Discussion #8 2/9/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

The Invertible Matrix Theorem, Lay 2.3 Theorem 8 (p. 145), is very very useful. You don't need to memorize all the parts of it, but you should know what they mean.

Determinants are

Problems

1. Let A be a square matrix. List at least four different statements which are equivalent to the statement
$$\det(A) \neq 0.$$

2. Let A and B be $n \times n$ matrices. Answer the following *True* or *False*. If *False* give a counterexample.
   (a) $\det(AB) = \det(A) \cdot \det(B)$
   (b) $\det(AB) = \det(BA)$
   (c) $\det(A + B) = \det(A) + \det(B)$
   (d) If A and B are both invertible, then AB is invertible.
   (e) $\det(A) = \det\left(A^T\right)$
   (f) If A is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$.
   (g) If c is a scalar then $\det(cA) = c^n \det(A)$.

3. What is the volume of a parallelogram with vertices at $(0, 0)$, $(4, 1)$, $(3, 5)$, $(7, 6)$?

4. Suppose A is a 3×3 matrix with determinant 5. What is $\det(3A)$? $\det(A^{-1})$? $\det(2A^{-1})$? $\det((2A)^{-1})$?

5. Compute the following:
$$\text{(a)} \quad \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} \qquad \text{(b)} \quad \begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} \qquad \text{(c)} \quad \begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

6. By inspection, evaluate the following determinants:
$$\text{(a)} \quad \begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix} \qquad \text{(b)} \quad \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 9 \\ 0 & 16 & 25 & 36 \\ 49 & 64 & 81 & 100 \end{vmatrix} \qquad \text{(c)} \quad \begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix}$$

7. Show that if A is a square matrix with a row of zeros, then $\det(A) = 0$. What if A has a column of zeros?

8. (a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as

$$(A - I)\,\mathbf{x} = 0,$$

where A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

(b) Use part (a) to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(c) Solve $A\mathbf{x} = 4\mathbf{x}$.