A Tutorial on Mean-payoff and Energy Games

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Abstract. In this tutorial, we first review the definitions of mean-payoff and energy games. We then present simple and self-contained proofs of their main properties. We also provide a pseudo-polynomial time algorithm to solve them. Pointers to more advanced materials in the literature are also provided to the reader.

1. Introduction

Two-player mean-payoff games are infinite duration games played on weighted game arenas. Those arenas are integer weighted graphs in which every edge between two states (vertices) has an integer weight. States are partitioned into states of Eve and states of Adam. The game is played starting in a state for an infinite number of rounds, each round being as follows: if the current state is an Eve state, Eve chooses the successor state from the set of outgoing edges; if it is an Adam state, Adam chooses the successor state from the set of outgoing edges. Then the game continues from the new state. Such an interaction between Eve and Adam leads to an infinite path through the graph. The long-run average of the edge-weights along this path, called the *value of the play*, is won by Eve and lost by Adam. This is thus a *zero-sum* game.

The decision problem for mean-payoff games asks, given a state s and a rational number $\nu \in \mathbb{Q}$, if Eve has a strategy to win a value larger than or equal to ν when the game starts in state s. The associated strategy synthesis problem is to construct a strategy for Eve that ensures a value at least ν, if such a strategy exists. Mean-payoff games have been first studied by Ehrenfeucht and Mycielski in [AJ79] where it is shown that memoryless strategies are sufficient to achieve optimal values. Memoryless optimality implies that the decision problem for these games lies in $\mathsf{NP} \cap \mathsf{coNP}$. Despite many efforts, no polynomial-time solution is known so far.

Apart from the theoretical interest that those games attract, due mainly to the current complexity status of their decision problem, mean-payoff games are also attractive from a practical point of view. They are relevant for the synthesis, analysis and verification of reactive systems. Examples of applications include various kinds of scheduling, finite-window on-line string matching, or more generally, analysis of online algorithms [UM96]. Furthermore, mean-payoff games have also important connections with central problems in game theory and logic: parity games and the modal mu-calculus model-checking [ECA93] can be reduced in polynomial time to mean-payoff games.

Energy games are also played on weighted game arenas. In an energy game, given an initial energy level $c_0 \in \mathbb{N}$, the objective of Eve is to maintain the sum of the weights (the energy level) positive at all time. The decision problem for energy games asks, given a weighted game arena and a state s, if there exists an initial energy level $c_0 \in \mathbb{N}$ for which Eve wins from s. Energy games have been first studied in [CdHS03], and their close connection to mean-payoff games has been discovered in [BFL$^+$08] where it is shown that energy and mean-payoff games are log-space inter-reducible.

In this tutorial, we first review the definitions of mean-payoff and energy games. Then we provide simple and self contained proofs of the main properties of those games. First, we show how to solve the decision problems associated to those games with a conceptually simple algorithm based on the unfolding of the weighted game arena into a finite tree over which we define reachability game. From this algorithm, we obtain an elementary proof that those games are determined and that finite memory strategies are sufficient for both Eve and Adam to play optimally. Second, we show how those problems can be solved with a fixed point algorithm that executes in pseudo-polynomial time. This algorithm has the potential of being useful in practice. Again, we provide a self-contained proof of the correctness and completeness of this algorithm. Finally, we show why memoryless determinacy is a corollary of monotony properties of the algorithm. This tutorial provides alternative and arguably *simpler* proofs than those that can be found in the literature.

Structure of the paper Sect. 2 recalls preliminaries. Sect. 3 and 4 recall the definitions of mean-payoff and energy games respectively. Sect. 5 is concerned with finite tree reachability games and a basic theorem due to Zermelo on finite duration games. Sect. 6 introduces the notion of first-cycle unfolding and establishes basic properties that relate a game arena and its first-cycle unfolding. Sect. 7 shows how the first-cycle unfolding can be used to solve decision problems defined on mean-payoff and energy games. Sect. 8 defines a fixed point algorithm to solve energy games and mean-payoff games, and establishes memoryless optimality for both players in those games. Sect. 9 provides pointers to relevant related works.

2. Preliminaries

Let S be a finite set, let $w : \mathbb{N} \to S$, also noted $w \in S^\omega$, be a infinite sequence of elements in S, then $\mathsf{occ}(w) = \{s \in S \mid \exists i \geq 0 : w(i) = s\}$ is the subset of elements of S that occur along w, and $\mathsf{inf}(w) = \{s \in S \mid \forall i \cdot \exists j \geq i \geq 0 : w(j) = s\}$ is the subset of elements of S that occur infinitely often along w.

Definition 1 (Weighted game arena)**.** *A finite (turn-based) two-player weighted* game arena *is a tuple $\mathcal{A} = \langle S_\exists, S_\forall, \mathsf{E}, s_{\mathsf{init}}, \mathsf{w}\rangle$ where:*

- *$S_\exists$ is the finite set of states owned by* Eve*, $S_\forall$ is the finite set of states owned by* Adam*, $S_\exists \cap S_\forall = \emptyset$ and we denote $S_\exists \cup S_\forall$ by S.*

- $\mathsf{E} \subseteq S \times S$ *is a set of transitions, we say that* E *is* total *whenever for all states* $s \in S$*, there exists* $s' \in S$ *such that* $(s, s') \in \mathsf{E}$ *(we often do this assumption w.l.o.g.).*
- $s_{\mathsf{init}} \in S$ *is the initial state.*
- $\mathsf{w} : \mathsf{E} \to \mathbb{Z}$ *is the weight function that assigns an integer to each edge of the weighted game arena.*

Unless otherwise stated we consider a fixed arena $\mathcal{A} = \langle S_\exists, S_\forall, \mathsf{E}, s_{\mathsf{init}}, \mathsf{w} \rangle$ for the rest of the paper.

A *play* in the arena $\mathcal{A}$ is an *infinite* sequence of states $\pi = s_0 s_1 \dots s_n \dots$ such that for all $i \geq 0$, $(s_i, s_{i+1}) \in \mathsf{E}$. A play $\pi = s_0 s_1 \dots$ is *initial* when $s_0 = s_{\mathsf{init}}$. We denote by $\mathsf{Plays}(\mathcal{A})$ the set of plays in the arena $\mathcal{A}$, and by $\mathsf{InitPlays}(\mathcal{A})$ its subset of initial plays. To each play π is associated an infinite sequence of weights, denoted $\mathsf{w}(\pi)$, and defined as follows:

$$\mathsf{w}(\pi) = \mathsf{w}(\pi(0), \pi(1))\mathsf{w}(\pi(1), \pi(2)) \dots \mathsf{w}(\pi(i), \pi(i+1)) \dots \in \mathbb{Z}^\omega.$$

A *history* ρ is a finite sequence of states that is a *prefix* of a play in $\mathcal{A}$. We denote by $\mathsf{Pref}(\mathcal{A})$ the set of prefixes of plays in $\mathcal{A}$, and the set of prefixes of initial plays is denoted by $\mathsf{InitPref}(\mathcal{A})$. Given an infinite sequence of states π, and two finite sequences of states ρ_1, ρ_2, we write $\rho_1 < \pi$ if ρ_1 is a prefix of π, and $\rho_2 \leq \rho_1$ if ρ_2 is a prefix of ρ_1. For a prefix of play $\rho = s_0 s_1 \dots s_n$, we denote by $last(\rho)$ its last state s_n, and for all i, j, $0 \leq i \leq j \leq n$, by $\rho(i..j)$ the infix of ρ between position i and position j, i.e. $\rho(i..j) = s_i s_{i+1} \dots s_j$, and by $\rho(i)$ the position i of ρ, i.e. $\rho(i) = s_i$. The set of prefixes that belong to Eve, noted $\mathsf{Pref}_\exists(\mathcal{A})$ is the subset of prefixes $\rho \in \mathsf{Pref}(\mathcal{A})$ such that $last(\rho) \in S_\exists$, and the set of prefixes that belong to Adam, noted $\mathsf{Pref}_\forall(\mathcal{A})$ is the subset of prefixes $\rho \in \mathsf{Pref}(\mathcal{A})$ such that $last(\rho) \in S_\forall$.

Definition 2 (Strategy)**.** *A* strategy *for* Eve *in the arena* $\mathcal{A}$ *is a function* $\sigma_\exists : \mathsf{Pref}_\exists \to S$ *such that for all* $\rho \in \mathsf{Pref}_\exists(\mathcal{A})$*,* $(last(\rho), \sigma_{\mathsf{Eve}}(\rho)) \in \mathsf{E}$*, i.e. it assigns to each play prefix of* $\mathcal{A}$ *that belongs to* Eve *a state which is a* E*-successor of the last state of the prefix. Symmetrically, a* strategy *for* Adam *in the arena* $\mathcal{A}$ *is a function* $\sigma_\forall : \mathsf{Pref}_\forall \to S$ *such that for all* $\rho \in \mathsf{Pref}_\forall(\mathcal{A})$*,* $(last(\rho), \sigma_{\mathsf{Adam}}(\rho)) \in \mathsf{E}$*.*

When we want to refer to a strategy of Eve or Adam, we write it σ. We denote by $\mathsf{Dom}(\sigma)$ the domain of definition of the strategy σ, i.e. for all strategies σ of Eve (resp. Adam), $\mathsf{Dom}(\sigma) = \mathsf{Pref}_\exists$ (resp. $\mathsf{Dom}(\sigma) = \mathsf{Pref}_\forall$).

A play $\pi = s_0 s_1 \dots s_n \dots$ is compatible with a strategy σ if for all $i \geq 0$ such that $\pi(0..i) \in \mathsf{Dom}(\sigma)$, we have that $s_{i+1} = \sigma(\rho(0..i))$. We denote by $\mathsf{Outcome}_s(\sigma)$ the set of plays that are starting in s and that are compatible with the strategy σ. Given a strategy $\sigma_\exists$ for Eve and a strategy $\sigma_\forall$ for Adam, and a state s, we write $\mathsf{Outcome}_s(\sigma_\exists, \sigma_\forall)$ the unique play that starts in s and which is compatible both with $\sigma_\exists$ and $\sigma_\forall$.

A strategy σ is *memoryless* if for all histories $\rho_1, \rho_2 \in \mathsf{Dom}(\sigma)$ such that $last(\rho_1) = last(\rho_2)$ then $\sigma(\rho_1) = \sigma(\rho_2)$, i.e. memoryless strategies only depend on the last state of the history and so they can be seen as (partial) functions

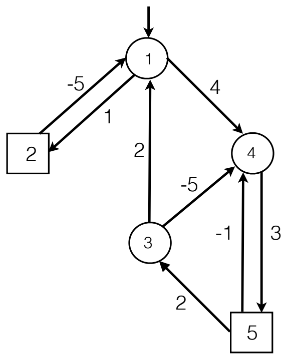

Figure 1. An example of a Mean-payoff game.

from S to S. A strategy σ is *finite memory* if there exists an equivalence relation $\sim \subseteq \mathsf{Dom}(\sigma) \times \mathsf{Dom}(\sigma)$ of *finite index* such that for all histories ρ_1, ρ_2 such that $\rho_1 \sim \rho_2$, we have that $\sigma(\rho_1) = \sigma(\rho_2)$. If the relation $\sim$ is *regular* (computable by a finite state machine) then the finite memory strategy can be modeled by a finite state transducer (a so-called *Moore* or *Mealy* machine).

3. Mean-payoff games

Let $\rho = s_0 s_1 \dots s_n$ be s.t. $(s_i, s_{i+1}) \in \mathsf{E}$ for all i, $0 \le i < n$, the mean-payoff of this sequence of edges is

$$\mathsf{MP}(\rho) = \frac{1}{n} \cdot \sum_{i=0}^{i=n-1} \mathsf{w}(\rho(i), \rho(i+1)).$$

I.e. the mean-value of the weights of the edges traversed by the finite sequence ρ. The *mean-payoff* of an (infinite) play π, denoted $\mathsf{MP}(\pi)$, is a real number defined starting from the sequence of weights $\mathsf{w}(\pi)$ as follows:

$$\mathsf{MP}(\pi) = \liminf_{n \to +\infty} \frac{1}{n} \cdot \sum_{i=0}^{i=n-1} \mathsf{w}(\pi(i), \pi(i+1)).$$

I.e. $\mathsf{MP}(\pi)$ is the limit of the infimums of running averages of weights seen along the play π. Note that we need to use $\liminf$ because the value of the running averages of weights may oscillates along π, and so the limit is not guaranteed to exist.

In a mean-payoff game, Eve want to maximize the average value of the edges that are traversed by the play while Adam tries to minimize this value.

Example 1. *Consider the example depicted in Fig. 1 (circles belong to* Eve *and squares to* Adam*). The game is started in state* 1*. There,* Eve *can choose to move either to state* 2 *or to state* 4*. Assume she decides for state* 4*. Then from there,*

Adam has only one choice: to move back to 1. At that point of the game, the prefix 121 has been created and the mean-payoff so far is equal to $\frac{1}{2} \cdot (1 + (-5)) = \frac{-4}{2} = -2$. The game continues from 1 for infinitely many rounds. It is easy to see that if Eve systematically moves from state 1 to state 2, then the resulting mean-payoff will be equal to -2. Clearly, Eve can play a better strategy in this game.

Indeed, in state 1, Eve should move to state 4 and then to state 5. In state 5, Adam can choose either to play back to 4 or to move to 3. In this last case, the strategy of Eve should be to go back to state 1. Let us analyse the outcome of those two scenarios. In the first case, the outcome of the game would be the infinite play $\pi_1 = (1453)^\omega$ with $\mathsf{MP}(\pi_1) = \frac{11}{4}$ and in the second case $\pi_2 = 1(45)^\omega$ with $\mathsf{MP}(\pi_2) = \frac{2}{2} = 1$. Clearly, as Adam want to minimize the mean-payoff of the resulting play, he should play according to the second scenario.

Exercise 1. *Using the mean-payoff game of Fig. 1, definite an infinite play for which the limit of the sequence of averages $(\frac{1}{n} \cdot \sum_{i=0}^{i=n-1} \mathsf{w}(\pi(i), \pi(i+1)))_{n \in \mathbb{N}}$ does not exist.*

Given a threshold value $\nu \in \mathbb{Q}$, the set of winning plays for Eve in the mean-payoff game defined by the weighted arena $\mathcal{A}$ is equal to $\{\pi \in S^\omega \mid \mathsf{MP}(\pi) \geq \nu\}$. We say that Eve wins this game from state s, if there exists a strategy $\sigma_\exists$, such that $\mathsf{Outcome}_s(\sigma_\exists) \subseteq \{\pi \in S^\omega \mid \mathsf{MP}(\pi) \geq \nu\}$. If, on the other hand, there exists a strategy $\sigma_\forall$, such that $\mathsf{Outcome}_s(\sigma_\forall) \cap \{\pi \in S^\omega \mid \mathsf{MP}(\pi) \geq \nu\} = \emptyset$, then we say that Adam wins the game from s.

Definition 3 (Mean-payoff decision problem)**.** *Given a threshold $\nu \in \mathbb{Q}$, a weighted game arena $\mathcal{A}$, decide whether there exists a strategy $\sigma_\exists$ for Eve such that*

$$\mathsf{Outcome}_s(\sigma_\exists) \subseteq \{\pi \in S^\omega \mid \mathsf{MP}(\pi) \geq \nu\}$$

, or if there exists a strategy $\sigma_\forall$ for Adam such that

$$\mathsf{Outcome}_s(\sigma_\forall) \subseteq \{\pi \in S^\omega \mid \mathsf{MP}(\pi) < \nu\}.$$

Remark 1. *In this decision problem, we can assume w.l.o.g. that the threshold $\nu = 0$. This is because, for any $\nu \in \mathbb{Q}$ and weighted game arena, we can do the following transformation: change the weight function w into $\mathsf{w}'(e) = \mathsf{w}(e) - \nu$ for all edges $e \in \mathsf{E}$. Clearly, for all plays $\pi \in \mathsf{Plays}(\mathcal{A})$, it is the case that $\mathsf{MP}(\mathsf{w}(\pi)) - \nu = \mathsf{MP}(\mathsf{w}'(\pi))$, and so $\mathsf{MP}(\mathsf{w}(\pi)) \geq \nu$ if and only if $\mathsf{MP}(\mathsf{w}'(\pi)) \geq 0$. Then, if the resulting weights are not integers but rationals, it remains to multiply them by the least common multiple of their denominators so that we obtain only integer weights in the weighted game arena.*

For all mean-payoff games, threshold $\nu \in \mathbb{Q}$, and state s, it is always the case that either Eve has a winning strategy or Adam has a winning strategy, that is *mean-payoff games* are *determined*, this is a consequence of a general result on the determinacy of turn-based games for Borel objectives [Mar75]. We will give a simple and self-contained proof of this result later in this tutorial.

For quantitative games, we are also interested by the maximal value that a player can force from a given state.

Definition 4. *The value for* Eve *in state s is noted and defined as follows:*

$$\mathsf{Val}_{\mathsf{Eve}}(s) = \sup_{\sigma_\exists \in S_\exists} \inf_{\sigma_\forall \in S_\forall} \mathsf{MP}(\mathsf{Outcome}_s(\sigma_\exists, \sigma_\forall)).$$

While the value for Adam *is defined as:*

$$\mathsf{Val}_{\mathsf{Adam}}(s) = \inf_{\sigma_\forall \in S_\forall} \sup_{\sigma_\exists \in S_\exists} \mathsf{MP}(\mathsf{Outcome}_s(\sigma_\exists, \sigma_\forall)).$$

For mean-payoff games, it is also known that $\mathsf{Val}_{\mathsf{Eve}}(s) = \mathsf{Val}_{\mathsf{Adam}}(s)$, i.e. mean-payoff games are *valued determined*.

Definition 5 (Mean-payoff value problem)**.** *Given a weighted game arena $\mathcal{A}$ and a state s, compute the maximal value $\mathsf{Val}_{\mathsf{Eve}}(s)$, i.e. the maximal value that* Eve *can force from state s.*

4. Energy games

In a mean-payoff game, the objective of Eve is to maximize the average weight of edges traversed along the play. In an energy game, starting from a given energy level, Eve pursues the objective to maintain the running sum of weights crossed along the play positive at all time. With this interpretation in mind, a negative weight models a loss of energy while a positive weight models a gain of energy, and the goal of Eve is to keep the energy level nonnegative all along the infinite play.

Let $\rho = s_0 s_1 \dots s_n \in \mathsf{Pref}(\mathcal{A})$, and $c_0 \in \mathbb{N}$. The *energy level* associated to ρ starting with initial energy level c_0 is noted and defined as follows:

$$\mathsf{EL}_{c_0}(\rho) = c_0 + \sum_{i=0}^{i=n-1} \mathsf{w}(s_i, s_{i+1}).$$

Given an initial energy level $c_0 \in \mathbb{N}$, a play $\pi = s_0 s_1 \dots$ is *winning* for Eve if for all positions $i \geq 0$, $\mathsf{EL}_{c_0}(\pi(0..i)) \geq 0$, i.e. if the energy level stays nonnegative in all the prefixes of π, otherwise, the play is winning for Adam. So the set of winning plays for Eve is

$$\{\pi \in S^\omega \mid \forall i \geq 0 \cdot \mathsf{EL}_{c_0}(\pi(0..i)) \geq 0\}.$$

Example 2. *Let us consider again the weighted game arena depicted in Fig. 1. If* Eve *always plays from 1 to 2, then no matter what is the initial energy level $c_0 \in \mathbb{N}$, the energy level will be negative at some point along the play. On the other hand, if she always plays from 1 to 4 and then from 3 to 1, then no matter what is the strategy of* Adam*, she will be able to maintain a nonnegative energy level even if she starts with initial energy level $0 \in \mathbb{N}$.*

We will consider two decision problems for energy games.

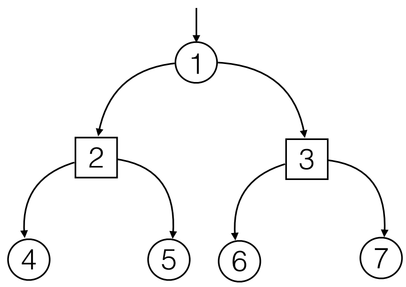

Figure 2. An example of a reachability game on a finite tree.

Definition 6 (Unknown initial energy level problem)**.** *Given a weighted game arena $\mathcal{A}$ and a state s, decide if there exists an initial energy level $c_0 \in \mathbb{N}$ and a strategy $\sigma_\exists$ for* Eve*, such that for all strategies $\sigma_\forall$ of* Adam*, for all prefixes ρ of* $\mathsf{Outcome}_s(\sigma_\exists, \sigma_\forall)$*,* $\mathsf{EL}_{c_0}(\rho) \geq 0$.

Definition 7 (Fixed initial energy level problem)**.** *Given a weighted game arena $\mathcal{A}$, a state s, and an (fixed) initial energy level $c_0 \in \mathbb{N}$, decide if there exists a strategy $\sigma_\exists$ for* Eve*, such that for all strategies $\sigma_\forall$ of* Adam*, for all prefixes ρ of* $\mathsf{Outcome}_s(\sigma_\exists, \sigma_\forall)$*,* $\mathsf{EL}_{c_0}(\rho) \geq 0$.

5. Finite Game Reachability Trees

While mean-payoff and energy games are games of *infinite duration*, we consider in this section games of *finite duration* played on finite trees.

Definition 8 (Finite Tree Arena)**.** *A (two-player)* finite tree arena *is a tuple $\mathcal{T} = (N, n_0, Tr, L)$ where:*

- *N is a finite set of* nodes *partitioned into nodes $N_\exists$ that belong to* Eve*, and nodes $N_\forall$ that belong to* Adam*;*
- *$n_0 \in N$ is the root of the tree;*
- *$Tr \subseteq N \times N$ is the* transition relation *of the tree;*
- *$L \subseteq N$ is the set of leaves of the tree (nodes without children), i.e. $\forall \ell \in L : \neg\exists n' \in N : (\ell, n') \in Tr$.*

The underlying graph (N, Tr) has a tree structure. The sequence of nodes from the root to a node is called a branch*. We sometimes identify nodes with their branches. We write $n < n'$ if n is an ancestor of n' as the branch down to n is a prefix of the branch down to n'.*

We define a *reachability game* on a finite tree by declaring a subset of its leaves as winning for Eve. Formally, a reachability game on a finite tree is defined by $\mathcal{G}_\mathcal{T} = (\mathcal{T}, \mathcal{L})$ where $\mathcal{L} \subseteq L$. A reachability game on a finite tree arena is played in rounds as follows: the game is started at the root of the tree, then in each round, the player that owns the current node chooses a successor of that node in Tr, and when the game reaches a leaf in L, it is stopped. Eve is the winner of the game if the leaf that is reached is in $\mathcal{L}$, otherwise it is Adam that wins the game. So, as mean-payoff and energy games, finite tree reachability games are zero-sum games.

We specialize the notion of *strategies* and *outcomes* for finite tree arenas as follows. A *strategy* for Eve in a finite tree arena $\mathcal{T} = (N, n_0, Tr, L)$ is a function $\sigma_\exists : N_\exists \setminus L \to N$ such that for all $n \in N_\exists \setminus L : (n, \sigma_\exists(n)) \in Tr$. Symmetrically, a *strategy* for Adam is a function $\sigma_\forall : N_\forall \setminus L \to N$ such that for all $n \in N_\forall \setminus L$: $(n, \sigma_\exists(n)) \in Tr$. The outcome of a strategy $\sigma_\exists$ is the set of leaves (identifying branches) that are *compatible* with $\sigma_\exists$ in the following sense:

$$\mathsf{Outcome}_{n_0}(\sigma_\exists) = \{\ell \in L \mid \forall \rho < \ell : \rho \in N_\exists \to \sigma_\exists(last(\rho)) \leq \ell\}.$$

I.e. it is the set of leaves with a branch that is compatible with the choices made by $\sigma_\exists$. The notion of outcome of a strategy for Adam is defined symmetrically.

Example 3. *Let us consider the finite tree game depicted in Fig. 2 (circles belong to* Eve *and squares to* Adam*). In this finite tree reachability game,* Eve *has a winning strategy to reach the set of leaves* $\{4, 5\}$*. Indeed at the root of the tree, she can decide to move the game to node* 2*. From there,* Adam *cannot avoid to enter the set* $\{4, 5\}$*. On the other hand,* Eve *does not have a strategy to reach the set of leaves* $\{4, 6\}$ *as* Adam *has a strategy to reach the set* $\{5, 7\}$*.*

The following classical result, due to Zermelo [SW01], states that finite tree reachability games are determined:

Theorem 2. *For all finite tree reachability games* $\mathcal{G}_\mathcal{T} = (\mathcal{T}, \mathcal{L})$*, it is the case that either* Eve *has a strategy to reach a leaf in* $\mathcal{L}$ *or* Adam *has a strategy to reach a leaf in* $L \setminus \mathcal{L}$*.*

Proof. The proof is by induction on the depth of tree $\mathcal{T}$. Base case. If the depth is one, then the root is a leaf and if this leaf is in $\mathcal{L}$, Eve wins otherwise Adam wins. Inductive case. By induction hypothesis, for every node n which is the root of a subtree of depth at most k, either Eve has a strategy from there to reach a leaf in $\mathcal{L}$ or Adam has a strategy from there to reach a leaf outside of $\mathcal{L}$. We now consider the following cases. First, assume that n belongs to Eve. Then n is winning for Eve if and only if there exists a child n' of n from which Eve has a winning strategy. Indeed, assume that Eve has a winning strategy from a child n', then in n, Eve chooses n' and continue to behave as prescribed by the winning strategy in n'. Clearly this allows Eve to win from n. Assume now that there is no successor n' from which Eve can win. In that case, by induction hypothesis, we know that in all children n' of n, Adam has a strategy to reach a leaf outside of $\mathcal{L}$. So, no matter what is the choice of Eve in n, Adam can reach a leaf outside $\mathcal{L}$, so this means that Adam has a winning strategy in n. The case where n belongs to Adam is symmetric. □

Clearly, this last proof suggests an algorithm for determining who is the winner in a finite tree reachability game. The algorithm works by propagating information starting from the leaves of the tree up to the root as follows. The leaves that are in $\mathcal{L}$ are marked winning for Eve, and the other leaves are marked winning for Adam. Then inductively, as in the proof, the nodes of the tree are marked as winning for Eve or Adam applying the following rule: a node is winning for its owner if and only if it has a child which is winning for this owner, otherwise it is winning for the other player. If the root of the tree is marked winning for Eve, then Eve has a winning strategy in the finite tree reachability game, otherwise the root is marked winning for Adam, and Adam has a winning strategy. This algorithm is classically called *backward induction.*

Example 4. *Let us consider again the finite tree reachability game depicted in Fig. 2. Let us consider the set of leaves* $\mathcal{L} = \{4, 5, 6\}$*. The backward labelling algorithm first labels leaves* 4*,* 5 *and* 6 *as winning for* Eve *and the leaf* 7 *as winning for* Adam*. Then, it proceeds to the upper level and declare* 2 *as winning for* Eve *because even if the node belongs to* Adam*, he does not have any possibility to avoid a winning node of* Eve *as both children of* 2 *are winning for* Eve*. On the contrary, node* 3 *that belongs to* Adam *is declared winning for* Adam*, as he has the option to choose to go from* 3 *to* 7*. The root belongs to* Eve *and is declared winning for* Eve *as she has the choice to move the game to node* 2 *which is winning for her. A winning strategy for* Eve *is thus to move the game from* 1 *to* 2*, from there* Adam *cannot avoid a winning leaf for* Eve*.*

6. First Cycle Unfoldings

Given a weighted game arena, we consider its unfolding up to a first cycle and associated to it a finite tree reachability game.

A *simple cycle* in the weighted game arena $\mathcal{A}$ is a finite sequence $s_0 s_1 \dots s_n$ such that $s_0 = s_n$ and for all $0 < i < j \leq n$, $s_i \neq s_j$. We write $\mathsf{Cy}(\rho)$ if ρ is a simple cycle. In the sequel, we simply say cycle instead of simple cycle. The first cycle unfolding is formally defined as follows.

Definition 9 (First cycle unfolding)**.** *Given a weighted game arena* $\mathcal{A} = \langle S_\exists, S_\forall, \mathsf{E}, s_{\mathsf{init}}, \mathsf{w}\rangle$*, its* first cycle unfolding *from state* s_{init} *is the finite tree* $\mathsf{FCU}(\mathcal{A}) = (N, n_0, Tr, L)$ *where:*

- *The set of nodes* N *is the set of prefixes of initial histories of* $\mathcal{A}$ *stopped at the first occurence of a cycle, i.e. the set*

  $$\mathsf{Pref}\{\rho = \rho_1 \cdot \rho_2 \in \mathsf{InitPref}(\mathcal{A}) \mid \mathsf{Cy}(\rho_2) \wedge \forall 0 \leq i < i' < |\rho| - 1 : \rho(i) \neq \rho(i')\}.$$

  We partition the nodes of the tree into nodes $N_\exists = \{\rho \in N \mid last(\rho) \in S_\exists\}$ *that belong to* Eve *and nodes* $N_\forall = \{\rho \in N \mid last(\rho) \in S_\forall\}$ *that belong to* Adam*;*
- $n_0 = s_{\mathsf{init}}$*, the root of the unfolding is the initial state of* $\mathcal{A}$*;*
- Tr *contains exactly the pairs of nodes* $(n, n') \in N \times N$*, such that if* $n = \rho$ *and* $n' = \rho'$ *then* $|\rho'| = |\rho| + 1$*,* $\rho < \rho'$*, and* $(last(\rho), last(\rho')) \in \mathsf{E}$*;*
- $L = \{n \in N \mid \neg\exists n' \in N : (n, n') \in Tr\}$*.*

For all $\rho \in L$, $\mathsf{Cycle}(\rho)$ denotes the cycle that closes the branch ρ.

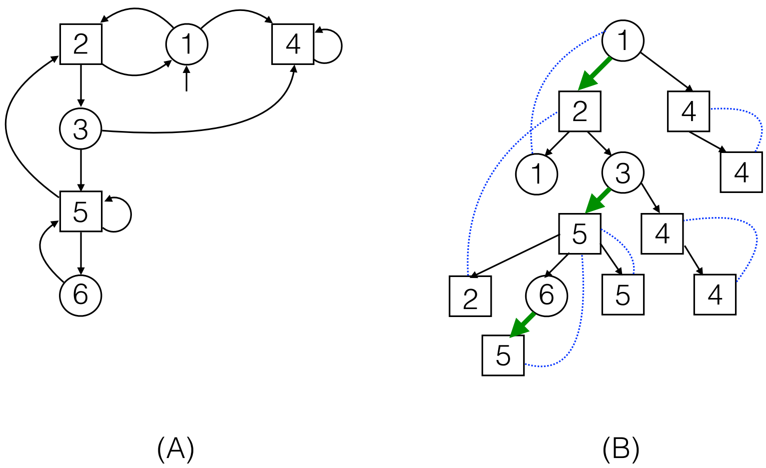

Figure 3. An weighted game arena and its first-cycle unfolding.

Example 5. *The* FCU *of the game arena of Fig. 3(A) is given in Fig. 3(B). Each dotted edge depicts the cycle that is used to stop the unfolding of the branch on which appears the cycle.*

Cycle decomposition of plays and prefixes The cycle decomposition of a prefix of play ρ, noted $\mathsf{dec}(\rho) = (\mathsf{C}(\rho), \beta(\rho))$ is a decomposition of ρ into a sequence simple cycles $\mathsf{C}(\rho)$ and a acyclic part $\beta(\rho)$, called the *residue*. It is defined inductively as follows: for a single state prefix $\rho = s$, $\mathsf{dec}(\rho) = (\emptyset, s)$, and for a prefix $\rho' = \rho \cdot s$:

- if $s \in \beta(\rho)$ and $\beta(\rho) = \rho_1 \cdot s \cdot \rho_2$, then $\mathsf{C}(\rho') = \mathsf{C}(\rho); s \cdot \rho_2 \cdot s$ and $\beta(\rho') = \rho_1 \cdot s$.
- if $s \notin \beta(\rho)$ then $\mathsf{C}(\rho') = \mathsf{C}(\rho)$ and $\beta(\rho') = \beta(\rho) \cdot s$.

The cycle decomposition above can also be applied to an infinite play π by applying it to its successive prefixes.

The following properties of cycle decompositions hold (proofs are left as exercises.)

Proposition 1 (Residues)**.** *Let $\mathcal{A}$ be a weighted game arena, for all plays $\pi \in \mathsf{Plays}(\mathcal{A})$, let $\mathcal{S} = \cup_{i \in \mathbb{N}} \{s \mid s \in \beta(\pi(0..i))\}$, i.e. the set of states that appear in residues along the cycle decomposition of π, then $\mathcal{S} = \mathsf{occ}(\pi)$, i.e. this set of states is exactly equal to the set of states that appear along π.*

Proposition 2 (Cycles)**.** *Let $\mathcal{A}$ be a weighted game arena, for all plays $\pi \in \mathsf{Plays}(\mathcal{A})$, let C be the set of cycles c such that there exists infinitely many positions $i \in \mathbb{N}$ in π such that c is added to $\mathsf{C}(\pi(0..i))$, then the following equality holds $\mathsf{inf}(\pi) = \cup_{c \in C} \mathsf{occ}(c)$.*

Example 6. *Let us illustrate those two propositions on an example using the game arena of Fig. 3(A) and its* FCU *given in Fig. 3(B). Let us consider the finite path* 12123521 *in the game arena. The cycle decomposition of this path is as follows. In the decomposition process,* β *can be seen as being a stack. Initially,* β *contains* 1*, then it contains* 12*, and then* 121*. At that point a cycle is detected, added to* C *and remove from the stack, i.e. the stack* β *now only contains* 1*. Then* 2*,* 3*,* 5*, and* 2 *are added to* β *which is now equal to* 12352*, at that point, the cycle* 2352 *is added to* C*, and removed from* β *which is now equal to* 12*. Finally,* 1 *is added to* β *which contains now* 121*. A new cycle is thus detected and added to* C*. So,* $\mathsf{C}(12123521) = 121; 2352; 121$ *and* $\beta(12123521) = 1$*. Now, if we consider the infinite path* $(12123521)^\omega$*, the cycles* 121 *and* 2352 *will be added infinitely often in the cycle decomposition and so the union of the sets of states that compose those two cycles, i.e. the set* $\{1, 2, 3, 5\}$ *correspond to the set of states that appear infinitely often along* $(12123521)^\omega$ *as expected. This is also the set of states that appear in the stack* β *along the decomposition, so it corresponds exactly to the set of states that appear along* $(12123521)^\omega$*.*

Now, if the cycle decomposition is applied to the infinite path $1235652(12)^\omega$*, then the set of states that appear in cycles that are removed infinitely many times during the cycle decomposition is exactly* $\{1, 2\}$*, while the set of states that appear in the stack* β *is equal to* $\{1, 2, 3, 5, 6\}$*.*

Strategy transfer Let $\mathcal{A}$ be a weighted game arena. Let σ be a strategy of Eve (resp. Adam) in the first-cycle unfolding $\mathsf{FCU}(\mathcal{A})$ of $\mathcal{A}$ from s_{init}, we associate to σ a strategy σ^* in the weighted game arena $\mathcal{A}$ which is defined for all initial prefixes $\rho \in \mathsf{InitPref}_\exists$ (resp. $\rho \in \mathsf{InitPref}_\forall$):

$$\sigma^*(\rho) = \sigma(last(\beta(\rho))).$$

and arbitrary for non initial histories.

Example 7. *Let us explain the strategy transfer with the Fig. 3. The strategy* σ *that we consider is depicted with bold arrows in the* FCU*. It is thus a strategy of* Eve*. The corresponding strategy* σ^* *in the weighted game arena is obtained as follows:* σ^* *plays in the weighted game arena exactly as* σ *in the* FCU *up to the point a leaf is reached in the* FCU*. At that point, the strategy* σ^* *continues to follow the strategy* σ *but from the ancestor of the leaf which is responsible for stopping the branch in the unfolding (dotted arrows in the picture), the two nodes of the tree are thus labelled with the same state of the weighted game arena. From there, we repeat this each time that we reach a leaf. Clearly, the current state in the* FCU *is exactly the one identified by the* β *decomposition of the current history in the weighted game arena, and so the process described above to define* σ^* *matches the formal definition.*

Let us illustrate this on an example of interaction between Eve *and* Adam *when* Eve *plays as the strategy depicted in the* FCU *of Fig. 3(B). From* 1*,* Eve *plays* 2 *as suggested by the strategy in the* FCU*. Now assume that* Adam *decides to play* 3*. In history* 123*,* Eve *plays* 5 *as suggested by the strategy in the* FCU*. Assume now that from there* Adam *plays* 2*. Then the history of the play is now*

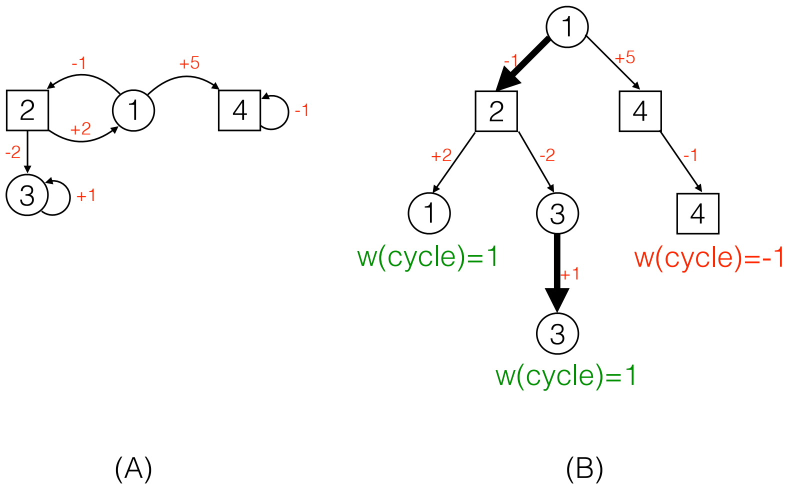

Figure 4. An weighted game arena and its first-cycle unfolding.

12352 *and a leaf is reached in the* FCU*. Note that* $\beta(12352) = 12$*, so* Eve *will continue to play as defined in the* FCU *but from node* 12*.*

Clearly, all strategies σ^* obtained from strategies in the FCU are finite memory strategies as the number of β decompositions of histories is finite (bounded by $|S|!$). Additionally, the strategies that are transferred from the first-cycle unfolding to the weighted game arena have the following properties.

Lemma 1. *The set of states in* $\mathsf{FCU}(\mathcal{A}) = (N, n_0, Tr, L)$ *that are compatible with strategy* σ *is equal to the set of states that are visited by outcomes of* σ^* *in* $\mathcal{A}$*, i.e.*

$$\{last(\rho) \mid \rho \in \mathsf{Outcome}_{n_0}(\sigma)\}$$
$$= \{s \in S \mid \exists \pi \in \mathsf{Outcome}_{s_{\mathsf{init}}}(\sigma^*) \cdot \exists i \geq 0 : s = \pi(i)\}$$

Lemma 2. *The set of cycles in* $\mathsf{FCU}(\mathcal{A})$ *that are compatible with strategy* σ *is equal to the set of cycles that are visited by outcomes of* σ^* *in* $\mathcal{A}$*, i.e.*

$$\{\mathsf{C}(\rho) \mid \rho \in \mathsf{Outcome}_{n_0}(\sigma)\}$$
$$= \{\rho \mid \exists i, j \geq 0 \cdot \exists \pi \in \mathsf{Outcome}_{s_{\mathsf{init}}}(\sigma^*) \text{ } s.t. \text{ } \pi(i..j) = \rho \wedge \mathsf{Cycle}(\rho)\}$$

7. FCU for Solving Mean-payoff and Energy Games

The FCU technique can be used to solve the decision problem for MP games as follows. Let $\mathcal{A}$ be a weighted game arena and w.l.o.g. let the threshold $\nu = 0$. We consider $\mathsf{FCU}(\mathcal{A}) = (N, n_0, Tr, L)$ and the following reachability objective for Eve: $\mathcal{L} = \{\rho \in L \mid \mathsf{MP}(\mathsf{C}(\rho)) \geq 0\}$.

The following lemma tells us that the winner in the tree is the same as the winner in the mean-payoff game.

Lemma 3. *Let $\mathcal{A}$ be a weighted game arena. Let* $\mathsf{FCU}(\mathcal{A})$ *be the first-cycle unfolding of $\mathcal{A}$ and let $\mathcal{L} = \{\rho \in L \mid \mathsf{MP}(\mathsf{C}(\rho)) \geq 0\}$, then the following holds:*

- *if σ is a winning strategy for* Eve *in the tree than σ^* is a winning strategy for* Eve *in the mean-payoff game played on $\mathcal{A}$ for threshold* 0.
- *if σ is a winning strategy for* Adam *in the tree than σ^* is a winning strategy for* Adam *in the mean-payoff game played on $\mathcal{A}$ for threshold* 0.

Proof. Let σ be a winning strategy for Eve in the $\mathsf{FCU}(\mathcal{A})$. Let σ^* be the transfer of this strategy in $\mathcal{A}$. Then all (simple) cycles obtained during the cycle decomposition of any outcome compatible with σ^* in the weighted game arena $\mathcal{A}$ have a mean-payoff larger than or equal to 0. This is a direct consequence of Lemma 2 (cycle transfer lemma). So, the running sum of all prefixes is bounded from below by $-mW$ where $m = |S|$ is the number of states in the weighted game arena and $W = \max_{e \in \mathsf{E}} |\mathsf{w}(e)|$ is the largest absolute value of weights that appear in the weighted game arena, i.e. $-mW$ is a bound on the maximal negative value of the residue along the cycle decomposition. As a consequence the mean-payoff of the play is nonnegative, so Eve wins $\mathsf{MP} \geq 0$.

Let us now consider the other possibility. Let σ be a winning strategy for Adam in the $\mathsf{FCU}(\mathcal{A})$. Let σ^* be the transfer of this strategy in $\mathcal{A}$. By Lemma 2, all (simple) cycles obtained during the cycle decomposition of any outcome compatible with σ^* in the weighted game arena $\mathcal{A}$ have a sum of weights which is less than or equal to -1. So, in this case, the sequence of running sums of the prefixes tends to $-\infty$ and each cycle has a mean-payoff less than or equal to $\frac{-1}{n}$. As a consequence, the mean-payoff of the play is less than or equal to $\frac{-1}{n}$ (as the finite residue on the stack can be neglected in the long run), and so the mean-payoff games is won by Adam. ☐

As a consequence of this reduction and Zermelo's Theorem, we can conclude that mean-payoff games are determined. We can even conclude a stronger result as if Adam has a winning strategy then he has one that forces a negative value bounded away from 0 as stated in the following theorem.

Theorem 3 (MPG strong determinacy)**.** *For all weighted game arena $\mathcal{A}$, for all states s: either there exists a (finite memory) strategy $\sigma_\exists$ for* Eve *such that*

$$\mathsf{Outcome}_s(\sigma_\exists) \subseteq \{\pi \in \mathsf{Plays}(\mathcal{A}) \mid \mathsf{MP}(\pi) \geq 0\}$$

or there exists a (finite memory) strategy $\sigma_\forall$ for Adam *such that*

$$\mathsf{Outcome}_s(\sigma_\forall) \subseteq \{\pi \in \mathsf{Plays}(\mathcal{A}) \mid \mathsf{MP}(\pi) \leq -\frac{1}{n}\}.$$

From Theorem 2, Lemma 3 and Theorem 3, we can also conclude that winning in the game implies winning in the tree:

Corollary 1. *Let $\mathcal{A}$ be a weighted game arena and we consider w.l.o.g. that the threshold $\nu = 0$. Let* $\mathsf{FCU}(\mathcal{A})$ *be the first-cycle unfolding of $\mathcal{A}$ and let $\mathcal{L} = \{\rho \in L \mid \mathsf{MP}(\mathsf{C}(\rho)) \geq 0\}$, then the following holds:*

- *if* Eve *has a winning strategy in the mean-payoff game played on $\mathcal{A}$ for threshold* 0 *then* Eve *has a winning strategy in the reachability game defined on* $\mathsf{FCU}(\mathcal{A})$*.*
- *if* Adam *has a winning strategy in the mean-payoff game played on $\mathcal{A}$ for threshold* 0 *then* Adam *has a winning strategy in the reachability game defined on* $\mathsf{FCU}(\mathcal{A})$*.*

Example 8. *Let us consider the mean-payoff games depicted in Fig. 4 together with its first-cycle unfolding. The bold arrows in the Fig. 4(B) depict a strategy $\sigma_\exists$ of* Eve *that forces to reach the set of leaves* $\{121, 1323\}$*. Each leaf in this set is associated with a cycle which has a nonnegative mean-payoff. It is thus a winning strategy of* Eve *in the* FCU *and thus, according to Lemma 3, $\sigma^*_\exists$ is a winning strategy in the mean-payoff game for threshold* 0*.*

Let us now turn our attention to energy games and show that the $\mathsf{FCU}(\mathcal{A})$ can also be used to solve energy games with unfixed initial energy level. In fact, let us inspect the proof of Lemma 3 and observe that this proof allows us to conclude that a winning strategy σ in the tree for Eve is transferred into a winning σ^* for Eve in the weighted game arena for an initial energy level larger than or equal to mW. Similarly, a winning strategy σ for Adam, is transferred into a winning strategy σ^* in the weighted game arena for Adam as the running sum for all plays compatible with σ^* is not bounded from below and tends to $-\infty$, and so this strategy is winning whatever the initial energy level is. So, we conclude:

Theorem 4 (Determinacy Energy Games)**.** *For all weighted game arena $\mathcal{A}$, for all states s:*

- *either Player 1 wins the energy game from s with initial energy level mW,*
- *or Player 2 has a strategy from s to win the energy level, no matter what is the initial energy level.*

Exercise 2. *Write the details of the proofs of the theorem above.*

We can also formulate the following corollary.

Corollary 2 (Equivalence MPG-EG)**.** *For all weighted game arena $\mathcal{A}$, for all states s,* Eve *has a strategy to force a nonnegative mean-payoff if and only if there exists an initial energy level and a strategy for* Eve *to win the energy game.*

So, by using a simple reduction a to finite tree reachability game, we have proven that mean-payoff and energy games with unfixed initial energy level are inter-reducible. Each player can win in those games by playing strategies that are finite memory (as they are transferred from the finite FCU). A stronger result is known in the literature: in every mean-payoff and energy game, either Eve has a *memoryless* strategy to win or Adam has a memoryless strategy to win. Unfortunately, the use of FCU and Zermelo's theorem are not sufficient to establish this result. The following example shows that there are sets of cycles that can be forced by finite memory strategies but not by memoryless strategies.

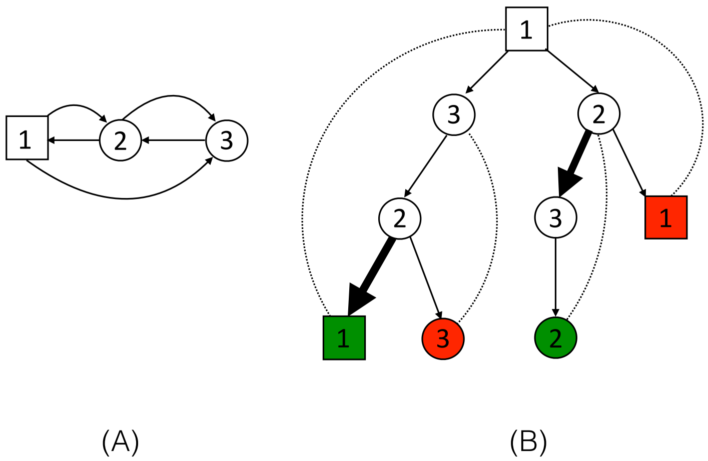

Figure 5. An weighted game arena and its first-cycle unfolding with winning memoryless strategies.

Example 9. *Let us consider Fig. 5 that depicts a weighted game arena together with its* FCU *from state* 1 *and a partition of the leaves into leaves that are good for* Eve $\{1321, 1232\}$*, and good for* Adam $\{1323, 121\}$*. The strategy* $\sigma_\exists$ *of* Eve *in the tree depicted with bold arrows is winning in the three and its transfer* $\sigma^*_\exists$ *in the weighted game arena allows* Eve *to force the set of cycles* $\{1321, 232\}$*. Clearly, strategy* $\sigma^*_\exists$ *is not a memoryless strategy and it is easy to see that there is no memoryless strategy for* Eve *that is able to force the set of cycles* $\{1321, 232\}$*.*

This fact is known and the subclass of games for which winning strategies in the FCU can be transformed into winning memoryless strategies in the weighted game arena has been studied in [AR14,HS07]. Those papers give an alternative to the classical proof [AJ79] of the memoryless determinacy of mean-payoff games. In this note, we propose yet another proof for this result which is based on a fixed point algorithm for solving energy games that was proposed in [BCD$^+$11] and which is exposed in the following section.

To finish this section, we propose the following exercise.

Exercise 3. *Propose and justify the correctness and completeness of an algorithm based on the* FCU *to solved energy games with* fixed *initial energy level.*

8. Pseudo-polynomial Time Algorithm to Solve EG and MPG

We present in this section a fixed point algorithm that determines for each state of a weighted game arena what is the minimal initial energy level that is necessary for Eve to win the energy game (if such an energy level exists), or if the state is winning for Adam no matter what is the initial energy level. Before defining this algorithm, we recall some basic facts about *finite* complete lattices and fixed points of monotonic functions defined on such sets. We refer the interested reader to [Bir67] for more advanced materials on lattice theory and the theory of fixed points.

Elements of lattice and fixed points theory Let X be a finite set of elements and $\leq \subseteq X \times X$ be a *partial order* on this set, i.e. a relation which is *reflexive*, *transitive* and *antisymmetric*. The partially ordered set $(X, \leq)$ forms a *complete lattice* if every subset $A \subseteq X$ has both :

- a *greatest lower bound*, i.e. a value $x_\sqcap \in X$ such that (i) $x_\sqcap \leq y$ for all $y \in A$, and (ii) for all $y \in X$ such that $y \leq y'$ for all $y' \in A$ then $y \leq x_\sqcap$. We denote the greatest lower bound of A by $\sqcap A$.
- a *least upper bound*, i.e. a value $x_\sqcup \in X$ such that (i) $x_\sqcup \geq y$ for all $y \in A$, and (ii) for all $y \in X$ such that $y \geq y'$ for all $y' \in A$, then $y \geq x_\sqcup$. We denote the least upper bound of A by $\sqcup A$.

A *fixed point* of a function $f : X \to X$ is a value $x \in X$ s.t. $f(x) = x$. The set of fixed points of f over X is denoted by $\mathsf{Fix}(f)$. The function f is *monotone* if for all $x, y \in X$ such that $x \leq y$, we have that $f(x) \leq f(y)$. The following lemma recalls classical results over monotone functions over finite complete lattice and their set of fixed points:

Lemma 4. *Let $(X, \leq)$ be finite complete lattice and f be a monotone function over $(X, \leq)$, then the following properties holds:*

1. *the set of fixed points of f is non empty, i.e.* $\mathsf{Fix}(f) \neq \emptyset$,
2. *there always exists a* least fixed point *to f, noted $f^\sqcap$ which is equal to* $\sqcap\mathsf{Fix}(f)$, *i.e. the least element of* $\mathsf{Fix}(f)$.
3. *the set of elements $x \in X$ such that $f(x) \leq x$ are such that $f^\sqcap \sqsubseteq x$, i.e. elements x on which f is decreasing are larger than the least fixed point of f.*
4. *the sequence of values $x_0 = \sqcap X$, and for all $i \leq 1$, $x_i = f(x_{i-1})$ stabilizes after a finite number of iteration on a value y which is equal to $f^\sqcap$.*

Point 4 of the above Lemma is particularly important from an algorithmic point of view as it gives us an effective way to compute the least fixed point of f on the finite set X whenever this function is computable.

Exercise 4. *Prove lemma 4.*

k-bounded energy level functions The fixed point algorithm to solve energy game is based on the notion of *k-bounded energy level functions*. Let $k \in \mathbb{N}$, then we denote by $\mathcal{F}(k)$ the set of functions $f : S \to \{0, 1, \dots, k, +\infty\}$, $+\infty$ being larger than any integer. Given two functions $f_k^1, f_k^2 \in \mathcal{F}$, we write $f_k^1 \sqsubseteq f_k^2$ if and only if for all states $s \in S$, we have that $f_k^1(s) \leq f_k^2(s)$.

The set of k-bounded energy level functions $(\mathcal{F}, \sqsubseteq)$ forms a finite complete lattice with:

- minimal element f_k^{min} where $f_k^{min}(s) = 0$ for all $s \in S$,
- maximal element f_k^{max} where $f_k^{max}(s) = +\infty$ for all $s \in S$,
- least upper bound operator $\sqcup$, where $f_1 \sqcup f_2$ is the function f such that $f(s) = \max(f_1(s), f_2(s))$, and

- greatest lower bound operator $\sqcap$ where $f_1 \sqcap f_2$ is the function f such that $f(s) = \min(f_1(s), f_2(s))$.

Also, we note that all $\sqsubseteq$-increasing chains in $(\mathcal{F}, \sqsubseteq)$ are bounded and contain at most $|S| \cdot (k+2)$ elements.

In the sequel, we use the following non-standard operation $\ominus_k$:

$$\ominus_k : \{0, 1, \ldots, k, +\infty\} \times \mathbb{Z} \to \{0, 1, \ldots, k, +\infty\}, \text{ where:}$$

$$a \ominus_k b = \begin{cases} a - b & \text{if } a, b \in \mathbb{Z} \wedge 0 \leq a - b \leq k \\ 0 & \text{if } a, b \in \mathbb{Z} \wedge a - b < 0 \\ +\infty & \text{if } (a, b \in \mathbb{Z} \wedge a - b > k) \vee a = +\infty. \end{cases}$$

We use k-bounded energy functions to specify for each state of the game what is the minimal energy level necessary for Eve to win. If $f(s) = c \in \mathbb{N}$, this means that an initial energy level of c is sufficient for Eve to win the energy game from state s. While if $f(s) = +\infty$, this means that no matter what is the initial energy level, Adam can force the energy level to become negative. The fixed point algorithm is parameterized by a maximal energy level $k \in \mathbb{N}$ that we track: when this level is exceeded, then we consider that Eve looses. So, for a fixed $k \in \mathbb{N}$, we put more constraint on Eve, but we also show that when $k \in \mathbb{N}$ is taken large enough then this method is *complete.*

Controllable predecessors We define an operator $\mathsf{CPre}_k : \mathcal{F}(k) \to \mathcal{F}(k)$ starting from the following intuition. Let $f \in \mathcal{F}(k)$, and assume that f defines for each state $s \in S$ the minimal energy level $f(s)$ that Eve needs to keep the energy level nonnegative for i steps starting in s. Then $\mathsf{CPre}_k(f)$ specifies for each state $s \in S$ the energy level that is necessary to keep the energy level positive for $i+1$ steps, or it declares that the energy level is too high if it exceeds the bound k. The function $\mathsf{CPre}_k : \mathcal{F}(k) \to \mathcal{F}(k)$ is defined formally as follows:

$$\mathsf{CPre}_k(f)(s) = \begin{cases} \min_{(s,s') \in \mathsf{E}} \quad f(s') \ominus_k \mathsf{w}(s, s') & \text{if } s \in S_\exists \\ \max_{(s,s') \in \mathsf{E}} \quad f(s') \ominus_k \mathsf{w}(s, s') & \text{if } s \in S_\forall \end{cases}$$

The following proposition states that the operator CPre_k is monotone for $\sqsubseteq$.

Proposition 3. *For all $f_1, f_2 \in \mathcal{F}(k)$, if $f_1 \sqsubseteq f_2$ then $\mathsf{CPre}_k(f_1) \sqsubseteq \mathsf{CPre}_k(f_2)$.*

Exercise 5. *Prove the above proposition.*

As CPre_k is monotone on $\mathcal{F}(k)$ and $\mathcal{F}(k)$ is a finite complete lattice, then by Lemma 4 (point 2), we conclude that CPre_k has a least fixed point on $\mathcal{F}(k)$, that we note $\mathsf{CPre}_k^{\sqcap}$, and by point 4 it is the limit of the following increasing sequence of k-bounded energy level functions:

- $f^0 = f_k^{min}$, and for all $i \geq 0$,
- $f^i = \mathsf{CPre}_k(f^{i-1})$.

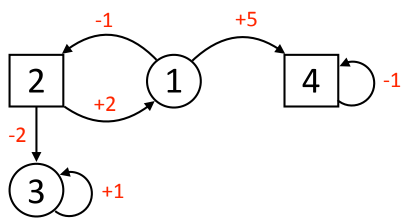

Figure 6. An energy game.

Because $\sqsubseteq$-chains in $\mathcal{F}(k)$ have length bounded by $|S| \cdot (k+2)$, the sequence of k-bounded energy level functions stabilizes in at most such a number of iterations.

Example 10. *Let us consider the example given in Fig. 6. Let us fix the maximal tracked value to be $k = 5$, then starting from $f_0 = (0,0,0,0)$, i.e. $f_0(s) = 0$ for all $s \in S$, and we get the following sequence of 5-bounded energy functions while applying the* CPre_5 *operator:*

- $f_0 = (0,0,0,0)$,
- $f_1 = \mathsf{CPre}_5(f_0) = (0,2,0,1)$,
- $f_2 = \mathsf{CPre}_5(f_1) = (0,2,0,2)$,
- $f_3 = \mathsf{CPre}_5(f_2) = (0,2,0,3)$,
- $f_4 = \mathsf{CPre}_5(f_3) = (0,2,0,4)$,
- $f_5 = \mathsf{CPre}_5(f_4) = (0,2,0,5)$,
- $f_6 = \mathsf{CPre}_5(f_5) = (0,2,0,+\infty)$,
- $f_7 = \mathsf{CPre}_5(f_6) = (1,2,0,+\infty)$,
- $f_8 = \mathsf{CPre}_5(f_7) = (2,2,0,+\infty)$,
- $f_9 = \mathsf{CPre}_5(f_8) = (3,2,0,+\infty)$,
- $f_{10} = \mathsf{CPre}_5(f_9) = (3,2,0,+\infty) = f_9 = \mathsf{CPre}_5^{\sqcap}$.

So, from state 1, Eve *wins in the energy game for any initial energy level larger than or equal to 3, initial energy level 2 is sufficient from state 2 and 0 from state 3. On the other hand, from state 4, the information collected by* CPre_5 *tells us that when tracking energy level up to value 5, no initial energy level is winning for* Eve*. It turns out that for that particular game, the information collected by* CPre_5 *is exact: there is no initial energy level which is sufficient for* Eve *to win from state 4 no matter what is the energy level that is tracked, this is because in 4 one unit of energy is lost during each round of the game. Theorem 5 below tells us that there always exists a natural number $k \in \mathbb{N}$ which is large enough to collect the exact information about the minimal energy levels that are winning for* Eve*.*

The following theorem states the correctness and completeness of the information obtained with the least fixed point of the CPre_k operator. The value $k \in \mathbb{N}$ for which completeness is proved establishes the pseudo-polynomial time complexity of the fixed point algorithm.

Theorem 5. *For all weighted game arenas $\mathcal{A}$ with m states and $W = \max_{e \in \mathsf{E}} |\mathsf{w}(e)|$, we have that:*

(1) *For all $k \in \mathbb{N}$, for all $s \in S$, if $\mathsf{CPre}_k^{\sqcap}(s) \in \mathbb{N}$ then there exists a memoryless winning strategy for* Eve *in the energy game started in state s with initial energy level $c_0 \in \mathbb{N}$ larger than or equal to $\mathsf{CPre}_k^{\sqcap}(s)$.*
(2) *For all $k \in \mathbb{N}$ such that $k \geq 2mW$, for all $s \in S$, if $\mathsf{CPre}_k^{\sqcap}(s) = +\infty$ then there exists a winning strategy for* Adam *in the energy game started in state s no matter what is the initial energy level $c_0 \in \mathbb{N}$.*

Proof. To prove (1), we proceed as follows. We fix the following strategy for Eve: each time the game enters state $s \in S_\exists$, Eve chooses a state s' such that the value

$$s' = \arg\min \mathsf{CPre}_k^{\sqcap}(s') \ominus_k \mathsf{w}(s, s')$$

So this strategy is a *memoryless* strategy. Now, let us prove that when Eve plays this strategy from state s with a energy level $c_0 \in \mathbb{N}$ such that $c_0 \geq \mathsf{CPre}_k^{\sqcap}(s)$ then all possible outcomes

$$(s_0, c_0)(s_1, c_1), \dots, (s_n, c_n), \dots$$

from $s = s_0$, with $c_i = c_{i-1} + \mathsf{w}(s_{i-1}, s_i)$ for all $i \geq 1$, are such that for all $i \geq 0$, $c_i \geq \mathsf{CPre}_k^{\sqcap}(s_i)$, and so $\mathsf{CPre}_k^{\sqcap}(s_i) \neq +\infty$. This will prove that the energy level always stays nonnegative as the function $\mathsf{CPre}_k^{\sqcap}$ only returns nonnegative values.

We prove that by induction. The property holds for $i = 0$, as $c_0 \geq \mathsf{CPre}_k^{\sqcap}(s_0)$ by definition of $c_0 \in \mathbb{N}$. Assume now, by induction hypothesis, that the property holds up to position $i - 1$, and let us show that it holds for position i. We need to consider two cases.

First, let us consider the case $s_{i-1} \in S_\exists$. By induction hypothesis, we have that $c_i = c_{i-1} + \mathsf{w}(s_{i-1}, s_i) \geq \mathsf{CPre}_k^{\sqcap}(s_{i-1}) + \mathsf{w}(s_{i-1}, s_i)$. By definition $\mathsf{CPre}_k^{\sqcap}(s_{i-1}) + \mathsf{w}(s_{i-1}, s_i) = \min_{(s_{i-1}, s') \in \mathsf{E}}(\mathsf{CPre}_k^{\sqcap}(s') \ominus_k \mathsf{w}(s, s')) + \mathsf{w}(s_{i-1}, s_i)$, and as Eve plays to minimize the expression $\mathsf{CPre}_k^{\sqcap}(s') \ominus_k \mathsf{w}(s, s')$, we have that $c_i = \mathsf{CPre}_k^{\sqcap}(s_i) \ominus_k \mathsf{w}(s_{i-1}, s_i) + \mathsf{w}(s_{i-1}, s_i)$ which is larger than or equal to $\mathsf{CPre}_k^{\sqcap}(s_i)$ by definition of the $\ominus_k$, because $\mathsf{CPre}_k^{\sqcap}(s_i)$ is different from $+\infty$, as by induction hypothesis $\mathsf{CPre}_k^{\sqcap}(s_{i-1})$ is different from $+\infty$.

Second, let us consider the case $s_{i-1} \in S_\forall$. By induction hypothesis, we have that $c_i = c_{i-1} + \mathsf{w}(s_{i-1}, s_i) \geq \mathsf{CPre}_k^{\sqcap}(s_{i-1}) + \mathsf{w}(s_{i-1}, s_i)$, which by definition of $\mathsf{CPre}_k^{\sqcap}$ is equal to $\max_{(s_i, s') \in \mathsf{E}}(\mathsf{CPre}_k^{\sqcap}(s') \ominus_k \mathsf{w}(s_{i-1}, s')) + \mathsf{w}(s_{i-1}, s_i)$. So, we have that $c_i \geq \mathsf{CPre}_k^{\sqcap}(s_i) \ominus_k \mathsf{w}(s_{i-1}, s_i) + \mathsf{w}(s_{i-1}, s_i)$ which is larger than or equal to $\mathsf{CPre}_k^{\sqcap}(s_i)$ by definition of $\ominus_k$, because $\mathsf{CPre}_k^{\sqcap}(s_i) \neq +\infty$ as by induction hypothesis $\mathsf{CPre}_k^{\sqcap}(s_{i-1}) \neq +\infty$.

For (2) the proof is as follows. We consider the contrapositive. We show that if Eve wins the energy game from state s, then $\mathsf{CPre}_k^{\sqcap}(s) \neq +\infty$ for all $k \geq 2mW$.

First, we note that if $k' \geq k$ then for all $s \in S$, $\mathsf{CPre}_{k'}^{\sqcap}(s) \leq \mathsf{CPre}_k^{\sqcap}(s)$. As a consequence, if $\mathsf{CPre}_k^{\sqcap}(s) \neq +\infty$ then $\mathsf{CPre}_{k'}^{\sqcap}(s) \neq +\infty$. So, it is sufficient to consider the case where $k = 2mW = K$.

Now, assume that Eve wins the energy game from state s. Then we know by corollary 1 that Eve has a winning strategy in the FCU of the game from state s. Let us consider this FCU and a winning strategy $\sigma_\exists$ for Eve in this unfolding, and $T^{\sigma_\exists}$ the subtree compatible with $\sigma_\exists$.

We annotate the nodes of $T^{\sigma_\exists}$ as follows: the root is annotated by the (initial) energy level mW and all the other nodes are annotated by the energy level on the branch up to the node starting with energy level mW in the root. Then all the nodes n compatible in $T^{\sigma_\exists}$ with $\sigma_\exists$ have received an energy level $\mathsf{EL}(n) = v$ such that $0 \leq v \leq K$, and if n is not a leaf, then $W \leq \mathsf{EL}(n) \leq K - W$. This is because the depth of the tree is at most $m = |S|$ and on each transition in the tree, we gain at most energy W or loose at most energy $-W$. As a consequence, all the nodes n, n' in $T^{\sigma_\exists}$ such that $Tr(n, n')$ (so n is not a leaf), we have that:

$$\mathsf{EL}(n') \ominus_K \mathsf{w}(\mathsf{label}(n), \mathsf{label}(n')) = \mathsf{EL}(n') - \mathsf{w}(\mathsf{label}(n), \mathsf{label}(n')) = \mathsf{EL}(n) \quad (*)$$

Now, we define an energy function associated with the energy annotation of the nodes of $T^{\sigma_\exists}$. Let $f^{\sigma_\exists}(s) = \min_{n|\mathsf{label}(n)=s} \mathsf{EL}(n)$, i.e. we associate to s the minimum energy level of a node labelled with that state in the tree $T^{\sigma_\exists}$, and $f^{\sigma_\exists}(s) = +\infty$ if s does not appear in $T^{\sigma_\exists}$. Additionally to this function, we define a function $g : S \to N$ such that for all states s that labels a node of $T^{\sigma_\exists}$, $g(s)$ returns, if $f^{\sigma_\exists}(s) \neq +\infty$, a node that realizes the value associates to s, i.e. $g(s) = \arg\min_{n|\mathsf{label}(n)=s} \mathsf{EL}(n)$.

Now, we note that for all states s such that $f^{\sigma_\exists}(s) \neq +\infty$, we can assume that $g(s) \notin L$. This is because, as $\sigma_\exists$ is a winning strategy in the FCU then each leaf in $T^{\sigma_\exists}$ has an ancestor with the same label and a *smaller* or an *equal* energy level $(**)$. As a consequence, for all states $s \in S_\exists$ such that $f^{\sigma_\exists}(s) \neq +\infty$, $\sigma_\exists(g(s))$ is well defined: it is a node in $T^{\sigma_\exists}$. Also, for all states $s \in S_\forall$ such that $f^{\sigma_\exists}(s) \neq +\infty$, $g(s)$ returns a node which has a child per state s' s.t. $(s, s') \in \mathsf{E}$. Note also that the following property holds for all nodes $n \in N_\forall$ of Adam in $T^{\sigma_\exists}$ with a child n':

$$\mathsf{EL}(n') - \mathsf{w}(\mathsf{label}(n), \mathsf{label}(n')) = \mathsf{EL}(n) \quad (***)$$

This is a direct consequence of the definition of EL.

Now, we establish that $\mathsf{CPre}_K^{\sqcap} \sqsubseteq f^{\sigma_\exists}$, i.e. $f^{\sigma_\exists}(s)$ is larger than or equal to the least fixed point of CPre_K. To prove that it suffices, according to Lemma 4 point 3, to show that $\mathsf{CPre}_K(f^{\sigma_\exists}) \sqsubseteq f^{\sigma_\exists}$, i.e. to show that $\mathsf{CPre}_K^{\sqcap}$ is decreasing on $f^{\sigma_\exists}$.

We consider now two cases. First, the case $s \in S_\exists$

$$\begin{array}{ll}
\mathsf{CPre}_K(f^{\sigma_\exists})(s) & \\
= \min_{(s,s') \in \mathsf{E}} f^{\sigma_\exists}(s') \ominus_K \mathsf{w}(s, s') & \text{def. } \mathsf{CPre}_K \\
\leq f^{\sigma_\exists}(\mathsf{label}(\sigma_\exists(g(s))) \ominus_K \mathsf{w}(s, \mathsf{label}(\sigma_\exists(g(s)))) & \\
= f^{\sigma_\exists}(\mathsf{label}(\sigma_\exists(g(s))) - \mathsf{w}(s, \mathsf{label}(\sigma_\exists(g(s)))) & \text{by } (*) \\
\leq \mathsf{EL}(\sigma_\exists(g(s))) - \mathsf{w}(s, \mathsf{label}(\sigma_\exists(g(s)))) & \text{def. of } f^{\sigma_\exists} \\
= \mathsf{EL}(g(s)) & \text{by def. of } \mathsf{EL} \\
= f^{\sigma_\exists}(s) & \text{def. of } f^{\sigma_\exists}
\end{array}$$

Second, the case $s \in S_\forall$.

$$\begin{array}{lr}
\mathsf{CPre}_K(f^{\sigma_\exists})(s) & \\
= \max_{(s,s')\in \mathsf{E}} f^{\sigma_\exists}(s') \ominus_K \mathsf{w}(s,s') & \text{def. } \mathsf{CPre}_K \\
\leq \max_{(g(s),n') \mid Tr(g(s),n')} \mathsf{EL}(n') \ominus_K \mathsf{w}(s, \mathsf{label}(n')) & \text{def. of } f^{\sigma_\exists} \\
= \max_{(g(s),n') \mid Tr(g(s),n')} \mathsf{EL}(n') - \mathsf{w}(s, \mathsf{label}(n')) & \text{by } (*) \\
= \mathsf{EL}(g(s)) & \text{by } (***) \\
= f^{\sigma_\exists}(s) & \text{def. of } f^{\sigma_\exists}
\end{array}$$

□

In the previous proof, we have established that Eve can play optimally with memoryless strategies in energy games. We now establish that it is also the case for Adam, and that the same holds for mean-payoff games.

To prove this result, we draw yet another link between mean-payoff and energy games in the following statement.

Lemma 5. *For all weighted game arena $\mathcal{A}$, for all states $s \in S$:*

(1) *for all memoryless strategies $\sigma_\exists$ of* Eve*, $\sigma_\exists$ wins the energy game from s with unknown initial energy level if and only if $\sigma_\exists$ wins the mean-payoff objective $\{\pi \mid \mathsf{MP}(\pi) \geq 0\}$ from s.*

(2) *for all memoryless strategies $\sigma_\forall$ of* Adam*, $\sigma_\forall$ wins the energy games with unknown initial energy level from s if and only if $\sigma_\forall$ wins the mean-payoff objective $\{\pi \mid \mathsf{MP}(\pi) < 0\}$ from s.*

Proof. For (1), we fix a memoryless strategy $\sigma_\exists$ for Eve in the weighted game arena $\mathcal{A}$. When doing so, we obtain a graph where the choices of Eve are fixed. Any infinite path in that graph is determined by a strategy of Adam. It is easy to see that $\sigma_\exists$ is winning for Eve both in the energy game and in the mean-payoff game from state s if and only if all the simple cycles that can be reached from s in the graph have a sum of weight which is nonnegative. The reasoning to obtain a proof of statement (2) is similar. □

Theorem 6. *Both players can play optimally with memoryless strategies in energy and mean-payoff games.*

Proof. We have already proved that Eve can play optimally with memoryless strategies in energy games. As winning strategies for energy games are winning strategies for mean-payoff games with threshold $\nu = 0$ by Lemma 5, the memoryless optimality holds for Eve in mean-payoff games. Remember also that the threshold of a mean-payoff game can always be supposed to be equal to 0 without lost of generality (see Remark 1). To prove the property for Adam, we do the following reasoning: Adam can enforce a mean-payoff less than or equal to $\frac{-1}{n}$ in $\mathcal{A}$ if and only if Adam can enforce a mean-payoff larger than or equal to 0 in the arena $\mathcal{A}'$ which is the arena $\mathcal{A}$ in which the weight function w is replaced by $-\mathsf{w} - \frac{1}{n}$ in $\mathcal{A}$. So Adam can also play optimally with a memoryless strategy in $\mathcal{A}$. By Lemma 5, this also shows that Adam can play optimally with memoryless strategies in energy games. □

As both players can play optimally with memoryless strategies, we deduce that value associated to a state s in a mean-payoff game (remember Definition 5) is determined by two memoryless strategies, one for Eve and one for Adam, and the value is equal to the mean-value of the simple cycle induced by those two strategies. So, we obtain the following corollary:

Corollary 3. *For all weighted game arenas $\mathcal{A}$ with m states and $W = \max_{e \in \mathsf{E}} |\mathsf{w}(e)|$, for all states $s \in S$, the value of state s belongs to the following set:*

$$\left\{ \frac{p}{q} \mid -mW \leq p \leq mW \wedge 1 \leq q \leq m \right\}.$$

So, using binary search the computation of the value of a state can be reduced to a polynomial number of calls to the threshold problem.

9. Related Works and Further Results

All known deterministic algorithms to solve mean-payoff and energy games have either exponential time complexity in the number of states of the weighted game arena or have pseudo-polynomial time complexity (polynomial in the number of states and in W the largest absolute weight in the weighted game arena), see [VAL88,UM96,N. 99,YD07,Sch09,BCD+11].

Multidimensional mean-payoff games have been studied in [CDHR10,CRR14, VCD+15]. Conjunction of *lim inf* mean-payoff objectives are coNP-complete, conjunctions of *lim sup* mean-payoff objectives are in NP $\cap$ coNP. The general case of Boolean combinations of mean-payoff objective is undecidable [Vel15]. Multidimensional energy games with unfixed initial energy level are coNP-complete [CDHR10], and with fixed initial energy level, they are 2EXPTIME-complete [JLS15]. Generalization of those games with imperfect information have been studied in [DDG+10]: energy games with unfixed initial credit problem are undecidable while energy games with fixed initial credit problem are decidable, and mean-payoff games with imperfect information are undecidable.

While the mean-payoff objective considers the long-run average of the weights along the whole play, the window mean-payoff (WMP) objective introduced in [CDRR15] considers weights over a local window of a given size sliding along the play. The objective is now to ensure that the average weight is larger than or equal to a given threshold ν over every bounded window. This is a strengthening of the mean-payoff objective: winning for the WMP objective implies winning for the (classical) mean-payoff objective. Also, any finite-memory strategy that forces a mean-payoff value larger than $\nu + \epsilon$ (for any $\epsilon > 0$), is also winning for the WMP objective for threshold ν provided that the window size is taken large enough. Aside from their naturalness, WMP objectives are algorithmically more tractable than classical mean-payoff objectives as they are solvable in polynomial time and WMP games with imperfect information are decidable [HPR15].

Other quantitative games with measures like *discounted sum*, inf, lim inf, sup, lim sup have been studied in the literature, the interested reader is referred to [UM96] for results about discounted sum and to [CdHS03,CDH10] for the other measures.

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