of S_i*
- *From symmetry, without loss of generality we can take the strategy of all S_i's neighbors to be* 0
- *The v player can guarantee for itself an expected utility of at least $l - 1$ by playing a mixed strategy of* $(1/2, 1/2)$*. In the same manner, all the x players can guarantee themselves an expected utility of $l - 2$, and the w player can guarantee itself* $1/2$
- *By the previous, in every equilibrium, the welfare of S_i is at least $l^2 - 2l + 2.5$*
- *The strategy where the w and v players pick* 0 *and all the x players pick* 1 *is an optimum for this set. Hence, $|U_{\text{OPT}}| = 2(l^2 - 2l + 1.5)$*
- *Putting everything together we get that the local price of anarchy of the set is not less than* $1/2$.

Observation 7. *For the set S_{k+1}, by the same reasoning as before, one can show that the induced price of anarchy for the set is at least* $1/2$.

Therefore, the cover $\mathcal{S}$ we described is a $(1/2, k + 1)$-cover. Thus, applying Theorem 2 completes the proof of Proposition 4.3. $\square$

Remarks. We note that when all agents play $(1/2, 1/2)$, the result is a Nash equilibrium that gives each agent half of its optimal utility. Since most of the nodes can guarantee themselves $1/2$ of their optimal utility, the actual price of anarchy of the game is around $1/2$ (this bound can be obtained immediately from Theorem 12). From a qualitative perspective, every set $S_{i \leq k}$ responds well to its environment and thus the whole game behaves well. We could of course, complicate the game significantly and still get the same phenomenon. In particular, we could have imposed on the surrounding cliques, games in which the total welfare of every clique is the same as before but no player can guarantee itself a constant fraction of its optimal utility.

4.3. A tight example

Our final example shows that in general, the α/β bound of the basic theorem is essentially tight. For simplicity we focus on pure Nash equilibria.

Example 8 (*Biased Consensus Game*). Let $\gamma > 1$ be a parameter. In a *biased consensus game* the players set of strategy is $\{0, 1\}$. The utility of each player i is defined by:

$$u_i = \begin{cases} 1 & \text{if } i \text{ and all its neighbors play } 1 \\ 1/\gamma & \text{if } i \text{ and some of its neighbors play } 1 \\ 1/\gamma & \text{if } i \text{ and all its neighbors play } 0 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, if a player has a neighbor playing 1 it should play 1 as well, if all its neighbors are playing 0, it should play 0 too. We assume that the graph of the game is connected.

Observation 9. *For the biased consensus game the following properties hold:*

1. *When all agents play* 0 *the game is in Nash equilibrium.*
2. *Every player can guarantee a payoff of* $1/\gamma$*, therefore the Nash where all players play* 0*, is a global worst Nash equilibrium.*
3. *Every player can only get* 1 *as a maximum payoff, therefore the (Nash) where all players play* 1*, is a global optimum.*
4. *By the previous, the global price of anarchy is* $1/\gamma$*.*

The next observation states that the local PoA of any connected set S is obtained when all its neighbors play 0. We will see that the local PoA is equal to the ratio between the case where all the members of S are playing 0 and the case where all of them play 1.

Observation 10. *Consider the biased consensus game and let S be a connected set of agents. Then the following properties hold:*

1. *If at least one of S's neighbors plays* 1*, the only Nash equilibrium occurs when all the players in S are playing* 1*. In this case, this is also optimal for S.*
2. *When all the neighbors of S are playing* 0*, then the worst Nash equilibrium is when all the players in S are playing* 0 *and the best is when all of them play* 1*.*

The following proposition shows the tightness of Theorem 2.

Proposition 4.4. *For every $\epsilon > 0$, there exists a graphical game G, and an (α, β)-cover $\mathcal{S}$, where:*

$$\frac{\alpha}{\beta} \le GPoA \le (1+\epsilon)\frac{\alpha}{\beta}.$$

Proof. Consider the biased consensus game played on a d-regular graph and the cover by all balls of radius 1. By Observation 10, the local PoA of such a ball S is obtained when all its neighbors are playing 0. In this case, the worst local Nash equilibrium occurs when all the players in the ball are playing 0. This yields a utility of $1/\gamma$ to every member of S. In the optimal strategy for S all its members play 1. This strategy vector results in a utility of $1/\gamma$ for each of the d boundary nodes of S, and a utility of 1 for the inner node. Therefore, the local price of anarchy of S equals: $\alpha_S = \frac{\frac{1}{\gamma}(d+1)}{1+d/\gamma} = \frac{d+1}{d+\gamma}$. Since G is d-regular, $\alpha = \frac{d+1}{d+\gamma}$ as all the balls have $d+1$ nodes. Since $\beta = d+1$, we have that $\alpha/\beta = \frac{1}{(d+\gamma)}$. Thus, if we set $\gamma > d/\epsilon$ we get:

$$(1+\epsilon)\frac{\alpha}{\beta} = \frac{1+\epsilon}{d+\gamma} > \frac{1+\epsilon}{\gamma\epsilon+\gamma} = 1/\gamma = \text{GPoA(G)}. \quad \square$$

5. Refinements of the basic bound

5.1. Averaging the parameters

In the biased consensus game (Example 8), all the induced sub-games of the cover have the same local price of anarchy. Most games do not possess this property, and the basic theorem is thus often wasteful (as $\text{LPoA}_S(G)$ is the *minimum* local PoA of the sets in the cover). Similarly, β is the *maximum* width. For this purpose we generalize the definitions of the local price of anarchy and the width to be an *average* instead of the minimum and maximum, respectively. We introduce improved bounds on the global price of anarchy using the new definitions.

Definition 5.1 (*Average Local Price of Anarchy*). Let G be a graphical game and let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover of $V[G]$ such that $\mathcal{S}^{(-)}$ is also a cover. Let α_i be the local PoA of S_i. The *average local price of anarchy* of G w.r.t. $\mathcal{S}$, $\overline{\text{LPoA}_{\mathcal{S}}(G)}$, is the weighted average of α_i by the maximum utilities of $S_i^{(-)}$, that is $\overline{\text{LPoA}_{\mathcal{S}}(G)} = \frac{\sum_{i=1}^{l} \alpha_i U_{\text{MAX}}(S_i^{(-)})}{\sum_{i=1}^{l} U_{\text{MAX}}(S_i^{(-)})}$.

Theorem 11. *Let G be a graphical game and let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover of $V[G]$ such that $\mathcal{S}^{(-)}$ is also a cover and $\mathcal{S}$ is of width β. Let $\alpha = \overline{LPoA_{\mathcal{S}}(G)}$, then GPoA$\ge \alpha/\beta$.*

Proof (sketch). The proof resembles the one of Theorem 2, and we thus only sketch it.
Let $S_i \in \mathcal{S}$. If we follow the steps of the proof of Claim 3.2 in the proof of Theorem 2, with the new definition of α_i, we will get: $U_{\mathrm{WN}}(S_i) \geq \alpha_i U_{\mathrm{MAX}}(S_i^{(-)})$. Now:

$$\sum_{i=1}^{l} U_{\mathrm{WN}}(S_i) \geq \sum_{i=1}^{l} \alpha_i U_{\mathrm{MAX}}(S_i^{(-)}) = \alpha \sum_{i=1}^{l} U_{\mathrm{MAX}}(S_i^{(-)})$$

where the 2nd equality is due to the definition of $\overline{\mathrm{LPoA}_{\mathcal{S}}(G)}$, and the first is just a summation of the former.

Like in Claim 3.4, since $\mathcal{S}$ is of width β we have that $\beta \sum_{i=1}^{n} U_{\mathrm{WN}}(i) \geq \sum_{i=1}^{l} U_{\mathrm{WN}}(S_i)$. Since $\mathcal{S}^{(-)}$ is a cover we have that $\sum_{i=1}^{l} U_{\mathrm{MAX}}(S_i^{(-)}) \geq |U_{\mathrm{OPT}}|$ (Similarly to Claim 3.3 in the proof of Theorem 2).

Putting all the inequalities together we conclude that:

$$\sum_{i=1}^{n} U_{\mathrm{WN}}(i) \geq 1/\beta \sum_{i=1}^{l} U_{\mathrm{WN}}(S_i) \geq 1/\beta \sum_{i=1}^{l} \alpha_i U_{\mathrm{MAX}}(S_i^{(-)}) \geq \alpha/\beta \sum_{i=1}^{l} U_{\mathrm{MAX}}(S_i^{(-)}) \geq \frac{\alpha}{\beta} |U_{\mathrm{OPT}}|. \quad \square$$

The above refinement is also interesting for the algorithmic task of finding a good cover. This is because one can look for sub-games with a high *average* PoA instead of a cover with a high *minimum* PoA. Next, we consider a weighted version of the width parameter.

Theorem 12. *Let G be a graphical game and let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover for $V[G]$ such that $\mathcal{S}^{(-)}$ is a cover, and the width of node $i \in V[G]$ in $\mathcal{S}$ is β_i. Define $\overline{\beta}$ as the average of β_i weighted by the agent's utilities in a predefined global worst Nash equilibrium, that is $\overline{\beta} = \frac{\sum_{i=1}^{n} \beta_i U_{\mathrm{WN}}(i)}{\sum_{i=1}^{n} U_{\mathrm{WN}}(i)}$. Let $\alpha = \overline{LPoA_{\mathcal{S}}(G)}$. Then $GPoA(G) \geq \alpha/\overline{\beta}$.*

Proof (sketch). We proceed according to the proof of Theorem 11 and the definitions: $\overline{\beta} \sum_{i=1}^{n} U_{\mathrm{WN}}(i) = \sum_{i=1}^{n} \beta_i U_{\mathrm{WN}}(i) = \sum_{i=1}^{l} U_{\mathrm{WN}}(S_i)$. $\square$

Going back to the star-of-cliques (Example 4), one can see now that in this case $\overline{\beta} = 1+\epsilon$ for a small $\epsilon = \epsilon(k, l)$ whereas $\beta = k+1$ is the non-weighted width. In the proposed cover, the center w is of width $k+1$, the k vertices of type v are of width 2, and all the $k(l-1)$ vertices of type x are of width 1, and the weights are roughly the same. Thus, Theorem 12 yields a bound of GPoA(G) $\geq \frac{1}{2(1+\epsilon)}$, instead of the much weaker bound of $\frac{1}{2(k+1)}$ of the basic theorem. As we noted before, it can be shown that the actual global price of anarchy is slightly greater than $1/2$, so the above bound is tight.

Note that in the last theorem we took the average according to the utilities of the agents in the global equilibrium. Computationally, a global equilibrium might not be easy to find. Therefore, averaging the β parameter may sometimes be less constructive. We address this issue in Section 5.3.

5.2. Nash expansion

The above methods are not always applicable. For example the width parameter may be computationally intractable. We now introduce a different local parameter that can help in analyzing games which are not well addressed by the previous theorems. This parameter resembles graph expansion parameters but refers directly to the equilibrium welfare so it cannot be deduced solely from the graph. Later we will define a *combinatorial* expansion parameter that can be deduced from the graph.

Definition 5.2 (*A Set Nash Expansion*)**.** Let G be a graphical game and $S \subseteq V[G]$. We say that the *Nash expansion* of S is ξ if, for all sets of strategies for the neighbors of S, for all Nash equilibria of S

$$\xi \leq \frac{\sum_{j \in S^{(-)}} u_j}{\sum_{j \in S} u_j}.$$

In other words, in every Nash equilibrium, the ratio between the welfare of $S^{(-)}$ and the welfare of its (external) boundary is bounded by $\frac{\xi}{1-\xi}$.

Definition 5.3 (*A Cover Nash Expansion*)**.** Let G be a graphical game and $\mathcal{S}$ a cover. We say that the *Nash expansion* of $\mathcal{S}$ is $\xi = \xi_G(\mathcal{S})$ if ξ is the minimum Nash expansion of a set $S_i \in \mathcal{S}$.

Observation 13. *Let G be a graphical game and $\mathcal{S}$ a cover. If the Nash expansion of $\mathcal{S}$ is at least $\xi = \xi_G(\mathcal{S})$ then:*

$$\frac{\sum_{S_i} U_{\mathrm{WN}}(S_i^{(-)})}{\sum_{S_i} U_{\mathrm{WN}}(S_i)} \geq \xi.$$

It is possible to show that if a cover $\mathcal{S}$, where $\mathcal{S}^{(-)}$ is a partition, and has a Nash expansion of ξ, then its weighted width $\overline{\beta}$ is bounded by $1/\xi$ as well. Hence the following can be derived as a corollary of Theorem 12:

Theorem 14. *Let G be a graphical game. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover with $\alpha = \overline{LPoA_{\mathcal{S}}(G)}$ and a Nash expansion ξ, such that $\mathcal{S}^{(-)}$ is a partition. Then $GPoA(G) \geq \alpha\xi$.*[2]

In the next Section 5.3, we discuss the properties of the expansion parameter further. Specifically, we show that if we can bound the maximum ratio between pairs of players' utilities in a global worst Nash equilibrium, then we can replace the Nash expansion parameter by a simple *combinatorial* parameter. This is appealing, for instance, from a computational point of view.

5.3. Balanced games and expansion

In many games it is natural that the utilities of the players will be relatively balanced. We now show that when this is the case, the Nash expansion parameter can be replaced by a simple *combinatorial* parameter. This can greatly assist in the analysis of many games of interest. For example, good bounds can be obtained without even finding any Nash.

Definition 5.4 (*Inequality Parameter*)**.** We say that the *inequality parameter* of a game is at least $\rho \leq 1$ if there exists a global worst Nash equilibrium[3] such that for every two players i, j, $U_{WN}(i) \geq \rho U_{WN}(j)$.[4]

Definition 5.5 (*Combinatorial Expansion*)**.** Let $\mathcal{S}$ be a cover for a graph G. The *combinatorial expansion* $\xi_{comb}(\mathcal{S})$ of $\mathcal{S}$ equals $\frac{\sum_{S_i} |S_i^{(-)}|}{\sum_{S_i} |S_i|}$.

In other words ξ_{comb} is the ratio between the sum of the *number* of elements in the sets without the boundary, and this sum of the whole sets. Note that this local parameter is purely combinatorial and does not refer to the utilities of the players. Let Ξ be the graph theoretic vertex expansion of a graph, then $\xi_{comb} = \frac{1}{1+\Xi}$.

Proposition 5.6. *Let G be a graphical game with an inequality parameter ρ. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover such that:*

1. $\mathcal{S}^{(-)}$ *is a partition*
2. $\alpha = \overline{LPoA_{\mathcal{S}}(G)}$
3. $\xi_{comb} = \xi_{comb}(\mathcal{S})$

Then GPoA$\geq \rho\alpha\xi_{comb}$.

Proof (sketch). We know that since α is the local price of anarchy, and $\mathcal{S}^{(-)}$ is a cover,

$$\sum_{S_i \in \mathcal{S}} U_{WN}(S_i) \geq \alpha \sum_{S_i \in \mathcal{S}} U_{MAX}(S_i^{(-)}) \geq \alpha |U_{OPT}|.$$

Claim 5.7. $\rho\xi_{comb} \sum_{S_i \in \mathcal{S}} U_{WN}(S_i) \leq \sum_{S_i \in \mathcal{S}} U_{WN}(S_i^{(-)})$.

Proof. By the definition of ξ_{comb}

$$\rho\xi_{comb} \sum_{S_i \in \mathcal{S}} U_{WN}(S_i) = \rho \frac{\sum\limits_{S_i} |S_i^{(-)}|}{\sum\limits_{S_i} |S_i|} \sum_{S_i \in \mathcal{S}} U_{WN}(S_i) = \rho \frac{\sum\limits_{S_i} |S_i^{(-)}|}{\sum\limits_{S_i} |S_i|} \sum_{S_i \in \mathcal{S}} \sum_{j \in S_i} U_{WN}(j).$$

Let $U_{WN}(\max)$ and $U_{WN}(\min)$ denote the highest and lowest players' utilities of the predefined global worst Nash equilibrium respectively. By the definition of ρ, $\rho U_{WN}(\max) \leq U_{WN}(\min)$. Thus,

$$\begin{aligned}
\rho \frac{\sum\limits_{S_i} |S_i^{(-)}|}{\sum\limits_{S_i} |S_i|} \sum_{S_i \in \mathcal{S}} \sum_{j \in S_i} U_{WN}(j) &\leq \rho \frac{\sum\limits_{S_i} |S_i^{(-)}|}{\sum\limits_{S_i} |S_i|} \sum_{S_i \in \mathcal{S}} \sum_{j \in S_i} U_{WN}(\max) \\
&\leq \rho U_{WN}(\max) \frac{\sum\limits_{S_i} |S_i^{(-)}|}{\sum\limits_{S_i} |S_i|} \sum_{S_i} |S_i| \\
&\leq U_{WN}(\min) \sum_{S_i} |S_i^{(-)}| \\
&\leq \sum_{S_i \in \mathcal{S}} U_{WN}(S_i^{(-)}). \quad \square
\end{aligned}$$

2 It is of-course possible to define ξ_i for every set S_i and obtain a similar theorem.

3 Note that it suffices that this condition holds for the set of all utilities and then it naturally holds for *global worst Nash* utilities. This way we avoid the need to know a *global worst Nash*.

4 Like in previous cases we can also 'average' this parameter, for example by defining ρ_S for every set S, and obtain similar results. We avoid doing it for the sake of simplicity.

By the fact that $\mathcal{S}^{(-)}$ is a partition

$$\sum_{S_i \in \mathcal{S}} U_{\mathrm{WN}}(S_i^{(-)}) \leq \sum_{i \in V} U_{\mathrm{WN}}(i).$$

We therefore conclude that

$$\sum_{i \in V} U_{\mathrm{WN}}(i) \geq \sum_{S_i \in \mathcal{S}} U_{\mathrm{WN}}(S_i^{(-)}) \geq \rho \xi_{\mathrm{comb}} \sum_{S_i \in \mathcal{S}} U_{\mathrm{WN}}(S_i) \geq \rho \xi_{\mathrm{comb}} \alpha |U_{\mathrm{OPT}}|. \quad \square$$

Since in the biased consensus game $\rho = 1$, if we take a cover $\mathcal{S}$ where $\mathcal{S}^{(-)}$ is a partition, and $\alpha = \overline{\mathrm{LPoA}_{\mathcal{S}}(G)}$, we will have, by Proposition 5.6, GPoA$\geq \alpha \xi_{\mathrm{comb}}$. We can show also that this proposition is tight. Formally:

Proposition 5.8 (*Tightness*). *For every $\epsilon > 0$, there exists a graphical game G and a cover $\mathcal{S}$, such that:*

1. *$\mathcal{S}^{(-)}$ is a partition*
2. *$\alpha = \overline{LPoA_{\mathcal{S}}(G)}$*
3. *$\xi_{\mathrm{comb}} = \xi_{\mathrm{comb}}(\mathcal{S})$*

and: $\alpha \xi_{\mathrm{comb}} \leq GPoA \leq (1 + \epsilon) \alpha \xi_{\mathrm{comb}}$.

Proof (sketch). Consider the biased consensus game (Example 8) played on a torus graph, and consider a cover by $k \times k$ grids (Example 3). Proposition 5.6 implies that $\alpha \xi_{\mathrm{comb}} \leq$GPoA. For the other direction, as noted before: $\xi_{\mathrm{comb}} = \frac{k^2}{k^2+4k}$. By Observation 10, $\alpha = \frac{k^2+4k}{\gamma k^2+4k}$ By choosing $\gamma = \frac{4}{k\epsilon}$ we will get that $\alpha \xi_{\mathrm{comb}} = \frac{\frac{4}{\epsilon\gamma}}{\gamma \frac{4}{\epsilon\gamma} + 4}$ By Observation 9, GPoA(G)$= 1/\gamma$. A simple calculation then shows that GPoA$= 1/\gamma \leq (1 + \epsilon)\alpha \xi_{\mathrm{comb}}$. $\square$

6. Monotonicity

This section studies whether the global price of anarchy is related in a monotone way to the local one. We demonstrate that the answer is negative. We then describe a different parameter that gives rise to such monotonicity. Unfortunately, in many cases, this parameter may yield only very weak bounds. Recall that the local price of anarchy of a subset $S \subseteq V[G]$ is denoted by α_S.

6.1. Non-monotonicity of local price of anarchy

Consider the following family of strict majority games.

Example 15 (*Strict Majority Game*). In a *majority game* each player has $\{0, 1\}$ as the set of strategies. The utility of player i is $u_i = a$ if it plays the same as the strict majority of its neighbors, and $u_i = b$ $(b < a)$ otherwise.

Proposition 6.1 (*Non-Monotonicity*). *For each of the following monotone properties below, there exists a game G and a cover $S = (S_1, \ldots, S_l)$ that contradicts it:*

1. *$\exists i$ s.t. GPoA(G) $\leq \alpha_{S_i}$*
2. *$\forall i$ GPoA(G) $\geq \alpha_{S_i}$*
3. *GPoA(G) $\geq \min_i \{\alpha_{S_i}\}$ even if $\mathcal{S}$ is a partition.*
4. *GPoA(G)$\leq \max_i \{\alpha_{S_i}\}$ even if $\mathcal{S}$ is a partition.*

Proof (sketch). We use the strict majority game from Example 15 to introduce counter examples for the above proposition. We let C_n denote a cycle graph with n nodes.

1. Consider a strict majority game on C_5. Let $S_i = \{i, (i + 1) \mod 5\}$ be a set of two adjacent vertices. Suppose that the two neighbors of S play the same, say without loss of generality 0. If the two nodes in S play 1, we have a local Nash equilibrium that yields a welfare of $2b$ for S_i. If both members play 0, the total utility will be $2a$. Therefore $\alpha_{S_i} \leq b/a$
   On the other hand, in every vector of pure strategies, there is always a pair of adjacent vertices with the same strategy. This means that GPoA(G)$> b/a$. Thus, $\forall i$, GPoA(G) $> \alpha_{S_i}$
2. Consider a strict majority game G on C_4. It is not difficult to verify that PoA(G)$= b/a$, but if $S_i = \{v_1\}$ then $\alpha_S = 1$ and PoA(G)$< \alpha_{S_i}$
3. Consider any game G where GPoA(G)$<$ 1. Consider the partition $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$, where $s_i = \{v_i\}$ are singletons. Since, in equilibrium, players always respond optimally to their environments, $\forall i$, PoA$_G(s_i) = 1$. Thus, PoA(G)$< \min_i \{\alpha_{S_i}\}$
4. Consider again a majority game G on C_5. Let $S_1 = \{v_5, v_1\}$ and $S_2 = \{v_2, v_3, v_4\}$. We already know that GPoA(G)$> b/a$ and LPoA$_G(S_1) = b/a$. We will show that LPoA$_G(S_2) \leq b/a$
   Consider a local Nash equilibrium on S_2 where its neighbors play 0, v_2 play 0 and v_3, v_4 play 1. It is a Nash equilibrium since no player can play like the strict majority of its neighbors. The welfare of S_2 in this equilibrium is $3b$. If all the members of S_2 play 0, the welfare would have been $3a$. Therefore LPoA$_G(S_2) \leq b/a$. Thus, we got that GPoA(G)$> b/a = \max_i \{\alpha_{S_i}\}$. $\square$

In other words, the local price of anarchy of the individual subsets $(S_1, \ldots, S_l)$ does not say much about the price of anarchy of the whole set $S = \bigcup_i S_i$. It is possible to construct examples in which the ratio between the α_{S_i}s and α_S is arbitrarily high. Thus, from an algorithmic perspective, it may be difficult to find good covers for general games.

6.2. A monotone local parameter

We now introduce another local parameter which is monotone.

Definition 6.2. For a game G and $S \subseteq V[G]$, define δ_S to be the ratio between welfare of the worst Nash equilibrium on S for every neighbors' strategy, denote by $U'_{\mathrm{WN}}(S)$ and the best utility that S can get, that is, $\delta_S = \frac{U'_{\mathrm{WN}}(S)}{U_{\mathrm{MAX}}(S)}$.

In other words, δ_S measures the ratio between the worst possible welfare of S and best welfare that S can hope for. Note that in general $U'_{\mathrm{WN}}(S) \leq U_{\mathrm{WN}}(S)$ since the former is not restricted for neighbors' strategies only from Nash. Of course, typically, δ_S is very wasteful.

Proposition 6.3. *Let $\mathcal{S} = \{S_1, S_2, \ldots, S_l\}$ be a cover for G, a graphical game. Then GPoA(G) $\geq \min_{S_i}\{\delta_{S_i}\}$.*

We next show that this bound is also tight. The proposition uses the biased consensus game but now for all covers.

Proposition 6.4. *For every graph G, and every cover $\mathcal{S}$, there is a graphical game for which*

$$GPoA \leq \min_{S_i}\{\delta_{S_i}\} < 1.$$

Corollary 6.5. *By the last proposition, and by Proposition 6.3, there exists a game G, for which for every cover $\mathcal{S} = \{S_i\}_i$*

$$GPoA(G) = \min_{S_i}\{\delta_{S_i}\}.$$

Unfortunately, it is not difficult to construct examples in which these δ values yield only very weak bounds on the global price of anarchy.

7. Conclusion

In real life, almost every game is embedded in a larger game and players are likely to be able to consider only their close vicinity. Thus, we view the investigation of the relations between local and global properties of games as a basic issue in the understanding of large games. This paper demonstrates that at least from the perspective of the price of anarchy, a good local behavior of a game implies a good global behavior. The converse is not necessarily true, and there are many non-trivial questions which are related to bounding the price of anarchy of graphical games. Of course, it is natural to investigate questions, similar to the ones which are studied here, in the context of other properties of games.

In general, we believe that models like graphical games provide an excellent opportunity to introduce many structural properties into games. We believe that such properties arise naturally in many contexts and can give rise to a lot of fruitful research on the border of game theory, combinatorics, and computer science.

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