Discussion #8 2/9/26 – Spring 2026 MATH 54
Linear Algebra and Differential Equations

The Invertible Matrix Theorem, Lay 2.3 Theorem 8 (p. 145), is very very useful. You don't need to memorize all the parts of it, but you should know what they mean.

Determinants are

Problems

1. Let A be a square matrix. List at least four different statements which are equivalent to the statement
$$\det(A) \neq 0.$$

   Solution:

   1. The matrix A is invertible.
   2. There is only one solution to $A\mathbf{x} = \mathbf{0}$.
   3. The columns of A are linearly independent.
   4. The matrix A is 1-1.
   5. The matrix A is onto.
   6. For each $\mathbf{b} \in \mathbf{R}^n$, there exists a unique solution to $A\mathbf{x} = \mathbf{b}$.

   $\vdots$

2. Let A and B be $n \times n$ matrices. Answer the following *True* or *False*. If *False* give a counterexample.

   (a) $\det(AB) = \det(A) \cdot \det(B)$

   Solution: **True**: This holds as shown in §3.2 Theorem 6.

   (b) $\det(AB) = \det(BA)$

   Solution: **True**: We have
$$\det(AB) = \det(A) \cdot \det(B) = \det(B) \cdot \det(A) = \det(BA).$$

   (c) $\det(A + B) = \det(A) + \det(B)$

   Solution: **False**: Consider $I \in \mathbf{R}^{2\times 2}$:
$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \quad \text{and} \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

and so

$$4 = \det(2I) \neq \det(I) + \det(I) = 1 + 1 = 2.$$

(d) If A and B are both invertible, then AB is invertible.

Solution: True: We have

$$\det(AB) = \det(A) \cdot \det(B) \neq 0$$

and so AB is invertible.

(e) $\det(A) = \det\left(A^T\right)$

Solution: True: This is Theorem 5 in §2.2.

(f) If A is invertible, then $\det(A^{-1}) = (\det(A))^{-1}$.

Solution: True: We have

$$\det(I) = 1 \quad \text{and} \quad \det(I) = \det(AA^{-1}) = \det(A) \cdot \det(A^{-1})$$

and so with $\det(A) \neq 0$

$$\det(A^{-1}) = (\det(A))^{-1}.$$

(g) If c is a scalar then $\det(cA) = c^n \det(A)$.

Solution: True: This is n applications of Theorem 3 part (c) in §3.2.

3. What is the volume of a parallelogram with vertices at $(0, 0)$, $(4, 1)$, $(3, 5)$, $(7, 6)$?

Solution: We take the determinant of the matrix with the vertices as the entries:

$$\det \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix} = 17. \tag{1}$$

4. Suppose A is a 3×3 matrix with determinant 5. What is $\det(3A)$? $\det(A^{-1})$? $\det(2A^{-1})$? $\det((2A)^{-1})$?

Solution: We have

$$\det(3A) = 3^3 \det(A) = 27 \cdot 5 = 135, \quad \det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{5},$$

and

$$\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{5}.$$

While

$$(2A)^{-1} = \frac{1}{2}A^{-1}$$

since

$$(2A)\frac{1}{2}A^{-1} = \frac{2}{2}AA^{-1} = I$$

so

$$\det((2A)^{-1}) = \det\left(\frac{1}{2}A^{-1}\right) = \frac{1}{2^3}\cdot\frac{1}{5} = \frac{1}{40}.$$

5. Compute the following:

$$\text{(a)}\quad \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} \qquad \text{(b)}\quad \begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} \qquad \text{(c)}\quad \begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Solution:

(a) Here

$$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 1\cdot 5 - 2\cdot 3 = -1.$$

(b) Here we expand about the second row

$$\begin{aligned}
\begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} &= (-1)^{2+1}\cdot 3\cdot \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix} + (-1)^{2+2}\cdot 0\cdot \begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} \\
&\quad + (-1)^{2+3}\cdot 1\cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} \\
&= (-3)\cdot((-1)\cdot 1 - (-2)\cdot 1) + 0 - (1\cdot 1 - (-1)\cdot(-1)) \\
&= (-3)\cdot 1 - 0 \\
&= -3.
\end{aligned}$$

(c) We expand about the 3rd row

$$\begin{aligned}
\begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} &= (-1)^{3+1}\cdot 0\cdot \begin{vmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \end{vmatrix} + (-1)^{3+2}\cdot 0\cdot \begin{vmatrix} \sin(\theta) & 0 \\ -\cos(\theta) & 0 \end{vmatrix} \\
&\quad + (-1)^{3+3}\cdot 1\cdot \begin{vmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{vmatrix} \\
&= 1\cdot(\sin^2(\theta) + \cos^2(\theta)) \\
&= 1.
\end{aligned}$$

6. By inspection, evaluate the following determinants:

(a) $\begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 9 \\ 0 & 16 & 25 & 36 \\ 49 & 64 & 81 & 100 \end{vmatrix}$ (c) $\begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix}$

Solution:

(a) We have

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix} = 3 \cdot 4 \cdot 9 = 54$$

because we have a lower triangular matrix.

(b) If we swap R_1 and R_4, the R_2 and R_3 we get an upper triangular matrix

$$\begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix} = \begin{vmatrix} 49 & 64 & 81 & 100 \\ 0 & 16 & 25 & 36 \\ 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 49 \cdot 16 \cdot 4 \cdot 1$$

and the two row swaps cancel out the sign change in the determinant. Thus this also gives the determinant of our matrix.

(c) We perform 2 row swaps, R_1 and R_3, then R_2 and R_3, which do not change the value of the determinant, and then get

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix} = 1 \cdot \pi \cdot (\sqrt{2}) \cdot 22.$$

7. Show that if A is a square matrix with a row of zeros, then $\det(A) = 0$. What if A has a column of zeros?

Solution: We can perform a row swap to get B with a row of zeros on the top then

$$\det(A) = -\det(B)$$

but then we expand $\det(B)$ along the top row and get

$$\det(B) = 0$$

and thus $\det(A) = 0$. If A has a column of zeros we can use

$$\det(A) = \det\left(A^T\right) = 0$$

since A^T now has a row of all zeros.

8. (a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as

$$(A - I)\,\mathbf{x} = 0,$$

where A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

Solution: To get matrix I to appear, we will not that $\mathbf{x} = I\mathbf{x}$. Then

$$\begin{aligned} A\mathbf{x} &= \mathbf{x} \\ A\mathbf{x} &= I\mathbf{x} \\ A\mathbf{x} - I\mathbf{x} &= \mathbf{0} \\ (A - I)\,\mathbf{x} &= 0. \end{aligned}$$

(b) Use part (a) to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution: Here

$$A - I = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

and so

$$\begin{aligned} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right] &\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 18 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \end{aligned}$$

tells us we only have the trivial solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(c) Solve $A\mathbf{x} = 4\mathbf{x}$.

Solution: Inspired by (a), we want to solve

$$(A - 4I)\mathbf{x} = \mathbf{0}$$

where

$$A - 4I = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & -1 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 3 \\ 1 & -2 & -1 \\ 2 & -2 & -3 \end{bmatrix}.$$

Now

$$\begin{aligned} \left[\begin{array}{rrr|r} -2 & 2 & 3 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & -2 & -3 & 0 \end{array}\right] &\sim \left[\begin{array}{rrr|r} -2 & 2 & 3 & 0 \\ 0 & -1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} -2 & 2 & 3 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} -2 & 0 & 4 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{aligned}$$

tells us that our solution is of the form

$$\mathbf{x} = x_3 \begin{bmatrix} 2 \\ 1/2 \\ 1 \end{bmatrix}$$

where $x_3 \in \mathbf{R}$.