Action-angle-based closed-form solution of the 1.5 post-Newtonian binary black hole system

Lecture Notes

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Contents

Chapter 1

Introduction

These lecture notes are based on Refs. [1, 2, 3] which aim to give closed-form solutions to the spinning, eccentric binary black hole dynamics at 1.5PN via two different equivalent ways: the standard way and the one which uses action-angle variables.

Also these notes lack rigor. The above papers assume certain level of familiarity with the symplectic geometric approach to classical mechanics, the non-fulfillment of which on the reader's part may make the papers appear esoteric. The purpose of these lecture notes is to give the reader this basic knowledge which the above papers assume on the reader's part. If the reader finds these notes to be incomplete or lacking rigor, they are welcome to refer to the sources cited in these notes as well as the above papers.

We use two kinds of filled boxes in these lecture notes

- The boxes with an explicit label "Box": These are meant to give the reader a reference to more advanced sources to supplement the material discussed herein.

- The boxes without an explicit label "Box": These simply summarize the important points.

Chapter 2

The setup

2.1 Statement of the problem

We start by describing the canonical variables and the dynamical setup used to study eccentric binaries of black holes with precessing spins in the post-Newtonian (PN) approximation. The BBH system under consideration is schematically displayed in Fig. 2.1, using its center-of-mass frame [4], to define the separation vector $\vec{R} \equiv \vec{R}_1 - \vec{R}_2$ and the linear momenta $\vec{P} \equiv \vec{P}_1 = -\vec{P}_2$ of a binary of black holes with masses m_1 and m_2. With these quantities, we build the Newtonian orbital angular momentum $\vec{L} \equiv \vec{R} \times \vec{P}$, and the total angular momentum $\vec{J} \equiv \vec{L} + \vec{S}_1 + \vec{S}_2$ which includes the BH spins $\vec{S}_1$ and $\vec{S}_2$. The individual BH masses are m_1 and m_2 and the total mass $M \equiv m_1 + m_2$. Additionally, the reduced mass is given by $\mu \equiv m_1 m_2/M$ and the symmetric mass ratio $\nu \equiv \mu/M$ is a function of the reduced mass. The constants $\sigma_1 \equiv 1 + 3m_2/4m_1$ and $\sigma_2 \equiv 1 + 3m_1/4m_2$ are used to build the effective spin

$$\vec{S}_{\text{eff}} \equiv \sigma_1 \vec{S}_1 + \sigma_2 \vec{S}_2 \,. \tag{2.1}$$

In terms of the scaled variables $\vec{r} \equiv \vec{R}/GM, \vec{p} \equiv \vec{P}/\mu$, the 1.5 post-Newtonian (PN) Hamiltonian is given by [5, 6, 7, 8, 9]

$$H = H_{\text{N}} + H_{\text{1PN}} + H_{\text{1.5PN}} + \mathcal{O}(c^{-4}) \,, \tag{2.2}$$

where

$$H_{\text{N}} = \mu \left(\frac{p^2}{2} - \frac{1}{r} \right) , \tag{2.3}$$

$$\begin{aligned} H_{\text{1PN}} = \frac{\mu}{c^2} \Bigg\{ & \frac{1}{8}(3\nu - 1)p^4 + \frac{1}{2r^2} \\ & - \frac{1}{2r} \left[(3 + \nu)p^2 + \nu(\hat{r} \cdot \vec{p})^2 \right] \Bigg\}, \end{aligned} \tag{2.4}$$

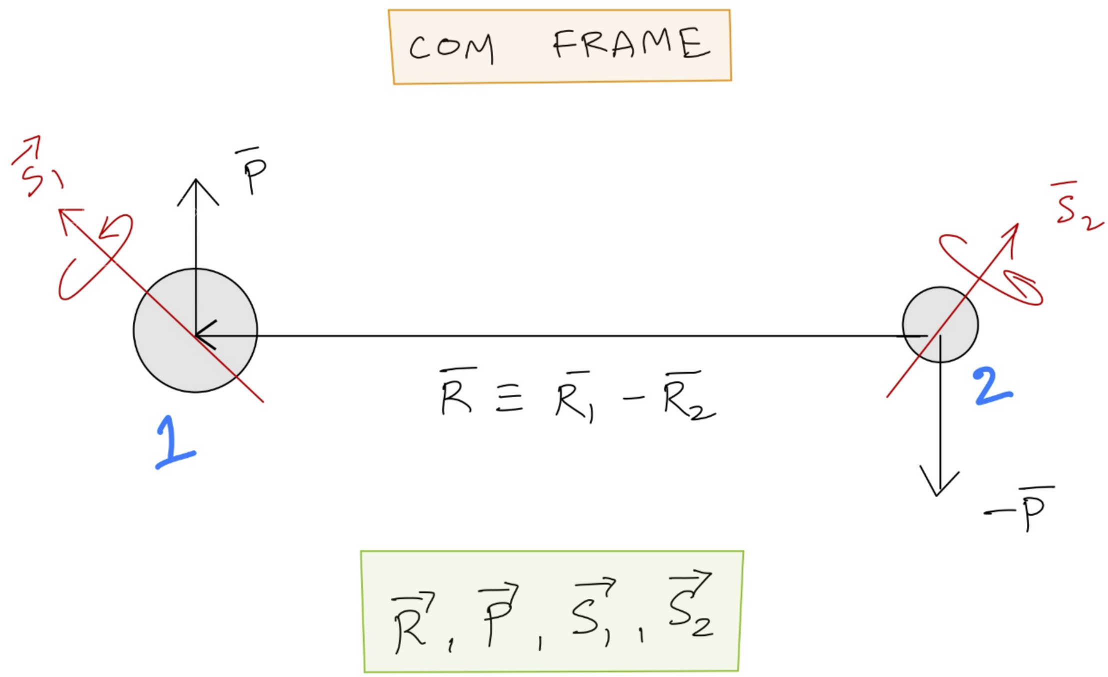

Figure 2.1: Schematic setup of a precessing black hole binary. All phase space variables are contained in the form of four 3D vectors $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$.

$$H_{1.5\mathrm{PN}} = \frac{2G}{c^2 R^3} \vec{S}_{\mathrm{eff}} \cdot \vec{L}, \; . \tag{2.5}$$

The problem: The above 1.5PN Hamiltonian is a function of the phase-space variables $\vec{R}(t), \vec{P}(t), \vec{S}_1(t)$ and $\vec{S}_2(t)$ only. Our challenge is to integrate the Hamilton's equations got from this Hamiltonian to obtain the solution $\vec{R}(t), \vec{P}(t), \vec{S}_1(t)$ and $\vec{S}_2(t)$. This is the subject of these lecture notes.

2.2 Defining the 1.5PN BBH system

In graduate level classical mechanics, specification of the Hamiltonian $H(\vec{p}, \vec{q})$ is considered equivalent to specifying the system, for the application of Hamiltons's equations [10]

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{2.6}$$

immediately furnishes the equations of motion (EOMs). Eqs. (2.6) further imply that any function $f(p, q)$ of the phase space variables obeys (Eq. 9.94 of Ref. [10])

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} = \{f, H\}\,, \text{ where} \tag{2.7}$$

$$\{f, g\} \equiv \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2.8}$$

where the last equality in Eq. (2.7) is valid if the Hamiltonian H is not a function of time (which will be the case for our BBH system, the subject of these lecture notes). $\{f, g\}$ is called the Poisson bracket (PB) between f and g. Actually,

$$\text{Eq. (2.6)} \Leftrightarrow \text{Eq. (2.7)},$$

i.e., the implication goes both ways; the two equations are equivalent.

Our point of view towards the PB-based EOMs for these lecture notes will be somewhat different from the above standard approach adopted by classical mechanics texts. For the BBH system with phase-space variables $\vec{R}(t), \vec{P}(t), \vec{S}_1(t)$ and $\vec{S}_2(t)$, the EOMs for any general function $f(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t))$ is still given by Eq. (2.7). But instead of Eq. (2.8), we define the PBs via

$$\{R_i, P_j\} = \delta_{ji} \quad \text{and} \quad \left\{S_A^i, S_B^j\right\} = \delta_{AB} \epsilon^{ij}_{k} S_A^k, \tag{2.9}$$

where the labels A and B refer to the two BHs. How to define the PB between any two functions f and g of the phase-space variables? We simply axiomatize the anti-commutativity, bilinearity, product, and the chain rules for the PBs

$$\{f, g\} = -\{g, f\}, \tag{2.10a}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, \quad a, b \in \mathbb{R}, \tag{2.10b}$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\}, \tag{2.10c}$$

$$\{f, g(v_i)\} = \{f, v_i\} \frac{\partial g}{\partial v_i}, \tag{2.10d}$$

where v_i represents any of the phase-space variables. The PB between any two phase space variables, i.e. components of $\vec{R}(t), \vec{P}(t), \vec{S}_1(t)$ and $\vec{S}_2(t)$ which does not fall under the purview of Eqs. (2.9) and (2.10) is assumed to vanish. Note that Eq. (2.8) implies Eqs. (2.10), but for these lecture notes, we will take the latter as definitions and totally disregard the former.

We therefore have defined our system of interest completely in that we can now write its EOM. This definition consists of

- specifying the Hamiltonian via Eq. (2.2).
- axiomatizing Eqs. (2.9), and (2.10). These enable us to evaluate the PB between any two functions of the phase-space variables.
- stating the EOM via Eq. (2.7).

Note that in this slightly different point of view, we don't try to define the PBs via partial derivatives as in Eq. (2.8). Also, note that it appears as if there is no way to recover the spin PB of Eq. (2.9) via Eq. (2.8). In this sense, this new way of defining the PB seems more general.

Exercise 1

Problem: Compute the PB

$$\{R_x, \sin P_x + P_x\}\,. \tag{2.11}$$

Solution:

$$\begin{aligned}
&\{R_x, \sin P_x + P_x\}\,, && (2.12)\\
&= \{R_x, \sin P_x\} + \{R_x, P_x\}\,, && (2.13)\\
&= \{R_x, P_x\}\frac{\partial \sin P_x}{\partial P_x} + \{R_x, P_x\}\,, && (2.14)\\
&= \cos P_x + 1. && (2.15)
\end{aligned}$$

Use has been made of the second and the fourth of Eqs. (2.10) in the above manipulations.

If we try to compute the PB between the azimuthal angle $\phi_A = \arctan(S_A^y/S_A^x)$ of the spin vector of a BH and its z-component, it can be checked (using Eqs. (2.9) and (2.10)) that it comes out to be

$$\{\phi_A, S_B^z\} = \delta_{AB}, \tag{2.16}$$

which upon comparison with Eqs. (2.9) makes us conclude that ϕ_A and S_A^z respectively "act like" position and momentum variables respectively. This is of big significance when we deal with the action-angle variables (AAVs) later. AAVs are the key to obtaining the closed-form solutions we have set out to seek. We end this section with a definition.

Commuting quantities: Two quantities commute if their PB vanishes.

Chapter 3

Integrable systems and action-angle variables

3.1 Definitions

We will focus our attention on systems which possess a time-independent Hamiltonian

Hamiltonian systems: A dynamical system possessing a Hamiltonian $H(\vec{p},\vec{q})$ and whose EOMs are given via Hamilton's equations.

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{3.1}$$

$$\tag{3.2}$$

Canonical transformation: For a Hamiltonian system with the Hamiltonian $H(\vec{p},\vec{q})$, a transformation $\vec{Q}(\vec{p},\vec{q}),\vec{P}(\vec{p},\vec{q})$ is called canonical if Hamilton's equations in the old coordinates imply Hamilton's equations in the new coordinates, i.e.

$$\dot{q}_i = \frac{\partial H}{\partial p_i},\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{3.3}$$

$$\Longrightarrow \dot{Q}_i = \frac{\partial K}{\partial P_i},\ \dot{P}_i = -\frac{\partial K}{\partial Q_i}, \tag{3.4}$$

where $K(\vec{P},\vec{Q}) = H(\vec{p}(\vec{P},\vec{Q}),\vec{q}(\vec{P},\vec{Q}))$.

Integrable system and action-angle variables: We will define integrable systems and action-angle variables (AAVs) in one shot. For integrable systems, canonical transformation $(\vec{p},\vec{q}) \leftrightarrow (\vec{\mathcal{J}},\vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ (or rather $\partial H/\partial\theta_i = 0$) and also $\{\vec{p},\vec{q}\}(\theta_i + 2\pi) = \{\vec{p},\vec{q}\}(\theta_i)$. The first equation means that the Hamiltonian depends only on the actions and not the angles. The last equation means that $\vec{p}$ and $\vec{q}$ are 2π-periodic functions of θ_i's. $\mathcal{J}_i$'s and θ_i's are respectively called the action and the angle variables.

3.2 Elementary properties of action-angle variables

The above definition of action-angle variables may seem ad-hoc but some very useful conclusions follow from this definition. First note that due to $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ being a canonical transformation, the actions $\vec{\mathcal{J}}$ act like the new momenta variable and the angles $\vec{\theta}$ act like the new position variables.

Writing Hamilton's equations in terms of these new momenta and positions (actions and angles), we get

$$\dot{\mathcal{J}}_i = -\frac{\partial H}{\partial \theta_i} = 0 \qquad\qquad \Longrightarrow \mathcal{J}_i \text{ stay constant }, \tag{3.5}$$

$$\dot{\theta}_i = \frac{\partial H}{\partial \mathcal{J}_i} \equiv \omega_i(\overrightarrow{\mathcal{J}}) \qquad\qquad \Longrightarrow \theta_i = \omega_i(\overrightarrow{\mathcal{J}})t. \tag{3.6}$$

We have chosen to call $\partial H/\partial \mathcal{J}_i$ the frequencies ω_i's since they denote the linear rate of increase of the corresponding angles θ_i's. These frequencies ω_i's are constants too since they are functions of only the constant $\mathcal{J}_i$'s.

More interesting and useful conclusions follow. It's clear that we know what actions and angles are at any later time given their values at an initial time (actions stay constants, angle increase linearly with time at a known rate). Therefore, *if we know how to switch back* from $(\vec{\mathcal{J}}, \vec{\theta})$ to $(\vec{p}, \vec{q})$, then we can have $\vec{p}(t), \vec{q}(t)$ for any later time t, that is to say, we can have the solution of the system.

We now summarize the above conclusions.

From the above definition of action-angle variables, it follows that

- Actions are constants.
- Angles increase linearly with time at constant rate $\omega_i = \partial H/\partial \mathcal{J}_i$.
- Having action-angle variables can help us have the solution $\vec{p}(t), \vec{q}(t)$ of the system.

3.3 Liouville-Arnold theorem

3.3.1 Statement of the theorem

Stated loosely, the Liouville-Arnold (LA) theorem says that if a Hamiltonian system with $2n$ phase-space variables (positions and momenta), possesses n constants of motion (including the Hamiltonian) which mutually commute among themselves, then the system is integrable and it possesses action-angle variables.

Box 3.1

For a more rigorous statement of the the theorem, along with its proof, the reader is to referred to Chapter 11 of Ref. [11].

3.3.2 Application of the Liouville-Arnold theorem to the spinning BBH system

To apply the LA theorem to the spinning BBH system, we need to determine $2n$, the total number of positions and momenta. The total number of coordinate appears to be 12; each vector $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$ contributes 3 components. Despite that, $2n \neq 12$. The reason we can't count all 12 components of vectors $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$ is because when it comes to spins $\vec{S}_1$ and $\vec{S}_2$, it is not clear which components are positions and which are momenta. Remember, $2n$ is supposed to be the total number of positions and momenta.

To delineate the positions and momenta clearly in the vectors $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$, we reproduce parts of Eqs. (2.9) and (2.16)

$$\{R_i, P_j\} = \delta_{ij} \quad \text{and} \quad \{\phi_A, S^z_B\} = \delta_{AB}, \tag{3.7}$$

which lets us see that ϕ_A (the azimuthal angle of $\vec{S}_A$) and S^z_B (the z-component of $\vec{S}_B$) act like position and momentum, respectively. It is in this sense that we need to count positions and momenta to determine "$2n$" for the application of the LA theorem. Every spin vector thus contributes 2 positions-momenta. Only 2 coordinates are needed to specify each spin since spin magnitudes are constants: $\dot{S_A} = \{S_A, H\} = 0$ (easily follows from Eqs. (3.7) and (2.10)). Therefore $2n$ for our spinning BBH is $3 + 3 + 2 + 2 = 10$, which means that we require $10/2 = 5$ mutually commuting constants of motion to establish integrability.

The PB between any two quantities among $R^i, P_i, \phi^i_A, S^z_{iA}$ that falls outside the purview of Eqs. (3.7) and (2.10) is 0.

Box 3.2

Rigorously speaking, integrability, action-angle variables and the LA theorem are built on the foundations of symplectic geometry (a branch of differential geometry). If the above process of counting the number of positions and momenta for the application of the LA theorem seems shaky to the reader due to lack of rigor, they are referred to [12, 13, 14, 15] for an introduction to symplectic manifolds and Darboux coordinates and the statement of the LA theorem against this mathematical backdrop.

For the spinning BBH, the five required commuting constants have been long-known [6]. They are

$$H,\ J^2,\ L^2,\ J_z,\ \vec{S}_{\text{eff}} \cdot \vec{L}. \tag{3.8}$$

These quantities have already been defined in Sec. 2.1. Hence the 1.5PN spinning BBH is integrable and it possesses action-angle variables.

Chapter 4

Hamiltonian flow

4.1 Basics

In this section, we will define curves and vector field following the approach adopted in Secs. 2.5, 2.6 and 2.7 of Ref. [16].

4.1.1 Curve

A curve C is a mapping from $\mathbb{R}$ to a manifold M, $C : \mathbb{R} \to M$. This is illustrated in Fig. 4.1. Note that two curves need not be the same even if they map to the same image in M. For example, the curves C_1 and C_2 are not the same in Fig. 4.1. This is so because their pre-images (elements of the domain $\mathbb{R}$ corresponding to an image in M) are different.

4.1.2 Vector field

Imagine a curve C defined by equations $\vec{x} = \vec{x}(\lambda)$ ($\vec{x}$ being the vector of coordinates on M and $\lambda \in \mathbb{R}$ being the pre-image of C). Then at a point P through which the curve passes, we can

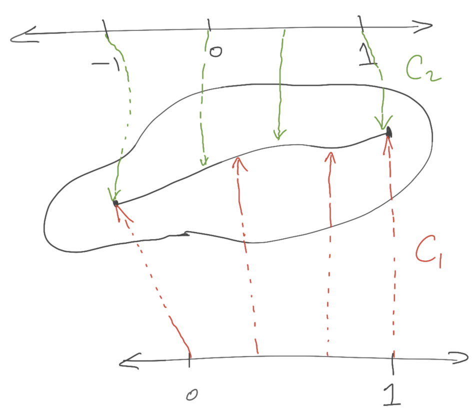

Figure 4.1: Pictorial depiction of the concept of a curve. We are showing two different curves C_1 and C_2 which have the same image in manifold M.

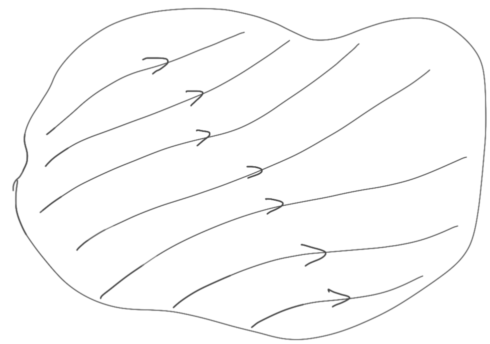

Figure 4.2: A manifold permeated with curves is a manifold with a vector field. These set of curves are also collectively refereed to as the flow of the associated vector field.

define a vector whose components are $(dx^i/d\lambda)$. This vector (tangent to curve $C = x^i(\lambda)$) is also denoted by $d/d\lambda$. After all, a vector is basically a derivative operator. This way we define vectors at every point of a curve C. If there are curves permeating the manifold M, this can help us define a unique vector field on M, provided these curves don't intersect. A pictorial representation is shown in Fig. 4.2.

4.2 Hamiltonian vector field and flow

Unless stated otherwise, $\vec{V}$ from now on represents a column vector which contains all the components of $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$. Now, at all the points of the phase-space (which is a legitimate manifold), we can define curves for a function $f(\vec{V})$ via (parameterized by λ)

$$\frac{d\vec{V}}{d\lambda} = \left\{\vec{V}, f\right\}. \tag{4.1}$$

It's implied that the above equation is to be interpreted in a component-wise manner. The resulting vector field from these curves is called the *Hamiltonian vector field* of f. Note that if $f = H$, then we will have the Hamiltonian vector field of the Hamiltonian. More generally, with this definition, the Hamiltonian vector field of position R_x, momentum P_z and the Hamiltonian H are all well defined mathematical entities. For brevity we may refer to Hamiltonian flow as only the flow.

The Mathematica notebook for evaluating the Poisson brackets is available at (...Git) and (... YouTube).

Box 4.1

Hamiltonian vector field: See Refs. [12, 13] for a more highbrow but equivalent definition of Hamiltonian vector fields using symplectic forms.

Exercise 2

Problem: Draw pictures representing the Hamiltonian flows of R_x, P_x, L_x, L^2, and J^2.

Solution:

For this exercise let $\vec{V}$ represent the totality of coordinates contained in $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$, unless stated otherwise. We will give the mathematical solution for only the first flow (in addition to the pictorial representation of the flow). Solutions for the other flows will not be given; they are quite simple to obtain. Also the associated figures representing the flows, the arrows denote the direction of increasing flow parameter, i.e. λ.

(a)

Under the flow of R_x, among all the coordinates of $\vec{V}$, only P_x changes via

$$\frac{dP_x}{d\lambda} = \{P_x, R_x\} = -1, \tag{4.2}$$

since the PB of R_x with all other variables is 0. The solution is $P_x - P_x(\lambda_0) = (\lambda_0 - \lambda)$. The corresponding flow diagram is shown in Fig. 4.3.

(b)

Under the flow of P_x, among all the coordinates of $\vec{V}$, only R_x changes via

$$\frac{dR_x}{d\lambda} = \{R_x, P_x\} = 1, \tag{4.3}$$

since the PB of R_x with all other variables is 0. The corresponding flow diagram is shown in Fig. 4.4.

(c)

The flow of L_x is encoded in

$$\frac{d\vec{R}}{d\lambda} = \hat{x} \times \vec{V}, \tag{4.4}$$

$$\frac{d\vec{P}}{d\lambda} = \hat{x} \times \vec{V}, \tag{4.5}$$

$$\frac{d\vec{S}_A}{d\lambda} = 0, \tag{4.6}$$

that is to say that $\vec{R}$ and $\vec{P}$ rotate around $\hat{x}$, whereas the spins don't move. The corresponding flow diagrams are shown in Figs. 4.5 and 4.6. Note that the latter figure is easier to understand because of the use of 3D vectors in the drawing.

(d)

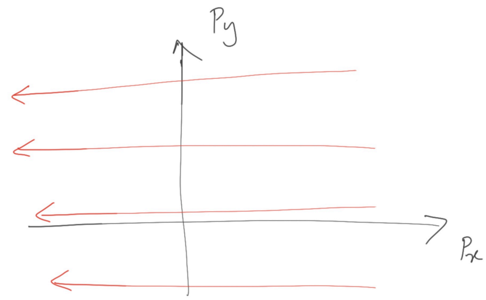

Figure 4.3: Pictorial representation of R_x flow.

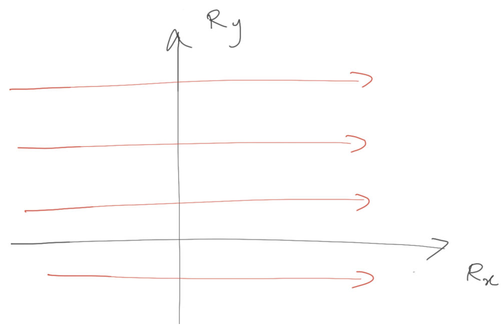

Figure 4.4: Pictorial representation of P_x flow.

The flow of L^2 is given by

$$\frac{d\vec{V}}{d\lambda} = \left\{\vec{V}, L^2\right\} = 2\vec{L} \times \vec{V} \tag{4.7}$$

$$\frac{d\vec{S}_A}{d\lambda_1} = 0, \tag{4.8}$$

where in the above two equations, $\vec{V}$ stands for only $\vec{R}$ and $\vec{P}$, and not the spin vectors. This means that $\vec{R}$ and $\vec{P}$ rotate around $\vec{L}$ (which itself remains fixed since its PB with L^2 is 0). The corresponding flow diagrams are shown in Figs. 4.7, where use of 3D vectors has been made, just like the previous case.

(e)

The flow of J^2 is given by

$$\frac{d\vec{V}}{d\lambda} = 2\vec{J} \times \vec{V} \tag{4.9}$$

that is to say that all the four 3D vectors rotate around $\vec{J}$ (which itself remains fixed since its PB with J^2 is 0). The corresponding flow diagrams are shown in Figs. 4.8.

General remark: Note that the flow diagrams show the trajectory of the flow but don't give us the information regarding the rate of flow. The rate has to be gleaned from the corresponding PB equations which encode the flow.

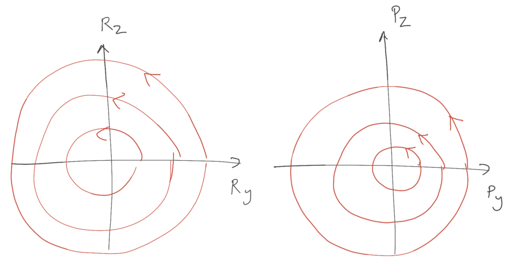

Figure 4.5: Pictorial representation of L_x flow.

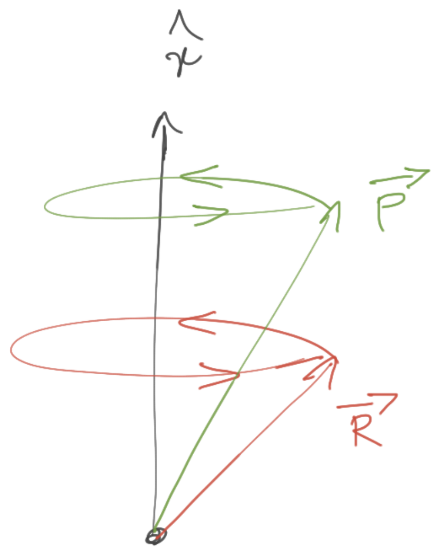

Figure 4.6: Pictorial representation of L_x flow.

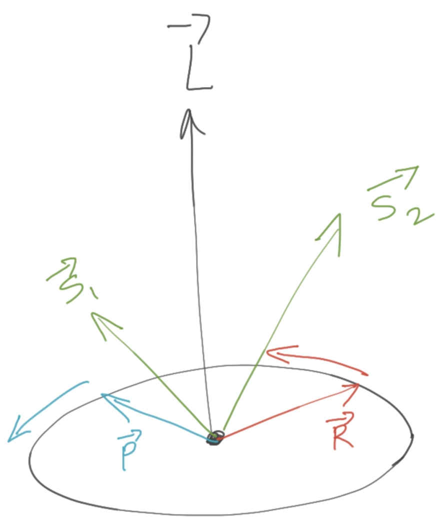

Figure 4.7: Pictorial representation of L^2 flow.

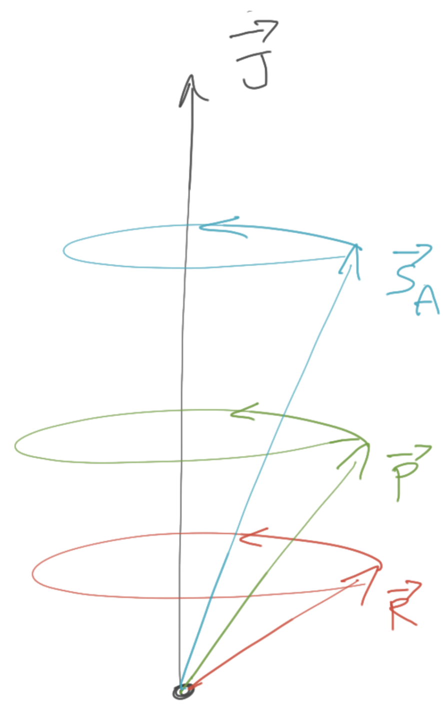

Figure 4.8: Pictorial representation of J^2 flow.

4.3 Hamiltonian flow of the Hamiltonian

Hamiltonian flow of the Hamiltonian is encoded in

$$\frac{d\vec{V}}{d\lambda} = \left\{ \vec{V}, H \right\}, \tag{4.10}$$

which upon comparison with Eq. (2.7) is found to be the EOM of the system. The flow parameter λ plays the role of time t. In this sense, the Hamiltonian flow of the Hamiltonian is indeed special for the flow of the Hamiltonian dictates the real time evolution of a system.

Chapter 5

Action-angle variables in more detail

5.1 Poisson bracket of canonical coordinates

Definition: Canonical coordinates are the ones which obey Hamilton's equations.
Theorem: Any general canonical coordinates $(\vec{r}, \vec{p})$ have the following PBs.

$$\{p_i, p_j\} = \{r_i, r_j\} = 0, \qquad \{r_i, p_j\} = \delta_{ij}. \tag{5.1}$$

Also, any set of coordinates obeying the above PBs are canonical coordinates.

Box 5.1

See Theorem 10.17 of Ref. [11] for a proof of the above statement.

5.2 Constructing action-angle variables

In this section we will try to form a strategy to construct the action-angle coordinates which satisfy the definition given in Sec. 3. The definition of action-angle variables necessitates that the action-angle variables satisfy Eqs. 5.1. We will break down our process of forming this strategy of constructing action-angle variables into a few steps.

Step 1:
Theorem: Consider the following integral

$$\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}, \tag{5.2}$$

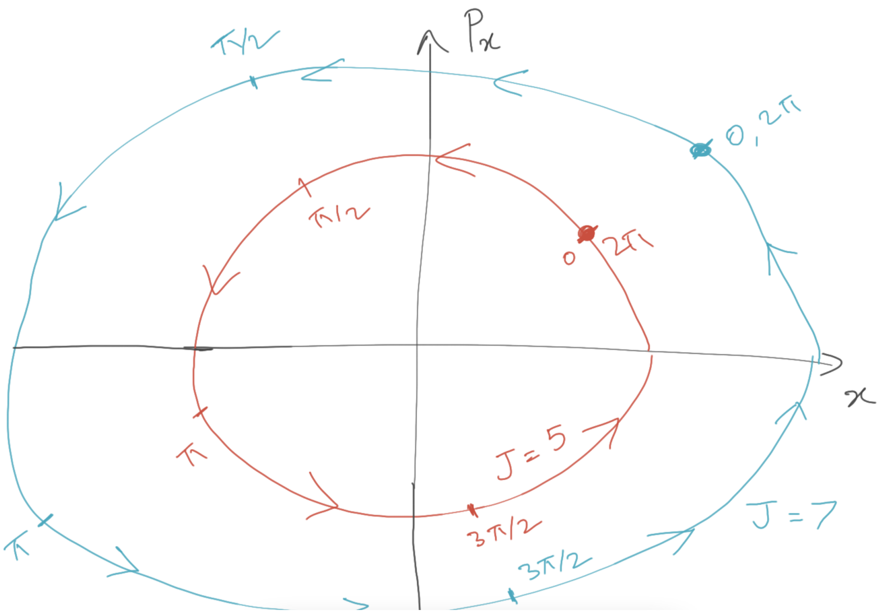

Figure 5.1: Action flow makes a loop. On this loop, stamp the angle coordinates via $d\theta_i = d\lambda$.

where the line integral is done over a loop γ_i which is on the n-dimensional sub-manifold defined by the constant values of the n commuting constants. The flow under $\mathcal{J}$ by an amount 2π forms another closed loop in the phase space on the above mentioned n-dimensional sub-manifold.

See Definition 11.6 and Theorem 11.6 of Ref. [11] for a proof of this statement. This is pictorially shown in Fig. 5.1. We will argue later that this integral is the action variable. Many textbooks take this expression of action as a definition [10].

Box 5.2

Note that there are two different loops in the picture when in our above discussions. The first is the γ_i loop (Loop 1) over which the action integral is evaluated, and the other is the one which is generated by flowing under the action (Loop 2).

In general, these two loops are not the same. Just like Loop 1, Loop 2 is also in the n-dimensional submanifold defined by the constants values of C_i's, since an action flow does not change the values of the C_i's. Also, it can be shown that the action integral evaluated on both the loops is the same.

Step 2:

Theorem: The $\mathcal{J}_i$ constructed above are mutually commuting,

$$\{J_i, J_k\} = 0. \tag{5.3}$$

Proof: Let's denote by $\vec{C}$, the vector of all n commuting constants. Now all the $\mathcal{J}_i$'s can be considered to be functions of only the C_i's, (apart from some other constants like the masses of the BHs). See the proof of Proposition 11.2 of Ref. [11] for a proof of this statement.

If that is so then it follows that

$$\{J_i, J_k\} = \sum_{l,m=1}^{n} \frac{\partial J_i}{\partial C_l} \frac{\partial J_k}{\partial C_m} \{C_l, C_m\} = 0. \tag{5.4}$$

In the above proof, we have used the chain rule for PBs (introduced in Eq. (2.10)). So, we have succeeded in ensuring the first equality of Eqs. (5.1) to hold true. (with $\vec{p} = \vec{\mathcal{J}}$). We now move on to ensure the last equality of Eqs. (5.1) holds true. We are talking about i.e. $\{r_i, p_j\} = \delta_{ij}$, or rather $\{\theta_i, \mathcal{J}_j\} = \delta_{ij}$.

Step 3: Construct the angle coordinates θ_i's the following way. Dictate that the way to increase the angle θ_i (associated to $\mathcal{J}_i$ via $\{\theta_i, \mathcal{J}_j\} = \delta_{ij}$), and keeping other (action-angle) coordinates fixed is to flow under $\mathcal{J}_i$, thus tracing a loop. Also, demand that under this flow, $d\theta_i = d\lambda$.

Now, is this construction consistent with the action-angle PB $\{\theta_i, \mathcal{J}_j\} = \delta_{ij}$? Yes. To see this, let $f = \mathcal{J}_i$ and $\vec{V}$ be θ_i in Eq. (4.1), which then becomes

$$\frac{d\theta_i}{d\lambda} = \{\theta_i, \mathcal{J}_i\} = 1. \tag{5.5}$$

Note that we equated the two quantities to 1 because of our demand $d\theta_i = d\lambda$. Under the same flow of $\mathcal{J}_i$, we also have

$$\frac{dV}{d\lambda} = 0, \tag{5.6}$$

where V stands for any of the action variables or any of the angle variables (except for θ_i). This is because of Eq. (5.3) and also the fact that we dictated that one and only one angle θ^i will change as we flow under $\mathcal{J}_i$. Note that this way of construction of angle coordinates applies only on the n-dimensional submanifold defined by the constant values of the commuting constants. We have not specified how to construct angle coordinates off this submanifold. Thus, the last equality of Eq. (5.1) is ensured by our construction of angle coordinates.

Step 4: We won't try to ensure the second equality of Eq. (5.1), i.e. $\{\theta_i, \theta_j\} = 0$, because flow under any of the θ_i's implies changing the corresponding action $(\theta_i, \mathcal{J}_j = \delta_{ij})$, and for real-time evolution (flow under the Hamiltonian), actions do not change (see Eq. (3.5)). So there is not much use in stamping the angle coordinates off the n-dimensional submanifold defined by the constant values of the actions or the commuting constants.

Step 5: One might worry that we can have infinite number of $\mathcal{J}_i$' since we can have infinite number of loops γ_i's over which the integral in Eq. (5.2) is to be performed. This is in conflict with our expectation that the number of actions and angles both has to be n, so that $n + n = 2n$ is the dimensionality of the phase-space.

Actually, this is not a cause of concern. No matter how many action integrals we compute using however many loops γ_i's, there will be only n independent action variables. The rest will be linear combinations of these n independent action variables. See Proposition 11.2 and 11.3 of Ref. [11] for a proof of this.

Step 6: Now let's try to see if the action-angle variables defined or constructed above match with the original definition given in Sec. 3.1 or not. We will again not work things out from scratch but rather outline the steps while referring to other sources.

We have already mentioned that all the $\mathcal{J}_i$'s can be considered to be functions of only the C_i's, (apart from some other constants like the masses of the BHs). All the C_i's can also be considered to be functions of the $\mathcal{J}_i$'s, i.e. $\mathcal{J}_i(\vec{C})$ is an invertible function, if the determinant of the Jacobian of transformation (via the inverse function theorem) is non-zero

$$\det\left(\frac{\partial J_i}{\partial C_j}\right) \neq 0. \tag{5.7}$$

The determinant is indeed non-zero for usual configurations. Now all this implies that the Hamiltonian (one of the C_i's) is a function of only the actions (in usual circumstances). This is one of the defining criteria of action variables (as per the definition given in Sec. 3.1).

Box 5.3

Inverse of a function

There needs to be some clarification in the context of the meaning of the $\mathcal{J}_i$'s being functions of C_j's, and vice-versa. To simplify matters, lets talk in terms of a function f of a single variable g.

Note that f being a function of g in the neighborhood of point $g = g_0$ does not imply that f has to be expressible in terms of g in closed-form using standard functions like sine, cosine, exponential or elliptic functions. f being a function of g in the neighborhood of point $g = g_0$ means is that the function $g(f)$ is an injective function in this neighborhood, so that the inverse $f(g)$ is clearly defined. It may take numerical root-finding to evaluate $f(g)$ but that's alright. See articles on implicit function theorem and inverse function theorem for more details.

For example, with the famous Kepler equation $l = u - e \sin u$, l is a function of u (clearly), but u is also a function of l for all l's since $l(u)$ is an injective function. Also, to evaluate $u(l)$, numerical root-finding is required.

Yet another example is $y = \sin x$, which does not have an inverse function in a neighborhood around $x = \pi/2$ because in this neighborhood, $y(x)$ is not injective due to the maxima at $x = \pi/2$. So, one can't define $x(y)$ in this neighborhood, although one can clearly define $x(y)$ in small neighborhood around some other point say $x = \pi/4$.

So, for a one variable function $y(x)$, the inverse function does not exist in a neighborhood containing a point x_0, where $dy/dx(x = x_0) = 0$. The generalization of this to multi-variable case is that the determinant of the corresponding Jacobian has to be non-zero for the multi-variable functions to be invertible (the inverse function theorem). We apply precisely this theorem in Eq. (5.7).

Now let's come to the other criterion of this definition of action-angle variables. The other criterion demands that $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$, i.e. $\vec{p}$ and $\vec{q}$ are 2π-periodic functions of θ_i's. This is clearly obvious from our construction of angle variables. As per this construction, changing only one of the angles is tantamount to flowing under the corresponding action and doing so by an amount 2π brings us back to where we started from, thus forming a loop. Hence, $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$ is indeed satisfied.

All in all, the take-home message is that (also shown in Fig. 5.1)

as we flow under $\mathcal{J}_i$ (one of the actions), we form a loop after flowing by $\Delta\lambda = 2\pi$. We can also stamp the angle coordinate θ^i on this loop by setting $\theta = \lambda + C_0$, where C_0 is some constant real offset.

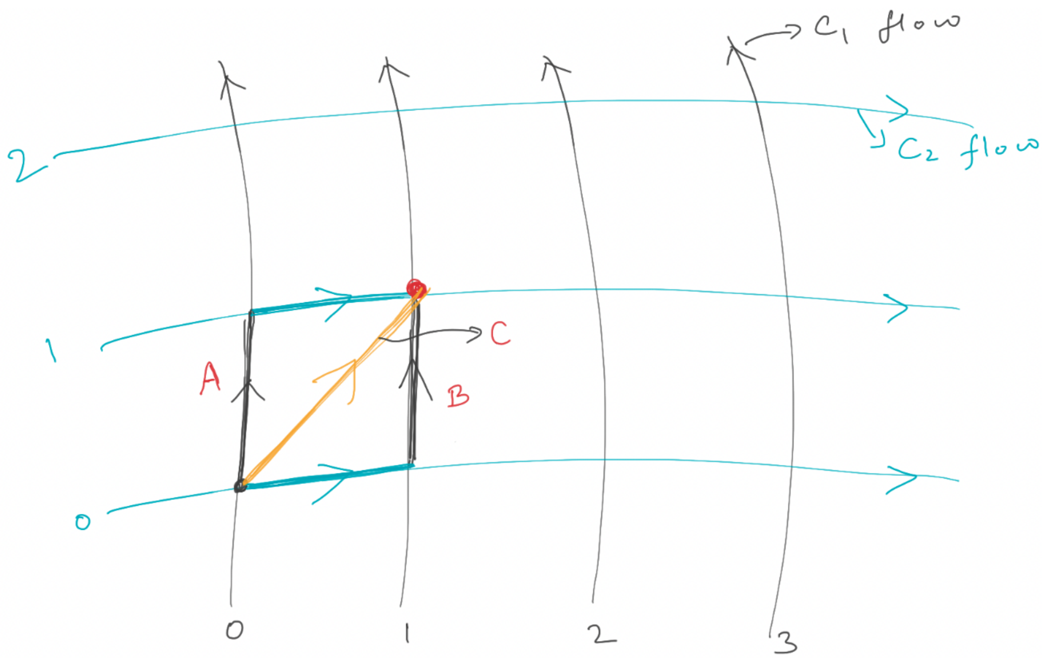

Figure 5.2: Setting up coordinates on the manifold using the flows of the commuting constants.

5.3 Action-angle variables of a simple harmonic oscillator

Without explicit calculations, we state that for a SHO, the action-angle variables are (with $\omega_0 = \sqrt{k/m}$ and $\theta_0 \in \mathbb{R}$)

$$\mathcal{J} = \frac{H}{\omega_0}, \tag{5.8}$$

$$\theta = \arctan \frac{m\omega_0 q}{p} + \theta_0. \tag{5.9}$$

The corresponding figure is again Fig. 5.1. Larger action values means larger energy H and hence a larger loop in the phase-space, as is evident from the figure.

5.4 How to flow under the actions?

5.4.1 Flowing under commuting constants

From Secs. 2.15 and 2.16 of Ref. [16], or Sec. 9.6 and Exercise 9.9 of Ref. [?], we learn that vector fields of commuting quantities can be used to set up coordinates on the manifold, the flow parameter being the coordinate. We have already seen the application of this above when we set up the angle coordinates along the flow of the corresponding actions in Sec. 5.2. Fig. 5.2 is a pictorial depiction of setting up coordinates this way.

Now we state two theorems:

- **Theorem:** Order of flow under two commuting quantities (need not be constants) does not matter.

Pictorially, flowing under two commuting quantities C_1 and C_2 in Fig. 5.2 by fixed amounts in different orders (paths A and B) starting from the same point point (black dot) leads us to the same ending point (red dot).

If we take for granted that vector fields of commuting quantities can be used to set up coordinates on the manifold (the flow parameter being the coordinate), then the proof of this theorem becomes almost self-evident from Fig. 5.2.

- **Theorem:** If $\{C_1, C_2\} = 0$, then a flow under $C_1 + C_2$ by a certain amount $\Delta\lambda$ is equivalent to a flow under one of them followed by the other, each by an amount $\Delta\lambda$.

  Here is the proof. Let $\vec{V}^i$ denote one of the components of $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$. Let λ_1 and λ_2 denote the coordinates generated C_1 and C_2 flows as shown in Fig. 5.2. So, we have $V^i = V^i(\lambda_1, \lambda_2)$ and from basic multivariable calculus

$$dV^i = \frac{\partial V^i}{\partial \lambda_1} d\lambda_1 + \frac{\partial V^i}{\partial \lambda_2} d\lambda_2, \tag{5.10}$$

$$\frac{dV^i}{d\lambda} = \frac{\partial V^i}{\partial \lambda_1} + \frac{\partial V^i}{\partial \lambda_2} \qquad \text{if } d\lambda_1 = d\lambda_2 = d\lambda, \tag{5.11}$$

$$\frac{dV^i}{d\lambda} = \left\{V^i, C_1\right\} + \left\{V^i, C_2\right\} = \left\{V^i, C_1 + C_2\right\}, \tag{5.12}$$

$$\implies \frac{\partial V^i}{\partial \lambda_1} d\lambda + \frac{\partial V^i}{\partial \lambda_2} d\lambda = \left\{V^i, C_1 + C_2\right\} d\lambda. \tag{5.13}$$

  Now the LHS of the above equation denotes the change in going from the black dot to the red dot in Fig. 5.2, which can be accomplished by flowing under C_1 and C_2 in any order (as already discussed after stating the previous theorem above). This, along with Eq. (5.13) finally proves the current theorem.

So, what we have learned is that

- The order of flows under commuting quantities does not matter.
- A simultaneous flow under commuting quantities by $\Delta\lambda$ can be broken down into the individual flows under them (by an amount $\Delta\lambda$ each).

5.4.2 Breaking down the action flow

Let us assume two things for the rest of the material in this chapter

- We have all the actions in terms of the commuting constants: $\vec{J}(\vec{C})$. We will compute $\vec{J}(\vec{C})$ in Chapters 6 and 7.
- We have closed-form solution for the flows under all the commuting constants C_i's. We will find these solutions for some of the commuting constants in Chapter 6. The solution of the flows under the Hamiltonian and $\vec{S}_{\text{eff}} \cdot \vec{L}$ is given in Refs. [3] and [1], respectively.

With the above two assumptions, how can we flow under an action? The answer is the chain-rule property of the PBs (Eq. (2.10)). So, we have

$$\frac{d\vec{V}}{d\lambda} = \left\{\vec{V}, \mathcal{J}_i\right\} = \left\{\vec{V}, C_j\right\} \left(\frac{\partial \mathcal{J}_i}{\partial C_j}\right). \tag{5.14}$$

If we have $\mathcal{J}(\vec{C})$, then the partial derivatives can be evaluated and they are simply constants because they will be functions of the C_i's. Apart from that, if we have the solution for flow under all the commuting constants, then the solution of flow under any of the actions can be easily had. Note that Eq. (5.14) is the equation for simultaneous flow under multiple commuting constants. But from Sec. 5.4.1, we know that we can flow under all these commuting constants one-by-one (order doesn't matter) and get the solution for the flow under $\mathcal{J}_i$ by any finite amount, provided the we have $\mathcal{J}(\vec{C})$ and flow solution under all the C_i's.

5.4.3 Computing frequencies

How do we compute the frequencies $\omega_i = \partial H/\partial \mathcal{J}_i$ given $\mathcal{J}_i(\vec{C})$, the Hamiltonian H being one of the C_i's? It's trivial to compute the Jacobian $\partial \mathcal{J}_i/\partial C_j$ using a computer algebra system as Mathematica.

Now, the frequencies $\omega_i = \partial H/\partial \mathcal{J}_i$ are elements of the the Jacobian of the inverse transformation, i.e. $\partial C_i/\partial \mathcal{J}_j$. From the inverse function theorem, we know that the Jacobian of the inverse transformation is the matrix inverse of the Jacobian of the original transformation.

5.5 Constructing the action-angle based solution of the spinning BBH system

Assuming that we have $\vec{\mathcal{J}}(\vec{C})$, and the solution for flow under all the C_i's, we now explicate the operational way to construct the action-angle based solution of the system. $\vec{V}$ represents the totality of the variables contained in the vectors $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$, and $\vec{V}_0$ denotes its initial value at time $t = 0$. We take $\vec{V}_0$ to correspond to all the angles being 0.

Suppose the solution at required at a later time t. So, we know that all the angles have changed to $\Delta\theta_i = \omega_i t$. We can easily have the numerical value of $\Delta\theta_i$ because we can compute all the ω_i's (see Sec. **??**). Remember from Step 3 of Sec. 5.2 that $\Delta\theta_i$, the change in the angles can be achieved by flowing under the corresponding actions by an equal amount $\Delta\lambda_i = \Delta\theta_i$. The order of action flows does not matter since $\{\mathcal{J}_i, \mathcal{J}_j\} = 0$. Now the problem is reduced to flowing under all the $\mathcal{J}_i$'s by specified amounts, which we have already figured out how to do in Sec. 5.4.2.

5.6 Afterthoughts and the plan ahead

Now, the only missing ingredients towards constructing closed-form solutions to the system are the $\vec{\mathcal{J}}(\vec{C})$ expressions and the solutions of the flows under all the C_i's. We won't discuss the solutions of the flows under all the C_i's, except for referring the reader to the relevant sources.

- The solution of the flow under J^2, L^2, and J_z is very simple; all the four 3D vectors $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$ keep their magnitudes fixed and rotate around some fixed vector at a constant rate, with the exception that spins don't move at all under the L^2 flow. We will discuss this to some degree in Chapter 6.

- The solution of the flow under the Hamiltonian is worked out in Ref. [3]. This paper ignores the 1PN Hamiltonian terms for succinctness. The authors deem the 1PN terms easy to deal with.

- The solution of flow under $\vec{S}_{\text{eff}} \cdot \vec{L}$ is worked out in Ref. [1].

So, in the remainder of these lecture notes, our focus will be on the only remaining task which is to obtain $\vec{\mathcal{J}}(\vec{C})$.

Chapter 6

Computation of the first four actions

Recall from Sec. 5.2 that action variables are given by

$$\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}, \tag{6.1}$$

where the integral is performed over a loop on the n-dimensional submanifold defined by constant values of the n commuting constants. One way to remain on this submanifold is to flow under any of the n commuting constants, for if you flow under any of the commuting constants, the other commuting constants don't change. This is because under the flow of C_j, C_i changes as (using Eq. (4.1))

$$\frac{dC_i}{d\lambda} = \{C_i, C_j\} = 0. \tag{6.2}$$

Owing to this observation, we will try to form loops for action integration while flowing under the commuting constants in the following sections.

6.1 Computation of $\mathcal{J}_1$

Let's flow under one of the commuting constants J^2, square of the magnitude of the total angular momentum $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$. Its flow equation is (with $\vec{V}$ standing for the column vector containing all the variables in $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$)

$$\frac{d\vec{V}}{d\lambda} = 2\vec{J} \times \vec{V}, \tag{6.3}$$

which implies that all the four 3D vectors rotate around the fixed $2\vec{J}$ vector (note that $\left\{\vec{J}, J^2\right\} = 0$). Using $\vec{n}$ to denote the vector around which all the four 3D vectors rotate, we see that $\vec{n} = 2\vec{J}$. In fact we had already worked this out and the pictorial representation has already been presented in the form of Fig. 4.8.

Now, Eq. (6.3) implies that we will arrive at where we started from (thus closing a loop) after we have flowed by an amount $\Delta\lambda = 2\pi/|\text{angular velocity}| = 2\pi/(2J) = \pi/J$.

Now, we break down the action integral as

$$\begin{aligned} \mathcal{J} &= \mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}} \\ \mathcal{J}^{\text{orb}} &\equiv \frac{1}{2\pi}\oint_{\mathcal{C}} \sum_i P_i dR^i \\ \mathcal{J}_A^{\text{spin}} &= \frac{1}{2\pi}\oint S_A^z d\phi_A. \end{aligned} \tag{6.4}$$

Let's tackle the orbital sector first. The orbital contribution to the action integral becomes

$$\mathcal{J}^{\text{orb}} = \frac{1}{2\pi}\int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi}\int_0^{\Delta\lambda} \vec{P}\cdot(\vec{n}\times\vec{R})d\lambda \tag{6.5}$$

$$= \frac{1}{2\pi}\int_0^{\Delta\lambda} \vec{n}\cdot\vec{L}d\lambda = \hat{n}\cdot\vec{L}. \tag{6.6}$$

The spin sector integral is

$$\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi}\oint S_A^z d\phi_A, \tag{6.7}$$

which does not appear to be SO(3) covariant, but it actually is. This means we can that this integral is insensitive to the rigid rotations of our coordinate axes.

Box 6.1

Using the language of symplectic forms and differential geometric version of the generalized Stokes' theorem, we can see that $\oint S_z d\phi = \int dS_z \wedge d\phi$, the integral on the LHS is a line integral, whereas that on the RHS is an area integral on the spin sphere. Area integrals are indeed SO(3) covariant.

So, we rotate our axes so that the z-axis points along $\vec{n}$. The spin sector integral is

$$\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi}\oint S_A^z d\phi_A = S_A^z = \hat{n}\cdot\vec{S}_A. \tag{6.8}$$

We have used the fact that S_A^z is constant on the loop of integration; this is so because $\vec{S}_A$ makes a constant angle with the z-axis (or $\vec{n}$ vector) while we perform the line integral.

Finally, combining Eqs. (6.6) and (6.8), our action integral becomes

$$\mathcal{J} = \hat{n}\cdot\left(\vec{L}+\vec{S}_1+\vec{S}_2\right) = \hat{n}\cdot\vec{J} = J. \tag{6.9}$$

6.2 Computation of $\mathcal{J}_2$ and $\mathcal{J}_3$

The procedure for computing the next two actions is very similar. Instead of flowing under J^2, we flow under L^2 and J_z, with the corresponding $\vec{n}$ being $2\vec{L}$ and $\hat{z}$, with the exception that under the L^2 flow, the spin vectors don't move; only orbital ones do. The amount of flow required to close the loop is still given by $\Delta\lambda = 2\pi/n$.

Doing similar calculations as above, we find that the corresponding actions turn out to be $\hat{n} \cdot \vec{J}$ which gives us

$$\mathcal{J}_2 = L, \quad \mathcal{J}_3 = J_z. \tag{6.10}$$

All in all we finally have

$$\mathcal{J}_2 = J, \quad \mathcal{J}_2 = L, \quad \mathcal{J}_3 = J_z. \tag{6.11}$$

From the expressions of the above three actions, we note two features that actions appear to possess

- Action variables are functions of the commuting constants: $\mathcal{J}(\vec{C})$. In fact, all actions are constants and are also mutually commuting.
- An action is a function of only those C_i's under which we need to flow to close the loop, the integral over which furnishes the action.

6.3 Computation of $\mathcal{J}_4$

We won't derive $\mathcal{J}_4$ because, this action also has a Newtonian limit which is derived in graduate level texts; see Eq. (10.139) of Ref. [10]. It's 1PN extension was worked out in Ref. [4], see Eq. (3.10) therein. The 1.5PN version of this action is given in Eq. (38) of Ref. [2]. The methods to achieve these PN versions of the fourth action is similar and chiefly involves complex contour integration technique invented by Arnold Sommerfeld.

For the reference of the reader, the fourth action is

$$\mathcal{J}_4 = -L + \frac{GM\mu^{3/2}}{\sqrt{-2H}} + \frac{GM}{c^2}\left[\frac{3GM\mu^2}{L} + \frac{\sqrt{-H}\mu^{1/2}(\nu - 15)}{\sqrt{32}} - \frac{2G\mu^3}{L^3}\vec{S}_{\text{eff}} \cdot \vec{L}\right] + \mathcal{O}\left(c^{-4}\right) \tag{6.12}$$

Chapter 7

Computation of the fifth action

7.1 Problems in trying to compute the spin sector action integral while flowing under $\vec{S}_{\text{eff}} \cdot \vec{L}$

Under the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow, we have the following EOMs.

$$\begin{aligned} \frac{d\vec{R}}{d\lambda} &= \vec{S}_{\text{eff}} \times \vec{R}, \\ \frac{d\vec{P}}{d\lambda} &= \vec{S}_{\text{eff}} \times \vec{P}, \\ \frac{d\vec{S}_a}{d\lambda} &= \sigma_a \left(\vec{L} \times \vec{S}_a\right), \end{aligned} \tag{7.1}$$

which further imply that

$$\frac{d\vec{L}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{L}. \tag{7.2}$$

We now try to compute the contribution to the action integral. We get

$$\begin{aligned} 2\pi\mathcal{J} &= 2\pi \left(\mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}\right) \\ &= \int_{\lambda_i}^{\lambda_f} \left(P_i dR^i + S_1^z d\phi_1^z + S_2^z d\phi_2^z\right) \\ &= \int_{\lambda_i}^{\lambda_f} \left(P_i \frac{dR^i}{d\lambda} + S_1^z \frac{d\phi_1^z}{d\lambda} + S_2^z \frac{d\phi_2^z}{d\lambda}\right) d\lambda \end{aligned} \tag{7.3}$$

Let's focus only on the orbital part.

$$2\pi\mathcal{J}^{\text{orb}} = \int_{\lambda_i}^{\lambda_f} \vec{P} \cdot \left(\vec{S}_{\text{eff}} \times \vec{R}\right) d\lambda = \int_{\lambda_i}^{\lambda_f} \left(S_{\text{eff}} \cdot L\right) d\lambda = \left(S_{\text{eff}} \cdot L\right) \Delta\lambda \tag{7.4}$$

where we have used $S_{\text{eff}} \cdot L$ to represent $\vec{S}_{\text{eff}} \cdot \vec{L}$. It could be pulled out of the integral because it is a constant during the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow, i.e. $d\vec{S}_{\text{eff}} \cdot \vec{L}/d\lambda = \left\{\vec{S}_{\text{eff}} \cdot \vec{L}, \vec{S}_{\text{eff}} \cdot \vec{L}\right\} = 0$.

So, the orbital sector of the integral was simple. But we have no idea how to do the spin sector integral. Note that the simplicity of the orbital sector integral is owed to $\vec{L}$ being written as a cross product $\vec{L} = \vec{R} \times \vec{P}$. The same is not true for the spins, which makes it impossible to do the spin sector integral in Eq. (7.3).

7.2 Computing the spin sector contribution to the action integral using the extended phase space

7.2.1 Introducing the fictitious variables

The main reason the spin sector contribution to the action integral while flowing under $\vec{S}_{\text{eff}} \cdot \vec{L}$ is that spins can't be written a cross product of some positions and some momenta, the way $\vec{L}$ can be. This motivates us to introduce unmeasurable, fictitious variables $\vec{R}_a$ and $\vec{P}_a$ with $a = 1, 2$ such that

$$\vec{S}_a \equiv \vec{R}_a \times \vec{P}_a. \tag{7.5}$$

With these fictitious variables, the spins are no longer the independent, fundamental coordinates but now they rather depend on the fictitious variables. We now introduce some simple terminology. The totality of $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$ forms the standard phase space (SPS). The totality of $\vec{R}, \vec{P}, \vec{R}_{1/2}$ and $\vec{P}_{1/2}$ forms the extended phase space (EPS). The vectors $\vec{R}_{1/2}$ and $\vec{P}_{1/2}$ form the sub-spin space.

Many things need to be put on a firm footing with the introduction of these new fictitious variables. We do so one by one.

- **Hamiltonian:** The Hamiltonian will now be seen as a function of the EPS coordinates, rather than the SPS ones.

- **Poisson brackets:** PBs need to be defined, for the EOMs are written in their terms. We propose

$$\{R_i, P_j\} = \delta_{ij}, \quad \{R_{ai}, P_{bj}\} = \delta_{ab}\delta_{ji} \tag{7.6}$$

- **EOM:** The EOMs are still given by the same familiar equation

$$\frac{df}{dt} = \{f, H\} \tag{7.7}$$

- **Equivalency of the SPS and the EPS pictures in terms of EOMs:** The only reason we can introduce the EPS picture is that the EPS picture is completely equivalent to the SPS one, i.e. the EOMs for all real (non-fictitious) variables are the same in the two pictures.

  Why do we say that? This is because the EPS PBs imply the SPS PBs, i.e. Eqs.(7.6) $\Longrightarrow$ Eqs. 3.7.

- **Equivalency of the SPS and the EPS pictures in terms of integrability:** The system is integrable even in the EPS picture. To show that we need to come up with $n = 2n/2 = 18/2 = 9$ commuting constants in the EPS picture. Five of them are the SPS commuting constants (see Eq. (3.8)) but considered functions of the EPS coordinates and not the SPS coordinates. The next four are $S^2_{1/2}$ and $\vec{R}_{1/2} \cdot \vec{P}_{1/2}$.

- **Sanity check 1:** All observables should depend only on non-fictitious variables alone. This is indeed the case. All five actions we derive depend only on $\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$ and the two masses; any dependence on the fictitious variables is through $\vec{S}_1$ and $\vec{S}_2$. Although actions are not observables, frequencies $\omega_i = \partial H/\partial \mathcal{J}_i$ are; and they are a functions of non-fictitious variables only.

- **Sanity check 2:** We have checked numerically that the fifth action (computed with the help of fictitious variables), when seen as a function of the SPS coordinates only, generates a flow which forms a closed loop after flowing by 2π within numerical errors.

- **An EPS action is also an SPS action:** If we succeed in computing an EPS action which is a function of ($\vec{R}, \vec{P}, \vec{S}_1$ and $\vec{S}_2$), then this should also serve as an SPS action. How? Our EPS action flow must make a loop in the EPS. The same action when seen as a function of the SPS coordinates must also make a loop in the SPS. This is because the PBs are the same in the SPS and the EPS and flow equations are written using PBs. Since the flow of the EPS action (when seen as a function of SPS coordinates) makes a closed loop in the SPS, it can also be considered as a legitimate action in the SPS.

Box 7.1

There is small hole in the argument presented above regarding the EPS action being also the SPS action. What if the EPS action loop (obtained by flowing under the fifth action by 2π) corresponds to a loop in the SPS which goes around more than once. We require actions to make a closed loop *just once* when we flow under them and the above possibility is undesirable. But this is not a cause for worry because in Ref. [1], we invoke some topology arguments to rule this out.

The method of inventing temporary spurious variables is not that uncommon. We use complex contour integration methods to compute integrals which are cast in terms of reals and whose result is also real. See Sec. 11.8 of Ref. [17]. In this case, the complex numbers can be thought of as temporary spurious variables invented to solve some problem which was cast in terms of fewer variables (reals).

7.2.2 Computing the spin sector of the action integral under the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow

In the EPS picture, under the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow the EOMs are

$$\begin{aligned}
\frac{d\vec{R}}{d\lambda} &= \vec{S}_{\text{eff}} \times \vec{R}, \\
\frac{d\vec{P}}{d\lambda} &= \vec{S}_{\text{eff}} \times \vec{P}, \\
\frac{d\vec{R}_a}{d\lambda} &= \sigma_a \left(\vec{L} \times \vec{R}_a\right), \\
\frac{d\vec{P}_a}{d\lambda} &= \sigma_a \left(\vec{L} \times \vec{P}_a\right).
\end{aligned} \tag{7.8}$$

The contribution to the action integral becomes

$$\mathcal{J}_k = \frac{1}{2\pi} \oint_{\mathcal{C}_k} \left(\vec{P} \cdot d\vec{R} + \vec{P}_1 \cdot d\vec{R}_1 + \vec{P}_2 \cdot d\vec{R}_2\right), \tag{7.9}$$

which further entails that

$$2\pi \mathcal{J}_{S_{\text{eff}} \cdot L} = 2\pi \left(\mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}\right) \tag{7.10}$$

$$= \int_{\lambda_i}^{\lambda_f} \left(P_i \frac{dR^i}{d\lambda} + P_{1i} \frac{dR_1^i}{d\lambda} + P_{2i} \frac{dR_2^i}{d\lambda}\right) d\lambda \tag{7.11}$$

$$= \int_{\lambda_i}^{\lambda_f} \left(\vec{P} \cdot \left(\vec{S}_{\text{eff}} \times \vec{R}\right) + \vec{P}_1 \cdot \left(\sigma_1 \vec{L} \times \vec{R}_1\right)\right. \tag{7.12}$$

$$\left. + \vec{P}_2 \cdot \left(\sigma_2 \vec{L} \times \vec{R}_2\right)\right) d\lambda \tag{7.13}$$

$$= 2 \int_{\lambda_i}^{\lambda_f} \left(S_{\text{eff}} \cdot L\right) d\lambda = 2 \left(S_{\text{eff}} \cdot L\right) \Delta\lambda_{S_{\text{eff}} \cdot L} \tag{7.14}$$

$$\mathcal{J}_{S_{\text{eff}} \cdot L} = \frac{\left(S_{\text{eff}} \cdot L\right) \Delta\lambda_{S_{\text{eff}} \cdot L}}{\pi}, \tag{7.15}$$

which is just twice the contribution of the orbital part given in Eq. (7.4).

Basically, with the introduction of the fictitious variables, and expanding the SPS to EPS, we bring the spins at the same mathematical footing as $\vec{L}$, in that all three of them can be written as cross products of some position with some momentum. This renders the otherwise insoluble spin sector contribution to the action integral under the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow rather trivial to deal with (the way orbital sector contribution is), as if it was meant to be.

7.3 Computing the fifth action

7.3.1 Setting up the stage

We will only outline the steps needed to compute the fifth action. For details, refer to Ref. [1]. To compute the fifth action, a flow only under $\vec{S}_{\text{eff}} \cdot \vec{L}$ is not enough; it won't make a closed loop. In the SPS space, to close the loop, we further need to flow under J^2 and L^2, whereas in the EPS space, we need two more flows (S_1^2 and S_2^2 flow) on the top of that because we have extra variables in the EPS. Because we know how to compute the action integral contribution corresponding to $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow only in the EPS, we will try to close the loop in the EPS. The successive flows we need are those of $\vec{S}_{\text{eff}} \cdot \vec{L}$, J^2, L^2, S_1^2 and S_2^2. At this point, we mention the result that under the flow by the last four constants of motion, the contribution to the action integral is

$$\begin{aligned}
\mathcal{J}_{J^2} &= \frac{J^2 \Delta\lambda_{J^2}}{\pi}, \\
\mathcal{J}_{L^2} &= \frac{L^2 \Delta\lambda_{L^2}}{\pi}, \\
\mathcal{J}_{S_1^2} &= \frac{S_1^2 \Delta\lambda_{S_1^2}}{\pi}, \\
\mathcal{J}_{S_2^2} &= \frac{S_2^2 \Delta\lambda_{S_2^2}}{\pi}.
\end{aligned} \tag{7.16}$$

These are easy to derive and can be derived in an analogous manner as contribution corresponding to the first constant $\vec{S}_{\text{eff}} \cdot \vec{L}$ was derived in Eq. (7.15). Therefore the fifth action becomes

$$\mathcal{J}_5 = \frac{1}{\pi} \left\{ (S_{\text{eff}} \cdot L)\, \Delta\lambda_{S_{\text{eff}} \cdot L} + J^2 \Delta\lambda_{J^2} + L^2 \Delta\lambda_{L^2} + S_1^2 \Delta\lambda_{S_1^2} + S_2^2 \Delta\lambda_{S_2^2} \right\}. \tag{7.17}$$

The problem of determining the fifth action thus reduces to determining the flow amounts needed (under various commuting constants) to close the loop.

7.3.2 Determining the flow amounts

$\vec{S}_{\text{eff}} \cdot \vec{L}$ flow

Under the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow the magnitudes of all the three 3D position and momentum vectors stay constant. So does the magnitude of all the three angular momenta. The effect of $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow on the three angular momenta is shown in Fig. 7.1. The triad formed by the three angular momenta acts like a lung and it "inhales" and "exhales". All three mutual angles between the angular momenta are periodic functions of the flow parameter with the same period $\Delta\lambda_{\vec{S}_{\text{eff}} \cdot \vec{L}}$. See Ref. [1] for the derivation of these results. So, naturally we want to flow under $\vec{S}_{\text{eff}} \cdot \vec{L}$ by exactly by a multiple of this period because if we did not, then there is very little hope of restoring all the coordinates to their initial state by flowing under other constants. This is so because other constants do not change these mutual angles between the three angular momenta (except for the Hamiltonian). So, we decide to flow under $\vec{S}_{\text{eff}} \cdot \vec{L}$ by an amount $\Delta\lambda_{\vec{S}_{\text{eff}} \cdot \vec{L}}$.

J^2 flow

It's shown in Ref. [1] that by an appropriate amount $\Delta\lambda_{J^2}$ of flow under J^2, we can restore all the three angular momenta to their initial states.

L^2, S_1^2, S_2^2 flow

Because all three angular momenta have been restored, the only way $\vec{R}, \vec{P}, \vec{R}_{1/2}$ and $\vec{P}_{1/2}$ off from their initial state is by a some finite rotation in the plane perpendicular to $\vec{L}$ (for $\vec{R}, \vec{P}$) and $\vec{S}_{1/2}$ (for $\vec{R}_{1/2}, \vec{P}_{1/2}$). To negate this offset, all we need to do is to flow under L^2, S_1^2 and S_2^2 by some appropriate amounts. This is also worked out in Ref. [1]. Once these amounts are determined, Eq. 7.17 finally yields the fifth action variable.

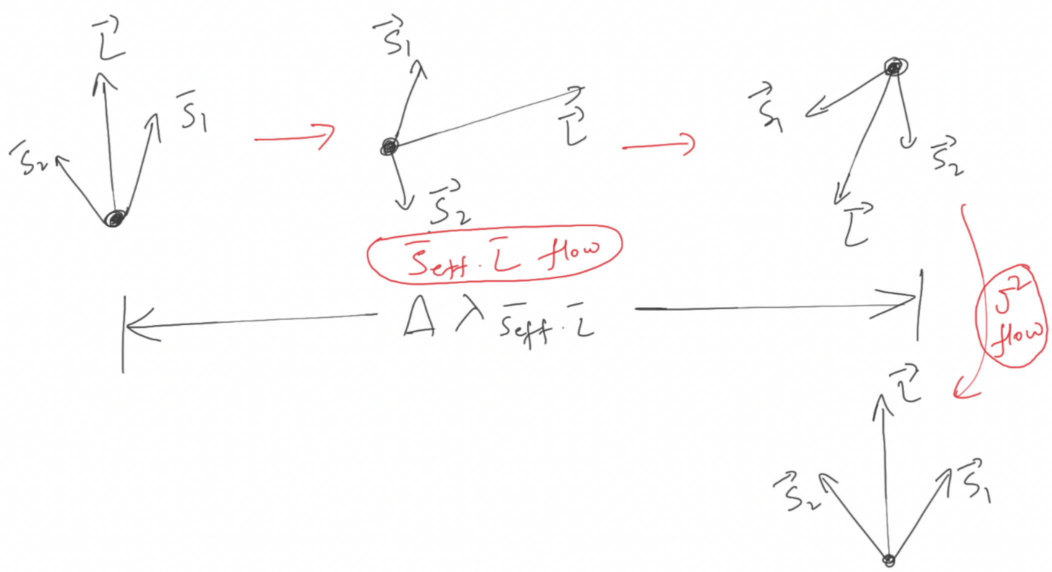

Figure 7.1: Behavior of the angular momenta under the $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow.

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