Dynamic Optimization for Goals-Based Wealth Management with Multiple Goals *

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Abstract

We develop a dynamic programming methodology that seeks to maximize investor outcomes over multiple, potentially competing goals (such as upgrading a home, paying college tuition, or maintaining an income stream in retirement), even when financial resources are limited. Unlike Monte Carlo approaches currently in wide use in the wealth management industry, our approach uses investor preferences to dynamically make the optimal determination for fulfilling or not fulfilling each goal and for selecting the investor's investment portfolio. This can be computed quickly, even for numerous investor goals spread over different or concurrent time periods, where each goal may allow for partial fulfillment or be all-or-nothing. The probabilities of attaining each (full or partial) goal under the optimal scenario are also computed, so the investor can ensure the algorithm accurately reflects their preference for the relative importance of each of their goals. These portfolio prescriptions are consistent with Prospect Theory.

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1 Introduction

Goals-based wealth management (GBWM) is an investment philosophy focused on attaining the desired goal or goals specified by an investor. (See Chhabra (2005); Nevins (2004); Browne (1997); Browne (1999); Brunel (2015); Pakizer (2017)). This investor based viewpoint corresponds to a new notion of risk. Traditionally, risk is defined as the volatility of the investments in an investor's portfolio. In contrast, for GBWM with a single goal, risk is defined as the probability that an investor does not meet that financial goal. So, for example, if an investor's portfolio is doing poorly compared to the investor's final goal for wealth, shifting the portfolio to asset classes with more expected return and more volatility increases the traditional notion of risk, but decreases the GBWM notion of risk.

For multiple goals in well funded portfolios, the GBWM notion of risk for a single goal translates easily to multiple goals. It becomes the probability that the investor will not be able to attain all their goals. But of considerably more importance is the complex question of what, optimally, should be done when limited investor resources necessitate prioritizing and then choosing among the investor's multiple competing goals.

Consider the simple case where an investor has just two goals: Let's say that in 5 years, the investor wants to take a nice vacation and in 10 years the investor wants to pay for their child's first car. If they don't have the money in their portfolio for the vacation at the end of five years, they must forgo the vacation goal of course. If they have a lot of money, they fulfill the vacation goal knowing that they will almost certainly be able to afford the car goal as well.

But what if the investor has a moderate amount of money in their portfolio? Do they take the vacation? It depends. It depends not only on the cost of the vacation and the car, but also on the relative importance of the two goals to the investor. If they have just enough money to afford the vacation but attaining the car goal, while more important, is likely out of reach due to being significantly more expensive, they should fulfill the vacation goal. If they have more money, they should switch to forgoing the vacation so as to optimize the chance of attaining the more important car goal, but if they have even more money they should again switch to taking the vacation because they can likely attain both goals. This leads to an obvious, key question: what values of portfolio wealth in five years optimally correspond to these switches regarding the decision to fulfill or not fulfill the vacation goal?

This simple two goal example quickly gets far more complicated when made more realistic. What if we consider partial goals like less expensive vacation choices instead of the nice vacation, or less expensive cars? What if we add additional goals like paying off a mortgage in 15 years or remodeling a kitchen in 5 years, concurrent with considering the vacation? Or, after 17 years, annually removing money from the portfolio for the 30 years after that to fund retirement? Optimally determining which goals to fulfill is not obvious, nor is it obvious what the optimal portfolio is at each time period to attain these goals. And if you add a projected income stream for the investment portfolio, how would that change these optimal decisions?

These are all important questions, but the Monte Carlo methods commonly used in practice for attaining and analyzing goals in the wealth management industry are, by their nature, incapable of answering them, except in extremely simplified cases. If we consider a static portfolio with a single goal or with multiple goals that must all be fulfilled, then Monte Carlo works well. Let's say there are five possible portfolios to consider. Running 10,000 Monte Carlo simulations for each of these five portfolios can be done in about $5 \times 0.01 = 0.05$ seconds on a personal computer. But if we want to optimize the portfolio choice even slightly dynamically by allowing it to change only once after five years, we have to consider each portfolio at each potential wealth value at five years. If we have a grid allowing, say, just 40 potential wealth values, that takes $5^{40} \times 0.01 = 9 \times 10^{25}$ seconds, which is 2.85 quintillion years! If we go back to the static portfolio, but require a decision to forgo versus take the cheap vacation versus take the expensive vacation at each wealth value, that takes less time: just $3^{40} \times 0.01 = 1.2 \times 10^{17}$ seconds — still over 3 billion years. Clearly, Monte Carlo is not feasible for optimization even in these simple cases, let alone the more complicated, realistic problems we wish to address.

In this paper, we will show that, in contrast, the method of dynamic programming can solve these more complicated problems, often within a few seconds. Even in the worst case we consider, which has a 60 year portfolio horizon with over 1000 potential wealth values and hundreds of competing goals, we can determine the optimal solution in under 40 seconds on a laptop/desktop computer. That is, in each case, we show how to optimally determine prioritizing which goals should and should not be fulfilled, while also dynamically determining the optimal portfolio for the investor at every time period given their portfolio's worth at that time.

In contrast, current GBWM applications for multiple goals in the marketplace use Monte Carlo methods, and because Monte Carlo methods are not well-suited for optimizing decisions, these marketplace applications suffer from deficiencies that do not arise in our approach. In these Monte Carlo approaches, a set of net desired cashflows over time (known as a "cashflow ladder") are first determined, based on forecasted infusions and withdrawals for goals. Monte Carlo simulations are then run for a fixed portfolio strategy (such as a 60/40 portfolio) to determine the probability of fulfilling the entire cashflow ladder, with a required probability for fulfillment chosen to be around 70-80%. If this required probability is not met, additional simulations are run to determine how much more wealth is needed in the portfolio at inception or how many goals must be removed to reach the required probability. Comparing this approach to ours, we note the following:

- The Monte Carlo approach assumes a fixed investment strategy, while ours determines how to optimally switch investment strategies, considering all possible future investment strategies and all possibilities for fulfilling or forgoing taking future goals.
- The Monte Carlo approach only evaluates the likelihood of meeting all its goals.

In reality some of these goals are more important to the investor than others, but Monte Carlo cannot compute the probability of meeting each goal separately. For example, the investor may not be comfortable with only an 80% chance of paying off their mortgage, but they are fine with their chance of a fancy vacation being much lower. The Monte Carlo approach can require a higher than 80% chance for both goals or for neither goal. It is unable to reflect the investor's priorities. In contrast, after our approach optimizes investment choices and taking goals using investor assigned weights to these goals, it then reports the probability of achieving each goal. This allows the investor to change these goal probabilities through assigning different weights to best fit their priorities.

- The Monte Carlo approach does not optimally exercise goals. The simulations assume that goals are taken if there are sufficient funds, otherwise the portfolio goes bankrupt. This prioritizes earlier goals over later ones, again without consideration of the investor's priorities. Because of this, the investor is more likely to fail being able to pay their final mortgage payments or their final long term care insurance payments, nullifying much of the advantage of paying the earlier, prioritized installments. Dynamic programming, on the other hand, is explicitly tailored to the investor's priorities and optimizes goal-taking strictly with respect to these priorities.

- Simulation approaches are de facto approximations whose accuracy improves in proportion to the square root of the number of simulations, which is slow, unlike dynamic programming. More importantly, as stated earlier, dynamic programing is designed for optimizing problems like ours and Monte Carlo is not, so, within even remotely similar computational timeframes, the Monte Carlo approach will generally result in substantial suboptimality in attaining an investor's goals compared to our dynamic programming approach.

Our GBWM model combines ideas from both modern/rational portfolio theory and behavioral finance. Modern portfolio theory (Markowitz (1952)) prescribes the static portfolio that optimizes expected return for a fixed level of volatility. The set of these optimal portfolios forms the efficient frontier. Staying on the efficient frontier minimizes traditional risk, since it minimizes volatility for a given expected return. Staying on the efficient frontier also minimizes GBWM risk since, for a fixed level of volatility, we want the optimum expected return to attain goals. We therefore, ideally, only consider portfolios on the efficient frontier, although our algorithm will also work just as effectively if we are restricted to a set of portfolios that are not on the frontier. This combination of mean-variance optimality and goals-based optimization addresses the criticisms of modern portfolio theory detailed in Muralidhar (2018).

Markowitz's question of how to optimize portfolio investment choices was extended to dynamic models via maximizing the utility of the investor's wealth (Merton (1969); Merton (1971)). The utility of a wealth value corresponds to its importance or use to the investor. Because Merton uses volatility as his notion of risk, his utility functions are always concave, since the degree of the utility function's concavity corresponds to the degree of the investor's aversion to volatility.

Behavioral finance work on the portfolio optimization problem incorporates how to embed behavioral considerations into the optimization. For example, the papers by Shefrin and Statman (2000), Das et al. (2010), and Alexander et al. (2017) considered how to maximize expected return subject to minimizing shortfall probability, while the papers by Wang et al. (2011) and Deguest et al. (2015) looked to maximize utility in a GBWM context by setting minimum goal probabilities as constraints.

The notion of utility will be a key idea in this paper, since we require the investor to determine the importance, that is, the utility, of each full or partial goal to them, in addition to its cost. Traditionally, determining the utility that corresponds to an investor's preferences is very difficult to determine. However, in the context of this paper, assigning appropriate utility values to the investor's goal preferences is comparatively easy, as we next explain.

As a coarse initial approximation, the investor may be asked to assign broad categories to each full or partial goal (for example, we could employ Brunel's categories of needs, wants, wishes, or dreams, see Brunel (2015)). Each category is assigned a specific utility value. Our algorithm first optimizes the expected total utility from fulfilled goals, and then computes the corresponding probabilities of attaining each full or partial goal.

Behavioral research (see, for example, Das et al. (2018)) shows that investors understand the notion of the probability of attaining a goal far better than most financial terms commonly used by wealth managers. This means they can understand the ramifications of their utility assignments to various goals and then alter them as finely as desired to best fit their desires and preferences. Specifically, if the investor finds a probability of attaining a specific goal too low, they can just increase the utility assigned to that goal, with the understanding that it will lower the probability of attaining most of the other goals. Should they wish to increase the probability of attaining all of their goals, they can decide to add cash infusions into the portfolio at any specific time period or collection of time periods, or they can expand the range of investment portfolio strategies available to them. The investor can use our algorithm to recompute the new optimized probabilities of attaining each goal. This procedure may be iterated as many times as desired.

This approach yields the correct extension to understanding GBWM risk in the context of multiple competing goals. It is no longer a single probability. It is now a collection of probabilities that compete with each other, and this overall risk is minimized by fulfilling as many of the investor's goals as possible, weighted by their importance to the investor.[1]

[1]Because GBWM risk differs from traditional volatility risk, we do not have strictly concave utility functions. If we consider the case of a single goal wealth, W_g, at the time horizon $t = T$, the utility function at that time is just a step function in the portfolio wealth, W, with the step occurring at $W = W_g$. For earlier times, $t < T$, this corresponds to the value function being convex (volatility seeking) for low wealth and concave (volatility avoiding) at high wealth, see Das et al. (2019). This is consistent with Prospect Theory (Kahneman and Tversky (1979)), where investors ascribe values over

In this paper we develop an algorithmic solution to the GBWM problem with multiple goals to quickly determine the optimal dynamic strategy both for the investor's portfolio and for determining when it is best to forgo, partially fill, or totally fill each of the investor's goals. Our solution has interesting behavioral and financial ramifications:

- *First*, GBWM intersects with mental accounting (Thaler (1985); Shefrin and Statman (2000); Das et al. (2010)) where investors have different goals in separate mental buckets and express different risk preferences for each mental account. In this paper, each partial or full goal is linked to a different utility, which determines its importance to the investor. Prior work assumed a separate portfolio for each mental account. However, optimizing a single portfolio that encompasses all of these separate goals, as we do here, only provides benefit to the investor, since having separate portfolios can prevent overfunded portfolios from optimally helping attain goals in underfunded portfolios.

- *Second*, because our method clearly and quickly connects the utility assignments made for each full and partial goal to the corresponding probabilities of attaining all the full and partial goals, we are able to connect traditional utility theory with behavioral theory in a way that can clearly inform investors. This has the important advantage of allowing traditional wealth advisors or robo-advisors to better work with each investor to create a tailored financial plan that optimally suits the investor's desires, where the investor clearly understands the trade-offs their choices imply.

- *Third*, as noted earlier, because the solution to the multiple goals problem is implemented via backward recursion through dynamic programming, we are able to get far better results than Monte Carlo methods, which simulate forwards in time and are not well adapted to optimization. Dynamic programming allows us to maximize the expected total utility across all goals by providing both the optimal portfolio strategy and the optimal decisions for exercising partial or entire goals at each point in time. In this last regard, it is analogous to optimally determining when to exercise or not to exercise a complicated Bermudan option on goals at each time period. Further, our method accommodates projected cash infusions into the portfolio, allowing investors to consider the effect of future savings on attaining their goals.

Our approach in this paper proceeds as follows: Section 2 explains how our problem is formulated over multiple and partial goals. Subsection 2.1 introduces our basic variables, notation, and setup. In Subsection 2.2, we consider all possible combinations of full or

losses and gains, which are based on over-achieving or under-achieving with respect to a reference point. In GBWM, this reference point is W_g at $t = T$. Prospect theory, like our GBWM approach in Das et al. (2019), defines a value function over outcomes that is convex in the region below the reference point and concave above. This is also consistent with the Disposition Effect of Shefrin and Statman (1985), the Cumulative Prospect Theory (CPT) of Tversky and Kahneman (1992), and the SP/A (Security-Potential/Aspiration) Theory of Lopes and Oden (1999). An excellent and detailed discussion of these behavioral theories may be accessed in the book by Shefrin (2008), particularly chapters 24 through 27.

partial goals from each of the concurrent goals to be addressed at a given time period. For each combination of concurrent goals, we sum the total dollar amount ("cost" c) to fulfill the combination's goals and attach it to the sum of the importance ("utility" u) of the combination's goals to the investor. After removing dominated cost-utility pairs, such as pairs where both the cost is higher and the utility is lower than the cost and utility in another pair, we form the set of desirable goal combinations at this time period. The dynamic programming method will later determine which one of these pairs is optimal. In Subsection 2.3, we determine a grid of feasible wealth values that we will use at all time periods in our algorithm.

Section 3 presents the dynamic programming formulation and solution over this wealth grid. In Subsection 3.1, we determine the utility assignment for the residual wealth in the last time period, $t = T$, noting that this time period may be after the investor's goals have been addressed, in which case the assignment corresponds to the importance of various levels of excess wealth to the investor. We then recursively work backwards in time, determining in Subsection 3.2 the transition probability over each time period between two wealth grid points given each portfolio choice and desirable cost-utility combination. This allows us to form the corresponding Bellman equation in Subsection 3.3, where we can decide the optimal portfolio and cost-utility combination to select at each wealth grid point. We work backwards through each time period until we reach the present, $t = 0$. Then, in Subsection 3.4, use our knowledge of the optimal strategy and its corresponding transition probabilities to work forward in time to compute the probability of attaining each goal, including the probability of being at each wealth grid point at each time t, so the investor can fully understand the effect of the optimal strategy on attaining their goals and minimizing their GBWM risk.

Section 4 demonstrates a variety of numerical examples from our algorithm. We present both simple examples to give insight into the nature of our solution, as well as complicated examples that are more realistic. In Subsection 4.1, we consider the interplay between just two all-or-nothing goals at different times. Subsection 4.2 considers an example with seven all-or-nothing goals at different times, as well as the effect on attaining these goals that occurs if we change the utilities assigned to these goals, change the initial investment size, add cash infusions at various time periods, change the available investment portfolios, or value excess cash that remains at the end of the portfolio horizon. Subsection 4.3 explores the results for concurrent goals and partial goals by looking at the examples introduced in Subsection 2.2. In Subsection 4.4, we use our algorithm to optimize the investing and goals taking strategy for a couple in their mid-thirties over the course of the next 60 years. This couple considers a number of competing annual goals of varying importance that include paying for mortgages, property tax, long-term care insurance, medical expenses, other everyday expenses, cars, house remodeling, trips, philanthropy, and, for their child, orthodontia, private high school tuition, college tuition, and wedding expenses.

Finally, in Section 5, we conclude with some final comments and future directions for this work.

2 Setup for Multiple Goals

2.1 Basic Variables and Notation

The basic quantities and variables in our setup are

- Time: We consider time periods $t = 0, 1, \ldots, T$ with an interval of h years between time periods. So if $h = 0.25$, then $t = 4$ corresponds to one year from the present, $t = 0$. The final time period, $t = T$, for the portfolio may or may not correspond to the projected date of death for the investor.

- Infusions: The investor can specify wealth infusions, $I(t) > 0$, that they will contribute to their portfolio at any time period or collection of time periods $t = 1, 2, ..., T-1$. The values taken by $I(t)$ over these time periods may be chosen to be identical or different. For example, automatic infusions from paychecks may be chosen to remain constant or they may be chosen to be indexed by inflation.

- Portfolio Evolution and Portfolio Investment Strategies: Our dynamic programming approach works with any Markovian stochastic evolution model for portfolios, but for simplicity we will use geometric Brownian motion for our evolution model in this paper. We assume that the investor has access to $l_{\max}$ different possible portfolio investment strategies, indexed by $l = 1, 2, \ldots, l_{\max}$. For our examples in Section 4, we will choose these different portfolios to be along the efficient frontier (see Markowitz (1952)) with the ordering $\mu_1 < \mu_2 < \cdots < \mu_{l_{\max}}$ for the portfolios' expected returns and $\sigma_1 < \sigma_2 < \cdots < \sigma_{l_{\max}}$ for the corresponding portfolios' volatilities. Should the selected portfolios not be on the efficient frontier, they will still conform to the same ordering. A portfolio that cannot fit this ordering should not be used, since it is guaranteed to have both a lower expected return and a higher volatility than at least one of the other portfolios, and therefore cannot minimize risk from a GBWM (or a traditional) viewpoint. At times we will use the notation $\mu_{\min}$ for μ_1, $\mu_{\max}$ for $\mu_{l_{\max}}$, $\sigma_{\min}$ for σ_1, and $\sigma_{\max}$ for $\sigma_{l_{\max}}$.

- Cost and Utility Vectors: Implementing full or partial goals at a given time period t results in a reduction in portfolio worth to pay for the goal, with an accretion in utility for the investor. As will be explained in Subsection 2.2, the potential costs and utilities from implementing combinations of various full and partial goals at a given time period t will be contained in the cost vector $\mathbf{c}(t)$ and the utility vector $\mathbf{u}(t)$, with corresponding components $c_k(t)$ and $u_k(t)$, where $k = 1, 2, \ldots, k_{\max}(t)$.

- Initial Wealth: We will denote the initial wealth, W, that an investor puts into their portfolio at $t = 0$ by $W(0)$.

- Wealth Grid: In Subsection 2.3, we will detail the grid of possible wealth values that will form the state space used by our dynamic programming model in Section 3. This wealth grid, which is the same at each time period, will contain $i_{\max}$ wealth values, where these wealth values, $W_{\min} = W_1 < W_2 < \ldots < W_{i_{\max}} = W_{\max}$, will have equal *logarithmic* spacing.

2.2 The Cost and Utility (Importance) of an Investor's Goals

2.2.1 Assigning a cost and a utility to each full or partial goal

As discussed in the introduction, each full or partial investor goal must have assigned to it both a cost (reduction in portfolio wealth) and a utility (defining the goal's importance to the investor). The cost can be projected from current prices and inflation rates or any other method deemed appropriate by the investor with their wealth manager.

The utility, as discussed in the introduction, can be approximated initially and fine tuned later. Perhaps we start with just four priority levels, where level one is of extreme importance to the investor and level four would create mild happiness. We might initially assign a utility of 1000 to level one, a utility of 300 to level two, 100 to level three, and 30 to level four. These numbers approximately equate the value to an investor of a goal on a given priority level with the combined value of three goals from the next lowest priority level.

For example, assume that an investor has a goal to upgrade to a top-notch electric car four years from now that will cost \$50,000, and this goal has a priority level of two for the investor. If h, the time step, is 0.5, then we have a goal at $t = 8$ (i.e., in four years) with a cost of $c = \$50,000$ and, at least initially, a utility of $u = 300$. This is an example of an all-or-nothing goal. Should the investor be open to alternatives, they can consider partial goals. Maybe they are open to a hybrid car with lots of features for \$32,000 or a version with fewer features for \$28,000. The investor might think of both of these as level three priorities, but, of course, would prefer more features, so we assign, say, $u = 125$ to the hybrid with lots of features and $u = 80$ with fewer features (just above and below the value $u = 100$ that we generically assign a level three priority). So, for Goal 1, upgrading the car, we have that

Goal 1 ($t = 8$):

Cost	0	28	32	50
Utility	0	80	125	300

,

where the cost is in thousand of dollars. Note that there are four possibilities here: forgoing the goal completely (with a cost and utility of zero), the two partial goals, and the full goal.

The investor may also have a level one goal of paying each semester's tuition for their child at a specific four year college. If tuition currently costs \$30,000 per year, which is projected to increase at a rate of 8% per year, and if the child is intending to start college at $t = 6$, we would have a semi-annual cost of \$30,000$\times 1.08^3 \div 2 =$\$18,895 at $t = 6$ and $t = 7$, a cost of \$30,000$\times 1.08^4 \div 2 =$\$20,407 at $t = 8$ and $t = 9$, and we then continue in this manner for the goals at $t = 10, 11, 12$, and 13. The utility at each of these times would (initially) be 1000. In particular, we note that utility does not need to have a growth factor over time like inflation for the cost. At times other than $t = 8$, this tuition goal is called Goal 1, but since we have the *concurrent* goal of upgrading the car at $t = 8$, the tuition goal at $t = 8$ is labelled to be Goal 2:

Goal 2 ($t = 8$):

Cost	0	20.407
Utility	0	1000

.

We make two short notes before we further examine how to address concurrent goals in our algorithm:

1. As stated in the introduction, at this initial stage we only need to approximate the utility values. Once we have determined the corresponding probabilities of attaining each full or partial goal using dynamic programming in Section 3, we can iteratively fine tune these utility assignments to suit the investor. An example of this fine tuning will be demonstrated in Subsection 4.2.1.

2. We allow goals to be considered at any time period, $t = 0, 1, \ldots, T - 1$, except, for expositional ease, the final time period, $t = T$. Instead, at this final time period, the investor may wish to consider the utility of any excess wealth they retain. Methods for assigning utility values to this excess wealth at the portfolio horizon will be discussed in Subsection 3.1.

2.2.2 The Cost and Utility Vectors for a Year's Concurrent Goals

Consider an investor who has three different concurrent goals that they hope to fulfill in a specific time period. The first goal is an all-or-nothing goal. The second two goals allow for being partially fulfilled. Specifically, we have the following for these three goals:

Goal 1:

Cost	0	7
Utility	0	100

Goal 2:

Cost	0	9	20
Utility	0	90	300

Goal 3:

Cost	0	10	20	30	40
Utility	0	40	250	400	500

.

Since there are two possibilities for Goal 1 (fulfill the goal or forgo it), three possibilities for Goal 2, and five possibilities for Goal 3, we have a total of $2 \times 3 \times 5 = 30$ possibilities for combined goal fulfillment at this time period.

From these 30 possibilities, we first create a table with 30 columns containing the total cost and the total utility for each possibility. These are shown in two rows below, the second row continuing on from the first:

Cost	0	10	20	30	40	9	19	29	39	49	20	30	40	50	60
Utility	0	40	250	400	500	90	130	340	490	590	300	340	550	700	800

7	17	27	37	47	16	26	36	46	56	27	37	47	57	67
100	140	350	500	600	190	230	440	590	690	400	440	650	800	900

.

We then re-order the table's columns so that the total cost is monotonically increasing. If there are multiple columns corresponding to the same cost amount, we only retain the column that corresponds to the highest utility for that cost amount. This reduces the 30 cases to 24:

Cost	0	7	9	10	16	17	19	20	26	27	29	30	36	37	39
Utility	0	100	90	40	190	140	130	300	230	400	340	400	440	500	490

40	46	47	49	50	56	57	60	67
550	590	650	590	700	690	800	800	900

.

Finally, starting with the second column, we remove any column where the preceding column has a higher (or equal) utility. We remove these columns because it never makes sense to obtain them, given that the previous column attains a higher (or equal) total utility at a lower cost. This reduces the 24 cases to 13. These 13 cases comprise our final cost and utility vectors:

Cost	0	7	16	20	27	36	37	40	46	47	50	57	67
Utility	0	100	190	300	400	440	500	550	590	650	700	800	900

.

We note that both the cost vector, $\mathbf{c}(t)$, and the utility vector, $\mathbf{u}(t)$, now contain strictly increasing sequences. As stated in Subsection 2.1, we will use the subscript k to denote the components of these vectors, where $k = 1, 2, \ldots, k_{\max}(t)$. So, in the above example, if $k = 3$, then $c_k(t) = 16$ and $u_k(t) = 190$. Also, $c_{k_{\max}(t)}(t) = 67$ and $u_{k_{\max}(t)}(t) = 900$, where $k_{\max}(t) = 13$.

We note that our computer program must retain information for how each of these $k_{\max}(t)$ entries corresponds to the original goals. For example, the computer must retain that the $k = 8$ entry, which has a cost of \$40 and a utility of 550, corresponds to not taking Goal 1, total fulfillment of the Goal 2, and partial fulfillment of the Goal 3 at a cost of \$20.

2.3 The Wealth Grid

Our solution to the multiple goals GBWM problem is implemented at every time period, t, on a grid of wealth values. This grid contains $i_{\max}$ wealth values, where $i_{\max}$ can be any desired number, although suggestions for choosing values of $i_{\max}$ that are not too small to lead to inaccuracies nor too large to unnecessarily slow computations can be found in Das et al. (2019), and we have suitably modified the grid construction approach taken in that paper for the different types of cashflows here. These $i_{\max}$ wealth values are spread between the wealth limits $W_{\min}$ and $W_{\max}$.

We begin by approximating $W_{\min}$ and $W_{\max}$, noting that, ideally, they should represent the lowest and highest possible wealth values reasonably attainable for a solvent investor. Since we have chosen to use geometric Brownian motion for our portfolio evolution model in this paper, we have that an initial wealth, $W(0)$, when affected by no additional external monetary events, will evolve by

$$W(t) = W(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z},$$

where Z is a standard normal random variable. The effect of any additional external monetary event, be it an infusion, $I(t)$, or the cost of attaining a goal, $c_k(t)$, will also evolve by geometric Brownian motion. We look at the combined effect at each time period of these three types of external monetary events under the best and worst possible scenarios, subject to the assumption that $-3 \le Z \le 3$. We then locate the lowest and highest computed wealth values from all time periods, which gives us the approximated lower and upper wealth limits:

$$\begin{aligned}\tilde{W}_{\min} = \min_{t\in\{0,1,2,\ldots,T\}} \Bigg[& W(0)e^{\left(\mu_{\min} - \frac{\sigma^2_{\max}}{2}\right)t - 3\sigma_{\max}\sqrt{t}} \\ & + \sum_{\tau=0}^{t} \left(I(\tau) - c_{k_{\max}(\tau)}(\tau)\right) e^{\left(\mu_{\min} - \frac{\sigma^2_{\max}}{2}\right)(t-\tau) - 3\sigma_{\max}\sqrt{t-\tau}} \Bigg] \qquad (1)\end{aligned}$$

$$\begin{aligned}\tilde{W}_{\max} = \max_{t\in\{0,1,2,\ldots,T\}} \Bigg[& W(0)e^{\left(\mu_{\max} - \frac{\sigma^2_{\min}}{2}\right)t + 3\sigma_{\max}\sqrt{t}} \\ & + \sum_{\tau=0}^{t} I(\tau) e^{\left(\mu_{\max} - \frac{\sigma^2_{\min}}{2}\right)(t-\tau) + 3\sigma_{\max}\sqrt{t-\tau}} \Bigg]. \qquad (2)\end{aligned}$$

Given that most investors will have a number of goals, it will be common for $\tilde{W}_{\min}$ to be negative at this stage. We do not want this, so if $\tilde{W}_{\min} < W_{bankrupt}$, where $W_{bankrupt}$ is a chosen positive wealth, we set $\tilde{W}_{\min} = W_{bankrupt}$. Initially, it might seem reasonable to think that $W_{bankrupt}$ should be selected to be one cent, but that can have significant negative implications from a computational point of view. Because we want the points in the wealth grid to have approximately equal *logarithmic* spacing, there will be a concentration of lower wealth values in the grid. This can mean, for example, that choosing $W_{bankrupt}$ to be \$100 instead of one cent can cut the computational time in half, even though having \$100 is essentially the same as being bankrupt from the point of view of an investor whose goals are in terms of thousands, not hundreds, of dollars.

To obtain the logarithmic spacing on the wealth grid, we define $\tilde{W}_i$ by

$$\ln(\tilde{W}_i) = \ln(\tilde{W}_{\min}) + \frac{i-1}{i_{\max}-1}\left(\ln(\tilde{W}_{\max}) - \ln(\tilde{W}_{\min})\right), \text{ where } i = 1, 2, \ldots, i_{\max}.$$

We need one of our grid points to equal the initial wealth, $W(0)$, so we shift the entire grid downwards as little as possible to accomplish this. That is, we first define ε to be the smallest non-negative value with the property that $\ln(\tilde{W}_i) - \varepsilon = \ln(W(0))$ for some value of i, then we define the wealth grid values, W_i, where $i = 1, 2, \ldots, i_{\max}$ by $\ln(W_i) = \ln(\tilde{W}_i) - \varepsilon$, or, after exponentiating,

$$W_i = \tilde{W}_i e^{-\varepsilon}.$$

This also generates our wealth limits: $W_{\min} = W_1$ and $W_{\max} = W_{i_{\max}}$.

3 Dynamic Programming for Optimizing in the case of Multiple Investor Goals

With our set up in place from Section 2 for notation, the cost and utility vectors, and the wealth grid, we are now prepared to optimize the solution to our GBWM problem with multiple goals by using dynamic programming. Dynamic programming evolves the solution backwards in time, starting from the portfolio's time horizon, $t = T$. For each successive time period at every point on the wealth grid, we determine the strategy for attaining and forgoing goals and for selecting investment portfolios that optimize the total expected utility ultimately collected by the investor. The value function, $V(W_i(t))$, equals this optimized total expected utility, and it is determined by the Bellman (1952) equation in Section 3.3. Once we have used the Bellman equation to determine the optimal strategy at every time and wealth grid point, we then work forwards in time, starting at the initial wealth, $W(0)$, to generate the probability of being at any wealth grid point at any time period. This also generates the probability of fulfilling every full or partial goal.

3.1 Final time set up

At the portfolio horizon $t = T$, the investor has no more goals to consider, so the only remaining task is to value any excess wealth that remains. This means defining a utility function, $U(W_i)$, for the excess wealth at this time. By definition, the value function at $t = T$ is equal to this utility function:

$$V(W_i(T)) = U(W_i).$$

If the investor only cares about their goals and not their excess wealth (for example, if $t = T$ corresponds to their projected date of death, and they do not care what happens to any excess money after they die), then $U(W_i) = 0$ for all values of i. We will generally choose $U(W_i) = 0$ in later examples so that we can focus instead on the effects of the multiple goals.

However, if the investor does care about their excess money when they die because, for example, they wish to bequeath as much money as possible to an heir or charity, then $U(W_i)$ should be an increasing function. Further, if the investor is risk averse, $U(W_i)$ should also be concave. There are a variety of classic increasing, concave utility functions with scaling constant k that may be selected in such cases, such as:

- Constant relative risk aversion: $U(W_i) = k\frac{W_i^a}{a}$, where $a < 1$, $a \neq 0$, and $k > 0$.
- $U(W_i) = k\ln(W_i)$, where $k > 0$, which comes from taking the limit as $a \to 0$ in the constant relative risk aversion case.
- Constant absolute risk aversion: $U(W_i) = k(1 - e^{-aW_i})$, where $a, k > 0$.

- $U(W_i) = k\left(\frac{1}{1+be^{-aW_i}} - \frac{1}{1+b}\right)$, where $a, b, k > 0$.

The investor may also wish to specify other competing goals for this final time period, such as burial expenses. Our dynamic programming algorithm can be altered to accommodate such additional goals,[2] however, as stated in Subsection 2.2.1, for expositional ease, we do not pursue additional competing goals at $t = T$ further here.

3.2 Transition Probabilities

We now look to generate the value function at earlier time periods. More specifically, we will use the fact that we know $V(W_i(t+1))$, that is, the value function at all points on the wealth grid at time $t+1$, to determine $V(W_i(t))$, the value function at all points on the wealth grid at time t. This will allow us to iterate backward from time period T to $T-1$, then to $T-2$, etc., until we end at time 0.

To accomplish this, we need to know the transition probabilities between grid points at time period t to grid points at time period $t+1$. That is, we want to know the probability of transitioning to each value on the wealth grid, $W_j(t+1)$, at time $t+1$ if we are currently at a specific wealth value, $W_i(t)$, at time t and we select a specific $c_k(t)$ from the cost utility vector and a specific investment portfolio corresponding to μ_l and σ_l.

Recall that we have assumed geometric Brownian motion for our evolution model:

$$W(t) = W(0)e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z}. \tag{3}$$

Define $\phi(z)$ to be the value of the probability density function of the standard normal random variable at $Z = z$. After adapting equation (3) to our circumstances (noting that we start at time period t with $W_i(t) + I(t) - c_k(t)$ dollars and progress h years to the time period $t+1$), we then isolate Z to obtain the approximate transition probabilites, $\tilde{p}$:

$$\tilde{p}(W_j(t+1)|W_i(t), c_k(t), \mu_l) = \phi\left(\frac{1}{\sigma_l\sqrt{h}}\left(\ln\left(\frac{W_j(t+1)}{W_i(t) + I(t) - c_k(t)}\right) - \left(\mu_l - \frac{\sigma_l^2}{2}\right)h\right)\right).$$

These are approximate because their sum over all the j nodes at time $t+1$ is not necessarily equal to one. To obtain the transition probabilities, p, which do sum to one, we just normalize the approximate transition probabilities, $\tilde{p}$:

$$p(W_j(t+1)|W_i(t), c_k(t), \mu_l) = \frac{\tilde{p}(W_j(t+1)|W_i(t), c_k(t), \mu_l)}{\sum_{J=1}^{i_{\max}} \tilde{p}(W_J(t+1)|W_i(t), c_k(t), \mu_l)}.$$

[2]For example, one simple way to accommodate this is to set the time interval h to zero or something very small just for the final time step.

3.3 Bellman Equation

The value function, V, at time $t+1$ is the highest possible expected sum of the utilities from fulfilled goals in time periods $t+1, t+2, \ldots, T-1$ plus the utility of the final excess wealth at $t = T$. The term "highest possible" means that at each time and point on the wealth grid, the optimal choices for the component k from the cost/utility vectors and for the investment portfolio l are used.

We know $V(W_i(T))$ from Subsection 3.1. Since we know the transition probabilities from Subsection 3.2, we can compute $V(W_i(T-1))$ by substituting $t = T-1$ into the following Bellman equation and solving it for each $i = 1, 2, \ldots, i_{\max}$:

$$V(W_i(t)) = \max_{k,l} \left[u_k(t) + \sum_{J=1}^{i_{\max}} V\big(W_J(t+1)\big) \cdot p\big(W_J(t+1) | W_i(t), c_k(t), \mu_l\big) \right].$$

The maximum values of k and l in the Bellman equation are the optimal choices that lead to the highest possible expected utility sum. We can then repeat this process for the time period $t = T-2$, then $t = T-3$, etc., stopping after the results at $t = 0$ are computed. This gives the value function, and more importantly the optimal strategy, at all times and all gridpoints. As we progress through this Bellman procedure, at each gridpoint, i, and time, t, we store these optimal strategy choices for k and l, labeling them $k_{i,t}$ and $l_{i,t}$.

The overall run time increases as $k_{\max}(t)$, $l_{\max}$, the number of grid points $i_{\max}$, and the number of periods T increase. Note that if $k_{\max}(t)$ and $l_{\max}$ are both large, optimizing the Bellman equation over $k_{\max}(t) \times l_{\max}$ possibilities slows down the algorithm, in which case it is computationally wiser to split each time period optimization into two parts: first over the $l_{\max}$ portfolios, then over the $k_{\max}(t)$ goal choices. This essentially leads to optimizing over $k_{\max}(t) + l_{\max}$ possibilities, instead of $k_{\max}(t) \times l_{\max}$ possibilities. For all but the last of our experiments in Section 4, the computational time was under 5 seconds, so this split optimization approach was unnecessary. However, the final example in Section 4 has numerous goals and 15 possible portfolios in each of 60 years. Without the split optimization approach, the computational time for this example was about 13 minutes, so we employed the split optimization approach, which reduced the computational time to under 40 seconds.

3.4 Probability Distribution for Wealth and Optimally Attaining Goals

To determine the probability distribution for the investor's wealth at future times, we use the transition probabilities and the optimal strategy information, $k_{i,t}$ and $l_{i,t}$, determined from the Bellman equation to evolve the probability distribution forward in time, starting with $t = 0$, then $t = 1$, and ending with $t = T-1$.

More specifically, at $t = 0$, define i_0 so that W_{i_0} is the wealth node that equals $W(0)$, therefore $p(W_{i_0}(0)) = 1$ and $p(W_i(0)) = 0$ for all $i \neq i_0$. To obtain the wealth probability distribution at $t = 1$, we set $t = 0$ in the following "forwards equation" and run it for each $j = 1, 2, \ldots, i_{\max}$:

$$p(W_j(t+1)) \;\; = \;\; \sum_{i=1}^{i_{\max}} p\big(W_j(t+1)|W_i(t), c_{k_{i,t}}(t), \mu_{l_{i,t}}\big) \cdot p\big(W_i(t)\big). \tag{4}$$

We then set $t = 1$ in equation (4) and again run it for each $j = 1, 2, \ldots, i_{\max}$, continuing in this manner until we finish with the $t = T - 1$ case. This gives the probability distribution for every point in the wealth grid at every time period of the portfolio.

Once this wealth probability distribution is calculated, we can determine the probability of attaining any specific full or partial goal at any given time t: At each time t for each $k = 1, 2, \ldots, k_{\max}(t)$, we sum $p(W_i(t))$ over every i where $k_{i,t} = k$. This gives the probability that each component k in the cost/utility vectors will be chosen. Once this is known, the cost/utility vectors are reconnected to their original goals, and the probability that each full or partial goal will be fulfilled is determined by summing over the components connected to that full or partial goal. For example, if the cost/utility vector components $k = 2, 5$, and 9 are the only entries corresponding to, say, total fulfillment of goal two, and their probabilities are 0.05, 0.07, and 0.04, then the probability of totally fulfilling goal two is $0.05 + 0.07 + 0.04 = 0.16$.

This goal probability information can then be given to the investor. If the investor finds these goal probabilities do not fit well with their preferences, they may change the utility values assigned to their goals or increase their infusions and then run the dynamic programming algorithm again to see if the new results sufficiently meet their desires. (See subsection 4.2.1 for an example of this.) This iterative process enables the investor to gain a thorough understanding of what the trade-offs are among their goals when using optimal goal fulfillment and optimal portfolio investment strategies.

4 Numerical Examples

In this section, we present examples that demonstrate the features of our approach, including how changing inputs affects the optimal strategy and other results determined by our algorithm. We start with simple cases where the features are clearer and then proceed to more complex, realistic situations.

In our examples, the portfolios that are available to the investor will be on the efficient frontier shown in Figure 1. This frontier was generated from historical returns in the 20 year period between January 1998 to December 2017 for index funds representing U.S. Bonds, International Stocks, and U.S. Stocks[3].

[3]The three index funds used are (i) Vanguard Total Bond Market II Index Fund Investor Shares (VTBIX), representative of U.S. Fixed Income (Intermediate-Term Bond), (ii) Vanguard Total International Stock Index Fund Investor Shares (VGTSX), representative of Foreign Equity (Large Cap Blend), (iii) Vanguard Total Stock Market Index Fund Investor Shares (VTSMX), representative of U.S. Equity (Large Cap Blend). These three funds have been chosen as representatives of their respective asset categories for illustrative purposes only.

Further, unless stated otherwise, for the examples in this section we will assume that

- We restrict the portfolios used on the frontier to be between $\mu_{\text{min}} = 0.0526$ (which corresponds to $\sigma_{\text{min}} = 0.0374$) and $\mu_{\text{max}} = 0.0886$ (which corresponds to $\sigma_{\text{max}} = 0.1954$). The point $(\sigma_{\text{min}}, \mu_{\text{min}})$ corresponds to the vertex on the frontier. The value of μ_{max} corresponds to the highest of the three component index fund returns, specifically, the return on U.S. Stocks. These three asset classes were chosen because they are widely used by wealth management firms in constructing target-date funds.

- We set the number of available investment portfolios, $l_{\text{max}} = 15$, and these 15 portfolios on the frontier have μ values that are equally spaced between μ_{min} and μ_{max}.

- The time step $h = 1$, so the value of t corresponds to the number of years after the initial investment at $t = 0$ is deposited.

- The combined cost of all the goals could bankrupt the investor, so W_{min} essentially equals $W_{bankrupt}$, where we have chosen $W_{bankrupt} = \$1$.

- The excess wealth at time T is not important to the investor, so $V(W_i(T)) = U(W_i) = 0$ for all wealth grid points W_i.

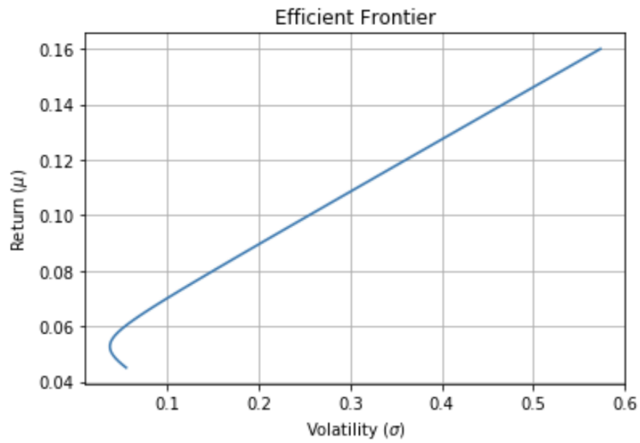
- There are no infusions, which means $I(t) = 0$ at all times t.

These assumptions may be easily changed if desired, as we will do in some examples later in this section.

Our dynamic programming algorithm is easily implemented on a single computer (desktop or laptop) and is coded up in the Python programming language. With the machine we used, which has 16 GB of RAM and an i5 Intel CPU, the run time is under five seconds, except in the complicated final example, where the runtime is 37 seconds.

4.1 Examples with Two All-Or-Nothing Goals

For the examples in this section, we have just two goals: one at $t = 5$ years, the other at $t = 10$. (We therefore use $T = 11$, since, as discussed in our setup, for simplicity we have chosen not to allow goals to be filled in the final year T.) We use an initial wealth of $W(0) = \$100$, which leads to $W_{\text{max}} = \$1834$. We have selected $i_{\text{max}} = 475$ for the number of nodes in the wealth grid between $W_{\text{min}} \approx 1$ and W_{max}. The runtime for this model is 1.5 seconds.



Portfolio number	Portfolio Weights U.S. Bonds	International Stocks	U.S. Stocks
1	0.9098	0.0225	0.0677
2	0.8500	0.0033	0.1467
3	0.7903	-0.0160	0.2257
4	0.7305	-0.0352	0.3047
5	0.6707	-0.0545	0.3837
6	0.6110	-0.0737	0.4628
7	0.5512	-0.0930	0.5418
8	0.4915	-0.1122	0.6208
9	0.4317	-0.1315	0.6998
10	0.3719	-0.1507	0.7788
11	0.3122	-0.1700	0.8578
12	0.2524	-0.1892	0.9368
13	0.1927	-0.2085	1.0158
14	0.1329	-0.2277	1.0948
15	0.0731	-0.2470	1.1738

Figure 1: Top: The efficient frontier generated from the returns of our three index funds. Bottom: The portfolio weights in the three index funds for each of the $l_{\max} = 15$ portfolios with equal spacing in μ between $\mu_{\min} = 0.0526$ and $\mu_{\max} = 0.0886$. As seen from the table, both long-only and long-short portfolios occur.

4.1.1 A Single Case with Two All-Or-Nothing Goals

We first consider a case where the goal at $t = 5$, if fulfilled, has a cost $c(5) = \$100$ and an assigned utility $u(5) = 1000$, while the goal at $t = 10$ has a cost $c(10) = \$150$ and utility $u(10) = 1000$. Because the utilities of the two goals are equal, it is clear what the investor should do regarding fulfilling the goal at $t = 5$: If the investor has the \$100 at $t = 5$, they should spend it to attain the $t = 5$ goal, because this guarantees a utility of 1000 with the possibility for an additional 1000 should they be able to obtain \$150 when $t = 10$. In contrast, had they chosen to forgo the goal at $t = 5$, they would, at most, have a total utility of 1000 from the $t = 10$ goal, which corresponds to a smaller total expected utility.

It is far less clear for this case which investment portfolio choices optimize the investor's total expected utility, but our algorithm generates these optimal choices, which are given in Figure 2, for all levels of wealth and values of time. In this figure, the darker the portfolio color at a given wealth and time, the higher the optimal portfolio is located on the efficient frontier, meaning the portfolio is more aggressive.

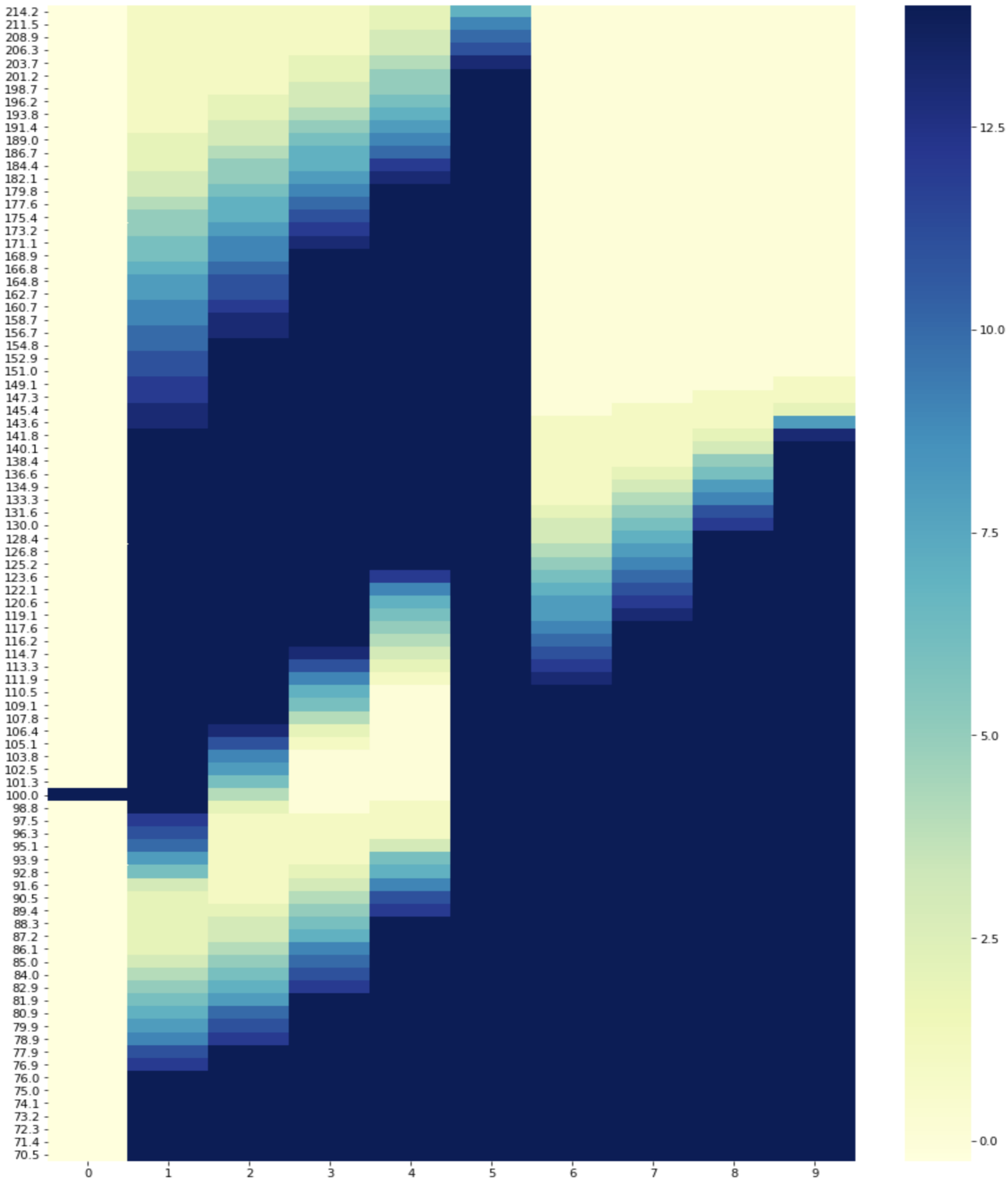


Figure 2: The grid of chosen portfolios for all levels of wealth and time. The darker the portfolio color, the higher up on the efficient frontier it is. The cost of the goals are \$100 at 5 years and \$150 at 10 years. Both goals have an assigned utility of 1000.

We see interesting strategy shifts in Figure 2 when $t < 5$. If the portfolio is doing poorly, the most aggressive portfolio is chosen, since that optimizes the chance of the investor getting back to \$100 and attaining the $t = 5$ goal. If the investor has a little more money, they optimally choose relatively safer portfolios to make sure they remain on-track to have at least \$100 at $t = 5$. But if they have even more than that, they optimally shift to more aggressive portfolios again since they are likely to attain the $t = 5$ goal and now hope to also attain the $t = 10$ goal. Finally, should they have even more money, they again move to safer portfolios, since they now want to preserve the wealth that will allow them to attain both goals.

At $t = 5$, the most aggressive portfolio is chosen for wealth levels below approximately \$201 because \$100 will be deducted if the investor has that much money leaving the need for a more aggressive portfolio to optimize the chance of growing the remaining \$101 or less into \$150 by $t = 10$. For $t > 5$, unsurprisingly, we see aggressive portfolios if reaching \$150 by $t = 10$ looks difficult and less aggressive portfolios when the investor has more money and is better off safeguarding their funds from losses that could reduce their wealth below \$150 by $t = 10$.

This optimal strategy leads to a probability of 0.893 for fulfilling the $t = 5$ goal and a 0.275 probability for fulfilling the $t = 10$ goal. The probability distributions for the wealth at each time are given in Figure 3. Note the figure's leftward shifts after $t = 5$ and $t = 10$ due to the payments made when fulfilling goals. Also, as expected, we see the optimized strategy create a clump in the distribution that just exceeds \$100 at $t = 5$ so as to attain the $t = 5$ goal and another clump exceed \$150 at $t = 10$. Because the probability of exercising the goal at 5 years is much higher than the one at 10 years, we see a bigger probability mass clump at 5 years. Further, note the probability mass clump located a little below \$250 at year $t = 5$, which occurs because the algorithm may be on track to attain both goals.

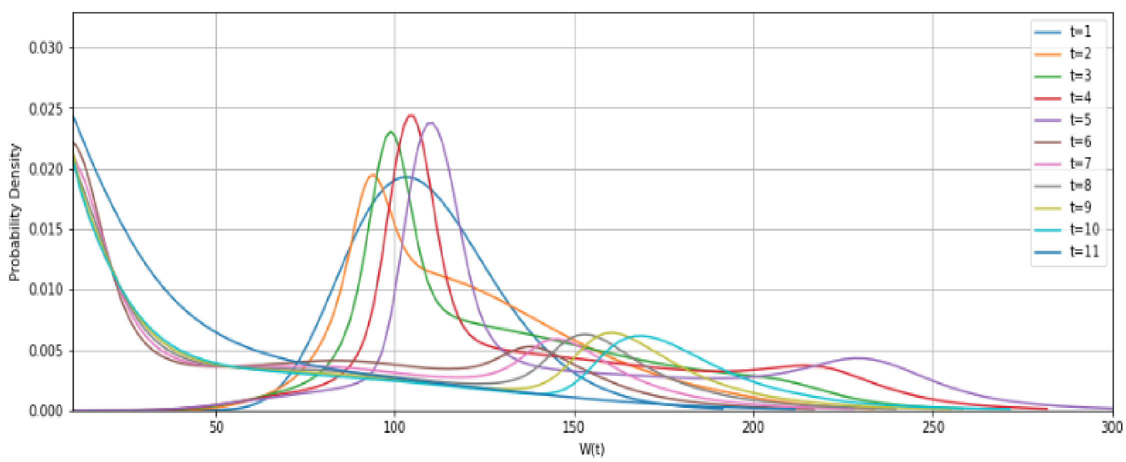


Figure 3: Probability distribution for wealth at different times, t, in the model. Attaining the goals at either $t = 5$ or $t = 10$ causes the distribution to shift to the left.

4.1.2 Altering the Two Goals' Assigned Utilities and Costs

The utility values assigned by the investor to the two goals have a significant effect on the optimal portfolio strategy. They also have a significant effect on deciding whether or not to exercise the option to fulfill or forgo the $t = 5$ goal. This decision is optimally determined by the dynamic programming algorithm by calculating which is larger: 1) the utility for the $t = 5$ goal plus the expected utility of the $t = 10$ goal starting from a wealth level that has been reduced due to paying for the $t = 5$ goal or 2) the utility of the $t = 10$ goal with no reduction in wealth because the $t = 5$ goal is not exercised.

In Panel A of Table 1, we see some of the changes and tradeoffs that occur as we vary the utilities of the two goals while keeping the costs fixed at $c(5) = \$100$ and $c(10) = \$150$. In Panel B of Table 1, we look at the same utility combinations, but we switch the costs so that $c(5) = \$150$ and $c(10) = \$100$. An analysis of the table reveals a number of properties concerning the optimal strategy:

- As we would expect, the relative utilities for the two goals determines their priority. Holding $u(5)$ constant and then increasing $u(10)$ from 1000 to 2000 to 3000, we see from the last two columns in the table that the probability of attaining the $t = 5$ goal decreases, while the probability for the $t = 10$ goal increases. This can be drastic. For example, in the case where $u(5) = 1000$ in Panel A, we see the probability of attaining the $t = 5$ goal decrease from 89.3% to 0.9% while the percentage for the $t = 10$ goal increases from 27.5% to 91.6%.

- Changing the cost has a similar effect. Comparing Panel A to Panel B, we see a considerable decrease in the probability of attaining the $t = 5$ goal and a corresponding increase in the probability of attaining the $t = 10$ goal caused by $c(5)$ increasing from \$100 to \$150 and $c(10)$ decreasing from \$150 to \$100.

- The third column gives the value function, V, at the initial condition, $W_{i_0}(0)$, which is equal to the optimal total expected utility. This value can be computed from the first two and last two columns; for example, for the second row in panel A, we can calculate $1855 = 1000 \times 0.123 + 2000 \times 0.866$. That is, the total expected utility is the sum of each goal's utility weighted by the probability of attaining that goal.

- For any set of goals, multiplying all the goals' utilities by the same constant has no effect on the optimal strategy. So, for our cases with two goals here, only the ratio of $u(5)$ to $u(10)$ matters, not the individual values of $u(5)$ and $u(10)$. For example, within each panel, when $u(5) = u(10)$ we see that the probabilities in the last three columns are identical. Further, when $u(5) = u(10) = 2000$ and then 3000, the corresponding values for the optimal total $E[u]$ in the third column are exactly twice and then thrice the value when $u(5) = u(10) = 1000$.

- If $u(5) \geq u(10)$, then the investor should always take the $t = 5$ goal if they have the money. This is due to the fact that taking the $t = 5$ goal guarantees an

Table 1: The tradeoffs between two goals, each of which is fulfilled or forgone. For various values of the utility, u, assigned to the goals at $t = 5$ and $t = 10$, we present the optimal expected total utility, $E[u]$, which is the value function at the initial wealth and time, $W_{i_0}(0) =$ \$100. Under this optimal strategy, we present the probability of having sufficient wealth to attain the goal at $t = 5$, the probability that the optimal strategy will fulfill the goal at $t = 5$, and the probability that the optimal strategy will attain the goal at $t = 10$. In Panel A, fulfilling the $t = 5$ goal costs \$100 and fulfilling the $t = 10$ goal costs \$150. These goal costs are switched in Panel B.

Panel A: c(5)=\$100, c(10)=\$150

$u(5)$	$u(10)$	Optimal $E[u]$ $= V(W_{i_0}(0))$	$Pr[t = 5$ wealth is $\geq \$100]$	Pr[fulfilling $t = 5$ goal]	Pr[fulfilling $t = 10$ goal]
1000	1000	1168	0.893	0.893	0.275
1000	2000	1855	0.917	0.123	0.866
1000	3000	2757	0.969	0.009	0.916
2000	1000	2110	0.969	0.969	0.171
2000	2000	2336	0.893	0.893	0.275
2000	3000	2886	0.849	0.298	0.764
3000	1000	3087	0.984	0.984	0.134
3000	2000	3259	0.950	0.950	0.205
3000	3000	3504	0.893	0.893	0.275

Panel B: c(5)=\$150, c(10)=\$100

$u(5)$	$u(10)$	Optimal $E[u]$ $= V(W_{i_0}(0))$	$Pr[t = 5$ wealth is $\geq \$150]$	Pr[fulfilling $t = 5$ goal]	Pr[fulfilling $t = 10$ goal]
1000	1000	1185	0.398	0.398	0.787
1000	2000	2137	0.358	0.186	0.976
1000	3000	3120	0.340	0.163	0.986
2000	1000	1631	0.493	0.644	0.493
2000	2000	2370	0.398	0.398	0.787
2000	3000	3306	0.373	0.210	0.962
3000	1000	2155	0.544	0.544	0.524
3000	2000	2792	0.449	0.449	0.723
3000	3000	3555	0.398	0.398	0.787

amount greater than or equal to the utility that may or may not happen from the $t = 10$ goal. This means the values in the fourth and fifth columns of the table are identical. When $u(5) < u(10)$, however, it may be best to forgo the $t = 5$ goal to optimize the chance of fulfilling the $t = 10$ goal. For example, in Panel A when $u(5) = 1000$ and $u(10) = 3000$, there is a 96.9% chance that the investor will have retained at least their original investment of \$100 by $t = 5$, but they optimally opt to spend this \$100 to fulfill the $t = 5$ goal only 0.9% of the time, because it is far more important to build up at least \$150 by $t = 10$ where they attain three times more utility. They only opt to fulfill the $t = 5$ goal if they have amassed so much money that they are confident they can obtain the $t = 10$ goal even after spending \$100 to obtain the $t = 5$ goal.

4.1.3 Determining the Wealth Intervals where the Investor Optimally Opts to Fulfill the $t = 5$ Goal

Recall from the introduction the case of an investor that wanted to take a nice vacation in 5 years and also wanted to buy their child a car in 10 years, where this second goal was more important and more expensive than the first goal. This corresponds to a situation like $c(5) = 100$, $c(10) = 150$, $u(5) = 2000$, and $u(10) = 3000$, given in Panel A of Table 1. In the introduction we pointed out that at $t = 5$, the investor should repeatedly shift their decision about whether or not to take the goal at $t = 5$ depending on their wealth, but it was not clear at which wealth values these decision switches should occur.

Our algorithm can determine these wealth values for switching the decision optimally. For example, take the case above where $c(5) = 100$, $c(10) = 150$, $u(5) = 2000$, and $u(10) = 3000$. At $t = 5$, if the investor has less than \$100, obviously they cannot take the $t = 5$ goal. If they have between \$100 and \$108 at $t = 5$, however, the investor should take the $t = 5$ goal, because their wealth is so low that the chance of being able to attain \$150 by $t = 10$ is too small to justify the additional utility they would obtain should they reach the $t = 10$ goal. Between \$108 and \$182, the situation flips: the investor should forgo the $t = 5$ goal, because it sufficiently increases the likelihood of attaining the $t = 10$ goal to justify the risk. Finally, if the investor has over \$182 at $t = 5$, they should again take the $t = 5$ goal, because they are sufficiently likely to be able to attain both goals.

4.2 Example with Multiple All-Or-Nothing Goals

In this section, we expand from two all-or-nothing goals to seven all-or-nothing goals whose details are listed in Table 2 below, where $T = 25$. We start with an initial wealth of $W(0) = \$30$ (thousand), which corresponds to $W_{\text{max}} = \$5026$, just over five million dollars. We have selected $i_{\text{max}} = 556$ for the number of nodes in the wealth grid between $W_{\text{min}} \approx 1$ and W_{max}. The runtime is approximately 4 seconds.

Table 2 also provides the results from our algorithm, which optimizes the total expected utility using the strategy shown in Figure 4. The value of the optimal total expected utility $E[u] = V(W_{i_0}(0))$ is 5419. As indicated in the previous section, this equals the dot product of the vectors comprising the bottom two rows of Table 2. Since the total utility from summing the elements in the third row is 10800, we see that the investor can adopt a strategy to attain goals that, at best, on average, achieves the fraction $5419/10800 = 0.5018$ of the utility corresponding to attaining all the investor's goals. This optimally achievable total expected utility fraction will also be referred to

Table 2: Base Case: Seven All-Or-Nothing Goals Under the Optimal Strategy where $W(0) = 30$.

	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
Goal time, t (in years)	5	8	10	11	17	22	24
Goal cost, c (in 1000s of \$)	25	17	15	80	50	60	130
Goal utility, u	1000	2500	500	1500	300	3000	2000
Probability of fulfilling goal	0.0435	0.9875	0.2085	0.0308	0.0190	0.7629	0.2311

as the "expected fraction of maximum attainable utility" or just the "expected utility fraction" in the tables, figures, and explanations below.

Should the investor not be happy with the probability of fulfilling a few of the individual goals, they may change the assigned utility of these goals, as we explore in the next subsection. Should they wish to increase the probabilities of fulfilling all their goals, they must increase their initial contribution, $W(0)$, or look to make cash infusions, $I(t) > 0$, or allow for a larger set of potential portfolios by, for example, relaxing their interval restrictions on the efficient frontier. We will consider each of these three possibilities in the three subsections following the next subsection. In the final subsection after that, we will consider the effect of valuing excess money at time $T = 25$, meaning that $V(W_i(T)) = U(W_i) > 0$.

4.2.1 The Effect of Changing the Assigned Utilities on Probabilities

The investor may look at the probabilities generated in Table 2 and decide that some are lower than they want. For example, let's say the investor feels that a 3.08% chance of attaining Goal 4 is too low. They may therefore decide to increase the utility assigned to Goal 4. Results generated from increasing the utility for Goal 4 are shown in Table 3. The table shows that increasing the utility assigned to Goal 4 increases the probability that Goal 4 is attained, but, in general, it also decreases the probability that the other goals are attained.

Note that this is not always the case, however. For example, in Table 3, we see that the probability of attaining Goal 3 *increases* when the utility for Goal 4 increases from 1500 to 2000. This is because the utility increase encourages more aggressive portfolios early on, which increases the chance that attaining Goal 3 at $t = 10$ is both possible and worthwhile, especially if Goal 4, which is at $t = 11$, remains unattainable, since the \$80 (thousand) Goal 4 cost is much higher than the \$15 (thousand) Goal 3 cost.

Because Goal 2 generates a high utility, 2500, at the low cost of \$17 (thousand), it remains heavily desirable until the utility assigned to Goal 4 increases past 5000. At that point, it becomes progressively more important to consider abandoning Goal 2, so as to maximize the chance of attaining Goal 4.

Note also that when the utility for Goal 4 increases from 10,000 to 100,000, the algo-

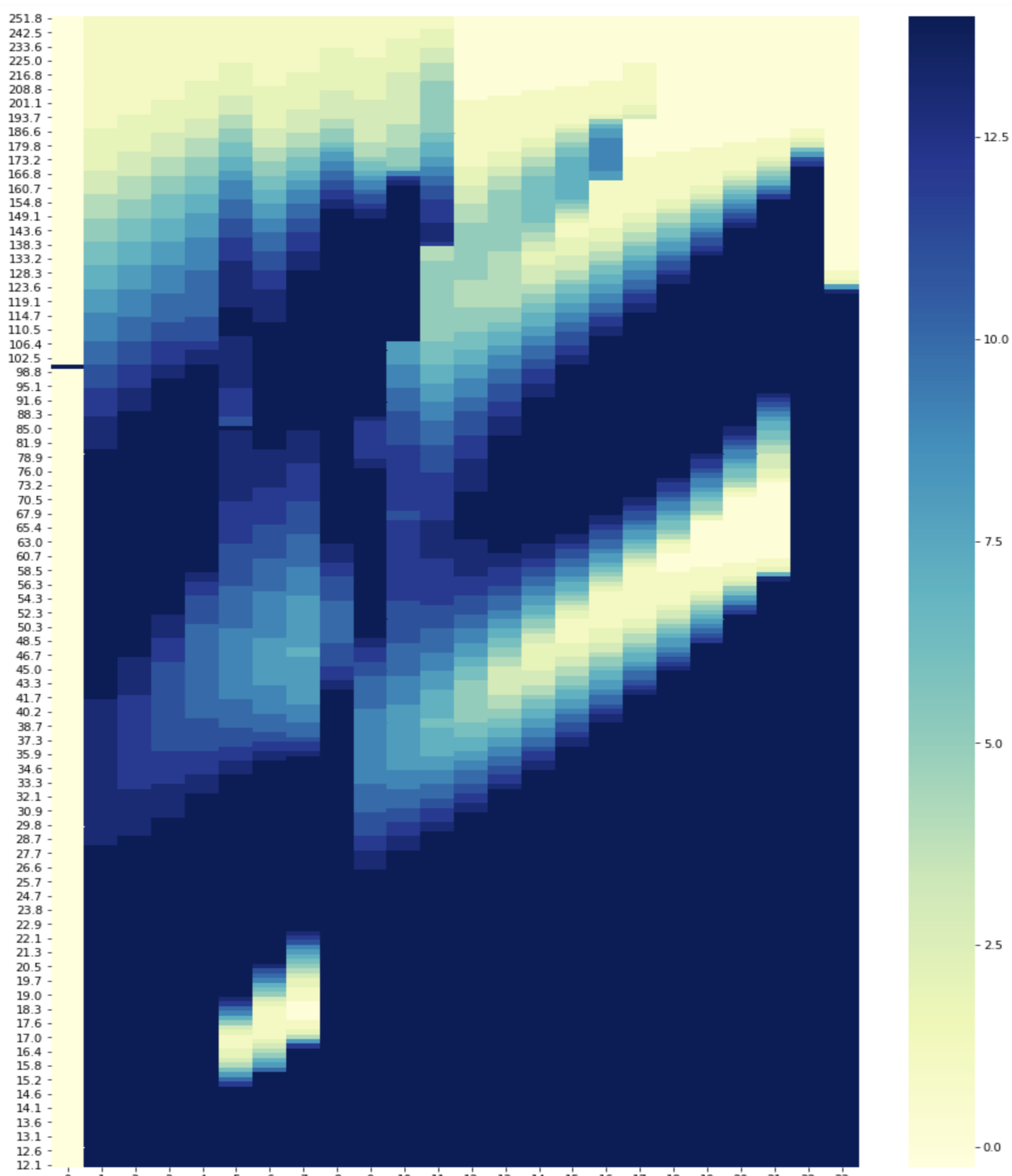


Figure 4: The grid of optimal portfolios for all levels of wealth and time. The darker the portfolio color, the higher up on the efficient frontier it is. Information for the seven goals at times 5, 8, 10, 11, 17, 22, and 24 used here are presented in Table 2.

Table 3: The effect of the investor increasing the utility value assigned to Goal 4 upon the base case specified in Table 2, reprised at the top of the table. This increases the chances of attaining Goal 4, but generally decreases the chance of attaining the other goals.

	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
Goal time, t (in years)	5	8	10	11	17	22	24
Goal cost, c (in 1000s of \$)	25	17	15	80	50	60	130
Goal utility, u	1000	2500	500	1500	300	3000	2000

Utility, u,	Probability of fulfilling goal						
for Goal 4	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
1500	0.0435	0.9875	0.2085	0.0308	0.0190	0.7629	0.2311
2000	0.0309	0.9860	0.2244	0.0596	0.0158	0.7526	0.2261
3000	0.0073	0.9854	0.1327	0.1314	0.0106	0.7329	0.2022
5000	0.0021	0.9853	0.0342	0.2341	0.0049	0.6561	0.1438
10000	0.0002	0.6972	0.0098	0.3857	0.0021	0.5234	0.0839
100000	0.0000	0.1258	0.0002	0.4776	0.0025	0.4997	0.1082

rithm adopts maximally aggressive portfolios early on. This increased volatility increases the chance of extremely high or low wealth values later on when the investor is looking to attain Goals 6 and 7. Attaining Goal 6 has priority over Goal 7 because it is much less expensive and has a significantly higher utility. Therefore, we see the approximate likelihood of attaining neither of these final goals due to low wealth increasing from $1 - 0.5234 = 0.4766$ to $1 - 0.4997 = 0.5003$ and the approximate likelihood of attaining both goals due to high wealth increasing from 0.0839 to 0.1082. This last increase corresponds to the numbers in Table 3 for the probability of attaining Goal 7, since Goal 6, with its higher priority, is generally attained whenever Goal 7 is attained. This increase in the probability of attaining Goal 7 is another example of where increasing the utility assigned to Goal 4 can increase, instead of decrease, the probability of attaining a goal other than Goal 4.

The investor, as much as they desire, can proceed to increase (or decrease) the assigned utilities for each goal and see the effect this has on the optimal probabilities of attaining their goals. At the end of the day, they will have an accurate understanding of the trade-offs between all of their goals and the limitations of their current portfolio under the set of available investment portfolios, when optimally run. Should they feel these limitations on their optimal probabilities are too severe, they have three options for lessening them, which correspond to the next three subsections.

4.2.2 The Effect of Changing the Initial Wealth, $W(0)$

Should the investor wish to increase their chances of obtaining all of their goals, they may decide to increase their initial investment, $W(0)$. Figure 5 shows how the optimally achievable total expected utility fraction depends upon $W(0)$. The red dot in the figure represents our base case from Table 2, where $W(0) = \$30$ and the expected utility fraction is 0.5018. Should the investor increase their original investment so that $W(0) = \$50$ instead of \$30, then, from the figure, we can see that the expected utility fraction will increase from approximately 1/2 to approximately 2/3. As $W(0)$ gets larger, the expected utility fraction approaches 1, of course, and, similarly, as $W(0)$ gets small, the expected utility fraction goes to 0. We note that because $W_{\max}$ in our algorithm is determined from the value of $W(0)$, to generate Figure 5 we had to make certain that $W_{\max}$ was sufficiently large to accommodate all the values of $W(0)$ used in the figure.

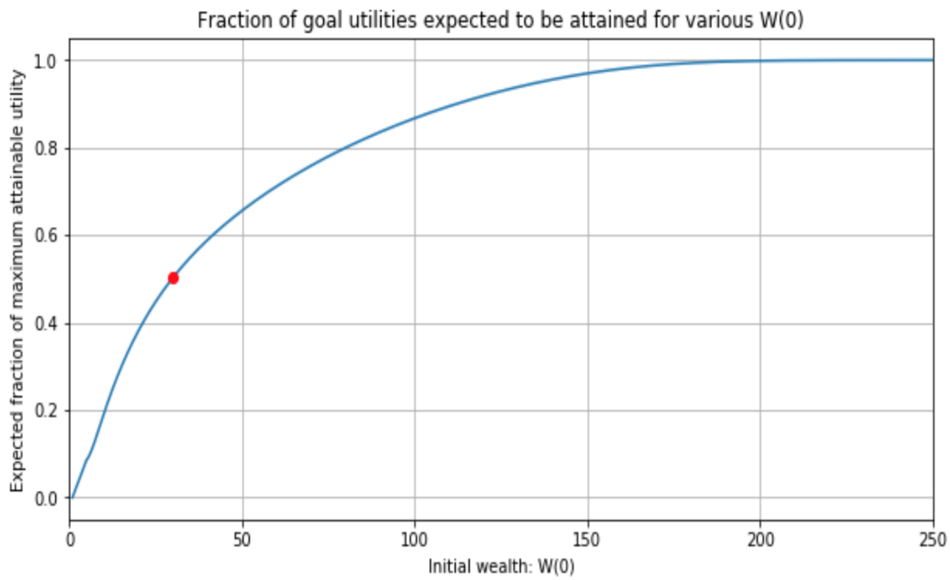


Figure 5: The effect of varying the initial wealth, $W(0)$, on attaining the investor's goals. Information for the seven goals used here is presented in Table 2. The initial wealth base case of \$30 corresponds to an optimally achievable total expected utility fraction of 0.5018, as represented by the red dot.

4.2.3 The Effect of Infusions

Should the investor not have the funds or the desire to increase the initial investment, they may decide instead to commit to future cash infusions, $I(t)$. The infusions can be for any desired amount. Typical infusion examples include being constant ($I(t) = k$), adjusted for inflation ($I(t) = k(1+r)^t$), or a one-time infusion ($I(t_0) = k$ while $I(t) = 0$ if $t \neq t_0$).

Panel A of Table 4 shows the beneficial effect on the base case in Table 2 of having constant annual infusions, starting at the beginning of year 1 and ending at the beginning of year 24. The first row in the panel has no infusions and is therefore the same as the base case in Table 2. As we increase the constant infusion amount, we see an increase not only in the expected utility fraction, as must be the case, but also in the probabilities of attaining each individual goal. We note that even small infusions have a substantial impact on achieving goals, making a strong mathematical argument in support of saving and investment. Further, our algorithm allows an investor to understand in clear, concrete terms how setting aside money on a regular basis directly impacts their (now quantifiable) chances of attaining their goals.

Panel B of Table 4 shows the effect on the base case in Table 2 of having one-time infusions. We note that the first and fourth rows correspond to infusions at $t = 0$, which is the equivalent of boosting $W(0)$ from its base case value of \$30. Therefore, from these rows we see that $W(0) = \$30 + \$30 = \$60$ induces an expected utility fraction of 0.7111 while $W(0) = \$30 + \$60 = \$90$ induces the fraction 0.8358. Both of these correspond to points on the graph in Figure 5.

Note that having an infusion helps attain goals, even if those goals are at times before the infusion. For example, having no infusion leads to a 4.35% chance of attaining Goal 1 at $t = 5$, and an infusion of an additional \$30 at $t = 0$ boosts this to a 50.51% chance, but if the \$30 infusion happens instead at $t = 10$, the chance of attaining Goal 1 at $t = 5$ is still boosted to 19.06%. This is because the investor is freed up from having to preserve some money from the initial investment to attain later goals.

The later the infusion occurs, the smaller its effect due to the investment having less time to grow. Note, for example, from inspecting the expected utility fractions that an infusion of 60 at $t = 10$ is approximately equal to an infusion of 30 at $t = 0$, and an infusion of 60 at $t = 20$ is approximately equal to an infusion of 30 at $t = 10$. Therefore, the probability of each individual goal generally goes down as the infusion time increases, but the timing of the infusion creates other effects that can counter this. For example, the probability of attaining Goal 6 generally *increases* as the infusion time increases. This makes some intuitive sense since Goal 6, which is at $t = 22$, has a cost of 60, so having an infusion of 60 at the later time of $t = 20$ helps attain this goal more than having the infusion earlier would, when it is more likely to be subject to losses.

4.2.4 The Effect of Changing the Available Portfolios

Finally, if the investor wishes to increase their ability to fulfill their goals without adding additional money to their portfolio, this can be accomplished by expanding the range of portfolios that are accessible on the efficient frontier by either decreasing $\mu_{\min}$ or increasing $\mu_{\max}$. This expanded set of investment possibilities can only increase the expected total utility of the investor. In our case, we have chosen the default value of $\mu_{\min}$ to be 0.0526, which corresponds to the vertex of the efficient frontier shown in Figure 1. Because it is the vertex, selecting lower values of $\mu_{\min}$ make no financial sense, since lower values of $\mu_{\min}$ correspond to higher values of $\sigma_{\min}$. On the other hand, in Table 5 we can see the effect of increasing the value of $\mu_{\max}$ from its default value of 0.0886.

Table 5 shows how having access to more aggressive portfolios increases the expected utility fraction, as it must. It also increases the probability of fulfilling most of the individual goals. The clear exception is Goal 2, whose probability decreases. This

Table 4: The effect of infusions on the base case shown in Table 2, reprised at the top of the table, for which $W(0) = \$30$. Panel A shows the effect of annual infusions. Panel B shows the effect of a single infusion at the given time $t = t_0$. The infusions' effects on increasing the probabilities of exercising the various goals are shown.

	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
Goal time, t (in years)	5	8	10	11	17	22	24
Goal cost, c (in 1000s of \$)	25	17	15	80	50	60	130
Goal utility, u	1000	2500	500	1500	300	3000	2000

Panel A: Annual infusions

Annual Infusion, $I(t)$	Expected Utility Fraction	Probability of fulfilling goal						
		Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
0	0.5018	0.0435	0.9875	0.2085	0.0308	0.0190	0.7629	0.2311
1	0.6059	0.1468	0.9979	0.4056	0.0709	0.0390	0.9344	0.3895
2	0.6824	0.3419	1.0000	0.6205	0.1080	0.0621	0.9852	0.5409
3	0.7416	0.6278	1.0000	0.7624	0.1494	0.0823	0.9963	0.6312
5	0.8418	0.9811	1.0000	0.9309	0.3386	0.1667	0.9994	0.7946
10	0.9915	1.0000	1.0000	0.9993	0.9996	0.7945	1.0000	0.9854

Panel B: Single Infusion at time $t = t_0$

Infusion time, t_0	Infusion Amount, $I(t_0)$	Expected Utility Fraction	Probability of fulfilling goal						
			Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
0	30	0.7111	0.5051	0.9998	0.6385	0.2354	0.1028	0.9603	0.5458
10	30	0.6351	0.1906	0.9975	0.4950	0.0636	0.0405	0.9724	0.4509
20	30	0.5849	0.1098	0.9902	0.3486	0.0456	0.0258	0.9579	0.3035
0	60	0.8358	0.8646	1.000	0.855	0.5050	0.2233	0.9883	0.7224
10	60	0.7164	0.4459	0.9974	0.9391	0.0930	0.0635	0.9896	0.6006
20	60	0.6388	0.1758	0.9966	0.5229	0.0540	0.0369	0.9986	0.4410

decrease happens because the early use of high volatility stocks means the investor is more likely to lose the money they need to safeguard in order to fulfill Goal 2. However, the bigger problem for an investor considering increasing $\mu_{\max}$ is the corresponding significant increase in $\sigma_{\max}$ shown in Table 5, as well as the required significant shorting and going very long for the component positions within such aggressive portfolios.

4.2.5 The Effect of Assigning Utility to Excess Money at $T = 25$

Up until this point, we have assumed that the investor has no use for any excess money left over when the fund is closed at $T = 25$. (That is, $V(W(T)) = U(W) = 0$.) We now consider the effect of valuing this excess money by assigning the final example utility

Table 5: The effect of increasing $\mu_{\max}$ — and therefore the available investment portfolios — on the base case shown in Table 2, reprised at the top of the table. The effect of expanding the range of investment portfolios on the probabilities of fulfilling the various goals are shown.

	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
Goal time, t (in years)	5	8	10	11	17	22	24
Goal cost, c (in 1000s of \$)	25	17	15	80	50	60	130
Goal utility, u	1000	2500	500	1500	300	3000	2000

Value for $\mu_{\max}$	Value for $\sigma_{\max}$	Expected Utility Fraction	Probability of fulfilling goal: Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
0.0886	0.1954	0.5018	0.0435	0.9875	0.2085	0.0308	0.0190	0.7629	0.2311
0.10	0.2551	0.5117	0.0613	0.9756	0.2269	0.0477	0.0238	0.7662	0.2679
0.15	0.5201	0.5462	0.1727	0.9498	0.3394	0.1057	0.0433	0.7665	0.3555
0.20	0.7861	0.6053	0.3747	0.8879	0.4281	0.2231	0.1054	0.8093	0.4670

function from Subsection 3.1 to these excess funds, which was

$$U(W) = k\left(\frac{1}{1+be^{-aW}} - \frac{1}{1+b}\right), \tag{5}$$

where $a, b, k > 0$. Note that $0 \le U(W) < k\frac{b}{1+b}$, no matter what the value of a is. As a increases, the investor becomes more risk averse. Generally, we have $a < 0.5$, since behavior where $a > 0.5$ becomes hyper-averse. See Figure 6 for examples of our utility function in equation 5, where $b = 1$ and $k = 2$. Note, as expected, that $U(W)$ is both increasing and concave in W. For the cases in the figure, when $a = 0.01$, $U(W)$ essentially reaches its maximum value of 1 by $W = 500$, whereas when $a = 0.05$, it is essentially reached by $W = 100$.

We implemented our algorithm with $U(W)$ being given by equation (5) with $a = 0.01$ and $b = 1$, so the maximum utility is $k/2$. Table 6 shows the effect of various values of the magnitude k on $E[W(T)]$, which is the expected final wealth, $E[U(W(T))]$, which is the expected utility from the final wealth that is guaranteed to be less than $k/2$, and the probabilities of attaining the seven goals in the base case from Table 2.

When $k = 500$, the maximum value of $U(W(T))$ is $500/2 = 250$, which is much smaller than the utilities associated to the seven goals, so there is little effect on the probabilities that the goals are attained. Indeed, the probability of attaining Goal 7 is actually increased. This is because the excess money at $T = 25$ is now valued, which pushes the investment portfolios at later times to be more aggressive. These aggressive portfolios lead to more available money, on average, near the end, which is better used for Goal 7, whose utility is 2000, than as excess money, which has a utility of 250 at the most.

As k increases, the expected worth of the excess money increases and the probability of attaining the first six goals decreases since it becomes more worthwhile to have excess

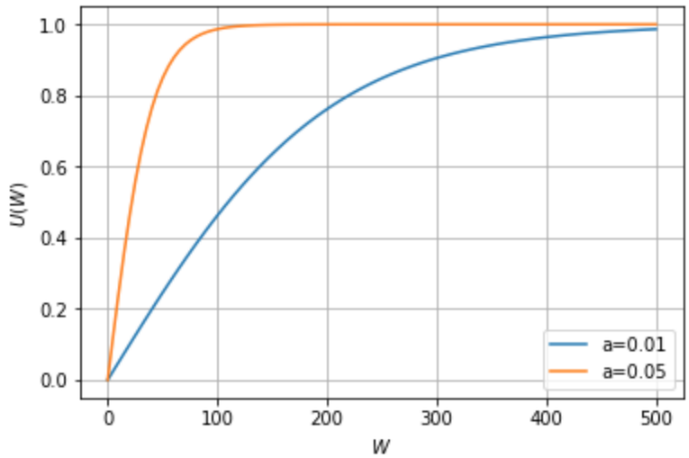


Figure 6: The utility function $U(W) = k\left(\frac{1}{1+be^{-aW}} - \frac{1}{1+b}\right)$, which we can use to value residual wealth, W, at the terminal horizon, T. In the figure, $b = 1$ and $k = 2$, so the maximum value of U is 1. As a increases, the investor becomes more risk averse.

Table 6: The effect of the investor valuing their excess wealth at $T = 25$ on the base case specified in Table 2, reprised at the top of the table. The utility of the excess wealth is based on equation (5), where we fix $b = 1$, $a = 0.01$, and vary the magnitude, k. As k increases, the investor progressively values having excess wealth. The expected excess wealth is $E[W(T)]$. The expected utility of this excess wealth is $E[U(W(T))]$. The utility corresponding to an infinite excess wealth is $k/2$.

	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
Goal time, t (in years)	5	8	10	11	17	22	24
Goal cost, c (in 1000s of \$)	25	17	15	80	50	60	130
Goal utility, u	1000	2500	500	1500	300	3000	2000

			Probability of fulfilling goal						
k	$E[W(T)]$	$E[U(W(T))]$	Goal 1	Goal 2	Goal 3	Goal 4	Goal 5	Goal 6	Goal 7
0	32.31	0	0.0435	0.9875	0.2085	0.0308	0.0190	0.7629	0.2311
500	36.01	42.70	0.0412	0.9874	0.2058	0.0304	0.0189	0.7608	0.2363
1000	37.70	88.68	0.0390	0.9872	0.2029	0.0300	0.0185	0.7582	0.2413
2000	42.14	193.7	0.0397	0.9861	0.1993	0.0299	0.0152	0.7510	0.2483
5000	61.98	655.9	0.0249	0.9833	0.1614	0.0231	0.0108	0.7248	0.2529
10000	90.35	1779	0.0087	0.9804	0.0918	0.0137	0.0065	0.6785	0.1879

funds. The probability for attaining Goal 7, however, continues to increase until the maximum utility from the excess money, $k/2$, exceeds 2000. Once this occurs, the probability of attaining Goal 7 decreases along with the other six goals.

4.3 Previous Examples with Concurrent Goals and Partial Goals

Sections 4.1 and 4.2 were concerned with all-or-nothing goals from different years. We now consider what happens when we allow for multiple goals to happen concurrently and when we allow for partial fulfillment of goals.

4.3.1 Results for the Concurrent and Partial Goals example in Subsection 2.2.2

We revisit the example with three concurrent goals described in Subsection 2.2.2. We will assume these three goals occur at year 5. To this, we add another goal in year 10 that may be taken partially at a cost of \$50 with a utility of 500 or taken fully at a cost of \$90 with a utility of 1000. We assume the initial wealth, $W(0)$, is \$50, so $T = 11$ and $W_{\max} = \$909$. We have selected $i_{\max} = 419$ for the number of nodes in the wealth grid between $W_{\min} \approx 1$ and $W_{\max}$. The runtime is approximately 2 seconds.

The optimal total utility is 1013 out of a maximum possible of 1900, so the expected utility fraction is $1013/1900 = 0.5330$. Table 7 shows the probability of reaching our various partial and full goals. The final three columns in Panel A show, respectively, the cost vector, $\mathbf{c}(5)$; the utility vector, $\mathbf{u}(5)$; and the probability of fulfilling the partial or full goals associated with the components of these vectors. The first six columns break down the cost and utility vectors' values by goal. This is key to creating Panel B, which gives the probabilities of attaining each full or partial goal as an investor would want to see. For example, the first row in Panel B gives the probability of not attaining Goal 1 in year 5. This is calculated by summing all the probabilities in Panel A that correspond to Goal 1 not being attained: $0.0198 + 0.1547 + 0.0672 + 0.1162 = 0.3579$.

The optimal decisions shown in Panel B make intuitive sense. At year $t = 5$, the investor is first attracted to fully realizing Goal 2, because its utility per cost ratio is very high. The investor may defer this goal so as to optimize the chance of attaining the very high utility values assigned to Goal 1 at $t = 10$, even though these high utility values also correspond to higher costs. If the cost of \$20 for fully attaining Goal 2 is too much, the investor may decide to partially fulfill Goal 2 and totally fill Goal 1 at $t = 5$ for a total cost of \$16. Should the investor have even less portfolio wealth and needs to choose between partially filling Goal 2 or totally filling Goal 1, it is better to choose the latter because it gives a higher utility (100 versus 90) at a lower cost (\$7 versus \$9). Should the investor have enough money at $t = 5$ to fully attain Goal 2 and spending more money at $t = 5$ will not sufficiently hurt the chances of attaining Goal 1 at $t = 10$, the algorithm will look to also fulfill Goal 1 at $t = 5$ and then progressively fulfill more and more of Goal 3 at $t = 5$.

Table 7: Partial and concurrent goals. We implement the three goals described in Subsection 2.2.2 as goals at year 5 and add to that a single goal in year 10 that may be taken partially at a cost of \$50 with a utility of 500 or fully at a cost of \$90 with a utility of 1000. The initial wealth, $W(0)$, is \$50. The detailed breakout of goals and probabilities at year 5 is shown in Panel A. Note that the seventh and eighth columns are, respectively, the cost vector, $\mathbf{c}(5)$, and the utility vector, $\mathbf{u}(5)$. In Panel B we show the costs and utilities for each partial or full goal at each time, along with the probability that the goal is fulfilled. The information in Panel A is summed to obtain the year 5 information shown in Panel B.

Panel A: Year 5

Cost			Utility			Total		
Goal 1	Goal 2	Goal 3	Goal 1	Goal 2	Goal 3	Cost	Utility	Probability
0	0	0	0	0	0	0	0	0.0198
7	0	0	100	0	0	7	100	0.0554
7	9	0	100	90	0	16	190	0.0012
0	20	0	0	300	0	20	300	0.1547
7	20	0	100	300	0	27	400	0.2240
7	9	20	100	90	250	36	440	0.0109
7	0	30	100	0	400	37	500	0.0267
0	20	20	0	300	250	40	550	0.0672
7	9	30	100	90	400	46	590	0.0073
7	20	20	100	300	250	47	650	0.0394
0	20	30	0	300	400	50	700	0.1162
7	20	30	100	300	400	57	800	0.1938
7	20	40	100	300	500	67	900	0.0834

Panel B

Year	Goal#	Cost	Utility	Probability
5	1	0	0	0.3579
	1	7	100	0.6421
	2	0	0	0.1019
	2	9	90	0.0194
	2	20	300	0.8787
	3	0	0	0.4551
	3	20	250	0.1175
	3	30	400	0.3440
	3	40	500	0.0834
10	1	0	0	0.4187
	1	50	500	0.2137
	1	90	1000	0.3676

4.3.2 Results for the Example in Subsection 2.2.1 with a New Car Goal and a College Tuition Goal

We revisit the example from Subsection 2.2.1 where an investor is considering buying a new car in four years, but also places a much higher utility on paying their child's tuition every semester for four years, starting in year three. Because the tuition is paid semi-annually, we use $h = 0.5$ instead of $h = 1$, so, for example, year 4 corresponds to $t = 8$. Using the method detailed in Subsection 2.2.2, we can determine the cost vectors, $\mathbf{c}(t)$, and the utility vectors, $\mathbf{u}(t)$, from the numbers given in Subsection 2.2.1. For year 4 (i.e., $t = 8$), these vectors, $\mathbf{c}(8)$ and $\mathbf{u}(8)$, are given, respectively, in the fifth and sixth columns of Table 8.

We assume the initial wealth, $W(0)$, is \$100, so $T = 14$ (for a horizon of 7 years) and $W_{\max} = \$868$. We have selected $i_{\max} = 594$ for the number of nodes in the wealth grid between $W_{\min} \approx 1$ and $W_{\max}$. The runtime is approximately 2.5 seconds. The optimal total utility is 6356 out of a maximum possible of 8300, so the expected utility fraction is $6356/8300 = 0.7658$. Table 8 shows the probability of reaching our various partial and full goals.

From Panel B of Table 8, we see that the tuition goal in every semester is fulfilled with a reasonable probability ($> 75\%$, except in the last year). However, it is almost certain that we will forgo the car goal in year 4, because it is costly and does not offer a utility that is mildly close to that of the tuition goal.

From Panel B of Table 8, we also see that the probability of attaining the tuition goal decreases as we move between years due to the tuition cost going up while the utility remains constant. However, as we change semesters within years 3, 4, and 5, the probability of attaining the tuition goal increases, since the later semester has an extra half a year for the investor's portfolio to grow. In contrast, as we change semesters within year 6, the probability decreases, instead of increases. This makes sense as well: Since the tuition goals at year 6 and at year 6.5 have the same utility, the goal at year 6 should always be taken if the investor has the money, because that guarantees a utility of at least 1000, whereas forgoing this goal guarantees a future utility of at most 1000 from the tuition goal at year 6.5. Since it is possible that fulfilling the tuition goal at year 6 means not having enough money to pay the last semester of tuition at year 6.5, we expect the probability of fulfilling the tuition goal at year 6.5 to be lower than the probability of fulfilling the tuition goal at year 6.

4.4 Example of Optimal Long Range Financial Planning with Numerous Lifetime Goals

We now consider a couple in their mid-thirties, say 35 years old, with one 5 year old child. The couple is looking to create a long term goals-based investment plan over the next 60 years, until they are 95. (So $T = 61$.) Their intent is to retire and take Social Security at the age of 70. As we have done in our previous examples, all of the goals

Table 8: Car and tuition goals example from Subsection 2.2.1. All cost and wealth numbers are in thousands of dollars. The initial wealth is $W(0) = \$100$. The format for this table is the same format used for Table 7. Goal 1 is for tuition, except at year 4, when Goal 1 is for the car and Goal 2 is for tuition.

Panel A: Year 4 (t=8)

Cost		Utility		Total		
Goal 1	Goal 2	Goal 1	Goal 2	Cost	Utility	Probability
0	0	0	0	0	0	0.2309
0	20.407	0	1000	20.407	1000	0.7633
28	20.407	80	1000	48.407	1080	0.0014
32	20.407	125	1000	52.407	1125	0.0021
50	20.407	300	1000	70.407	1300	0.0023

Panel B

Year	Goal#	Cost	Utility	Probability
3	1	0.000	0	0.1412
	1	18.895	1000	0.8588
3.5	1	0.000	0	0.0076
	1	18.895	1000	0.9924
4	1	0.000	0	0.9942
	1	28.000	80	0.0014
	1	32.000	125	0.0021
	1	50.000	300	0.0023
	2	0.000	0	0.2309
	2	20.407	1000	0.7691
4.5	1	0.000	0	0.0765
	1	20.407	1000	0.9235
5	1	0.000	0	0.2235
	1	22.039	1000	0.7765
5.5	1	0.000	0	0.1314
	1	22.039	1000	0.8686
6	1	0.000	0	0.2837
	1	23.802	1000	0.7163
6.5	1	0.000	0	0.5500
	1	23.802	1000	0.4500

and costs are in thousands of (after-tax) dollars.

The income sources (infusions) for the couple are:

1. Salary income: A combined annual salary of \$70, projected to increase annually at 4% until retirement at age 69 ($t = 34$). This includes money being saved for

retirement.

2. Retirement income: Social Security is assumed to pay \$120 at age 70 [4] ($t = 35$), which corresponds to around \$43 present day dollars, assuming a rate of inflation of 3%. We will also assume the rate of inflation continues at 3% for Social Security payouts between $t = 35$ and $t = 60$.

The couple's goals are:

1. Level one (highest priority) goals ($u \geq 1000$)

   (a) Mortgage: Assume the couple needs to pay a fixed annual rate of \$10 for the next 25 years ($t = 1$ through $t = 25$) to pay off what remains from a 30 year fixed mortgage. The assigned annual utility is $u = 2000$.

   (b) Property tax: Assume the annual cost is \$6, which goes up every year by 2%. $u = 1000$, which is half of the mortgage utility because the cost is about half of the mortgage cost, at least initially.

   (c) Long term care insurance: Starting at age 50 ($t = 15$), at a cost of \$7, which then goes up by 4% every year; $u = 1000$.

   (d) Medical expenses: Assume these are insignificant until the age of 75, after which they are approximated to be \$8 initially with an inflation rate of 10%. The high inflation rate reflects both progressively higher costs of medical care, as well as progressively higher needs for care as the couple ages. $u = 5000$ to emphasize their importance late in life.

   (e) Everyday expenses: Assume these start at \$60 per year and go up at a 3% rate of inflation. Because this is so much more expensive than the other high priority goals, we choose $u = 10{,}000$, since paying nothing in this category is unrealistic. Further, if necessary, these can be trimmed to \$50 per year by cutting costs that are not as crucial. We drop the utility associated to this partial goal from 10,000 to 8000.

2. Level two goals ($u \approx 300$ initially)

   (a) Orthodontia for the couple's child: Assume this costs \$3 each year when their child is 11, 12, and 13; that is, $t = 6, 7$, and 8. $u = 150$, which is on the smaller side because the cost is so small compared to other level two goals.

   (b) College tuition for the couple's child: Assume the annual cost is \$40 at $t = 13$, when the child is 18, and then goes up by 8% in each of the next three years. A partial goal that starts at \$25 and then goes up by 8% is also available. Since this goal is so expensive and important, the full goal has $u = 900$. The partial goal has $u = 750$.

[4] Approximated using Social Security's quick benefit estimate calculator at https://www.ssa.gov/cgi-bin/benefit6.cgi.

(c) Cars: Assume the couple would like to purchase a new car every five years at a cost of \$32. They may also opt for the partial goal of a used car for \$22. We assign $u = 300$ to the full goal and $u = 200$ to the partial goal.

3. Level three goals ($u \approx 100$ initially)

   (a) Remodeling the house at $t = 16$: A nice remodel will cost \$50 and is assigned $u = 100$. Less nice remodels at costs of \$40 and \$25 are also available at the reduced utility levels of $u = 70$ and 50, respectively.

   (b) Wedding expenses for the couple's child: Assume this costs \$70, with the couple approximating that their child will marry at age 28; that is, $t = 23$. A less expensive wedding costs \$55, but the couple believes their child might be very upset about this, so, where $u = 100$ for the expensive wedding, the less expensive wedding has only $u = 35$.

4. Level four goals ($u \approx 30$ initially)

   (a) Private high school for the couple's child: Assume the annual cost is \$25 at $t = 9$, when the child is 14, and then goes up by 5% in each of the next three years. $u = 90$, since the cost is so high.

   (b) A fancy trip: Assume the couple wishes to take a nice trip every 10 years at a cost of \$15 at $t = 10$, which increases at the annual inflation rate of 3% thereafter; $u = 50$, because the cost is a bit high.

   (c) Philanthropic gifts: Initially, \$5 every year or a partial goal of \$2.5 at half the utility, $u = 15$, instead of $u = 30$. This cost goes up at a 3% annual rate of inflation; $u = 40$ for the full goal; $u = 20$ for the half goal.

We assume the initial wealth, $W(0)$, is \$100, therefore $W_{\max} = \$18{,}000$ (that is, \$18 *million*). We have selected $i_{\max} = 1221$ for the number of nodes in the wealth grid between $W_{\min} \approx 1$ and $W_{\max}$. Because of the significant increase in the number of goals, the horizon time T, and the number of wealth grid points, the runtime increases from the few seconds needed in previous examples to 37 seconds. Table 9 shows the optimal probabilities of reaching our various partial and full goals using the optimal investment portfolio choices presented in Figure 7.

The optimal total utility is 869,230 out of a maximum possible of 871,560, so the expected utility fraction is 869,230/871,560 = 0.9973. It is reasonable to think at first that such a high fraction means a high likelihood of attaining all the couple's goals, but a quick overview of Table 9 shows that this is not the case.

All of the couple's level one (high priority) goals are met with a very high probability. Indeed, the few cases where any of these goals have a probability below 0.999 of being attained only occur at the very end of the portfolio horizon when t is near 60. Because the sum of these level one goals' utility values are so high compared to the sum of the other levels, the expected utility fraction is also high.

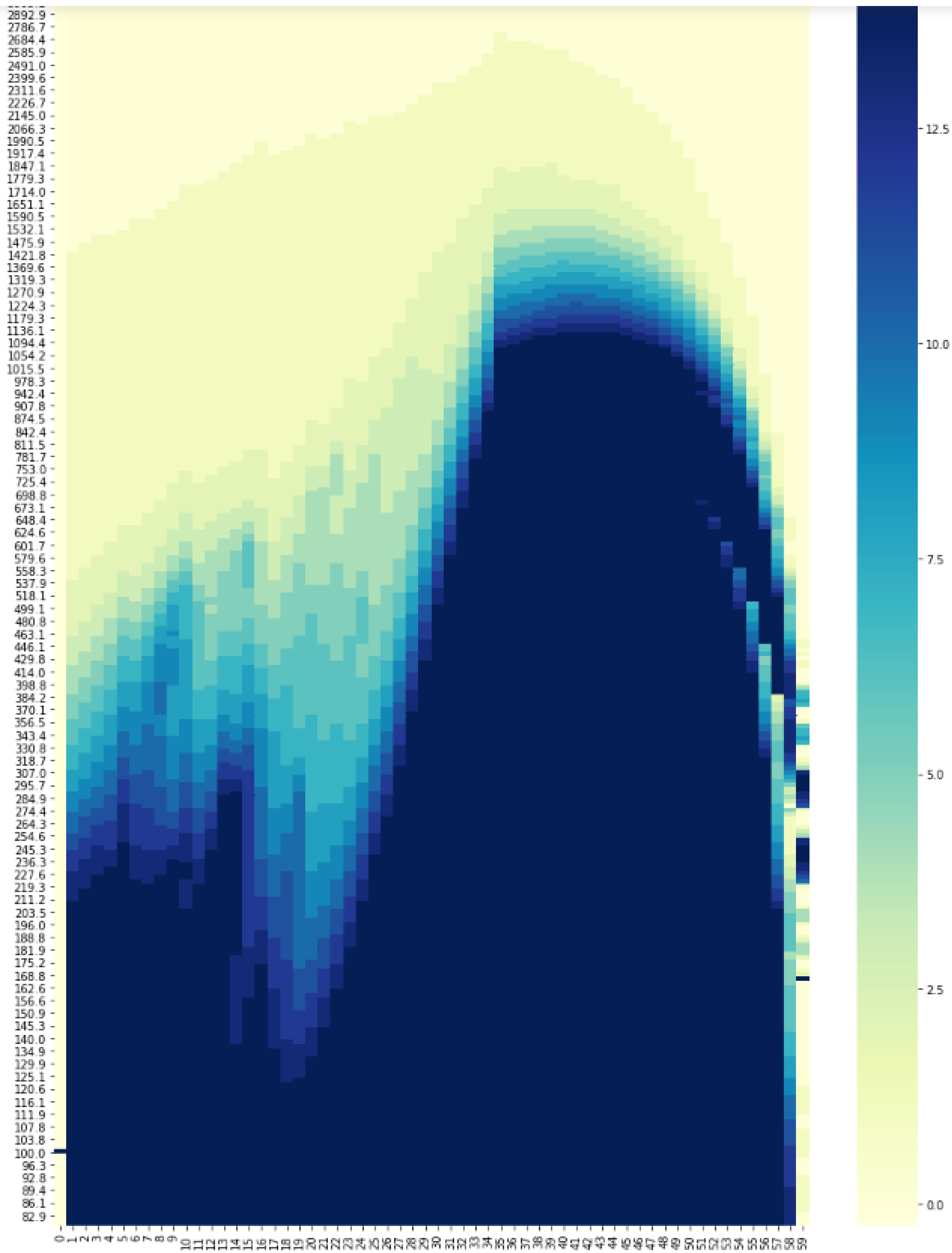


Figure 7: The optimal investment portfolio choices for the couple's goals in Subsection 4.4.

Table 9: Optimal probability ranges for the couple's goals in Subsection 4.4.

Goal	Goal Years (t)	Initial Cost	Inflation Rate	Utility	Range for optimal probability of fulfillment
1a: Mortgage	1–25	\$10	0%	2000	> 0.999
1b: Prop. tax	1–60	\$6	2%	1000	> 0.998
1c: LTC Insur.	15–60	\$7	4%	1000	> 0.98
1d: Med exp.	40–60	\$8	10%	5000	> 0.999
1e: Everyday	1–60	\$60	3%	10,000	> 0.997
1e: (partial)	1–60	\$50	3%	8,000	< 0.003
2a: Orthodon.	6–8	\$3	0%	150	> 0.999
2b: College	13–16	\$40	8%	900	0.76–0.82
2b: (partial)	13–16	\$25	8%	750	0.18–0.24
2c: Cars	5,10,...,55	\$32	0%	300	0.33–0.96
2c: (partial)	5,10,...,55	\$22	0%	200	< 0.005
3a: Remodel	16	\$50		100	0.10
3a: (partial)	16	\$40		70	0
3a: (partial)	16	\$25		50	0.02
3b: Wedding	23	\$70		100	0.21
3b: (partial)	23	\$55		35	0
4a: High Sch.	9–12	\$25	5%	90	0.06–0.11
4b: Trips	10, 20,...,50	\$15	3%	50	0.06-0.82
4c: Philanth.	1–60	\$5	3%	40	0–0.95
4c: (partial)	1–60	\$2.5	3%	20	< 0.04

However, the probability of attaining the lower level goals is far more mixed. For level two, the orthodontia goal is easy to fulfill because it costs so little compared to its utility. The more expensive college goal is met approximately 80% of the time. The less expensive college goal is almost always met otherwise. The car goal is a far more mixed bag: At first, at $t = 5$, the expensive car is purchased only 33% of the time. This increases to 96% by the time the car goal at $t = 55$ is reached. The less expensive car is almost never purchased. Since it is unrealistic to think that the same car can be kept in daily use for 30 years, the couple may decide to rerun the algorithm after possibly increasing the utility for the more expensive car and almost certainly after increasing the utility for the less expensive car. They may also decide to increase the assigned utility more at early years than later years.

The chances of attaining the level three goals are relatively poor. This is because they are expensive and occur relatively early on in the couple's plan. If the couple feels these are too low, they can alter the utilities associated to these goals, following Subsection 4.2.1. In particular, the couple may wish to rethink their attitude about a less fancy wedding for their child, given that the current scenario never selects it, which leaves an 80% chance that their wedding goal will be completely unfulfilled. Should the couple wish instead to increase the chance of attaining all of their goals, they can look to increase their initial contribution, increase their later infusions, or open up their range of potential portfolios to more aggressive positions, as discussed in Subsections 4.2.2–4.2.4.

Next, we look at the level four goals in Table 9. There is only a 6–11% chance of being able to fund private high school. This percentage increases from 6% to 11% over the four years because the expected return of the portfolio is generally higher than the 5% inflation rate for the cost of tuition. Similarly, there is only a 6% chance of attaining the couple's trip when $t = 10$, but this goes up to an 82% chance when $t = 50$, since the investment has had so much time to grow, compared to the 3% inflation rate. We see the same behavior for the philanthropy goal. Initially, the algorithm gives a zero percent chance of giving money to philanthropy. By the time $t = 30$, the chance of fully funding the annual philanthropy goal increases to 71%, and eventually it reaches 95% at $t = 60$. This fits reasonably well with the desires of most people, since they tend to give more to charities as they grow older when they are more able. The partial philanthropy goal is rarely exercised, suggesting the utility assigned to it should be increased.

In contrast, if we applied the standard Monte Carlo approach currently used in industry to this case, it would look to fulfill all of the above goals. Separately applying each of the 15 portfolios available from the bottom panel of Figure 1 using Monte Carlo, we find that the most aggressive, portfolio 15, has the highest probability of fulfilling all goals. However, this optimal probability is only 7%, and, indeed, its probability of even fulfilling the goals through year $t = 16$, when the couple's child leaves college, is only 11%. Using a target date fund allocation in this case leads to even worse results, since it has more weight on the more conservative bond allocation. For the example target date fund weights shown in Table 10,[5] the probability of fulfilling all goals is reduced to 4%, and the probability of fulfilling all goals through year $t = 16$ is reduced to 8%.

Table 10: The target date fund glide path used in Subsection 4.4

Age range	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–95
1. U.S. Stock	63%	63%	60%	55%	50%	45%	38%	29%	21%
2. International Stock	27%	27%	25%	23%	20%	18%	16%	12%	9%
3. U.S. Bond	10%	10%	15%	22%	30%	37%	46%	59%	70%

The issues presented by the Monte Carlo approach are clear. By fulfilling all the couple's early goals, even those that are less important, it hobbles the ability of the portfolio to fulfill the couple's more important later goals. The end result is that the couple, had they used Monte Carlo, would believe their financial situation was rather dire from the probabilities reported by that approach. In contrast, by using our optimal

[5]The three index funds used are (i) Vanguard Total Bond Market II Index Fund Investor Shares (VTBIX), representative of U.S. Fixed Income (Intermediate-Term Bond), (ii) Vanguard Total International Stock Index Fund Investor Shares (VGTSX), representative of Foreign Equity (Large Cap Blend), (iii) Vanguard Total Stock Market Index Fund Investor Shares (VTSMX), representative of U.S. Equity (Large Cap Blend). These three funds have been chosen as representatives of their respective asset categories for illustrative purposes only.

goal prioritization and optimal investment portfolio selection, which is individualized for the couple's needs, we see that the probability of the couple being able to meet their most important goals and also some of their less important goals is actually excellent.

5 Concluding Discussion

There is a great deal of interest in how to best apply a goals-based wealth management approach to investors hoping to attain as many of their goals as possible, weighted by their importance. It is standard in the financial industry to apply Monte Carlo methods to this problem. Monte Carlo methods can quickly determine which static portfolio optimizes the probability that the investor can attain all of their goals. However, in this paper we have shown that dynamic programming methods can go much farther to help investors than Monte Carlo methods, by their nature, are able to. For example, we have shown how to use dynamic programming methods to answer three key questions for investors pursuing multiple goals that Monte Carlo methods cannot answer:

1. When an investor has limited means, they must choose whether or not to fulfill or forgo each of their goals as they progress through time. Given the importance of each of the goals to the investor, we have shown how to optimally determine whether or not to fulfill or forgo each goal.

2. Further, we have shown how to optimally, dynamically determine which investment portfolio the investor should use at any time and any portfolio worth at that time. The term "dynamically" means we optimize with the knowledge that the investor can change their investment portfolio in the future. This is in contrast to the static case that assumes the investor will never change their investment portfolio, no matter how it performs over time.

3. Under the optimized goals taking and investment portfolio selection strategy in the preceding two points, we have shown how to determine the corresponding probabilities for attaining each of the investor's goals. Should the investor feel that these probabilities do not reflect their preferences, they may change the weights of the goals and rerun the algorithm until the generated probabilities match the investor's priorities.

We have also shown how our multiple goals method can easily accommodate concurrent goals (that is, multiple goals that occur at the same time) and partial goals (when the investor is open to options that cost less than the full goal, understanding that these partial goals will make the investor less happy than the full option). On a simple desktop or laptop computer, we are able to compute the optimal solution for a small number of multiple goals within a few seconds. For much more involved multiple goals scenarios, the computation still runs in under 40 seconds.

There are a number of potential additional features that we have not pursued in this paper, which we leave to future work. These include:

- Mandatory goals: These are goals that the investor does not believe to be optional. In some sense, these are like goals with an infinite utility, however, in this paper a goal, no matter how high its utility, will not be fulfilled if there is no money, whereas with mandatory goals, the investor will go bankrupt.

- Unexpected expenses: These are stochastic expenses, such as medical expenses due to an unforeseen accident or a reduction in pay due to an unforeseen job change. When such changes happen, the algorithm in this paper can be revisited, of course, by a re-optimization that includes the unexpected expense, but we can also consider how to optimize in light of these potential future occurrences before they occur.

- Infusions that depend on wealth in addition to time: In this paper, we assumed that the projected infusion amounts, $I(t)$, depended strictly on time. However, we could just as easily consider infusions, $I(W,t)$, that also depend on the portfolio worth. For example, if the investor has the ability to tap another account, they may choose to do so if their portfolio worth decreases sufficiently. That is I becomes larger when W gets small. On the other hand, the investor may wish to associate a reduction in worth, W, with a poor economy and less income, in which case I becomes smaller when W gets small.

- Stochastic expected returns, covariances, and volatility for investments: This encompasses models with stochastic efficient frontiers. For example, the model may be enhanced to have a chance of switching regimes between normal economic epochs and recessions, using a different efficient frontier for each regime. The expansion of the state-space in such a model may add further complementary insights to the ones in this paper.

- Taxes: This paper has assumed after-tax money in all of its calculations. Working within the tax code creates a variety of interesting complexities to our problem.

- Stochastic time of death: We have used a finite investment horizon, $t = T$, in this paper, however, given that this work applies to lifetime financial planning, the value of T becomes stochastic if it is associated to the investor's death.

- Dedicated accounts: If an investor has a 529 account or an HSA account, they can be applied to some goals, but not others. Some investors also wish to dedicate an account to a sole goal, as with mental accounts or with a bucket approach to financial planning.

In summary, we have prescribed a goals-based wealth management plan for multiple goals that may be implemented in a computationally efficient manner over a very long horizon. The goals may occur at different times or concurrently. The goals may be fulfilled partially, if the investor wishes this possibility, or fully. The algorithm maximizes an investor's expected utility, via the optimal exercise of goals and the optimal selection of investment portfolios. These optimal decisions are determined using dynamic programming, which optimally balances the trade-off between cost and utility for each goal in the plan. The algorithm provides the investor with the probability of achieving every goal so that the investor may adjust the plan in an iterative manner. Changes in the probability of achieving the goals may be tracked over time as an additional performance metric, to complement traditional metrics comparing risk-adjusted returns to a benchmark. Thus, this work offers a comprehensive approach to achieving multiple-goals in wealth management, thereby improving on (i) approaches that handle multiple goals in separate mental account portfolios and (ii) approaches that use Monte Carlo simulation.

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