Integrability in eccentric, spinning black hole binaries at ST: 1.5 and 2PN

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TODO

ST: Leo asked me to implement the concise 'coefficient notation [...](Series)' explained here: https://mathworld.wolfram.com/CoefficientNotation.html. But there are few places where it can be implemented.

ST: cite Hartl-Buonanno

ST: cite: https://link.springer.com/content/pdf/10.1007/s10714-011-1171-0.pdf

ST: Future work (write in paper): finding AAVs at 1.5PN will help find AAVs at 2PN via perturbation theory

ST: I don't know how to do away with the concept of Hamiltonian vector field (H-vector field) of a function and the associated flow for a couple of reasons:

1. Angle variables are parameters of a flow down the H-vector field around a loop. Hence, we can't define the angle variables without discussing H-vector fields.

2. The change of a quantity under the flow of certain other quantity (given via Eqs. (36), (38), and (40)) relies on Eqs. (8) and (9). How then do I do away with the concept of H-vector fields?

3. How to define Poisson bracket in terms of symplectic forms without using H-vector fields?

4. We do use the concept of H-vector fields in Appendix B.

5. Also, I could prove in a geometrical way (without using coordinates) that $\omega(X_f, X_g) = \omega^{-1} df dg$. I could not find any text which talks in terms of ω^{-1}.

So, please rewrite/remove the parts which talk about (or make use of) H-vector fields if you think it ought to be removed. I don't feel confident enough to write about it.

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I. INTRODUCTION

II. FORMALISM

For our system of BBHs, we first introduce a couple of definitions which read

$$\begin{aligned}
V &= \sqrt{V_i V^i} = |\vec{V}| \qquad \text{for any 3 vector } V^i,\\
m_1, m_2 &= \text{masses of the two black holes}\\
m &= m_1 + m_2,\\
\mu &= m_1 m_2 / m,\\
\nu &= \mu / m,\\
\vec{R}_A &= \text{position vector of the black hole A}\\
\vec{P}_A &= \text{momentum of the black hole A}\\
\vec{R} &= \vec{R}_1 - \vec{R}_2,\\
\vec{n} &= \vec{R}/R,\\
\vec{P} &= \vec{P}_1 = -\vec{P}_2,\\
\vec{r} &= \vec{R}/(Gm),\\
\vec{p} &= \vec{P}/\mu,\\
\vec{L} &= \vec{R} \times \vec{P},\\
\vec{J} &= \vec{L} + \vec{S}_1 + \vec{S}_2,\\
\sigma_1 &= \left(2 + \frac{3m_2}{2m_1}\right),\\
\sigma_2 &= \left(2 + \frac{3m_1}{2m_2}\right),\\
S^i_{\text{eff}} &= \sigma_1 S^i_1 + \sigma_2 S^i_2,\\
G &= \text{gravitational constant},\\
c &= \text{speed of light},\\
\epsilon &= 1/c^2,
\end{aligned} \tag{1}$$

where we have assumed Einstein's summation convention from now on unless stated otherwise.

A. Poisson brackets between the phase-space variables

The only non-vanishing Poisson brackets (PBs) between the phase-space variables R^i, P^i, S^i_1 and S^i_2 (with the indices i running over x, y, z) are

$$\left\{R^i, P_j\right\} = \delta^i_j \quad \text{and} \quad \left\{S^i_A, S^j_A\right\} = \epsilon^{ij}{}_k S^k_A. \tag{2}$$

With these canonical Poisson brackets and the following rules (sum, product, chain and anti-commutativity)

$$\{f, g+h\} = \{f, g\} + \{f, h\} \tag{3}$$
$$\{f, gh\} = \{f, g\}\, h + \{f, h\}\, g \tag{4}$$
$$\{f, g(U,V)\} = \{f, U\} \frac{\partial g}{\partial U} + \{f, V\} \frac{\partial g}{\partial V} \tag{5}$$
$$\{f, g\} = -\{g, f\}\,, \tag{6}$$

(f, g are functions of the phase-space variables and U, V denote any two phase-space variables), one can evaluate the PB between any two arbitrary functions of the phase-space variables. Finally, the evolution of any function f of the phase-space variables is dictated by the standard result

$$\dot{f} = \{f, H\}\,. \tag{7}$$

B. The symplectic manifold for the BBH

Here we try to cast our dynamical systems problem in the language of differential geometry. The reader is referred to Refs. [1, 2] for a more thorough discussion of the differential geometric viewpoint of dynamical systems which we employ below. From a mathematical point of view, Hamiltonian dynamics takes place on an (even-dimensional) symplectic manifold. A smooth manifold equipped with a closed non-degenerate differential 2-form ω (the symplectic form) is called a symplectic manifold. Some properties of ω are worth noting. Given a (scalar) function f on the manifold, its "H-vector field" X_f is defined so as to satisfy

$$\omega(\quad, X_f) = df, \tag{8}$$

and the the action of the symplectic form on H-vector fields X_f, X_g of two scalar functions f, g is

$$\omega(X_f, X_g) = \{g, f\}\,. \tag{9}$$

In literature, H-vector fields are commonly called "Hamiltonian vector fields". But we invent the terminology "H-vector fields" so as to avoid any confusion with the Hamiltonian function (to be introduced below), which determines the time-evolution of the system. As per these definitions, the Hamiltonian function also possesses its own H-vector field for it is a scalar function on the symplectic manifold. Finally we mention that we will sometimes refer to integral curves of the H-vector field of a certain quantity (a function of phase-space variables) as the "flow" under that quantity.

Apart from the orbital manifold (with the coordinates R^i, P^i), we have two more manifolds on which dynamics takes place: $\vec{S_1}$ and $\vec{S_2}$ manifold. Each of $\vec{S_1}$ and $\vec{S_2}$ manifolds are three-dimensional which may give an impression that defining a useful symplectic manifold would be a ST: hopeless idea since symplectic manifolds are even-dimensional. But since the magnitudes of the two spin vectors stay constant under both the 1.5PN and 2PN accurate Hamiltonians (to be presented in Eqs. (25)-(33)), we can forgo the two three-dimensional manifolds $\vec{S_1}$ and $\vec{S_2}$ in favor of two new symplectic manifolds with S_1^z, ϕ_1 and S_2^z, ϕ_2 as the coordinates on these manifolds, where ϕ_1 and ϕ_2 represent the azimuthal angles of the two spin vectors.

Eq. (9), helps us see that a certain Poisson-bracket structure imposes constraints on the form on the symplectic manifold. We verify in Appendix B that the symplectic manifold compatible with the PB relations of Eqs. (2) is given by

$$\omega = dP_i \wedge dR^i + dS_{1z} \wedge d\phi_1 + dS_{2z} \wedge d\phi_2. \tag{10}$$

With the construction of the symplectic form for our problem we can try to enumerate the number of coordinates of our symplectic manifold: $R^x, R^y, R^z, P_x, P_y, P_z, S_{1z}, \phi_1, S_{2z}, \phi_2$, which are ten in number.

C. A primer on post-Newtonian power counting

We devote this subsection to give the reader some general ideas regarding how to perform PN order counting. PN counting is a relative concept and hence PN order of an isolated term carries little to no relevance. The concept of PN counting is normally applicable to a sum of various terms where we want to determine the PN order of one term of the series wrt. ST: What do I replace 'wrt.' with? some other term.

1. *The variables involved in PN power counting*

Here we group some relevant variables for our BBH system are which will help us determine relative PN orders of terms of a PN series. The magnitudes and the PN order of all the variables belonging to a certain class are comparable.

1. Radial separation vector between the black-holes and derived quantities: $\vec{R}, R := \sqrt{R^i R_i}$

2. Momentum vector of the black-holes in the center-of-mass frame and derived quantities: $\vec{P} := \vec{P}_1 = -\vec{P}_2, P := \sqrt{P^i P_i}$.

3. Velocity vector associated with the above momentum vector and derived quantities $\vec{v} := \vec{P}/m, v := \sqrt{v^i v_i}$

4. A radial vector (of the order of the dimension of the black-holes) and related quantities: $\vec{R}_s, R_s := \sqrt{R_s^i R_{si}}$.

5. A momentum vector and derived quantities $\vec{P}_s, P_s := \sqrt{P_s^i P_{si}}$ such that the magnitude of this momentum vector is related to the magnitude of the above radial vector $\vec{R}_s$ via virial theorem

$$\frac{P_s^2}{m} \sim \frac{Gm^2}{R_s}. \tag{11}$$

This relation between R_s and P_s established via virial theorem is valid only for maximally spinning black-holes [3], but nevertheless we shall continue to assume so in this paper. ST: This is a fictitious momentum variable invented so that spin S is of the same order as $R_s P_s$

6. Spin angular momentum vectors of the black-holes and derived quantities: $\vec{S}_A, S_A = \sqrt{S_A^i S_{Ai}}$. Hence, S_A scales as

$$S_A \sim R_s P_s. \tag{12}$$

We will assume that the variables R_s, P_s and S_A for one of the BHs are comparable to those for the other BH. Throughout, we will also assume that m_1, m_2 and m are comparable: $m_1 \sim m_2 \sim m$.

Now R_s scales as

$$R_s \sim R\frac{v^2}{c^2}, \tag{13}$$

Relation (13) is easy to arrive at once we take note of the following two more order-of-magnitude relations

$$R_s \sim \frac{Gm}{c^2} \tag{14}$$

(relation for Schwarzschild radius)

$$R \sim \frac{Gm}{v^2}. \tag{15}$$

(relation between radius and speed for Newtonian orbits),

Also, S_A scales as

$$S_A \sim R_s P_s, \tag{16}$$

Finally, relation (16) with the aid of relations (11) and (14) implies that S_A scales as

$$S_A \sim \frac{Gm^2}{c}, \tag{17}$$

ST: Try to get this result from the standard result $a < M$ for Kerr metric.

2. *Order-of-magnitude comparison between the various variables*

As far as determining the PN order of a certain term is concerned, one can always replace a variable of a certain class (as defined in II C 1) with another variable of the same class since their PN orders are the same. One is also free to replace a variable with another belonging to a different class, provided they make use of the following order-of-magnitude relations among the variables which are given as follows

$$P \sim mv \tag{18}$$

(definition of momentum)

$$\frac{P^2}{m} \sim \frac{Gm^2}{R} \tag{19}$$

(virial theorem)

$$R \sim \left(\frac{Gm}{\omega^2}\right)^{1/2} \tag{20}$$

($\omega =$ ang. frequency of a Newtonian circular orbit)

$$S_A \sim \frac{Gm^2}{c}, \tag{21}$$

(same as Eq. (17))

ST: Only Eq. (21) is a repetition (of Eq. (17)). Eqs. (11) and (19) are different equations.

Now assume that we are given a series

$$y = y_1 + y_2 + y_3 + \ldots, \tag{22}$$

the relative PN order of whose terms is to be determined. We can choose any one of the three variables v, ω and R as our 'favorite variable' and rewrite the terms in the above series in terms of our chosen favorite variable by eliminating other variables in favor the favorite variable via the scaling relations (18-21). Note that substitutions done with the help of relations (18-21) are valid only as long as order-of-magnitude estimates are concerned, for the relations (18-21) are themselves order-of-magnitude estimates. The end result is that the series in Eq. (22) is recast into a power series in the dimensionless post-Newtonian parameter x

$$y = \mathrm{CF}(Y_1 x^0 + Y_2 x^k + Y_3 x^n + \ldots), \tag{23}$$

where CF ST: what's wrong with using CF? stands for a common factor that has been pulled out. Depending on whether our 'favorite variable' is v or ω or R, the PN parameter x will be

$$\frac{v^2}{c^2}, \left(\frac{Gm\omega}{c^3}\right)^{2/3} \text{ or } \frac{Gm}{c^2 R}, \tag{24}$$

respectively. With all this set up, we can finally state the following important punchlines as far as PN order counting is concerned.

- The PN order of a certain term say $(\mathrm{CF})Y_2 x^k$, in Eq. (23) wrt. the leading term $(\mathrm{CF})Y_1 x^0$ is k.

- There is however another way to do the PN order counting which relies on directly looking at the powers of $1/c$ alone. An extra power of $1/c^{2q}$ in a particular term relative to the leading order term earns a PN order of q wrt. the leading term. This is tied to the fact that all three PN parameters via expressions (24) contain the same power of c; i.e. -2. But one must be aware that via Eq. (17), the spin variables implicitly include a factor of $1/c$.

If the above ideas regarding performing PN order counting is new to the reader, then is strongly recommended to read two examples in Appendix C on how to determine PN orders of various terms in a given series.

D. Second post-Newtonian Hamiltonian with spins included

We now introduce the 2PN Hamiltonian we will be working with. To write the PN Hamiltonian we adopt the convention that $H_{n\mathrm{PN}}$ stands for the part of Hamiltonian which is of nPN order relative to the leading PN order term, also known as the Newtonian order. On the other hand $H^{n\mathrm{PN}}$ stands for the nPN accurate Hamiltonian which is composed of all the terms of the Hamiltonian starting from the Newtonian order up to the nPN order term. For example, the 2PN Hamiltonian of the BBH system in the center of mass frame is

$$H^{2\mathrm{PN}} = H_{\mathrm{N}} + H_{1\mathrm{PN}} + H_{1.5\mathrm{PN}} + H_{2\mathrm{PN}}, \tag{25}$$

where the various components read [3–5]

$$H_{\mathrm{N}} = \mu\left(\frac{p^2}{2} - \frac{1}{r}\right), \tag{26}$$

$$H_{1\mathrm{PN}} = -\frac{\epsilon}{8}\left(1-3\nu\right)p^4 - \frac{\epsilon}{2r}\left((3+\nu)p^2 + \nu(n^i p_i)^2\right), \tag{27}$$

$$H_{1.5\mathrm{PN}} = G\epsilon\frac{L_i S^i_{\mathrm{eff}}}{R^3}, \tag{28}$$

$$\begin{aligned} H_{2\mathrm{PN}} = {} & \frac{\epsilon^2\left(3\nu(n^i p_i)^2 + (8\nu+5)p^2\right)}{2r^2} + \frac{\epsilon^2\left(-3\nu^2(n_i p^i)^4 - 2\nu^2 p^2 (n_i p^i)^2 + \left(-3\nu^2 - 20\nu + 5\right)p^4\right)}{8r} \\ & + \frac{1}{16}\left(5\nu^2 - 5\nu + 1\right)p^6\epsilon^2 - \frac{(3\nu+1)\epsilon^2}{4r^3} + H_{\mathrm{SS}}, \end{aligned} \tag{29}$$

where H_{SS} comprises the spin-spin interaction term and is further given via

$$H_{\mathrm{SS}} = H_{\mathrm{S}11} + H_{\mathrm{S}22} + H_{\mathrm{S}12}, \tag{30}$$

$$H_{\mathrm{S}11} = \frac{Gm_2\epsilon\left(3({S_1}^i n_i)^2 - {S_1}^2\right)}{2m_1 R^3}, \tag{31}$$

$$H_{\mathrm{S}22} = \frac{Gm_1\epsilon\left(3({S_2}^i n_i)^2 - S_2^2\right)}{2m_2 R^3}, \tag{32}$$

$$H_{\mathrm{S}12} = \frac{G\epsilon(3\left(S_1^k n_k\right)\left(S_2^i n_i\right) - S_1^j S_{2j})}{R^3}, \tag{33}$$

The various terms in the above expressions have already been defined in Eqs. (1). Dropping $H_{2\mathrm{PN}}$ from $H^{2\mathrm{PN}}$ of Eq. (25), we are left with the 1.5PN Hamiltonian $H^{1.5\mathrm{PN}}$.

III. ACTION INTEGRALS AT 1.5PN

A. Completely integrable systems (Liouville sense)

ST: Please define integrability in terms of the existence of independent commuting constants. I don't want to write about something which I have not read first-hand in a textbook. A system is said to be completely integrable in the Liouville sense if there exists a canonical transformation from a given set of phase-space coordinates to action-angle coordinates [6]. For brevity, we will call such systems simply 'integrable'. Integrable systems are nice in that since they admit action-angle variables, perturbations to such systems can be handled via the classical perturbation theory. Also, integrable systems are not chaotic.

Liouville-Arnold theorem helps one determine whether a system is integrable or not [1, 2, 6]: On a symplectic manifold of dimension $2n$ (i.e. with n phase-space variables), if $\partial H/\partial t = 0$ and it admits n constants of motion which are mutually in involution such that the level manifold of these constants of motion is compact and connected, then the system is integrable. Two functions of phase-space variables are said to be in involution if their Poisson bracket vanishes. Level manifold of a set of quantities is the submanifold of the symplectic manifold where these quantities attain certain constant values. In light of this theorem and, we immediately see that the BBH system with the 1.5PN order Hamiltonian is integrable since there are 10 phase-space variables and 5 constants of motion: $H, L^2, J^2, J_z, S^i_{\text{eff}} \cdot L^i$, as has been long known [7]. In this paper, we will say that two quantities commute with each other if they are in involution.

B. Definition of action-angle variables

ST: Please define the action variables your way (without using H-vector fields). This is the definition I read in the books. Suppose we have a Hamiltonian system with n positions and n conjugate momenta. One is first required to form n independent linear combinations of the n independent constants of motion which are in mutual involution with each other (provided such constants they exist) such that the integral curves of the H-vector fields of these linear combinations close on to themselves thus forming a loop. Such n loops (denoted by C_i) will be non-homotopic (will have different topologies). With this setup, the n action variables are defined as [1, 2, 6],

$$J_k = \frac{1}{2\pi} \oint_{C_k} \sum_i P_i dQ^i, \tag{34}$$

where P_i and Q^i are canonical position and momentum variables which make up the symplectic form

$$\omega = \sum_i dP_i \wedge dQ^i, \tag{35}$$

where the index k in Eq. (34) refers to the kth loop. The corresponding n angle variables are the flow parameters of the n closed integral curves (or the loops) such that they run from 0 to 2π as the loop is traversed once. We also mention that the result of the above line integrals defining the action variables is invariant wrt. homotopic loops (see chapter 11 of Ref. [6]), thereby enabling one to perform the above line integral on a loop which is different but is homotopic to the originally chosen loop. We will make use of this fact later. ST: Leo will edit to comment on not having Liouville (canonical) form

C. Evaluation of the action-angle variables

In this subsection, $d/d\lambda$ will stand for the derivative wrt. the parameter of the flow along the H-vector field of a given function. Also, to evaluate the next three action integrals, we will employ results derived in Appendix D. This is so because under the flow of J_z, L^2, J^2 (corresponding to the next three action integrals), certain phase-space vectors rotate around some other fixed vector. It is situations of these kinds which Appendix D deals with and the results derived therein are applicable to the evaluation of the next three action integrals.

1. *Action variable under the flow under J_z*

With $\vec{V}$ representing any of the vectors $\vec{R}, \vec{P}, \vec{S_1}$ and $\vec{S_2}$, we have under the flow of J^z, that the rate of change of $\vec{V}$ wrt. the flow parameter λ is given by the Poisson bracket between $\vec{V}$ and J_z (from Eqs. (8) and (9)) ST: What's wrong with referring to these two equations

$$d\vec{V}/d\lambda = \left\{\vec{V}, J_z\right\} = \hat{z} \times \vec{V}, \tag{36}$$

thereby establishing that all these four vectors rotate around the z-axis at the same rate. This also shows that the integral curve under the flow of J_z closes onto itself thereby forming a loop. The associated action variable with the help of Eqs. (D7) and (D9) becomes

$$\mathcal{J}_1 = 2\pi \left(\vec{L} \cdot \hat{z} + \vec{S_1} \cdot \hat{z} + \vec{S_2} \cdot \hat{z}\right)/(2\pi) = \vec{J} \cdot \hat{z} = J_z. \tag{37}$$

2. *Action variable under the flow of L^2*

With $\vec{V}$ representing any of the two vectors $\vec{R}, \vec{P}$, we have under the flow of L^2,

$$d\vec{V}/d\lambda = \left\{\vec{V}, L^2\right\} = 2\vec{L} \times \vec{V}, \tag{38}$$

whereas the spin vectors $\vec{S_1}$ and $\vec{S_2}$ do not change thereby showing that the integral curve under the flow of L^2 also closes onto itself forming a loop. The action variable associated with the flow under L^2 with the help of Eq. (D7) becomes (with the replacement $\hat{n} \to \hat{L}$)

$$\mathcal{J}_2 = 2\pi \vec{L} \cdot \hat{L}/(2\pi) = \vec{L} \cdot \hat{L} = L. \tag{39}$$

3. *Action variable under the flow of J^2*

With $\vec{V}$ representing any of the vectors $\vec{R}, \vec{P}, \vec{S_1}$ and $\vec{S_2}$, we have under the flow of J^2,

$$d\vec{V}/d\lambda = \left\{\vec{V}, J^2\right\} = 2\vec{J} \times \vec{V}, \tag{40}$$

thereby establishing that all these four vectors rotate around $\vec{J}$ at the same rate. This also shows that the integral curve under the flow of J^2 closes onto itself forming yet another loop. The associated action variable with the help of Eqs. (D7) and (D9) becomes

$$\mathcal{J}_3 = (2\pi)\left(\vec{L}\cdot\hat{J} + \vec{S}_1\cdot\hat{J} + \vec{S}_2\cdot\hat{J}\right)/(2\pi) = \vec{J}\cdot\hat{J} = J. \tag{41}$$

4. The fourth action variable

To compute the fourth action integral, envisage a loop in the submanifold defined by constant values of $H, L^2, J^2, J_z, S^i_{\text{eff}}\cdot L^i$. The loop is such that on it, of all the phase-space variables, only R and P_r change and they are constrained by the constancy of the Hamiltonian $H^{1.5\text{PN}}$ which is displayed here in a slightly different form than the one in Eqs. (25)-(28) ST: It is different because I am trying to show the radial part of the momentum explicitly; notice the appearance of P_r. Why do I do that? Because we are trying to compute radial action integral

$$H^{1.5\text{PN}} = \frac{-2G\mu^2 mR + L^2 + P_r^2R^2}{2\mu R^2} + \mu\epsilon\left(\frac{G^2m^2}{2R^2} - \frac{Gm\left((\nu+3)\left(\frac{L^2}{\mu^2R^2}+\frac{P_r^2}{\mu^2}\right)+\frac{\nu P_r^2}{\mu^2}\right)}{2R} + \frac{(3\nu-1)\left(L^2+P_r^2R^2\right)^2}{8\mu^4R^4}\right) + \epsilon\frac{G\vec{S}_{\text{eff}}\cdot\vec{L}}{R^3}, \tag{42}$$

where P_r is defined via $P_r := \vec{n}\cdot\vec{P}$ so that $P^2 = P_r^2 + L^2/R^2$. Note that $H_{2\text{PN}}$ does not contribute to $H^{1.5\text{PN}}$. $H^{1.5\text{PN}}$ in the above equation is a sum of H_{N}, $H_{1\text{PN}}$ and $H_{1.5\text{PN}}$ of Eqs. (26)-(28). On this loop, $H^{1.5\text{PN}}$, L^2 and $\vec{S}_{\text{eff}}\cdot\vec{L}$ stay constant. Eq. (42) then lets us express P_r^2 in terms of R as a perturbative series ST: What's your objection here?

$$P_r^2 = 2E\mu + \epsilon E^2(1-3\nu) + \frac{\epsilon EG\mu(8-2\nu)m + 2G\mu^2 m}{R} + \frac{\epsilon G^2\mu^2(\nu+6)m^2 - L^2}{R^2} - \frac{\epsilon\left(2G\mu\vec{S}_{\text{eff}}\cdot\vec{L} + GL^2\nu m\right)}{R^3}, \tag{43}$$

where we have introduced $E := H^{1.5\text{PN}}$ to avoid writing superscript on the top of a superscript in the above equation. Hence, computing the associated action integral via the definition (34) amounts to computing an integral of the sort

$$\frac{2}{2\pi}\int_{R_{\min}}^{R_{\max}}\left(A + \frac{2B}{R} + \frac{C}{R^2} + \frac{D}{R^3}\right), \tag{44}$$

(with A, B, C, D being constants) whose evaluation methods have already been discovered by Damour and Schafer (see Eqs. (3.8) and (3.9) of Ref. [8]). There is a factor of 2 in the above equation relative to definition (34) because the limits of the integral are from one turning point to the other ($R_{\min}$ to $R_{\max}$) rather than being over the entire loop. Using the result of Eq. (3.8) of Ref. [8], one finds the radial action integral to be

$$\mathcal{J}_4 = -L + \frac{Gm\mu^{3/2}}{\sqrt{-2E}} + \epsilon\left(\frac{3G^2m^2\mu^2}{L} + \frac{\sqrt{-E}\;Gm\mu^{1/2}(-15+\nu)}{4\sqrt{2}}\right) - \frac{G^2m\vec{S}_{\text{eff}}\cdot\vec{L}\mu^3\epsilon}{L^3}. \tag{45}$$

It can be easily shown that the flow along an action makes a loop. How to know whether the loop lies on the torus or not? Either go along the flow of the vector field of one of the commuting constants or use the constancy of one of those constants to perform the line integration for the action. How to know if the loop is in a different homology? The action should be different (or maybe independent).

Note that we have not been able to give the quantity under the flow of whose H-vector field such a loop is got. ST: You think it does not matter. I thought it matters because only when I can have loops as flows under a certain quantity, I can define angle variables, just like in the first three cases above But nevertheless, we still can compute the associated action integral. The four loops considered above to compute the action integrals are necessarily non-homotopic for they give different values of action integrals. As already stated in subsection III B, if the loops were homotopic, then the action integral computed on them would have come out to be the same.

ST: Do our point of view regarding the evaluation of the above three action integrals change in light of our discussion with Samuel Lisi? We are yet to write something about the angle variables.

5. *Integral curves under the flow under $\vec{S}_{\text{eff}} \cdot \vec{L}$*

As for the flow under $S^i_{\text{eff}} L_i$, we managed to find the integral curves only partially and the evaluation of the action-angle variables so far remains out of reach. Despite this limitation, this calculation is important in its own right because it facilitates the analytical computation of $\vec{S_1}, \vec{S_2}$ and $\vec{L}$ of a circular binary in terms of elliptic functions. Due to the lengthy nature of these computations, we have devoted Sec. IV to this. ST: This section is totally empty right now .

IV. INTEGRAL CURVES FOR $\vec{S}_{\text{eff}} \cdot \vec{L}$

V. INTEGRABILITY AT 2PN

For this section we will work with the full 2PN Hamiltonian of Eq. (25) which are further defined via Eqs. (26-33).

A. Effect of Poisson brackets on the post-Newtonian order: the PBPN rule

ST: I have made this section much more succinct

We will devote this subsection to introduce an important general result which will have a crucial bearing on our results regarding discovery of mutually commuting constants of motion at 2PN order. Imagine we have a post-Newtonian series

$$y = y_1 + y_2 + y_3 \dots . \tag{46}$$

We define an operator [,] between two terms of the PN series to stand for their relative PN order

$$[y_i, y_j] = \text{PN order of } y_i - \text{ PN order of } y_j. \tag{47}$$

When we take the PB $\{y, z\}$ of the PN series y and another quantity z, we get yet another PN series the relative PN orders of whose terms we wish to determine. Consider the PB $\{y_i, z\}$ and $\{y_j, z\}$ each of which can be broken down the into a sum of terms using Eqs. (3-6) such that each term contains a Poisson bracket of the form given on the LHS of Eqs. (2) and hence can be readily evaluated using Eqs. (2). We also define $\overline{\{y_i, z\}}$ to stand for any one of the terms got by breaking down $\{y_i, z\}$ in the above mentioned way. As an example, $\overline{\{r^i p_j, r^k\}}$ could represent either $\{r^i, r^k\}\, p_j$ or $\{r^i, p_j\}\, r^k$

Now, $\{y_i, z\}$ and $\{y_j, z\}$ are terms of the PN series $\{y, z\}$. Breaking down $\{y_i, z\}$ and $\{y_j, z\}$ using Eqs. (3-6) yields $\overline{\{y_i, z\}}$ and $\overline{\{y_j, z\}}$ whose relative PN order is given by the following rules

$$[\overline{\{y_i, z\}}, \overline{\{y_j, z\}}] = [y_i, y_j] + N, \tag{48}$$

where

- $N = 0$ if both $\overline{\{y_i, z\}}$ and $\overline{\{y_j, z\}}$ are orbital or spin Poisson brackets
- $N = 1/2$ if $\overline{\{y_i, z\}}$ is an orbital Poisson bracket and $\overline{\{y_j, z\}}$ is a spin Poisson bracket.
- $N = -1/2$ if $\overline{\{y_i, z\}}$ is a spin Poisson bracket and $\overline{\{y_i, z\}}$ is an orbital Poisson bracket.

We will refer to this rule (encapsulated in the three bullets above) as the **PBPN** rule (standing for **P**oisson-**B**racket-**P**ost-**N**ewtonian rule). Of the many ways to see how does the PBPN rule come about, we present one. As mentioned in a bulleted point towards the end of subsection II C, the relative PN orders of terms in a PN series can also be determined solely by looking at the powers of $1/c^2$ and by also being cognizant of the fact that spin variables include a factor of $1/c$ implicitly (when compared against $L \sim RP$; c.f. Eq. (17)). As we can see clearly via Eqs. (2), wrt. the product of arguments of a spin PB (i.e. $S^i_A \times S^j_A$), there is one less factor of $1/c$ in the evaluated result of that PB (i.e. $\epsilon^{ij}{}_k S^k_A$). There is no such change in the number of factors of $1/c$ when an orbital PB is evaluated (wrt. the product of the arguments of an orbital PB) because neither the LHS, nor the RHS of the first of Eqs. (2) contain any $1/c$. And since each factor of $1/c$ accounts for a change in the PN power by $1/2$, we can now see how does the the PBPN rule come about.

If we are to speak loosely, we can restate this PBPN rule very succinctly by saying that *if for the effect of PBs on PN orders, we are to set as a benchmark the effect of orbital PBs on PN orders, then a spin PBs produces an output with a PN order lower by* $1/2$ *when compared against this benchmark.* Although succinct, the above statement of the PBPN rule can be misleading. That's why a real understanding of the PBPN rule merits a detailed bulleted statement as already made above.

B. Defining constancy, involution and integrability in the post-Newtonian scheme

From the Liouville-Arnold theorem stated earlier, the concept of integrability relies on the ideas of constancy (in time) and involution. Constancy of a quantity implies that the quantity gives a vanishing PB with the Hamiltonian and two quantities are said to be in involution if the their PB vanishes. Since we are working in the post-Newtonian scheme, we should introduce the post-Newtonian versions of the definitions of constancy and involution, which as we will see requires some care.

1. Definition of post-Newtonian constancy

Defintion 1: A simple definition that many articles in the literature follow is that a quantity f is constant up to nPN order if

$$\left\{f, H^{n\mathrm{PN}}\right\} = 0, \tag{49}$$

where $H^{n\mathrm{PN}}$ is the nPN Hamiltonian. The reader may think that the time-scale of variation of a quantity which is a constant of motion up to nPN order as per Definition 1 is at least mPN order larger than the Newtonian-Keplerian time-scale where $m > n$. By Newtonian-Keplerian time-scale, we mean the time-scale of variation of R^i or P^i vector under the Newtonian Hamiltonian which can be quantified as

$$T_{\mathrm{N}} \sim \frac{R^i}{\dot{R}^i} \sim \frac{R^i}{\{R^i, H^{\mathrm{N}}\}} \sim \frac{P^i}{\dot{P}^i} \sim \frac{P^i}{\{P^i, H^{\mathrm{N}}\}}. \tag{50}$$

When investigated closely, one finds that there are cases when the time-scale of variation of a quantity which gives a vanishing PB with the nPN Hamiltonian $H^{n\mathrm{PN}}$ varies at a time scale which is only nPN order larger than the Newtonian-Keplerian time-scale. This happens because $H_{n\mathrm{PN}}$, the nPN part of the Hamiltonian (remember the distinction between $H^{n\mathrm{PN}}$ and $H_{n\mathrm{PN}}$ as introduced in the beginning of subsection II D) may sometimes induce variations in a certain quantity at time-scales which are only $(n - 1/2)$PN order (rather than nPN order as one might expect) larger than the Newtonian time-scale T_{N}. A classic example is that of spin-precession of BBHs. The spins precess due to $H_{1.5\mathrm{PN}}$ at a time-scale which is only 1PN (and not 1.5PN) larger the Newtonian time scale T_{N}. ST: add a reference by Yunes and Klein . The resolution of this interesting paradox can be traced to the PBPN rule, or more specifically N of Eq. (48) being $\pm 1/2$ instead of being 0. If N of Eq. (48) could take no value other than 0, then a vanishing PB of a certain quantity f with the nPN Hamiltonian $H^{n\mathrm{PN}}$ would necessarily imply that the time-scale of variation of f is atleast qPN order larger than the Newtonian time-scale T_{N} where $q > n$.

This teaches us an important lesson. If a general quantity f contains spin variables, it is the vanishing of the PB of f with $H^{(n+1/2)\mathrm{PN}}$ (rather than $H^{n\mathrm{PN}}$) that guarantees that the time-scale of variation of f is atleast mPN order larger than T_N, with $m > n$. If f does not contain spin variables, then the vanishing of the PB of f with $H^{n\mathrm{PN}}$ suffices. The reason we have to check the PB of f with $H^{(n+1/2)\mathrm{PN}}$ rather than $H^{(n)\mathrm{PN}}$ in the former case can be traced to N of Eq. (48) being $\pm 1/2$ instead of being 0, where we made the statement of the PBPN rule. *In other words, if a quantity f contains (does not contain) spin variables, then a prior knowledge of $H^{(n+1/2)\mathrm{PN}}$ ($H^{n\mathrm{PN}}$) is required to show its constancy at the nPN order.* All this discussion motivates us to introduce a more physically relevant definition of PN constancy.

Defintion 2: If the time scale T_f of variation of a quantity f scales as

$$\frac{T_{\mathrm{N}}}{T_f} \sim x^m, \tag{51}$$

with $m > n$, then we say that f is constant up to the nPN order. T_f can be determined as

$$T_f \sim \frac{f}{\dot{f}} \sim \frac{f}{\{f, H\}}. \tag{52}$$

The contrast between PN Definitions 1 and 2 of constancy is brought about further once we note that the time scale of variation of $S^i_{\mathrm{eff}} L_i$ is 1.5PN (not 2PN) order longer than the Newtonian time-scale T_{N}, although it's exactly conserved under the 1.5PN Hamiltonian $H^{1.5\mathrm{PN}}$. To see this, we state that (without giving the detailed calculations for brevity)

$$d(S^i_{\mathrm{eff}} L_i)/dt = \left\{S^i_{\mathrm{eff}} L_i, H_{2\mathrm{PN}}\right\} \sim \frac{PRS_A^2}{c^2 R^3} \tag{53}$$

$$\dot{R}^i \sim \left\{R^i, H_{\mathrm{N}}\right\} \sim \frac{P}{m}, \tag{54}$$

which lets us determine that

$$\frac{T_{\mathrm{N}}}{T_{S^i_{\mathrm{eff}} L_i}} \sim \frac{R^i}{\dot{R}^i} \frac{d(S^i_{\mathrm{eff}} L_i)/dt}{S^i_{\mathrm{eff}} L_i}, \tag{55}$$

which with the aid of Eqs. (53) and (54) and PN order counting rule given in subsection II C give

$$\frac{T_{\mathrm{N}}}{T_{S^i_{\mathrm{eff} L_i}}} \sim x^{1.5}. \tag{56}$$

where x is the PN parameter of Eq. (24), thereby establishing that $S^i_{\mathrm{eff} L_i}$ varies at a time-scale which is 1.5PN larger than than the Newtonian time-scale. Thus $S^i_{\mathrm{eff} L_i}$ is not constant at the 1.5PN order according to Definition 2 of constancy but is a constant at 1.5PN order as per Definition 1.

2. A formal definition of post-Newtonian involution

ST: I think this discussion will become less verbose and compact if we define 'fundamental PBs' as the ones in Eqs. (2) and the fundamental form of a PB as the one got by breaking the PB into the fundamental PBs using Eqs. (3-6). Here we attempt to give a more mathematically rigorous definition of PN involution of any two quantities f and g. The basic idea will involve counting the powers of $1/c^2$ since we mention towards the end of II C 2 (in the second bulleted point) that counting the powers of $1/c^2$ alone can let us determine the relative PN orders of terms of a PN series, provided of course that certain order-of-magnitude relations have already been used, namely Eqs. (18)-(21). Right now, here is the rough idea on which we would want to base our definitions of PN involution. Let's say $\{f, g\}$ actually scales as $(1/c^2)^m$ but would have been expected to scale as $\sim (1/c^2)^n$ if none of the PBs got by breaking down $\{f, g\}$ (using Eqs. (3-6)) would have vanished. Then the degree of PN involution between f and g would be $m - n$. We will make this vague idea more precise in the following paragraphs and finally encapsulating it in a concrete mathematical form by Eq. (62).

First we define that if two PN series exactly commute then they are in involution up to infinite orders. The following discussion is devoted to defining the PN order up to which two PN series commute when their PB does not vanish exactly. To this end, we start off by defining the operator $\mathcal{C}$ acting on a certain quantity f to reflect the power n of $1/c^2$ so that $f \sim (1/c^2)^n$. As examples, $\mathcal{C}$ acting on all real numbers, R^i, P^i, m_1, m_2 and G are defined to be 0 since they don't contain c. Now, due to Eq. (21), we define $\mathcal{C}(S_A) \equiv 1/2$ (see also [9]). With these rules, the $\mathcal{C}$ of any combination of phase-space variables R^i, P^i, S^i_A and the constants G, m_1, m_2, c are now well defined. As another example, we have

$$\mathcal{C}\left(\frac{RP}{c^4 m} + \frac{GP}{c^6 R}\right) = 2. \tag{57}$$

Now we introduce the operator $\bar{\mathcal{C}}$ acting on any general PB to give the order of $1/c^2$ that 'one would anticipate' the PB to scale if the concerned PB did not vanish ' even to the minimal possible extent'. To make these nebulous ideas more precise, we define $\bar{\mathcal{C}}$ acting on the PBs of Eq. (2) to yield

$$\bar{\mathcal{C}}\left(\{R^i, P_j\}\right) = 0 \qquad \bar{\mathcal{C}}\left(\left\{S^i_A, S^j_B\right\}\right) = \frac{1}{2}, \tag{58}$$

where the latter equality is motivated by our earlier definition $\mathcal{C}(S_A) \equiv 1/2$ and the labels A, B refer to the individual black holes. Also, $\bar{\mathcal{C}}$ acting on a multiple of the two kinds PBs displayed in Eq. (2) is defined as

$$\bar{\mathcal{C}}(M\{f, g\}) = \mathcal{C}(M)\bar{\mathcal{C}}(\{f, g\}), \tag{59}$$

where $\{f, g\}$ is one of the PBs in Eq. (2) and M is a coefficient which itself is not manifestly written in a form which involves PBs. Finally, $\bar{\mathcal{C}}$ acting on a linear combination of PBs displayed in Eq. (2)

$$\bar{\mathcal{C}}\left(\sum_i M_i \{f_i, g_i\}\right), \tag{60}$$

is equal to the lowest value that $\bar{\mathcal{C}}$ yields when acting on any individual term of the above summation. Finally, $\bar{\mathcal{C}}$ acting on any general PB $\{f, g\}$ is given by

$$\bar{\mathcal{C}}\{f, g\} = \bar{\mathcal{C}}\left(\sum_i M_i \{f_i, g_i\}\right), \tag{61}$$

where $\{f, g\}$ has been broken down into the form $\sum_i M_i \{f_i, g_i\}$ using the rules of Eqs. (3-6).

At long last, we define that two quantities f and g are in involution up to mPN order if

$$\mathcal{C}(\{f, g\}) - \bar{\mathcal{C}}(\{f, g\}) = m. \tag{62}$$

Some comment on this choice of definition is surely warranted. $\mathcal{C}(\{f, g\})$ counts the powers of $1/c^2$ in $\{f, g\}$ and thereby is a measure of how well f and g commute. A higher value of $\mathcal{C}(\{f, g\})$ signifies a higher degree of commutation. On the other hand, $\bar{\mathcal{C}}(\{f, g\})$ is the 'expected' the power of $1/c^2$ in $\{f, g\}$ 'if f and g were not to commute at all'. This is actually the point of introducing $\bar{\mathcal{C}}$ of a PB.

3. Definition of post-Newtonian integrability

The following is the procedure to follow if we want to define the integrability of a Hamiltonian dynamical system up to nPN order with $2m$ phase-space variables (m DOFs). Including the $(n + 1/2)$PN Hamiltonian $H^{(n+1/2)\mathrm{PN}}$ suppose that altogether we have m quantities in a PN series form which satisfy the following criteria

- They are all independent.
- They are all in mutual involution up to atleast nPN order (the exact vanishing of PBs is to be considered an involution up to an infinite order as per V B 2).

Then the dynamical system under consideration is defined to integrable up to nPN order.

A few remarks are in order. The reader may wonder why have we included $H^{(n+1/2)\mathrm{PN}}$ instead of $H^{n\mathrm{PN}}$ in our list of mutually commuting constants to define nPN integrability. The reason is that $H^{(n+1/2)\mathrm{PN}}$ being one of the mutually commuting constants is what guarantees that all other commuting constants are indeed constants in the sense of Definition 2 rather than Definition 1.

C. Constants of motion at the 2PN order

Integrability at the 1.5PN order: In subsection III A, we concluded that the BBH system is integrable at the 1.5PN order because it possesses $5 = 10/2$ independent exact mutually commuting constants of motion for the 1.5PN Hamiltonian $H^{1.5\mathrm{PN}}$: $H^{1.5\mathrm{PN}}, L^i L_i, J^i J_i, J_z, S^i_{\mathrm{eff}} \cdot L^i$. This conclusion basically adopts "Definition 1" of the constancy of a certain quantity at the nPN order as presented in V B 1. The involutions among the constants are exact.

But as discussed a little later, a more useful definition of the PN constancy of a certain quantity, from a practical point of view is "Definition 2" according to which $S^i_{\mathrm{eff}}.L^i$ is not a constant of motion even at the 1.5PN order, thereby making us not be able to claim that the BBH system is integrable at the 1.5PN order, as of now. We will provide corrections to L^2 and $S^i_{\mathrm{eff}} L_i$ in V C 2 and V C 3 so that the corrected quantities along with H, J^2 and J^z are all in mutual involution up to 2PN thereby establishing the integrability of the BBH system up to 2PN order.

1. Approach to find commuting constants of motion at 2PN

J^2 and J_z always remain exact constants of motion under a Hamiltonian of any order and with the Hamiltonian they form a set of three independent mutually commuting constants of motion ST: Leo wants to comment that this inheritance is due to SO(3) invariance. I don't know how to write it . We need to add two more quantities to this set to prove the integrability. We first propose that the two required constants of motion (call them $\widetilde{L^2}$ and $\widetilde{S^i_{\mathrm{eff}} L_i}$) are the exact constants of motion under the 1.5PN Hamiltonian plus a small correction

$$\widetilde{L^2} = L^2 + \delta(L^2) \equiv L^2 + \delta L^2, \tag{63}$$

$$\widetilde{S^i_{\mathrm{eff}} L_i} = S^i_{\mathrm{eff}} L_i + \delta(S^i_{\mathrm{eff}} L_i), \tag{64}$$

where the second terms on RHS of Eqs. (63) and (64) are the sought after corrections. As the reader will soon discover, our method to find the commuting constants of motion is based on informative guesses, trials and luck rather than an established, well-researched approach. It nevertheless gives yields us the sought after constants of motion.

Focusing our attention on $\widetilde{L^2}$, we write

$$\begin{aligned}
&\left\{\widetilde{L^2}, H^{2\mathrm{PN}}\right\} \\
&= \left\{L^2 + \delta L^2, H^{1.5\mathrm{PN}} + H_{2\mathrm{PN}}\right\} \qquad (65)\\
&= \cancel{\left\{L^2, H^{1.5\mathrm{PN}}\right\}} + \left\{L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H^{1.5\mathrm{PN}}\right\} \\
&+ \left\{\delta L^2, H_{2\mathrm{PN}}\right\} \qquad (66)\\
&= \left\{L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H_{\mathrm{N}} + H_{1\mathrm{PN}} + H_{1.5\mathrm{PN}}\right\} \\
&+ \left\{\delta L^2, H_{2\mathrm{PN}}\right\} \qquad (67)\\
&= \left\{L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H_{\mathrm{N}}\right\} \\
&+ \left\{\delta L^2, H_{1\mathrm{PN}} + H_{1.5\mathrm{PN}} + H_{2\mathrm{PN}}\right\} \qquad (68)\\
&= \left\{L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H_{\mathrm{N}}\right\} \\
&+ \left\{\delta L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H_{1\mathrm{PN}} + H_{1.5\mathrm{PN}}\right\} \qquad (69)
\end{aligned}$$

With the hope that the second term in the Eq. (69) cancels the first term, the PBPN rule (introduced in subsection V A) implies that δL^2 is 2PN order higher than L^2. It's easy to see why is that so. Application of Eq. (48) to second and third term of Eq. (69) implies that the latter is 2PN order higher than the former (since $N = 0$ for both are orbital PBs). Since the first and the second term of Eq. (69) are meant to cancel each other, they have the same PN order. This implies that the third term is also 2PN order higher than the first one. Finally, the application of the PBPN rule (Eq. (48)) again to the first and third term finally tells us that δL^2 is 2PN order higher than L^2. This is so because N of Eq. (48) is 0 because we are again dealing with two orbital PBs. If δL^2 is 2PN order higher relative to L^2, then it also means that the third and fourth PBs in Eq. (69) will be of a higher PN order than the first two (again by the PBPN rule) and that's why we choose to ignore it for now.

So, we now try to construct the correction term δL^2 such that

$$\left\{L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H_{\mathrm{N}}\right\} = 0. \tag{70}$$

Since δL^2 is 2PN order higher than L^2, we try to make an ansatz for δL^2 which is a sum of terms of 2PN order relative to L^2. We require these terms which compose our ansatz to be made up of some basic tensors associated with our BBH system. The reason we want our ansatz to be composed of tensors is that we expect our sought-after constants of motion to be geometrical objects which exist independent of any coordinate system and that the mathematical form of the statement of their constancy (wrt. time) should look the same in every coordinate system. This motivates us to propose that the sought-after constants of motion are to be composed of only tensors (and not tensor densities). Let's enumerate some tensors that we have at our disposal.

- Kronecker delta tensor δ^i_j.
- metric tensor g_{ij}.
- Levi-Civita tensor ϵ_{ijk} (not the Levi-Civita symbol)

- position vector R^i
- momentum vector P^i
- spin vectors S^i_A
- unit radial vector $n^i := R^i/R$.

We decided to add to the list the vector n^i by noticing that the 2PN Hamiltonian $H^{2\mathrm{PN}}$ is built out of it (see Eq. (29)). One could as well think of including P^i/P into the consideration. But since $H^{2\mathrm{PN}}$ does not include the vector P^i/P, we did not take it into consideration based on our hunch ST: Is this sentence OK? . It later turned out that including vector P^i/P isn't necessary, for the 2PN constants of motion we found out was not built out of it. This is another instance of the uncertain and trial-and-error nature of our approach as we alluded to earlier. ST: I think this paragraph is important. There is no reason apriori to not consider P^i/P. We let the reader know why we got the hunch to not take P^i/P into consideration.

Since L^2 is a scalar, we propose our ansatz for δL^2 to be a sum of various scalars formed by contracting the seven tensors listed in bullets above. An example of our scalar ansatz which is 2PN order higher than L^2 is

$$\begin{aligned}\delta L^2 =&\alpha_1\left(\frac{(R^k S_{2k})(R^l S_{2l})}{(R^m R_m)^{3/2}}\right)\\ &+\alpha_2\left(\frac{S_2^l S_{2l}}{(R^m R_m)^{1/2}}\right)\\ &+\alpha_3 (P^i S_{2i})(P^j S_{2j})\end{aligned} \tag{71}$$

where αs represent constant quantity to be determined by demanding that Eq. (70) be true. One can form contractions of increasing complexities built out of the seven tensors to serve as an ansatz for δL^2. We start off with simple ansatzes ST: ansatzes is a valid word, indeed for δL^2 which can potentially satisfy Eq. (70) and proceed towards more complicated ones if the simpler ones don't fit the bill.

We performed contractions and made our ansatzes using the `AllContractions` and `MakeAnsatz` commands of the popular `Mathematica` package `xAct`, a popular package used in the gravitational physics community to perform tensor calculations ST: Add xAct reference here .

2. The fourth commuting constant of motion: perturbation to L^2

The fourth constant of motion (apart the already obvious three: $H^{2\mathrm{PN}}, J_i J^i, J_z$) got by following the procedure described above reads (with the relative PN orders displayed)

$$\widetilde{L^2} = \underbrace{L^2}_{\text{Newtonian}} + \underbrace{\delta L^2}_{\text{2PN}}, \tag{72}$$

where

$$\begin{aligned}\delta L^2 =& -\frac{\epsilon(m_2\ P^i S_{1i} + m_1\ P^i S_{2i})^2}{m_1^2\ m_2^2}\\ &+\frac{2G\epsilon(m_2\ R^i S_{1i} + m_1\ R^i S_{2i})^2}{(m_1+m_2)R^3}\\ &+\epsilon\left(\frac{P^i P_i}{m_1 m_2} - \frac{2Gm_1 m_2}{(m_1+m_2)R}\right) S_{1a} S_2^a,\end{aligned} \tag{73}$$

with the cancellations taking place as

$$\left\{L^2, H_{2\mathrm{PN}}\right\} + \left\{\delta L^2, H_{\mathrm{N}}\right\} = 0. \tag{74}$$

3. The fifth commuting constant of motion: perturbation to $\vec{S}_{\mathrm{eff}}.\vec{L}$

Employing the guidelines set in subsection V C 1, we found the fifth commuting constant of motion at the 2PN order to be (with the relative PN orders also indicated)

$$\widetilde{S^i_{\mathrm{eff}} L_i} = \underbrace{S^i_{\mathrm{eff}} L_i}_{\text{Newtonian}} + \underbrace{\delta_1(S^i_{\mathrm{eff}} L_i)}_{\text{0.5PN}} + \underbrace{\delta_2(S^i_{\mathrm{eff}} L_i)}_{\text{1.5PN}}, \tag{75}$$

with $\delta_1(S^i_{\mathrm{eff}} L_i)$ and $\delta_2(S^i_{\mathrm{eff}} L_i)$ standing for

$$\delta_1(S^i_{\rm eff}L_i) = \frac{1}{2}\left(S^a_1 S_{2a}\right) \tag{76}$$

$$\begin{aligned}
\delta_2(S^i_{\rm eff}L_i) =& \frac{\epsilon\left(P^a S_{1a}\right)^2}{m_1^2} + \frac{3m_2\epsilon\left(P^a S_{1a}\right)^2}{4m_1^3} - \frac{2Gm_2^2\epsilon\left(R^a S_{1a}\right)^2}{(m_1+m_2)\left(R_a R^a\right)^{3/2}} \\
&- \frac{3Gm_2^3\epsilon\left(R^a S_{1a}\right)^2}{2m_1\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} + \frac{3\epsilon\left(P^a S_{1a}\right)\left(P^a S_{2a}\right)}{4m_1^2} + \frac{3\epsilon\left(P^a S_{1a}\right)\left(P^a S_{2a}\right)}{4m_2^2} \\
&+ \frac{2\epsilon\left(P^a S_{1a}\right)\left(P^a S_{2a}\right)}{m_1 m_2} + \frac{3m_1\epsilon\left(P^a S_{2a}\right)^2}{4m_2^3} + \frac{\epsilon\left(P^a S_{2a}\right)^2}{m_2^2} - \frac{3Gm_1^2\epsilon\left(R^a S_{1a}\right)\left(R^a S_{2a}\right)}{2\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} \\
&- \frac{4Gm_1 m_2\epsilon\left(R^a S_{1a}\right)\left(R^a S_{2a}\right)}{\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} - \frac{3Gm_2^2\epsilon\left(R^a S_{1a}\right)\left(R^a S_{2a}\right)}{2\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} \\
&- \frac{2Gm_1^2\epsilon\left(R^a S_{2a}\right)^2}{\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} - \frac{3Gm_1^3\epsilon\left(R^a S_{2a}\right)^2}{2m_2\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}}.
\end{aligned} \tag{77}$$

The cancellations happen as

$$\underbrace{\left\{S^i_{\rm eff}L_i, H_{\rm SS}\right\}}_{\substack{\text{both orbital and spin PBs;}\\ \text{both Newtonian and 0.5PN}}} + \underbrace{\left\{\delta_1(S^i_{\rm eff}L_i), H_{\rm 1.5PN}\right\}}_{\substack{\text{spin PBs;}\\ \text{Newtonian}}} + \underbrace{\left\{\delta_1(S^i_{\rm eff}L_i), H_{\rm SS}\right\}}_{\substack{\text{spin PBs;}\\ \text{0.5PN}}} + \underbrace{\left\{\delta_2(S^i_{\rm eff}L_i), H_{\rm N}\right\}}_{\substack{\text{orbital PBs;}\\ \text{Newtonian}}} = 0 \tag{78}$$

where $H_{\rm SS}$ is defined via Eq. (30) and below every Poisson bracket of Eq. (78), we indicate what kind of PBs (orbital or spin) can the four PBs of Eq. (78) be broken down into using Eqs. (3-6). We mention this because this information with the help of PBPN rule helps one determine the relative PN orders of the four PBs of Eq. (78) which have also been displayed.

Along with the Hamiltonian H, J^2 and J^i, the quantities given via Eqs. (72) and (75) now form a set of 5 independent, mutually commuting constants of motion at the 2PN order, thereby establishing the integrability of the BBH system at the 2PN order. ST: Do verify thoroughly that the five quantities are 2PN commuting constants of motion. Perform all kinds of tests.

Let's take a moment to reflect the significance of this result. $S^i_{\rm eff}L_i$ although being an exact constant of motion under $H^{\rm 1.5PN}$ is not a constant of motion at the 1.5PN as per "Definition 2" of constancy introduced in subsection V B. In Eq. (75) above, we have added corrections to $S^i_{\rm eff}L_i$ so that the corrected quantity is now a constant of motion at the 2PN order as per "Definition 2" of constancy, in addition to fixing the constancy at the 1.5PN order. Moreover, our new set of commuting constants $H, J^i, J^2, \widetilde{L^2}$ and $\widetilde{S^i_{\rm eff}L_i}$ are in mutual involution with each other up to 2PN, thereby establishing the integrable nature of the system at the 2PN order in the sense of PN integrability defined in V B 3.

VI. SUMMARY AND FUTURE WORK

ACKNOWLEDGMENTS

We would like to acknowledge XXX for helpful discussion.

Appendix A: Analytic solutions for quasi-circular spin precession

Appendix B: Symplectic manifold for the BBH system

The aim of this appendix is to verify that the symplectic form given via Eq. (10) is consistent with the PB relations of Eqs. (2). We start off with an assumed form of the symplectic form

$$\begin{aligned}
\omega =& \alpha\ dP^i \wedge dR^i + \beta_1\ dS_1^z \wedge d\phi_1 \\
&+ \beta_2\ dS_2^z \wedge d\phi_2
\end{aligned} \tag{B1}$$

with α, β_1 and β_2 being proportionality constants to be determined.

A few comments are in order before we proceed to determine the constants α, β_1 and β_2. To begin with, we consider $dP^i \wedge dR^i$ to be living in the space of wedge product of elements of the cotangent space of the manifold whose coordinates are R^i and P^i (call it V_O). Similarly, $dS_1^z \wedge d\phi_1$ and $dS_2^z \wedge d\phi_2$ exist in separate separate vector spaces (V_{S_1} and V_{S_2}). We then define a manifold which is the product manifold of the orbital and the two spin manifolds with coordinates $R^i, P^i, S_1^z, \phi_1, S_2^z, \phi_2$. It is this product manifold in which the Hamiltonian dynamics of the BBH system takes place. Now, the vector space of two-forms on this product manifold is actually the direct sum of V_O, V_{S_1} and V_{S_2}. The two-forms whose linear combination forms the RHS of Eq. (B1) live in this direct sum vector space. The 1-forms in Eq. (B1) exist in the cotangent space of the product manifold.

This also implies that the inner product of a 1-form and a vector (which now live in the tangent and cotangent space of the product manifold) that have been promoted from the tangent space and the cotangent space of different submanifolds to the direct-sum space vanishes

$$dR(X_{S_1^z}) = d\phi_1(X_R) = ... = 0. \tag{B2}$$

It can be verified that $\alpha = \beta_1 = \beta_2 = 1$ if Eqs. (2, 8, 9) are imposed thereby justifying the use of the particular symplectic form in this paper.

Appendix C: Examples on how to do post-Newtonian counting

Below we consider two simple examples to illustrate how to do PN counting.

Post-Newtonian Hamiltonian: Imagine a fictitious Hamiltonian to be given by

$$H = \frac{P^2}{2m} - \frac{P^4}{m^3c^2}, \tag{C1}$$

and by eliminating P in favor of v, via relation (18) we get

$$H \sim \frac{P^2}{2m}\left(1 - \frac{v^2}{c^2}\right), \tag{C2}$$

or rather if we employ relation (19)

$$H \sim \frac{P^2}{2m}\left(1 - \frac{Gm}{c^2R}\right), \tag{C3}$$

indicating that

$$H \sim \frac{P^2}{2m}(1 + (\ldots)x), \tag{C4}$$

where x is v^2/c^2 or $Gm/(c^2R)$ depending on whether we are considering Eq. (C2) or Eq. (C3). In any case, the second term is of 1PN order relative to the leading term.

Total angular momentum: We assume black-holes in a BBH system to be maximally spinning and hence the magnitude of the spin angular momentum S_A of a certain black-hole of the BBH system is of the order (see Eq. (21))

$$S_A \sim \frac{Gm^2}{c}, \tag{C5}$$

$$\tag{C6}$$

which when added to L gives the order-of-magnitude of total angular momentum as

$$J^i = L^i + S^i \sim RP + \frac{Gm^2}{c} \tag{C7}$$

$$= RP\left(1 + \frac{Gm^2}{RPc}\right) \tag{C8}$$

$$\sim RP\left(1 + \frac{v}{c}\right), \tag{C9}$$

implying that the spin of a *maximally rotating* Kerr BH is 0.5PN order higher than L. In the last relation, we have used relations (18) and (19) to eliminate r, p in favor of v.

In closing, we also point out that we could have also done the PN order counting and obtained the same conclusions by simply looking at the powers of $1/c$, using the procedure mentioned as a bulleted point towards the end of II C 2.

Appendix D: General formulae concerning action integrals

The nature of flow under J^z, J^2, L^2 is such that certain vectors like $\vec{R}, \vec{P}, \vec{S}_{1z}$ and $\vec{S}_{2z}$ rotate around a fixed vector (say $\hat{n}$). Evaluation of action integrals then requires evaluation of $\oint P_i dQ_i$ and $\oint S_z d\phi$ under such a flow. In this appendix, we derive the expressions of these two integrals when $\vec{R}, \vec{P}, \vec{S}_{1z}$ and $\vec{S}_{2z}$ rotate about a certain fixed $\hat{n}$.

Without loss of generality, let the axes be chosen in such a way that $\vec{R}$ and $\vec{P}$ rotate around the $\hat{z}$ with the frequency ω, such that the angle of rotation $\theta = \omega t$ (see Fig. 1). The components of $\vec{P}$ and $d\vec{R}$ rotate in the following way:

$$\begin{bmatrix} P_x \\ P_y \end{bmatrix} = \begin{bmatrix} cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} P_{x0} \\ P_{y0} \end{bmatrix} \tag{D1}$$

$$\begin{bmatrix} dX \\ dY \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} dX_0 \\ dY_0 \end{bmatrix} \tag{D2}$$

The quantities with a nought in the subscript in the above two equations stand for the x and y components of the momentum and differential displacement vector at the initial time $t = 0$.

Using Eqs. (D1) and (D2), and noting that $d\vec{R}$ has no z-component, we evaluate with elementary means $\oint(P_x dX + P_y dY + P_z dZ)$ to be $2\pi P_{y0} X_0$, subject to the initial condition that the projection of $\vec{R}$ in the x-y plane is along the x direction and this projection is rotating

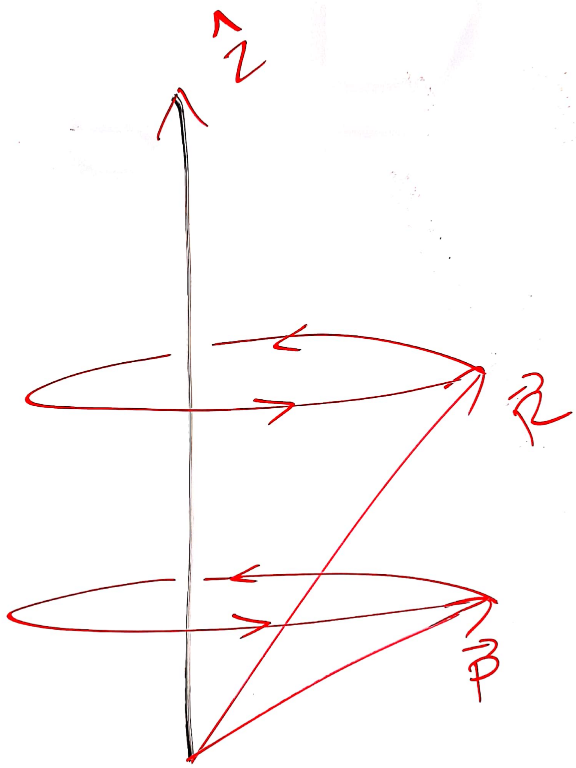

FIG. 1.

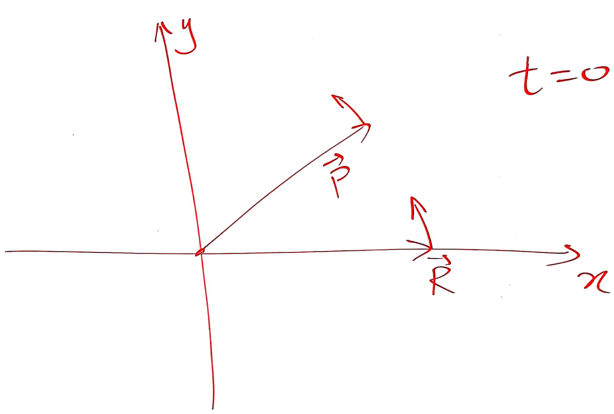

FIG. 2.

towards the y direction; i.e. $d\vec{R}$ is along the y-direction (cf. Fig. 2).

Let's try to cast the result of this particular evaluation $\int(P_x dX + P_y dY + P_z dZ) = 2\pi P_{y0} X_0$ into a geometrical form. For the particular initial condition which we have adopted, $Y_0 = 0$ and we can write

$$2\pi P_{y0} X_0 = 2\pi(P_{y0} X_0 - P_{x0} Y_0) \tag{D3}$$
$$= 2\pi(\vec{R_{0p}} \times \vec{P_{0p}}) \cdot \hat{z}, \tag{D4}$$

where the subscript p denotes the operation of taking the projection of the concerned vector in the $x - y$ ST: This is not hyphen plane. A simple calculation also reveals that

$$2\pi \left(\vec{R_{0p}} \times \vec{P_{0p}}\right) \cdot \hat{z} = 2\pi(\vec{R_0} \times \vec{P_0}) \cdot \hat{z}. \tag{D5}$$

Finally, since both the vectors $\vec{R}$ and $\vec{P}$ are rotating about the z-axis, the quantity $(\vec{R_0} \times \vec{P_0}) \cdot \hat{z}$ does not change with time, thereby allowing us to remove the noughts in the subscripts and write the orbital action integral as

$$2\pi(\vec{R} \times \vec{P}) \cdot \hat{z}. \tag{D6}$$

And finally, if we generalize from $\hat{z}$ to an arbitrary unit vector $\hat{n}$ about which $\vec{R}$ and $\vec{P}$ rotate, our orbital action integral becomes

$$2\pi(\vec{R} \times \vec{P}) \cdot \hat{n} = 2\pi\vec{L} \cdot \hat{n}. \tag{D7}$$

There is one more kind of action integral involved which pertains to the spin vectors. If a generic spin vector $\{S_x, S_y, S_z\}$ undergoes a rotation about the z-axis, the associated action integral $\oint P_i dQ_i$ becomes (remembering S_z and the azimuthal angle ϕ are canonical momentum and position coordinates; see Eq. (**??**)) ST: fix it

$$\oint S_z d\phi = 2\pi S_z. \tag{D8}$$

If this rotation of the spin vector were to happen about any general direction $\hat{n}$, the integral would then have become

$$2\pi\vec{S} \cdot \hat{n}. \tag{D9}$$

Eqs. (D7) and (D9) are the main results of this appendix.

[1] V.P.R.J.V. Jose, J.V. José, and E.J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge University Press, 1998).
[2] V.I. Arnold, K. Vogtmann, and A. Weinstein, *Mathematical Methods of Classical Mechanics*, Graduate Texts in Mathematics (Springer New York, 2013).
[3] J. Steinhoff, "Canonical formulation of spin in general relativity," Annalen der Physik **523**, 296–353 (2011), https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.201000178.

[4] B. M. Barker and R. F. O'Connell, "Gravitational Two-Body Problem with Arbitrary Masses, Spins, and Quadrupole Moments," Phys. Rev. **D12**, 329–335 (1975).

[5] Michael D. Hartl and Alessandra Buonanno, "The Dynamics of precessing binary black holes using the post-Newtonian approximation," Phys. Rev. **D71**, 024027 (2005), arXiv:gr-qc/0407091 [gr-qc].

[6] A. Fasano, S. Marmi, and B. Pelloni, *Analytical Mechanics: An Introduction*, Oxford Graduate Texts (OUP Oxford, 2006).

[7] Thibault Damour, "Coalescence of two spinning black holes: an effective one-body approach," Phys. Rev. **D64**, 124013 (2001), arXiv:gr-qc/0103018 [gr-qc].

[8] T. Damour and G. Schafer, "Higher-order relativistic periastron advances and binary pulsars." Nuovo Cimento B Serie **101B**, 127–176 (1988).

[9] Luc Blanchet, "Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries," Living Rev. Rel. **17**, 2 (2014), arXiv:1310.1528 [gr-qc].