The Leaning Tower of Lire

A Physical Manifestation of the Harmonic Series

Independent Project Report

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Motivation

The primary motivation for this project is to bridge the abstract mathematical concepts discussed in Prof. Amber Habib's **MAT1002: Calculus II** lectures with tangible physical phenomena in classical mechanics.

Calculus II introduces the rigorous study of infinite series, convergence tests, and specifically the counter-intuitive divergence of the harmonic series. While the mathematical proof that $\sum \frac{1}{n} \to \infty$ is standard, it often remains an abstract curiosity without a "real-world" visualization.

This report seeks to demonstrate that the harmonic series is not merely a mathematical construct but an **emergent property** of nature. By analyzing the center-of-mass constraints of a simple stack of wooden planks, we observe that the stability conditions naturally force the geometry of the stack to adopt the harmonic sequence. This project serves as a physical proof of divergence, illustrating how a finite set of forces can sum to create an infinite geometric reach.

Problem Statement

The **Leaning Tower of Lire** (also known as the Book Stacking Problem) poses the following question:

> *Is it possible to stack identical uniform blocks on top of each other, shifted forward, such that the top block is arbitrarily far from the base, without the stack collapsing?*

We aim to investigate:

1. Whether this arrangement can theoretically extend to infinite horizontal distance.
2. What the precise mathematical sequence of the overhang distances d_n is for a system in equilibrium.

Theoretical Basis: The Harmonic Series

The problem is deeply rooted in the properties of the harmonic series.

3.1 Definition

The harmonic series is defined as the sum of reciprocals of the positive integers:

$$S_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{1}$$

The infinite series is given by:

$$S_\infty = \sum_{i=1}^{\infty} \frac{1}{i} \tag{2}$$

3.2 Proof of Divergence

We prove divergence by contradiction, using the comparison test method (Oresme's proof).

Assumption: Assume the series converges to a finite limit S.

Consider the partial sum S_{2n}:

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots + \frac{1}{2n} \tag{3}$$

We group the terms from $n+1$ to $2n$:

$$S_{2n} = S_n + \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right) \tag{4}$$

In this grouping, there are n terms. The smallest term is $\frac{1}{2n}$. Therefore, the sum of this group must be greater than or equal to $n \times \frac{1}{2n}$:

$$\left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right) \geq n \cdot \frac{1}{2n} = \frac{1}{2} \tag{5}$$

Thus, we arrive at the inequality:

$$S_{2n} \geq S_n + \frac{1}{2} \tag{6}$$

If we take the limit as $n \to \infty$, and assuming $S_{2n} \to S$ and $S_n \to S$:

$$S \geq S + \frac{1}{2} \tag{7}$$

This implies $0 \geq \frac{1}{2}$, which is impossible. The initial assumption is flawed; hence, the harmonic series diverges.

The Physical Problem: Inductive Solution

We analyze the statics of the system using rigid body mechanics. We define a system of n planks, numbered from the top down, such that Plank 1 is the topmost plank and Plank n is the bottommost plank.

4.1 Assumptions and Definitions

To simplify the physical model while retaining the core mechanics, we make the following assumptions:

1. **Identical Planks:** All planks have the same mass M and length L.
2. **Uniform Mass Density:** The mass is distributed evenly, meaning the Center of Mass (COM) of any individual plank is located at its geometric center ($L/2$).
3. **Zero Height:** We treat the planks as having negligible thickness. This eliminates the need to calculate vertical torque components or structural shearing, isolating the problem to horizontal equilibrium.

4.2 Stability Criterion

For a stack of k planks to remain stable on top of the $(k+1)$-th plank, the net torque about the pivot point (the edge of the $(k+1)$-th plank) must be zero. This is equivalent to requiring that the Center of Mass (x_{cm}) of the subsystem (planks $1 \dots k$) lies exactly vertically above the edge of the support plank.

4.3 Inductive Derivation

Case $n = 1$: The Topmost Plank

Consider Plank 1 resting on Plank 2. Let the right edge of Plank 2 be at $x = 0$. For maximum overhang d_1, the COM of Plank 1 must align with the edge of Plank 2.

$$x_{cm}^{(1)} = \frac{L}{2} \tag{8}$$

Since the plank is uniform, its COM is at its center. Thus, it can extend half its length past the edge:

$$d_1 = \frac{L}{2} \tag{9}$$

Case $n = 2$: Two Planks on a Third

We now consider the system of Plank 1 and Plank 2 resting on Plank 3. Let the edge of Plank 3 be the origin $x = 0$.

- **Plank 2:** Mass M, overhang d_2. Its COM is at $d_2 - L/2$.

- **Plank 1:** Mass M. From the previous step, we know it sits at the very edge of Plank 2. Therefore, its COM acts at the edge of Plank 2, which is at position d_2.

For equilibrium, the combined Center of Mass must be at the pivot ($x = 0$):

$$X_{cm}^{system} = \frac{M(d_2) + M(d_2 - \frac{L}{2})}{2M} = 0 \tag{10}$$

Solving for d_2:

$$2Md_2 - \frac{ML}{2} = 0 \tag{11}$$

$$2d_2 = \frac{L}{2} \tag{12}$$

$$d_2 = \frac{L}{4} \tag{13}$$

Case $n = k$: General Recursive Step

Assume a stable stack of k planks. We place this stack on the $(k + 1)$-th plank. Let d_k be the overhang of the k-th plank relative to the $(k + 1)$-th plank. We balance moments about the edge of the $(k + 1)$-th plank (origin 0).

1. The k-th plank has mass M and COM at $d_k - L/2$.
2. The stack above (planks $1 \ldots k - 1$) has total mass $(k - 1)M$. Because the stack above is already optimized, its combined weight acts effectively at the edge of the k-th plank, position d_k.

Torque Balance Equation $\sum \tau = 0$:

$$\tau_{net} = [(k - 1)M \cdot g \cdot d_k] + \left[M \cdot g \cdot \left(d_k - \frac{L}{2}\right)\right] = 0 \tag{14}$$

Dividing by Mg and solving for d_k:

$$(k - 1)d_k + d_k - \frac{L}{2} = 0 \tag{15}$$

$$k \cdot d_k = \frac{L}{2} \tag{16}$$

$$d_k = \frac{L}{2k} \tag{17}$$

This proves the general term for the overhang of the n-th plank is $\frac{L}{2n}$.

4.4 Total Overhang Calculation

The total horizontal distance D_N spanned by N planks is:

$$D_N = \sum_{n=1}^{N} d_n = \sum_{n=1}^{N} \frac{L}{2n} = \frac{L}{2} \sum_{n=1}^{N} \frac{1}{n} \tag{18}$$

Since $\sum \frac{1}{n}$ diverges, $\lim_{N\to\infty} D_N = \infty$.

Computational Verification

To verify the analytical derivation, we implement a Python simulation. The simulation does not use the harmonic series formula directly. Instead, it simulates the physics of the stacking process by iteratively calculating the center of mass (COM) of the system and placing the next block to maintain equilibrium.

5.1 Python Simulation Code

```
import numpy as np

M = 1
L = 1

def com_sequence(n):
    xs = np.array([L/2])
    coms = np.array([L/2])
    m = M
    mx = M*xs[0]

    for k in range(2, n+1):
        # Physics Step: Place new block such that its edge supports the previous COM
        new_x = (mx/m) + L/2

        # Update system state
        xs = np.append(xs, new_x)
        mx += M*new_x
        m += M
        coms = np.append(coms, mx/m)

    return coms, xs

n = int(input("Enter number of blocks: "))

coms, xs = com_sequence(n)
print("COMs:\n", coms)
print("Block centers:\n", xs)

# Calculate emergent overhangs
overhangs = np.diff(coms, prepend=L/2)
overhangs[0] = L/2
print("Individual overhangs:\n", overhangs)

# Theoretical comparison
def HarmonicSeries(n):
    return 1/np.arange(1, n+1)

HS = HarmonicSeries(n)
print("Harmonic Series:\n", HS)
WHS = (L/2)*HS
```

```
print("Weighted Harmonic Series:\n", WHS)

# Error Analysis
diff = np.zeros(n)
for i in range(n):
    diff[i] = abs(WHS[i] - overhangs[i])

s = 0
for i in range(n):
    s += abs(diff[i])

print("absolute error:\n", s/n)
```

Listing 1: Computational Simulation of the Harmonic Stack

5.2 Detailed Algorithm Explanation

The significance of the Python script presented in Section 5.1 is that it operates as a "blind" physics engine. The Harmonic Series formula ($1/n$) is **never hard-coded** into the logic. Instead, the sequence emerges naturally from the iterative application of basic torque and center-of-mass laws. The algorithm proceeds in three distinct logical blocks:

Block 1: Initialization and Coordinate Definition

Lines 6–10 set up the physical environment. The system tracks the topmost block (Block 1) first.

- `xs`: This array stores the geometric center of every block in the stack. We define the local coordinate system of the first block such that its left edge is at 0. Consequently, its Center of Mass (COM) is initialized at position $L/2$.
- `mx`: This variable tracks the total moment (mass $\times$ position) of the current stack relative to the origin.

Block 2: The Emergent Physics Loop

Lines 12–14 contain the core recursive logic. The loop simulates adding a new block to the bottom of the existing stack.

- **Calculate System COM:** First, the code calculates the Center of Mass of the *entire existing stack* using the standard formula $x_{com} = \frac{\sum m_i x_i}{\sum m_i}$ (represented by `mx/m`).
- **Enforce Stability:** The critical physical constraint is applied here. For the system to be stable and achieve maximum overhang, the right edge of the new support block must lie exactly underneath the calculated COM of the stack above it.
- **Placement:** The code places the new block such that its center is at `mx/m + L/2`. This step relies purely on the equilibrium condition $\sum \tau = 0$, without any knowledge of the harmonic series.

Block 3: Verification and Error Analysis

Lines 30–54 act as the verification layer.

- The code computes the "emergent overhangs" by taking the difference between the centers of mass of successive layers (`np.diff`).
- It then compares these physically derived values against the theoretical harmonic sequence $L/2n$.
- The absolute error is computed to verify that the deviation is within the limits of floating-point arithmetic.

5.3 Simulation Results (Test Cases)

We validated the simulation against four specific test cases ($N = 1, 20, 100, 1000$) to confirm that the emergent behavior tracks with theoretical predictions across varied scales. The detailed results are discussed below.

Test Case 1: The Base Case ($N = 1$)

- **Scenario:** A single block is placed on a flat surface.
- **Physics:** A uniform block can overhang a support by exactly half its length before its center of mass crosses the pivot point.
- **Result:** The simulation returned an overhang of `0.5000`, which matches the theoretical $L/2$. This confirms the initialization logic is correct.

Test Case 2: Intermediate Recursion ($N = 20$)

- **Scenario:** A stack of 20 blocks.
- **Physics:** This tests the algorithm's ability to handle recursive stability conditions over a moderate number of iterations. For the 20th block, the overhang should be exactly $\frac{L}{2(20)} = \frac{1}{40} = 0.025$.
- **Result:** The simulation returned `0.02500...` with negligible error. This confirms the recursive logic holds over repeated iterations.

Test Case 3: Large Scale Consistency ($N = 100$)

- **Scenario:** A stack of 100 blocks.
- **Physics:** At this scale, the stack relies on the precise summation of moments from 99 previous blocks. The expected overhang is $\frac{L}{200} = 0.005$.
- **Result:** The simulation returned `0.00500...` with an error of $\approx 10^{-18}$. This confirms the stability of the system calculation.

Test Case 4: Extreme Stress Test ($N = 1000$)

- **Scenario:** A stack of 1000 blocks.
- **Physics:** This serves as a stress test for numerical drift and algorithmic precision. With 1000 terms, small floating-point errors could accumulate. The expected overhang is $\frac{L}{2000} = 0.0005$.
- **Result:** The simulation returned `0.000500...` with negligible error. This confirms that the harmonic series is indeed the robust mathematical solution to the physical problem, regardless of scale.

Table 1: Emergent Overhangs vs Theoretical Predictions

Test Case (n)	**Simulated Overhang**	**Theory ($\frac{L}{2n}$)**	**Absolute Error**
1	0.5000000000	0.5000000000	0.00
20	0.0250000000	0.0250000000	0.00
100	0.0050000000	0.0050000000	8.67×10^{-19}
1000	0.0005000000	0.0005000000	2.23×10^{-18}

Conclusion

In this project, we have investigated the mechanical properties of the Leaning Tower of Lire and established a rigorous connection between physical stability and mathematical series.

Our derivation demonstrated that the condition for rotational equilibrium naturally gives rise to the sequence of overhangs $d_n = \frac{L}{2n}$. This result is profoundly non-intuitive; common sense suggests that a structure extending infinitely far beyond its base must inevitably collapse due to an unbounded moment arm. However, the rapidly decreasing mass of the extended portion (relative to the counterbalance of the stack) allows the Center of Mass to remain perpetually supported.

Critically, the computational verification served to establish the harmonic series as an **emergent property**. The simulation engine contained no knowledge of the harmonic sequence; it was governed solely by the fundamental laws of torque and static equilibrium. Yet, when tasked with maximizing overhang, the system converged exactly to the harmonic distribution.

The stability of the solution is absolute. For any finite N, the stack is not merely a theoretical limit but a physically realizable static system. The total overhang, $D_{total} = \frac{L}{2} \sum \frac{1}{n}$, is unbounded because the harmonic series diverges. Thus, the Leaning Tower of Lire serves as a powerful physical manifestation of divergence, transforming an abstract mathematical concept into a tangible, if paradoxical, reality.

References

[1] Habib, A. (2025). *MAT1002: Calculus II - Lecture Notes*. Shiv Nadar Instiute of Eminence.

[2] Jerison, D. (2007). *Lec 38 — MIT 18.01 Single Variable Calculus, Fall 2007*. [Video]. YouTube. `https://www.youtube.com/watch?v=wOHrNt9ScYs`