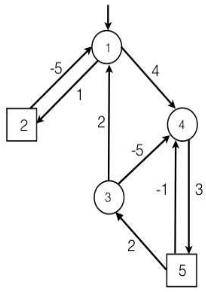

Example 4. *What are the sequence of cycles output by the cycles decomposition of*

$$(1,2)(2,1)(1,2)(2,1)(1,4)[(4,5)(5,3)(3,4)]^\omega$$

Algorithm 1 Cycles-Decomposition $CD(s,\pi)$

Require: s is a finite (possibly empty) simple path ▷ initial stack content
Require: π is a finite or infinite path $\pi_1\pi_2\cdots$ ▷ the path to decompose
Require: If s is non-empty then $\mathsf{trg}(s) = \mathsf{src}(\pi)$ ▷ $s\pi$ must form a path

```
step = 1
while step ≤ |π| do                                  ▷ Start a step
    Append π_step to s                               ▷ Push current edge into stack
    Say s = e_1 e_2 ··· e_m
    if ∃i : e_i e_{i+1} ··· e_m is a cycle then      ▷ If stack has a cycle
        Output e_i e_{i+1} ··· e_m                   ▷ output the cycle
        s := e_1 ··· e_{i-1}                         ▷ Pop the cycle from the stack
    end if
    step := step + 1                                 ▷ advance to next input edge
end while
```

Definition 12. *Let Y be a cycle property. For an infinite path π, let $first(\pi)$ be the first cycle output by $CD(\epsilon,\pi)$. The* first-cycle objective based on Y, written *$FC(Y)$ is the set of plays such $\lambda(first(\pi)) \in Y$.*

Define $N_z(\pi) \in \mathbb{N} \cup \{\infty\}$ to be the index of the first edge that starts with z, if one exists. Define $head_z(\pi)$ to be the prefix of π before $N_z(\pi)$, and $tail_z(\pi)$ to be the suffix of π starting at $N_z(\pi)$.

Formally, $N_z(\pi) := \infty$ if z does not occur on π, and otherwise $N_z(\pi) := \min\{j : \mathsf{src}(\pi_j) = z\}$. Also, $head_z(\pi) := \pi[1, N_z(\pi)-1]$ and $tail_z(\pi) := \pi[N_z(\pi), |\pi|]$. By convention, if $N_z(\pi) = \infty$ then $head_z(\pi) = \pi$ and $tail_z(\pi) = \epsilon$.

We now define a game, that is very similar to the first-cycle game, except that one of the nodes z of the arena is designated as a "reset" node... the first time play sees z (if at all), the history is erased:

Definition 13. *Fix an arena A, a vertex $z \in V$, and a cycle property Y. Define the objective $FC_z(Y)$ to consist of all plays π satisfying the following property: if $head_z(\pi)$ is not a simple path then $first(\pi) \in Y$, and otherwise $first(tail_z(\pi)) \in Y$.*

Playing the game with objective $FC_z(Y)$ is like playing the first-cycle game over Y, however, if no cycle is formed before reaching z for the first time, the prefix of the play up to that point is ignored. Thus, in a sense, the game is reset. Also note that if play starts from z, then the game reduces to a first-cycle game.

Definition 14. *Fix Y. An arena is Y-*resettable *if for every $i \in \{0,1\}$, and every node z, we have that $WR^i(A, FC_z(Y)) = WR^i(A, FC(Y))$.*

Theorem 6 (Resettability implies memoryless determinacy)**.** *Suppose that every arena A is Y-resettable. Then every game $(A, FC(Y))$ is memoryless determined.*

The proof relies on the fact that we may assume that a strategy of $(A, FC_z(Y))$ makes the same move every time it reaches z.

Definition 15 (Forgetful at z from v)**.** *Fix an arena A, a vertex $v \in V$, a Player $i \in \{0,1\}$, and a vertex $z \in V_i$ belonging to Player i. We call a strategy T for Player i* forgetful at z from v *if there exists $z' \in V$ such that $(z,z') \in E$ and for all $\pi \in plays(T,v)$, and all $n \in \mathbb{N}$, if $\mathsf{src}(\pi_n) = z$ then $\mathsf{trg}(\pi_n) = z'$.*

Lemma 2 (Forgetful at z from v)**.** *Fix an arena A, a vertex $v \in V$, a Player $i \in \{0,1\}$, and a vertex $z \in V_i$ belonging to Player i. In the game $(A, FC_z(Y))$, if Player i has a strategy S that is winning from v, then Player i has a strategy T that is winning from v and that is forgetful at z from v.*

Sketch. The second time z appears on a play, the winner is already determined, and so the strategy is free to repeat the first move it made at z. On the other hand, the first time z appears on a play, the strategy can make the same move regardless of the history of the play before z, because the winning condition ignores this prefix. □

Sketch of Theorem 6. A node $z \in V$ is a *choice node* of an arena B, if there are at least two distinct vertices $v', v'' \in V$ such that $(z,v') \in E^B$ and $(z,v'') \in E^B$.

For each $i \in \{0,1\}$, we induct on the number of choice nodes of player i, i.e., the kth inductive hypothesis says that in every arena with k choice nodes of player i, if player i has a winning strategy from v, player i also has a memoryless winning strategy from v.

Base case ($k = 0$): this is immediate since there is a single strategy for player i, which is memoryless.

Inductive step ($k > 0$). Consider an arena A with $k > 0$ choice nodes for player i, and suppose player i has a winning strategy in $(A, FC(Y))$ from v. Let z be a choice node for Player i.

- By the resettability assumption applied to A, Player i has a winning strategy from v in $(A, FC_z(Y))$.
- By Lemma 2, Player i has a strategy S that is winning from v and that is also forgetful at z from v. Thus we may form a sub-arena B of A by removing all edges from z that are not taken by S. Observe that S is winning from v in $(B, FC_z(Y))$.
- Applying the resettability assumption to B, Player i has a winning strategy from v in $(B, FC(Y))$.

- But B has less choice nodes for Player i, and thus, by induction, Player i has a memoryless winning strategy from v in $(B, FC(Y))$.

This memoryless strategy is also winning from v in A (since we only removed choices of player i, which are not used in this memoryless strategy). □

Connection with infinite duration games

We now define the connection between first-cycle games and games of infinite duration (such as parity games, etc.), namely the concept of Y-greedy games. We then prove the Strategy Transfer Theorem, which says, roughly, that for every arena, the winning regions of the First-Cycle Game over Y and a Y-greedy game coincide, and that memoryless winning strategies transfer between these two games.

Definition 16 (Greedy)**.** *Fix an arena A. The* all-cycle objective based on Y, written $AC(Y)$ *is the set of plays π of A such that the labelling of every cycle output by $CD(\epsilon, \pi)$ is in Y.*

Say that a game (A, O) is Y-greedy *if*

$$AC(Y) \subseteq O \text{ and } AC(\neg Y) \subseteq \neg O.$$

The following lemma says that only finitely many edges in a path are pushed but never popped. In particular, at most $|V| - 1$ edges:

Lemma 3. *For every path π in arena A with vertex set V, there are at most $|V| - 1$ indices i such that π_i does not appear in any of the cycles in cycles(π).*

Example 5.
1. *$(A, AC(Y))$ is Y-greedy.*
2. *Every parity game is Y-greedy where Y says "the largest occuring color is even".*
3. *Every game with mean-payoff winning condition is Y-greedy where Y says "the average of the cycle is non-negative" (HW)*

We state the Strategy Transfer Theorem:

Theorem 7 (Strategy Transfer)**.** *Let (A, O) be a Y-greedy game, and let $i \in \{0, 1\}$.*

1. *$WR^i(A, O) = WR^i(A, FC(Y))$ (so, in particular, (A, O) is determined).*
2. *For every memoryless strategy S for Player i, and vertex $v \in V$: S is winning from v in the game (A, O) if and only if S is winning from v in the game $(A, FC(Y))$.*

To prove the Strategy Transfer Theorem we need a lemma that states that one can pump a strategy S that is winning for the first-cycle game to get a strategy $S^{\circlearrowleft}$ that is winning for the all-cycles game.

The strategy $S^{\circlearrowleft}$ says to **follow S until a cycle is formed, remove that cycle from the history, and continue**.

Roughly, $S^{\circlearrowleft}(u) = stack(u)$ where $stack(u)$ is the stack content of CD after processing path u. The fact that every winning strategy in the first-cycle game of Y can be pumped to obtain a winning strategy in a Y-greedy game, is why we call such games "greedy".

First, we need notation to go from edge-paths to node-paths and back:

- For simple path $\pi = e_1 e_2 \cdots e_l \in E^*$ write $nodes(\pi)$ for $\mathsf{src}(e_1)\mathsf{src}(e_2)\cdots\mathsf{src}(e_l)\mathsf{trg}(e_l) \in V^*$.
- For path $u = v_1 v_2 \cdots v_l \in V^*$ with $l \geq 2$, write $edges(u)$ for $(v_1, v_2)(v_2, v_3)\cdots(v_{l-1}, v_l) \in E^+$.

Instead of writing these transformations we assume them implicitly.

Definition 17 (Pumping Strategy)**.** *For a finite path $\pi \in E^*$, the stack content at the end of $CD(\epsilon, \pi)$ (Algorithm 1) is denoted $stack(\pi)$.*

Fix an arena A, a Player $i \in \{0, 1\}$, and a strategy S for Player i. Define the pumping strategy of S *be the strategy $S^{\circlearrowleft}$ on history $u = u_1 \ldots u_k$ ending in a node of player i, as follows:*

$$S^{\circlearrowleft}(u) = \begin{cases} S(u_1) & \text{if } k = 1 \\ S(u_k) & \text{if } k > 1, stack(u) = \epsilon \\ S(stack(u)) & \text{otherwise.} \end{cases}$$

Note that $S^{\circlearrowleft}$ is well-defined since: if $stack(u) \neq \epsilon$ then $stack(u)$ ends with $u_k \in V_i$ and so $stack(\pi)$ is in the domain of S.

Lemma 4 (equal)**.** *If S is memoryless then $S^{\circlearrowleft} = S$.*

Proof. We need to show that $S^{\circlearrowleft}(u) = S(u)$ for all u. If $|u| = 1$ then this is immediate by construction. If $stack(u) = \epsilon$ then $S^{\circlearrowleft}(u) = S(u_k) = S(u)$ since S is memoryless. If $stack(u) \neq \epsilon$ then $last(stack(u)) = last(u)$. So $S^{\circlearrowleft}(u) = S(stack(u)) = S(u)$ since S is assumed memoryless. □

Lemma 5 (pumping)**.** *Fix Player $i \in \{0, 1\}$ and let (A, O) be a Y-greedy game. If S is a strategy for Player i that is winning from v in $(A, FC(Y))$ then $S^{\circlearrowleft}$ is winning from v in (A, O).*

Proof. The strategy $S^{\circlearrowleft}$ says to follow S, and when a cycle is popped by CD, remove that cycle from the history and continue. Thus, for every cycle C that is popped, let l be the time at which the first edge of C is being pushed, and note that the stack up to time l followed by C is a path consistent with S whose first cycle is C.

Thus if S is a strategy of player 0 that is winning from v in the game $(A, FC(Y))$ then for every play $\pi \in plays(S^{\circlearrowleft}, v)$, every cycle in $cycles(\pi)$ is in Y. By definition of Y-greedy, this means that $S^{\circlearrowleft}$ is winning from v in the game (A, O). The case of player 1 is symmetric. □

Proof of Strategy Transfer Theorem. Let Y be a cycle property and A an arena. Suppose that (A, O) is Y-greedy. We begin by proving the first item. Use Lemma 5[pumping] to get that for $i \in \{0, 1\}$,

$$\mathrm{WR}^i(A, FC(Y)) \subseteq \mathrm{WR}^i(A, O).$$

Since first-cycle games are determined, the winning regions $\mathrm{WR}^0(A, FC(Y))$ and $\mathrm{WR}^1(A, FC(Y))$ partition V. Thus, since $\mathrm{WR}^0(A, O)$ and $\mathrm{WR}^1(A, O)$ are disjoint, the containments above are equalities, as required for item 1.

We prove the second item. Suppose S is a memoryless strategy for player 0 and recall that $S = S^{\circlearrowleft}$ (by Lemma 4[memless]).

- Suppose S is winning from v in the game $(A, FC(Y))$. Then it is winning from v in the game (A, O) (By Lemma 5).
- Suppose S is not winning from v in the game $(A, FC(Y))$. Since S is memoryless, plays of A consistent with S are exactly infinite paths in the induced sub-arena $A^{||S}$. Hence, there is a path π in the induced solitaire arena $A^{||S}$ for which the first cycle, say C, satisfies $\neg Y$. Define the infinite path π' which pumps this cycle forever. Being a path in $A^{||S}$, it is a play of A consistent with S. Moreover, π' has the property that every cycle in its cycles-decomposition (i.e., C) satisfies $\neg Y$. Since (A, O) is Y-greedy, S is not winning from v in the game (A, O).

The case that S is a strategy for player 1 is symmetric. □

Putting together we get:

Theorem 8. *If every arena is Y-resettable then every Y-greedy game is memoryless determined.*

Recipe for positional determinacy

Question 14. *How to check if every arena is Y-resettable?*

Definition 18. *Fix an arena A. Let $TAC(Y)$ consist of all plays π of A such that some suffix of π is in $AC(Y)$. An arena A is Y-*unambiguous *if there is no play of A that is in $TAC(Y) \cap TAC(\neg Y)$.*

Example 6. *If A is Y-unambigous then $(A, TAC(Y))$ is Y-greedy. Why? clearly $AC(Y) \subseteq TAC(Y)$. Also $AC(\neg Y) \subseteq TAC(\neg Y) \subseteq \neg TAC(Y)$ by assumption.*

Theorem 9 (unambigous implies resettable)**.** *Every arena that is Y-unambiguous is also Y-resettable.*

Proof. First, if A is Y-unambigous then $(A, TAC(Y))$ is Y-greedy (above). Thus by Theorem 7 the winning regions of $(A, FC(Y))$ and $(A, TAC(Y))$ coincide.

Second, we now show that the winning regions of $(A, FC_z(Y))$ and $(A, TAC(Y))$ coincide. As usual, it is sufficient to show containment, i.e., $WR^i(A, FC_z(Y)) \subseteq WR^i(A, TAC(Y))$ for $i = 0, 1$.

So, suppose player i has a winning strategy S from v in $(A, FC_z(Y))$.

It is sufficient to define a strategy T for player i such that every play consistent with T is in $TAC(Z)$ where $Z = Y$ if $i = 0$ and $Z = \neg Y$ if $i = 1$ (why? if $i = 0$ then this is what we want. if $i = 1$ then every play consistent with T is in $TAC(\neg Y) \subseteq \neg TAC(Y)$, as required).

There are two cases:

1. there is no simple path consistent with S that ends in z. Thus S is winning for $FC(Z)$, so $T = S^{\circlearrowleft}$ is winning for $AC(Z)$ and thus for $TAC(Z)$.
2. o/w let h be a simple path consistent with S ending in z. Define

$$T(u) := \begin{cases} S^{\circlearrowleft}(u) & \text{if } z \text{ does not appear on } u, \\ (S_z)^{\circlearrowleft}(tail_z(u)) & \text{otherwise.} \end{cases}$$

where $S_z(u) = S(hu)$ if u starts in z (and, otherwise arbitrarily).

In words, T behaves like the pumping strategy $S^{\circlearrowleft}$. Once (and if) z is reached, T erases all its memory and switches to $(S_z)^{\circlearrowleft}$ (which itself is winning from z in the FCG). Note that T is winning $TAC(Z)$: if z never appears then every cycle is in Z; if z does appear, then starting at that point in time, every cycle is in Z. ☐

Question 15. *Ok, so how to check if A is Y-unambigous?*

Definition 19. *A* winning condition *is a set $W \subseteq \mathbb{U}^\omega$. On an edge-coloured arena $(A, \lambda : E \to \mathbb{U})$ it determines an objective, i.e., $O(W)$ consisting of all plays π in A such that $\lambda(\pi) \in W$.*

Example 7.

1. *The parity winning condition over $\mathbb{U} = [N]$ consists of all sequences $\alpha \in [N]^\omega$ such that $maxinf(\alpha)$ is even.*
2. *The mean-payoff winning condition over $\mathbb{U} = [-N, N]$ consists of all sequences α such that $\lim_n \inf avg_n(\alpha) \geq 0$.*

Lemma 6. *Suppose W and $\neg W$ are prefix-independent, and consider an arena A and a cycle property Y. If $(A, O(W))$ is a Y-greedy game then A is Y-unambiguous.*

Proof. Suppose not. Take a play π of some arena such that some suffix $\pi' \in AC(Y) \subseteq O(W)$ and some suffix $\pi'' \in AC(\neg Y) \subseteq \neg O(W)$. But one of these is a suffix of the other, say π' is a suffix of π'' (the other case is symmetric). Then $\lambda(\pi') \in W$ is a suffix of $\lambda(\pi'') \notin W$. Contradiction. ☐

But prefix-independence is typically easy to check!

To conclude that every game with winning condition W is memless determined:

1. Check that W and $\neg W$ are prefix independent.
2. Finitise the winning condition W to get a cycle property Y.
3. Show that every game $(A, O(W))$ is Y-greedy.

Example 8. *Apply the recipe to Büchi, parity-, mean-payoff winning conditions. e.g., Let W be the buchi condition. it and its complement are prefix-independent.*

What if the winning-condition is not prefix independent?

1. Say that Y is *closed under cyclic permutations* if $ab \in Y$ implies $ba \in Y$, for all $a \in \mathbb{U}, b \in \mathbb{U}^*$.
2. Say that Y is *closed under concatenation* if $a \in Y$ and $b \in Y$ imply that $ab \in Y$, for all $a, b \in \mathbb{U}^*$.

Theorem 10. *Let Y be a cycle property. If Y is closed under cyclic permutations, and both Y and $\neg Y$ are closed under concatenation, then every arena is Y-unambigous (and therefore Y-resettable).*

To conclude that $(A, O(W))$ is uniform memless determined:

1. Finitise the winning condition W to get a cycle property Y.
2. Check that Y is closed under cyclic permutations and both Y and $\neg Y$ are closed under concatenation.
3. Check that $(A, O(W))$ is Y-greedy.

We conclude with a more sophisticated use of the recipe, applied to the initial credit problem of energy games.

Definition 20. *The* Energy winning-condition with initial credit r, *written* $W_r \subseteq \mathbb{Z}^*$, *consists of all sequences* α *such that* $r + \sum_{1 \leq i \leq n} \alpha_i \geq 0$ *for all* n.

Theorem 11. *Either there is an initial credit with which Player* 0 *(the "energy" player) wins, or for every initial credit Player* 1 *wins. In both cases, we show that the winner can use a memoryless strategy.*

Proof. Finitise "energy" as $Y \subset \mathbb{Z}^*$ which says that the sum of the numbers is non-negative. This property is clearly closed under cyclic permutations and concatentation, and its complement (the sum is negative) is also clearly closed under concatenation. Also $(A, AC(Y))$ is Y-greedy, and thus memoryless determined.

Claim: if σ is a winning strategy for player 0 in $(A, AC(Y))$ then σ is a winning strategy in the energy game on A for some initial credit r.

Indeed, let $r = -t(|V| - 1)$, where t is the minimum amongst the negative weights of the arena. Let π be consistent with σ, and consider some prefix π' of it. By Lemma 3, at most $|V| - 1$ edges are not on $cycles(\pi')$, and thus the energy level at the end of π' is at least the initial credit plus $t(|V| - 1)$. Hence, an initial credit of r suffices for π to be winning for Player 0 also in the energy game.

Claim: if σ is a winning strategy for player 1 in $(A, AC(Y))$ then σ is a winning strategy in the energy game on A for all initial credits.

Indeed, observe that a winning strategy for player 1 in $(A, AC(Y))$ is also winning $(A, FC(Y))$; and thus by pumping is also winning in $(A, EC(Y))$ where $EC(Y)$, the objective for player 0, says that some cycle should be in Y. Thus every play consistent with this strategy has all cycles not in Y, and thus in the energy game the energy along the play tends to $-\infty$, and so for every initial credit player 1 wins. $\square$

7 Games with multiple players

Outline Games with multiple players require different types of solutions. We will define Nash equilibria on games on graphs, and show how to decide their existence (and compute an equilibrium, if it exists) for certain objectives that we have already encountered.

8 Different approaches to solving games

1. PDL satisfiability (Racer, FaCT, Pellet)
2. simulation based techniques
3. safety games, GR(1), ATL
4. Directed search (Stroeder and Pagnucco 09)
5. Planning
6. Antichain
7. Incremental
8. DES
9. parity games: progress measures, rank-based, zielonka

Reflections: 1 or 2 pars email to me or to florian on reflections on the course: what worked well for you, what didn't, level of interactive, good pace, enjoyable to come to class, connected to other material you learned, learn more than the syllabus e.g., how to formalise intuitions, etc.