

Discussion #15 3/2/26 – Spring 2026 MATH 54

Linear Algebra and Differential Equations

The **rank** of a matrix is the dimension of the column space. It is the number of pivot columns of the matrix (after row reducing). The **nullity** of a matrix is the dimension of the null space. It is the number of *non-pivot* column of the matrix (after row reducing).

To calculate the dimension of a subspace, find a spanning set for the subspace, put it into a matrix, row reduce, and count the number of pivot columns.

Problems

1. Answer the following *True* or *False*. Justify your answers.
 - (a) $\dim(\mathbf{P}_5) = 5$.
 - (b) $C[0, 1]$ is infinite dimensional. (Hint: See Theorem 10, page 241 of Lay.) If this part is confusing, skip it until later.
 - (c) If W is a subspace of V , and S is a basis for W , then we can add vectors to S to form a basis for V .
 - (d) If W is a subspace of V , and S is a basis for V , then some subset of S is a basis for W .
2. (a) Give an example of a 3×3 matrix whose null space has dimension 1.
(b) Give an example of a 3×3 matrix whose column space has dimension 1.
(c) Does there exist a 3×3 matrix whose null space and column space both have dimension 1?
3. Let A be an $m \times n$ matrix.
 - (a) If A is onto, what is the rank of A ?
 - (b) If A is one-to-one, what is the nullity of A ?
4. Suppose A is an invertible $n \times n$ matrix.
 - (a) What is $\text{rank}(A)$?
 - (b) What is $\dim(\text{Nul}(A))$?
5. Show that if A is not square, then either the rows of A or the columns of A are linearly dependent.
6. Let $W = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$.
 - (a) Find a matrix A such that $\text{Nul}(A) = W$.
 - (b) Use part (a) to find a basis for W and determine its dimension.
7. Let V be the vector space consisting of all polynomials $p(x)$ of degree 3 or less, satisfying $p(1) = 0$. What is $\dim(V)$? Give a basis for V .

8. (a) Find a basis for the vector space of all 3×3 matrices. What's the dimension?
- (b) Find a basis for the vector space of all 3×3 symmetric matrices. What is the dimension of this vector space?