Automatic Structures

Sasha Rubin

310 Malott Hall
Cornell University
Ithaca
NY 14853-4201 USA
srubin@math.cornell.edu

Abstract

The field of automatic structures concerns those mathematical structures (such as groups, orders, algebras, etc) that can be computed by automata. A structure is called *automatic* if it can be coded in such a way that the induced domain and relations and operations can be recognised by automata. There are as many types of automatic structures as there are models of automata, eg., automata operating (a)synchronously on (in)finite words or trees, possibly with oracle.

This chapter discusses fundamental logical, computational and algebraic properties of the main types of automatic structures.

Equivalent ways of defining automatic structures are via formulas (in the language of first-order or monadic-second order logic) or as solutions of equations. Automatic structures provide a framework in which to view other classes of infinite structures, eg. context-free graphs. The central themes in the literature include the classification of classes of automatic structures and identifying essentially different presentations of a given structure.

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1 Introduction

1.1 From computable structures

Automatic structures are instantiations of the more general notion of computable structure but have better properties including decidability (of the first-order theory) and closure under natural operations on structures (like product).

1.2 Graph-grammars and other automata based presentations

Traditionally graph-grammars describe sets of finite-graphs. However they may also be used to describe single, typically infinite, graphs. Standard formalism include HR-equational graphs and VR-equational graphs. Other approaches are ground-term rewriting systems. Tree-interpretable structures are those that are MSO-definable in the full binary tree. All of these are examples of previously-studied robust classes of tree-automatic structures.

1.3 Other notions of automaticity

Specic automatic presentations have been employed in other mathematical elds: computational group theory, symbolic dynamics, numeration systems (of integers or reals), and infinite sequences represented in natural numeration systems (eg. morphic words).

1.3.1 Thurston's automatic groups

Motivated by work of Cannon on hyperbolic groups, Thurston introduced 'automatic groups'. These are finitely generated groups displaying tractable algorithmic properties that are undecidable in the general case.

In a seminal paper Khoussainov and Nerode introduce automatic presentations as a generalisation of Thurston's automatic groups.

2 Definitions

2.1 Synchronous automata

(Recall or cite) definition and closure properties of automata operating on finite or infinite words or trees (with oracle).

2.2 Automatic presentations

Definition of finite-word, infinite-word, finite-tree, infinite-tree automatic structures.

2.3 Examples

Include examples of reducts of arithmetics, (well)-orders, trees, Boolean algebras, universal structures.

3 Basic properties

3.1 The fundamental theorem

Explanation of first-order logic.

Theorem 3.1 (Fundamental Theorem)**.** The first-order theory of an automatic structure is decidable.

3.1.1 Extensions of the fundamental theorem

Brief explanation of generalised quantifiers (cardinality, ramsey, etc.) and the extended theorem. Limitations (eg. MSO, fixed-point operators).

3.2 Equivalent definitions

Explain MSO, interpretability. State equivalent definition: those structures interpretable in universal structures (for MSO and FO). Explain VRS operations (give example) and state equivalent definition of word-automatic as VRS-equational structures.

3.3 Generalisations of the automaton model

3.3.1 Automata with oracle

Define automata with (tree) oracle. State extension of fundamental theorem and the Colcombet-Löding theorem.

3.3.2 Asynchronous automata

Define rational graphs. Give basic properties and relationship to traces.

4 Central themes

4.1 Proving non-automaticity

Usually it is quite straightforward to show that a structure has an automatic presentation (assuming it does indeed have one). Showing that a structure has no automatic presentation is a significant challenge. Here are the basic techniques.

4.2 Classification of classes of automatic structures

In some cases a class of similar structures (such as groups or linear orders) may be classically described by (full) invariants (often a natural number, or a sequence of natural numbers). We seek to identify those invariants that correspond to the automatic members of the class. Examples: well-orders, Boolean algebras, trees, linear orders, groups.

4.2.1 The isomorphism problem

Deciding whether two presentations present isomorphic objects is called the isomorphism problem. The complexity of this problem (in the analytic hierarchy) is a measure of the complexity/richness of a class of structures.

Examples: general case, well-orders, Boolean-algebras, trees, linear orders, groups.

4.3 Presentations up to equivalence

An automatic structure has many presentations. Discuss equivalent presentations and results of Bárány, Colcombet, Löding.

4.4 Summary of open questions

Specific questions and general directions (eg. automatic model theory).