

# Closed-form solutions of spinning BBHs at 1.5PN (using action-angle variables)

Lecture Workshop (Univ. of Illinois Urbana-Champaign)

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# References

- **RESEARCH PAPERS**

- The standard way of computing the solution (without 1PN part): <https://arxiv.org/abs/1908.02927>
- Action-angle-based solution: <https://arxiv.org/abs/2012.06586>, <https://arxiv.org/abs/2110.15351>

- **LECTURE NOTES**

- Lecture notes (latest): [https://github.com/sashwattanay/lectures\\_integrability\\_action-angles\\_PN\\_BBH/blob/gh-action-result/pdflatex/lecture\\_notes/main.pdf](https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/gh-action-result/pdflatex/lecture_notes/main.pdf)
- Lecture notes (for citation purposes): <https://arxiv.org/abs/2206.05799>

- **MATHEMATICA PACKAGE**

- <https://github.com/sashwattanay/BBH-PN-Toolkit>

- **YOUTUBE VIDEO**

- <https://youtu.be/aoiCk5TtmvE>

- **THIS PRESENTATION**

- [https://github.com/sashwattanay/lectures\\_integrability\\_action-angles\\_PN\\_BBH/blob/main/UIUC\\_workshop\\_presentation/uiuc\\_workshop\\_presentation.pdf](https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/main/UIUC_workshop_presentation/uiuc_workshop_presentation.pdf)

# Lecture plan

**Lecture style:** standing on the shoulders of giants (due to time constraints)

- **Lecture 1:**
  - Theory
  - Strategy to compute solution from action-angles
- **Lecture 2:**
  - Construct the solution

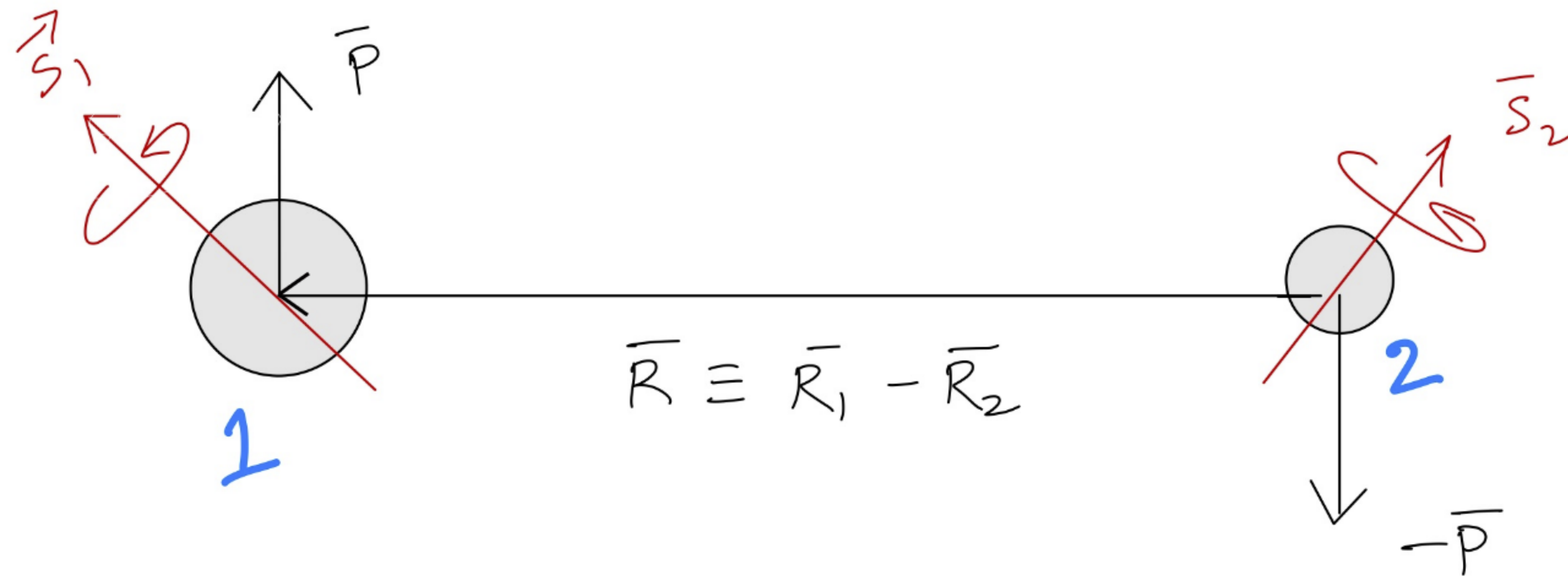
# Lecture plan

- **Lecture 1:**
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# Introduction to the system

## Spinning 1.5PN BBH system

COM FRAME



### Phase space variables

$\vec{R}(t)$ ,  $\vec{P}(t)$ ,  $\vec{S}_1(t)$  and  $\vec{S}_2(t)$

$\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$

# Statement of the problem

- The 1.5PN Hamiltonian is  $H = H_{\text{N}} + H_{1\text{PN}} + H_{1.5\text{PN}} + \mathcal{O}(c^{-4})$  with
- $H_{\text{N}} = \mu \left( \frac{p^2}{2} - \frac{1}{r} \right), \quad H_{1.5\text{PN}} = \frac{2G}{c^2 R^3} \vec{S}_{\text{eff}} \cdot \vec{L}.$
- Hamilton's equations  $\implies \frac{d \left( \vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t) \right)}{dt}.$
- **Problem:** Integrate Hamilton's eqns. to obtain  $\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t).$

# Historical context and the status quo

- The 1.5PN Hamiltonian is  $H = H_N + H_{1\text{PN}} + H_{1.5\text{PN}} + \mathcal{O}(c^{-4})$ .
- **1680s**: Issac Newton gave the Newtonian solution  $R = a(1 - e \cos u)$ .
- **1985**: Damour-Deruelle gave 1PN quasi-Keplerian solution.
- **2019**: Gihyuk Cho, H. M. Lee gave 1.5PN solution (1PN effects ignored for simplicity)
- **2020 & 2021**: We worked out an equivalent action-angle based solution (subject of these lectures).
- **Why action-angles?** Extendible to 2PN via canonical perturbation theory (Goldstein).
- **Do we even have the solutions?** See the plot of analytical and numerical solutions (via a Mathematica package) in the **YouTube video** @10:33 (in References)

# EOMs with Poisson brackets

## Standard approach

- Hamilton's eqns. are  $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$
- Leads to EOM  $\frac{df}{dt} = \{f, H\}$  with  $\{f, g\} \equiv \sum_{i=1}^N \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$



# EOMs with Poisson brackets for BBHs

## Our approach

- Define EOMs:  $\frac{df(t)}{dt} = \{f, H\}$  where  $f = f\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)$ .
- Define PBs:  $\left\{R_i, P_j\right\} = \delta_{ji}$   $\left\{S_A^i, S_B^j\right\} = \delta_{AB} \epsilon_k^{ij} S_A^k$ .

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, \quad a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\left\{f, g(v_i)\right\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

# PB Exercise 1

**Prob:** Compute  $\{R_x, \sin P_x + P_x\}$ .

**Sol:** Using the bilinearity and the chain rule (2<sup>nd</sup> and 4<sup>th</sup> rules) for PBs

$$\begin{aligned} & \{R_x, \sin P_x + P_x\} \\ &= \{R_x, \sin P_x\} + \{R_x, P_x\} \\ &= \{R_x, P_x\} \frac{\partial \sin P_x}{\partial P_x} + \{R_x, P_x\} \\ &= \cos P_x + 1. \end{aligned}$$

# PB Exercise 2

**Prob:** Show that  $\{\phi_A, S_B^z\} = \delta_{AB}$ , where  $\phi_A = \arctan(S_A^y/S_A^x)$  is the azimuthal angle of  $\vec{S}_A$ .

- Implies that  $\phi \sim \text{position}$ ;  $S^z \sim \text{momentum}$  upon comparison with  $\{R_i, P_j\} = \delta_{ji}$ .
- **Lingo:**  $f$  and  $g$  commute if  $\{f, g\} = 0$ .
- **How to evaluate general PBs quickly?:** Use the Mathematica notebook. See the [YouTube video](#) @14:22 (in References)

# Integrable systems and action-angles

- **Integrable system:** canonical transformation  $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$  exists such that  $H = H(\vec{\mathcal{J}})$  and  $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$ .

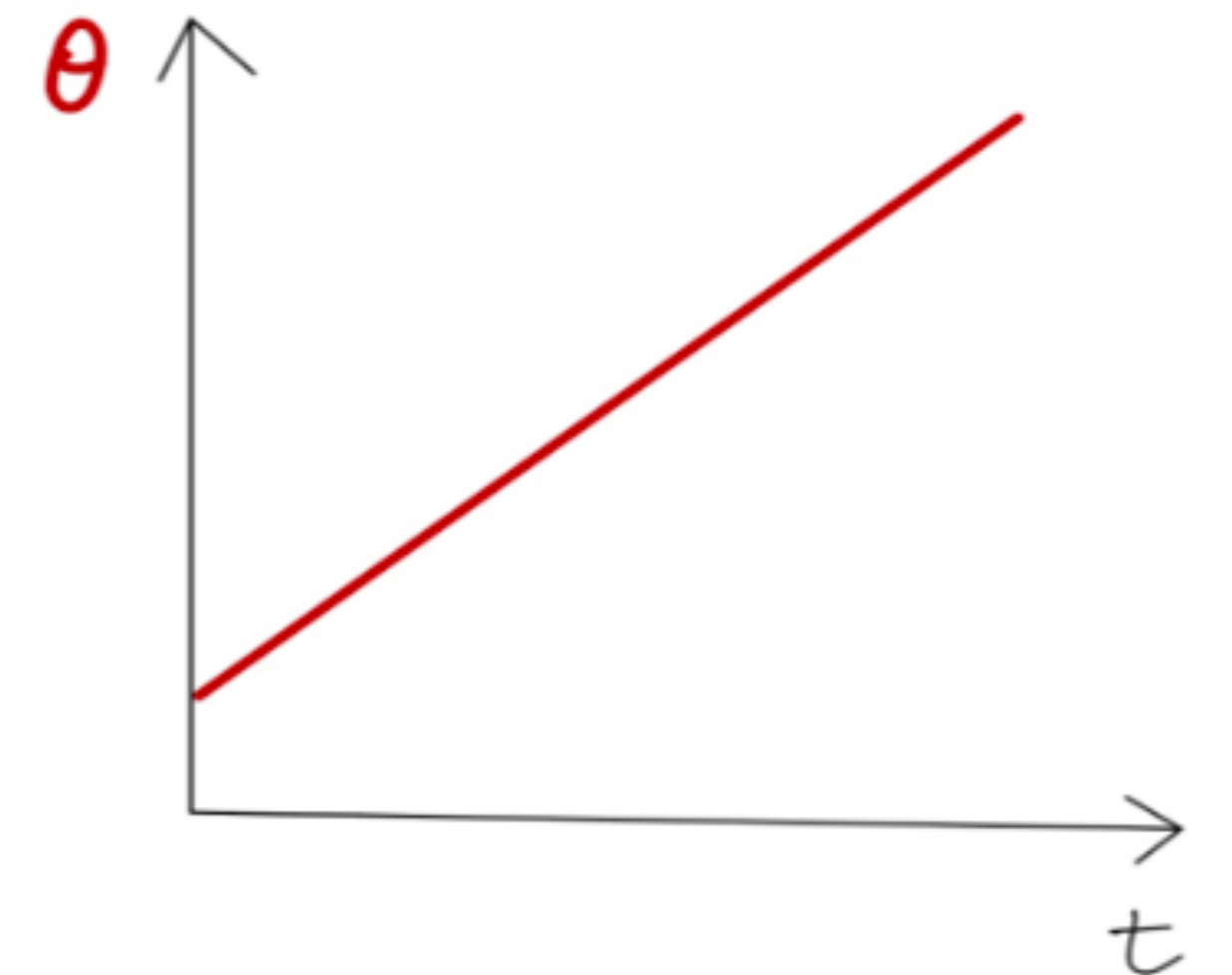
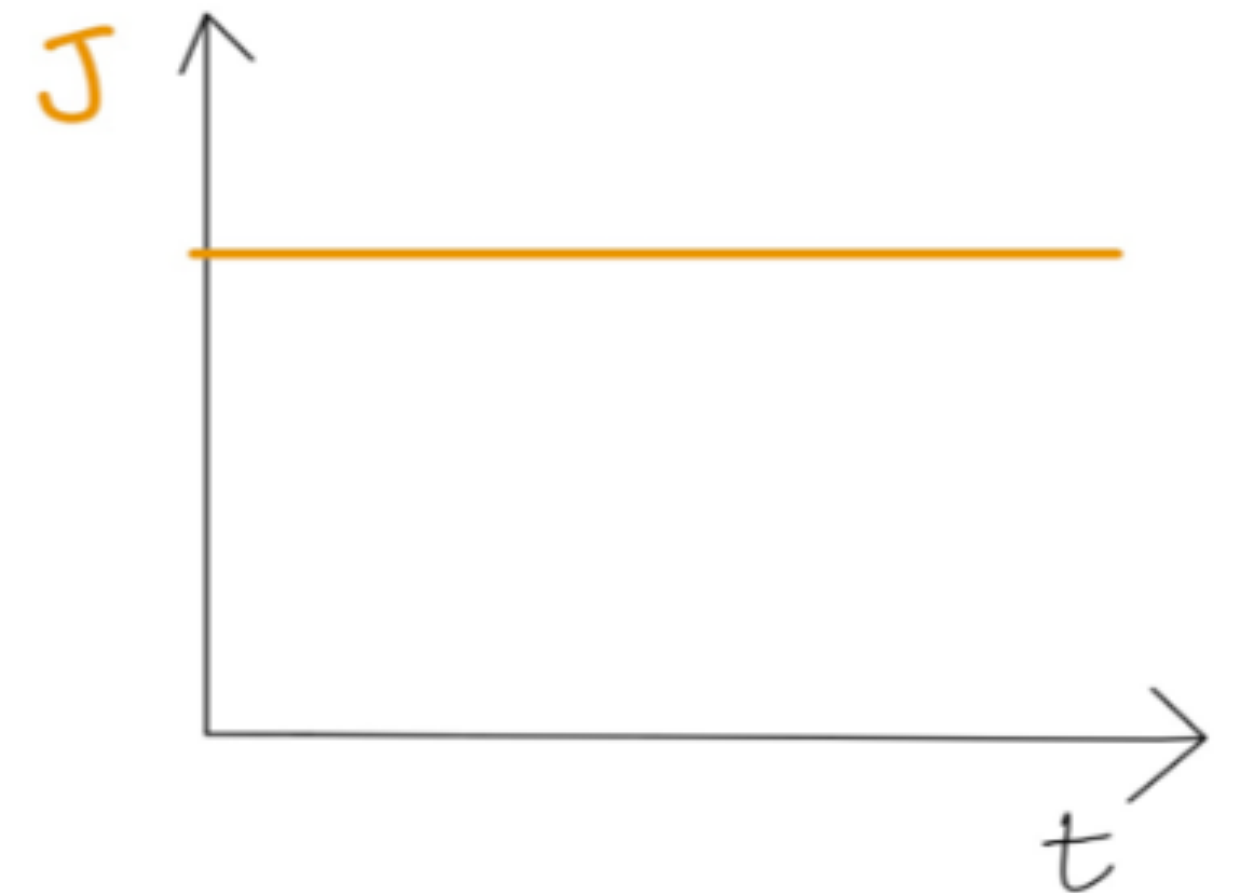
- Action  $\mathcal{J}_i \sim p$ ; angle  $\theta_i \sim q$ .

- Hamilton's equations  $\implies$

$$\dot{\mathcal{J}}_i = -\partial H / \partial \theta_i = 0 \quad \implies \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \quad \implies \theta_i = \omega_i(\vec{\mathcal{J}})t.$$

- Having action-angles  $\sim$  having closed-form solutions.



# Liouville-Arnold theorem

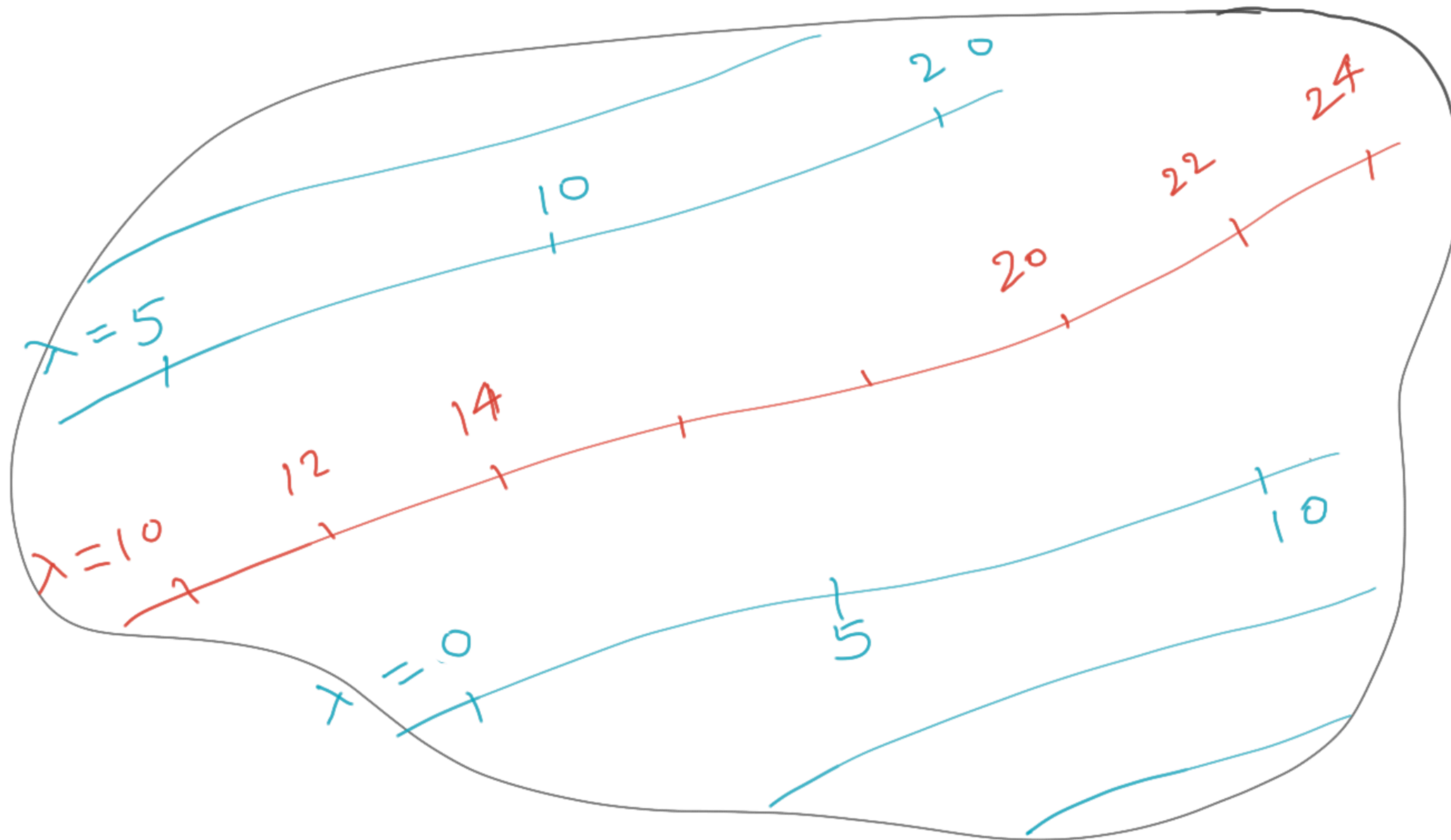
- **Theorem:**  $2n$  phase space variables &  $n$  commuting constants of motion  $\implies$  integrability.
- How to check if  $f$  is a constant of motion? Check if  $\dot{f} = \{f, H\} = 0$ .
- For BBH phase space  $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$ ,  $2n \neq 12$ . Positions-momenta delineation not clear for spins.
- Easy to check that  $\{R_i, P_j\} = \delta_{ij}$  and  $\{\phi_A, S_B^z\} = \delta_{AB}$  where  $\phi_A = \arctan(S_A^y/S_A^x)$ , the azimuthal angle for  $\vec{S}_A$ .
- $(\phi_A, S_A^z)$  are the **positions, momenta** of  $\vec{S}_A$ . Only 2 variables needed for  $\vec{S}_A$  since  $dS_A/dt = \{S_A, H\} = 0$ .
- Hence  $2n = 3 + 3 + 2 + 2 = 10 \implies 10/2 = 5$  commuting constants needed for integrability.

# Commuting constants for BBHs

- With  $m \equiv m_1 + m_2$ ,  $\mu \equiv m_1 m_2 / m$ ,  $\nu \equiv \mu / m$ ,  $\vec{L} \equiv \vec{R} \times \vec{P}$   
 $\sigma_1 \equiv (2 + 3m_2/m_1)$ ,  $\sigma_2 \equiv (2 + 3m_1/m_2)$ ,  $\vec{S}_{\text{eff}} \equiv \sigma_1 \vec{S}_1 + \sigma_2 \vec{S}_2$   
 $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$ .
- The 5 commuting constants are long known:  $H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}$ .
- Hence the 1.5PN BBH is integrable and has action-angles.

# Curves, vectors, vector fields and flows

- **Pictorial definition:** the vector is  $d/d\lambda$  (a derivative operator).





# Hamiltonian flow of a function $f(\vec{V})$

- $\vec{V} \equiv \{ \vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2 \}$ , unless states otherwise.
- “Hamiltonian flow” of  $f(\vec{V})$ :  $\frac{d\vec{V}}{d\lambda} = \{ \vec{V}, f \}$ .  $f$  **need not** be the Hamiltonian!
- Solution of the flow given in the form  $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda)$ .
- **Lingo:** Flow  $\equiv$  Hamiltonian flow (for brevity).
- Under  $f$  flow,  $g$  changes as  $\frac{dg}{d\lambda} = \frac{\partial g}{\partial V_k} \frac{\partial V_k}{\partial \lambda} = \frac{\partial g}{\partial V_k} \{ V_k, f \} = \{ g, f \}$ .



# Flow exercise 1

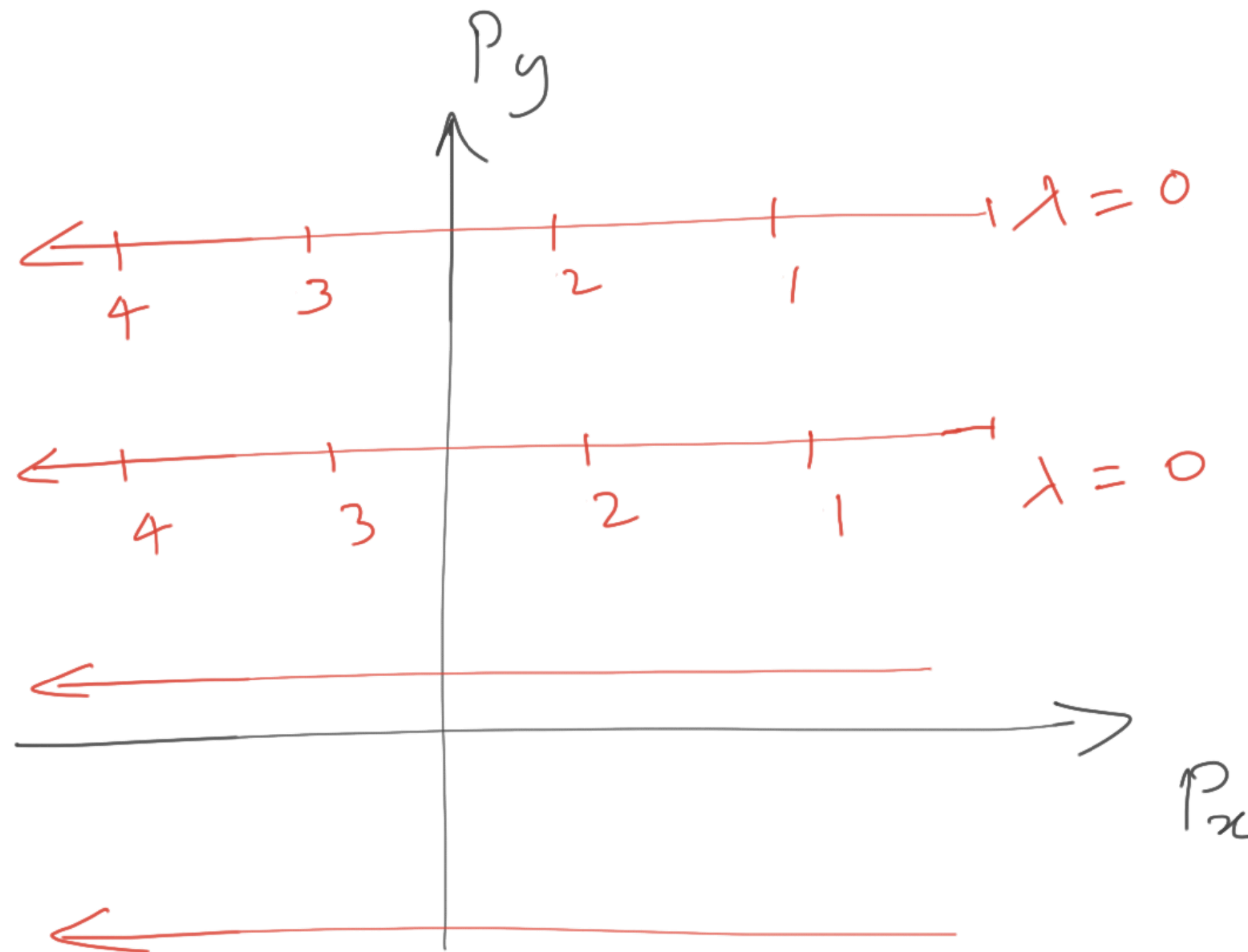
**Prob:** Solve the flow under  $R_x$  and draw pictures.

**Sol:** Under the  $R_x$  flow:

$$\frac{dP_x}{d\lambda} = \{P_x, R_x\} = -1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i\text{'s.}$$

$$\Rightarrow P_x - P_x(\lambda_0) = (\lambda_0 - \lambda).$$



# Flow exercise 2

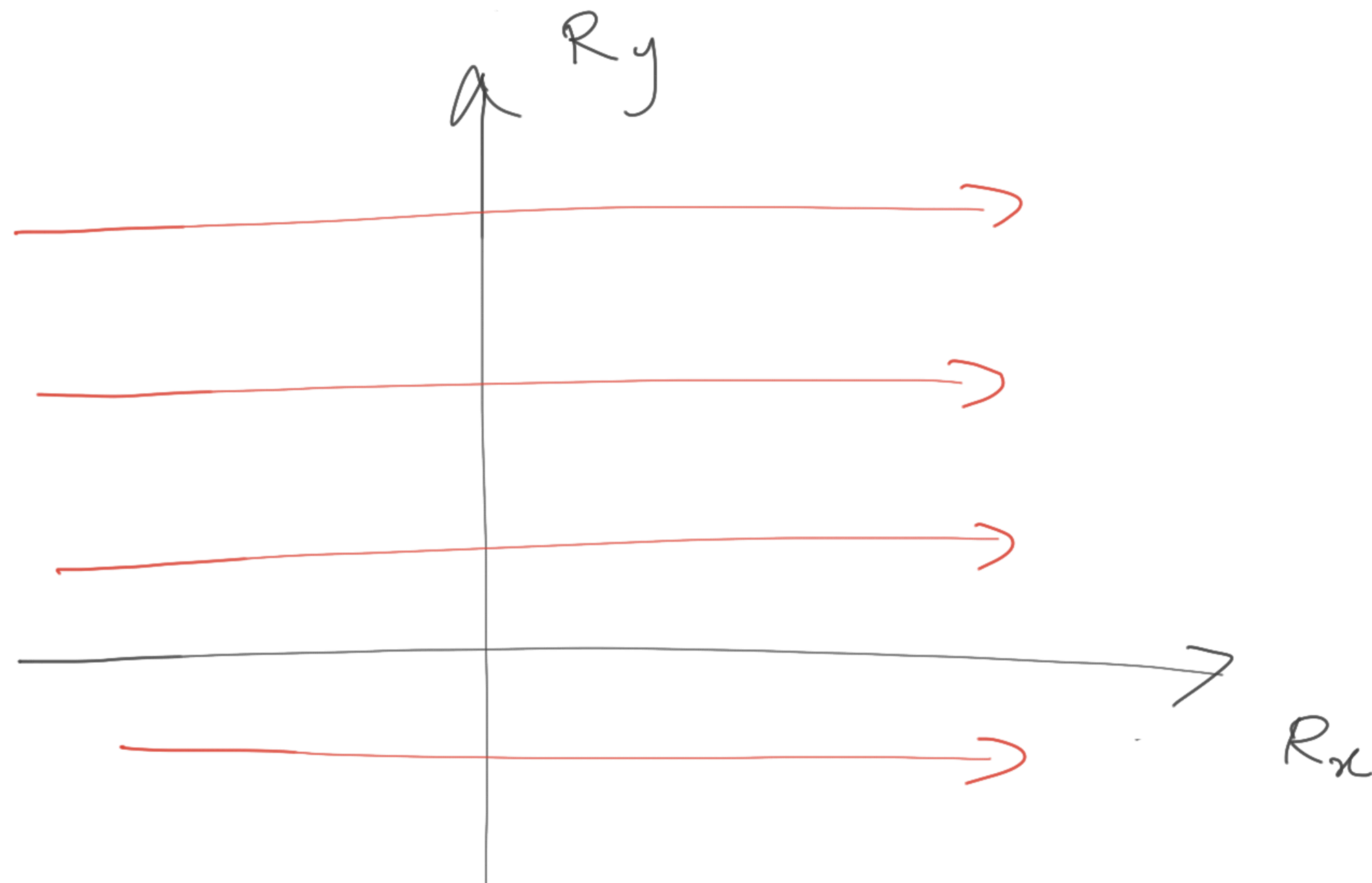
**Prob:** Solve the flow under  $P_x$  and draw pictures.

**Sol:** Under the  $R_x$  flow:

$$\frac{dR_x}{d\lambda} = \{R_x, P_x\} = 1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i\text{'s.}$$

$$\implies R_x - R_x(\lambda_0) = (\lambda - \lambda_0).$$



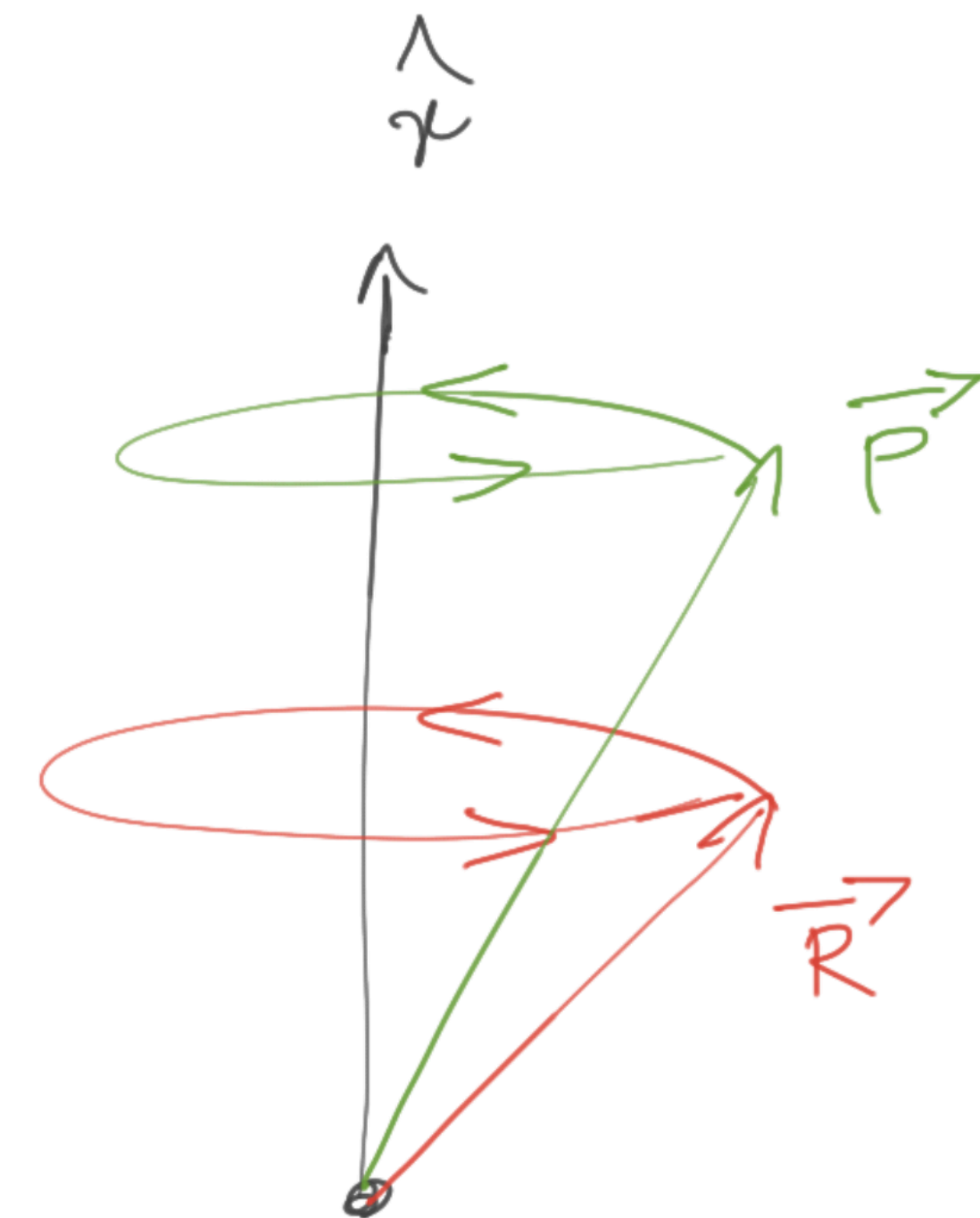
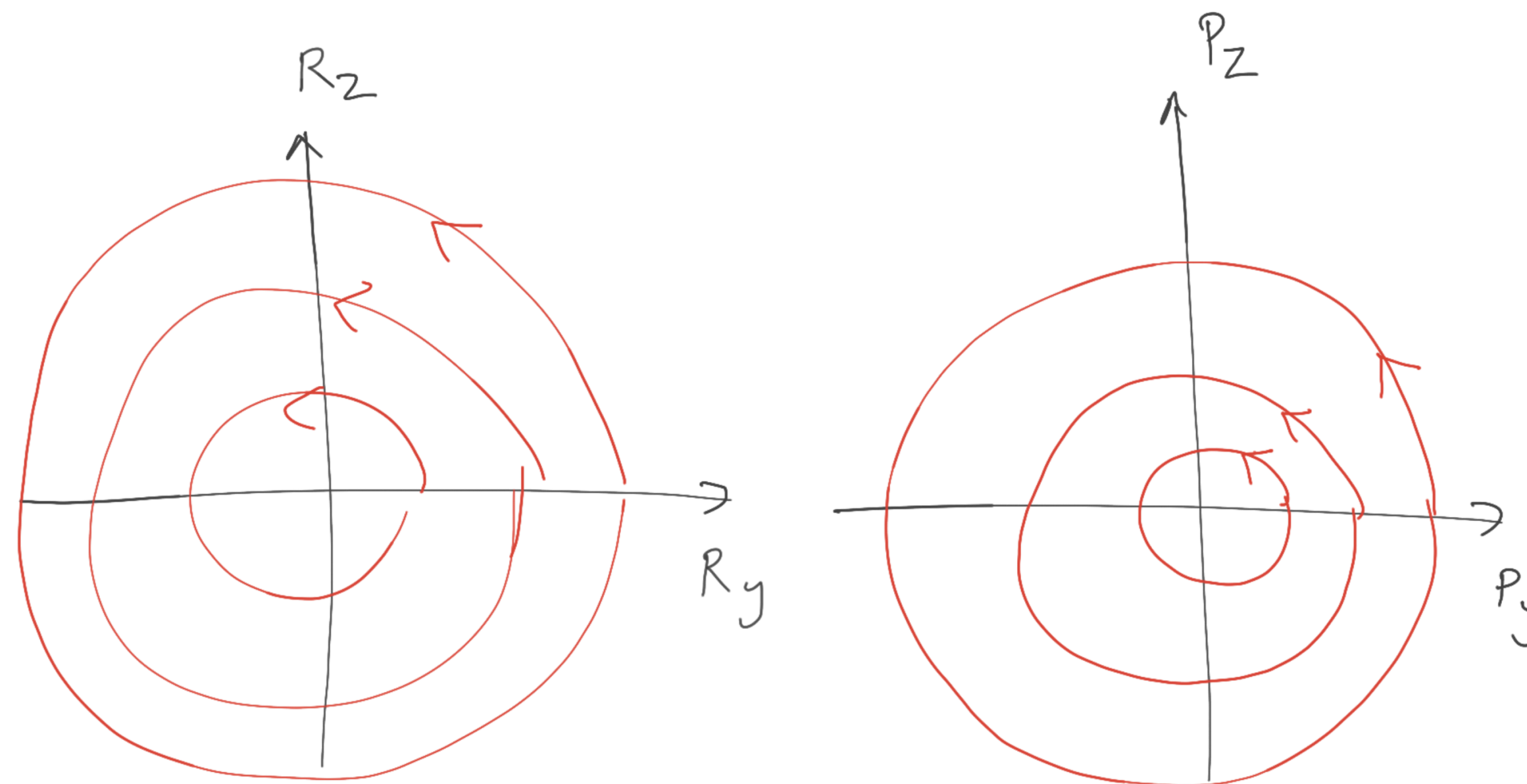
# Flow exercise 3

**Prob:** Draw pictures for  $L_x$  flow where  $\vec{L} = \vec{R} \times \vec{P}$ .

**Sol:** Under the  $L_x$  flow:

$$\frac{d\vec{R}}{d\lambda} = \hat{x} \times \vec{V}$$

$$\frac{d\vec{P}}{d\lambda} = \hat{x} \times \vec{V}$$

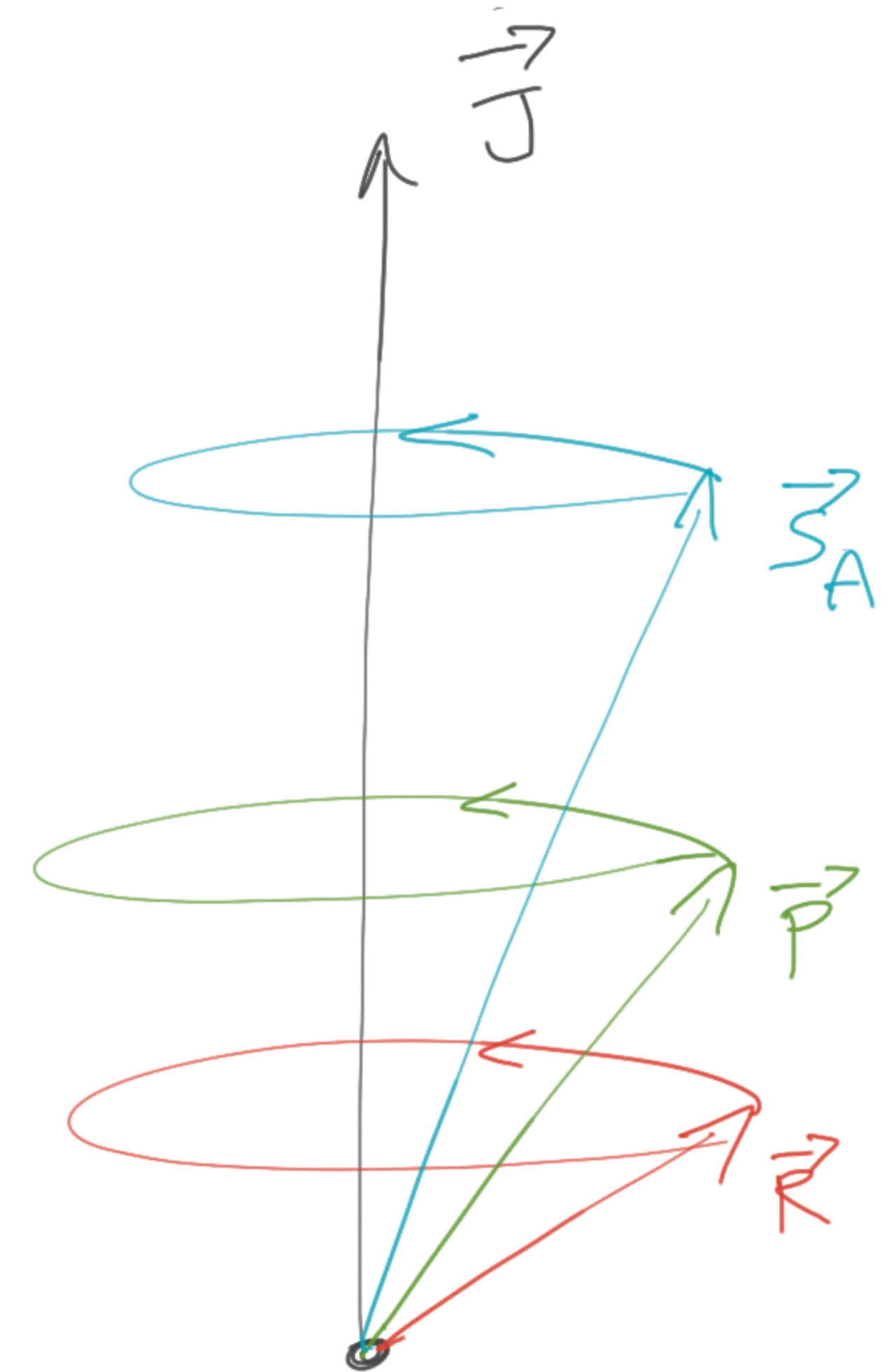
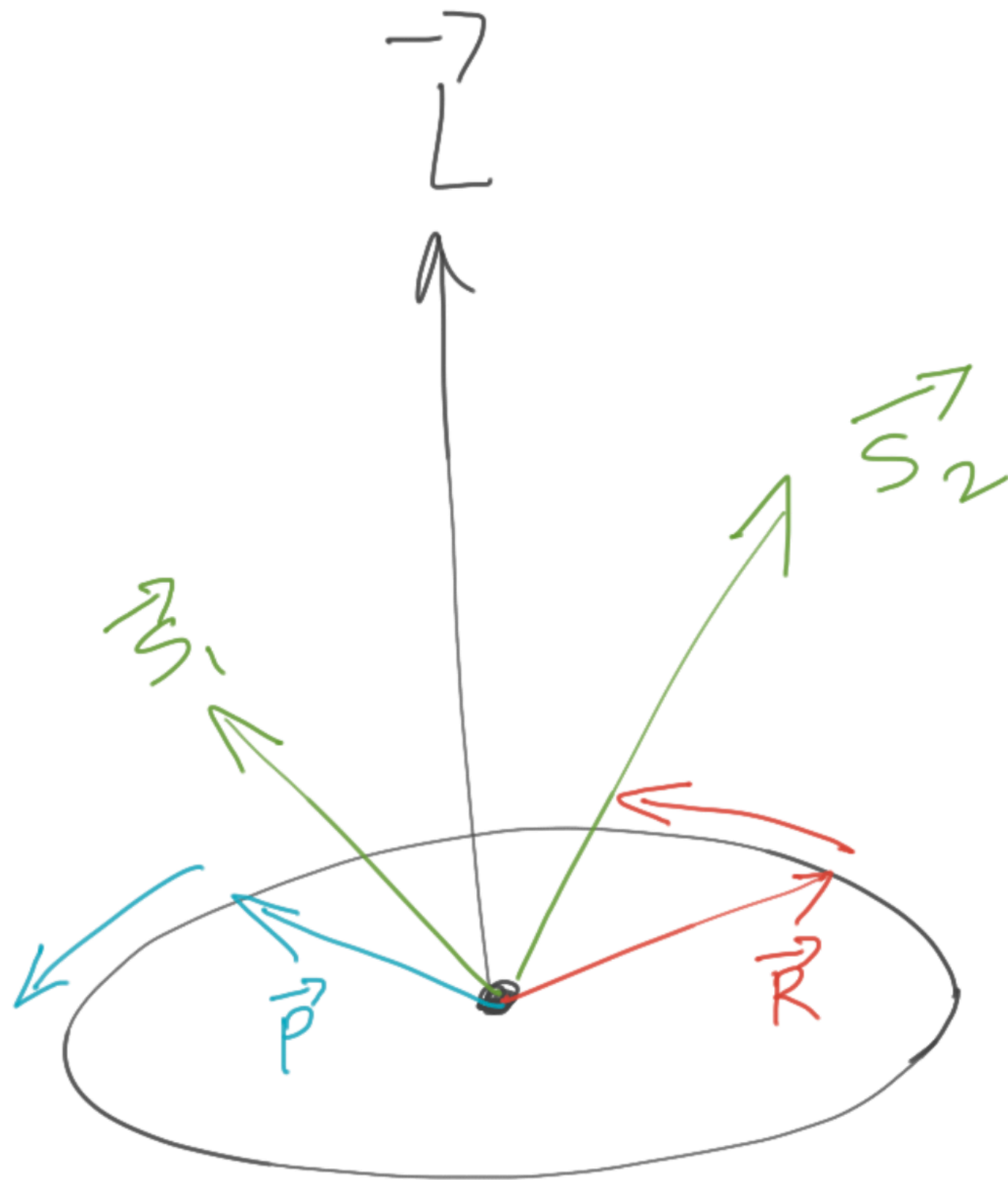


# Flow exercise 4

**Prob:** Draw pictures for  $L^2$  and  $J^2$  flow where  $\vec{L} = \vec{R} \times \vec{P}$  and  $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$ .

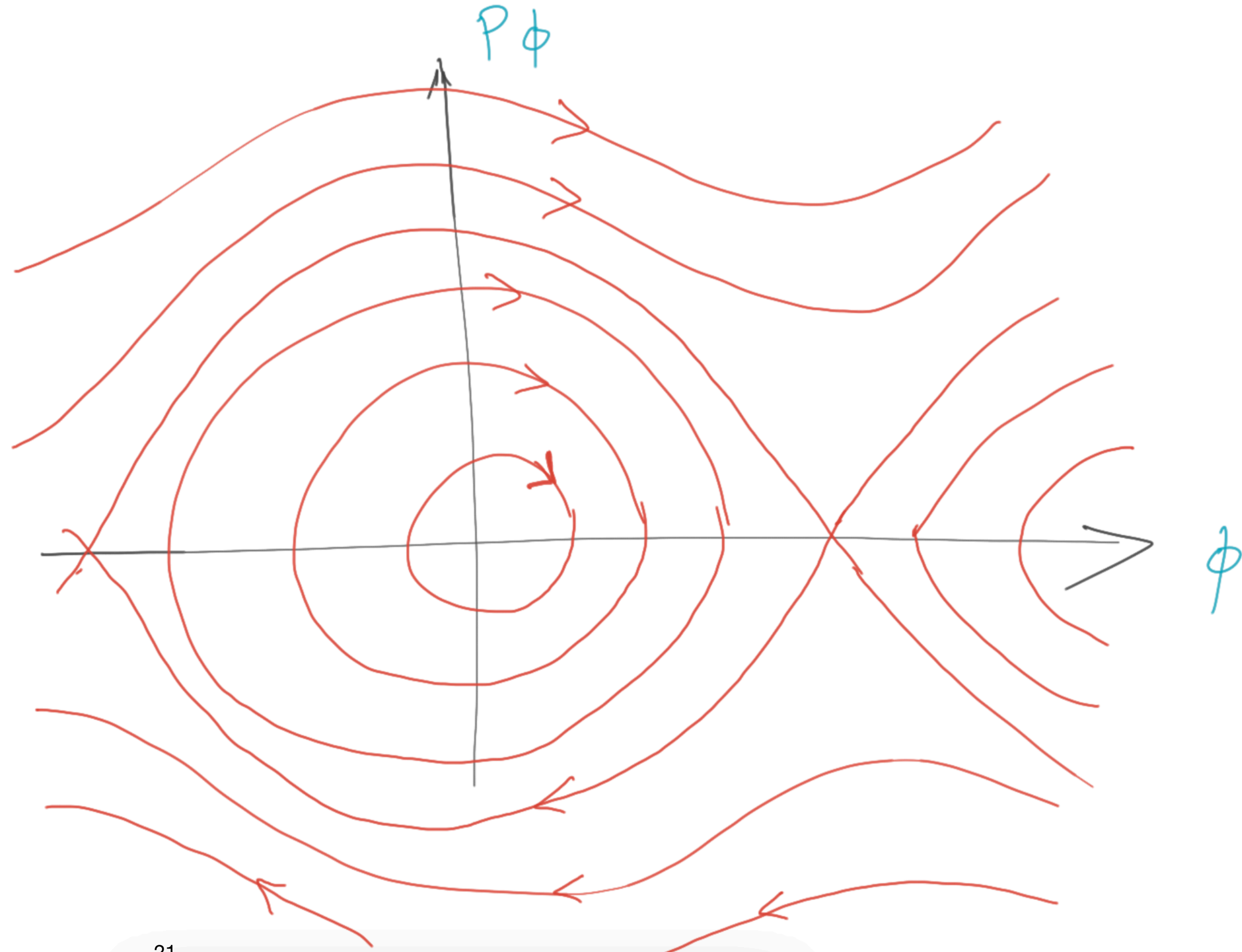
**Sol:**  $L^2$  flow  $\implies \{\vec{L}, L^2\} = \{\vec{S}_A, L^2\} = 0 \implies \vec{L}, \vec{S}_A$  remain fixed.

$J^2$  flow  $\implies \{\vec{J}, J^2\} = 0 \implies \vec{J}$  remains fixed



# Hamiltonian flow of the Hamiltonian $H$

- With  $\vec{V} \equiv \{ \vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2 \}$ , flow eqn. is  $\frac{d\vec{V}}{d\lambda} = \{ \vec{V}, f \}$ .
- Flow under  $H \implies \frac{d\vec{V}}{d\lambda} = \{ \vec{V}, H \}$ .
- This is the **EOM**. Gives the **real-time evolution**, unlike other flows.
- Hamiltonian flow of the Hamiltonian is special!
- **Example:** flow under  $H$  for a pendulum



**5 minute break**

**Coffee, questions?**



# Lecture plan

- **Lecture 1:**
  - Theory
  - Strategy to compute solution from action-angles
- **Lecture 2:**
  - Construct the solution

# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
(subject of the next lecture).

- How to combine  $\vec{C}$  flows?
- Construct action-angles.
- Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .
- How to flow along the actions  $\mathcal{J}_i$ ?
- Solution via action-angles.



# Action-angle-based solution: strategy

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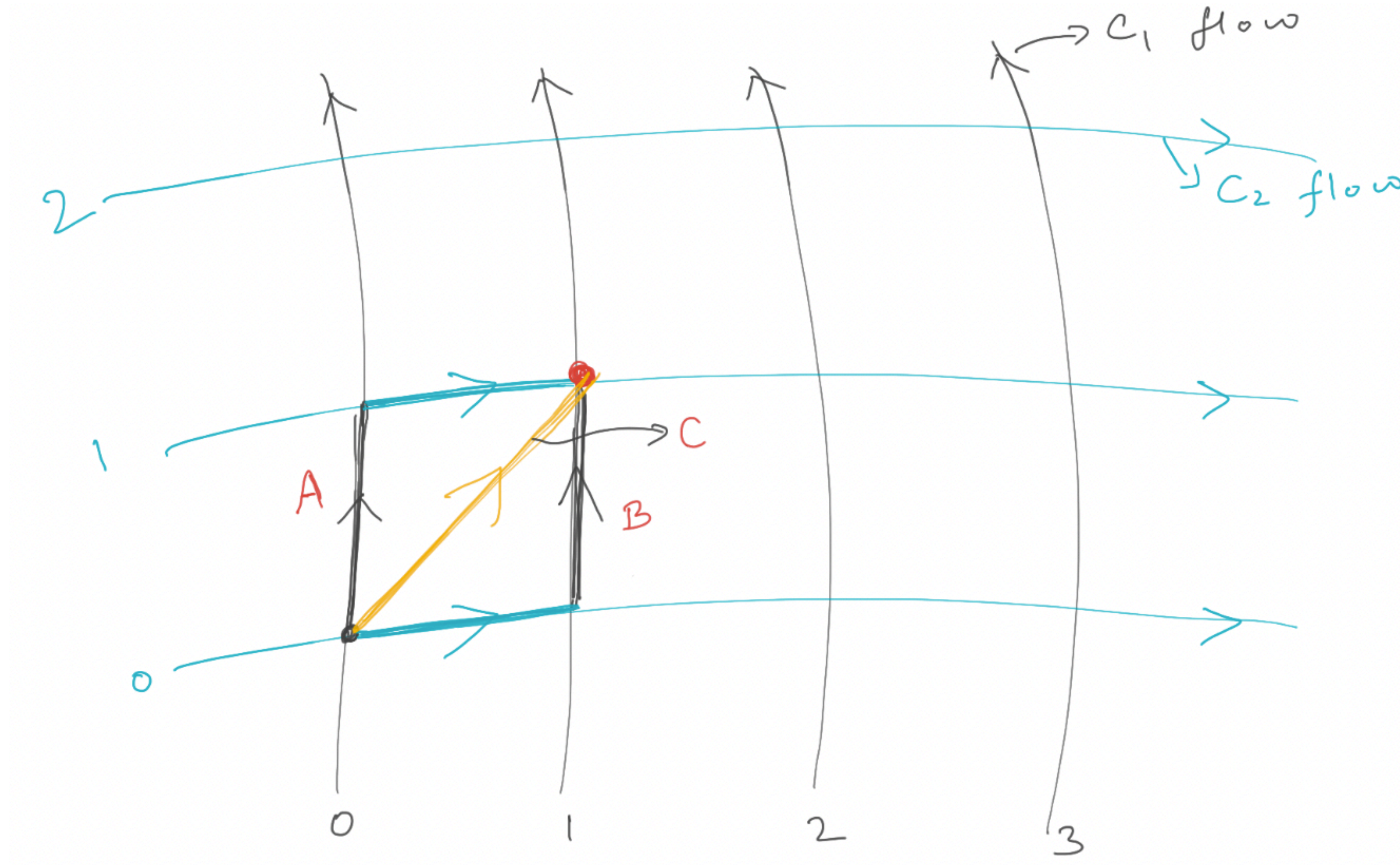
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# How to combine $\vec{C}$ flows?

- Assume  $C_i$ 's are commuting quantities (don't have to be constants).
- **Notation:** Output of  $C_i$  flow  $\frac{d\vec{V}}{d\lambda_i} = \{ \vec{V}, C_i \}$  denoted by  $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda_i)$ .
- **Result:** Order of flow does not matter, i.e.  $\vec{V}(\vec{V}_0, \Delta\lambda_1, \Delta\lambda_2) = \vec{V}(\vec{V}_0, \Delta\lambda_2, \Delta\lambda_1)$ .
- **Result:** Simultaneous flows can be made sequential:  $\frac{d\vec{V}}{d\lambda} = \{ \vec{V}, C_1 + C_2 \}$  by  $\Delta\lambda$  is  $C_1$  flow followed by  $C_2$  flow (both by  $\Delta\lambda$ ). Or in the reverse order.

# How to combine $\vec{C}$ flows?

Pictorial depiction of the two flow rules



# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
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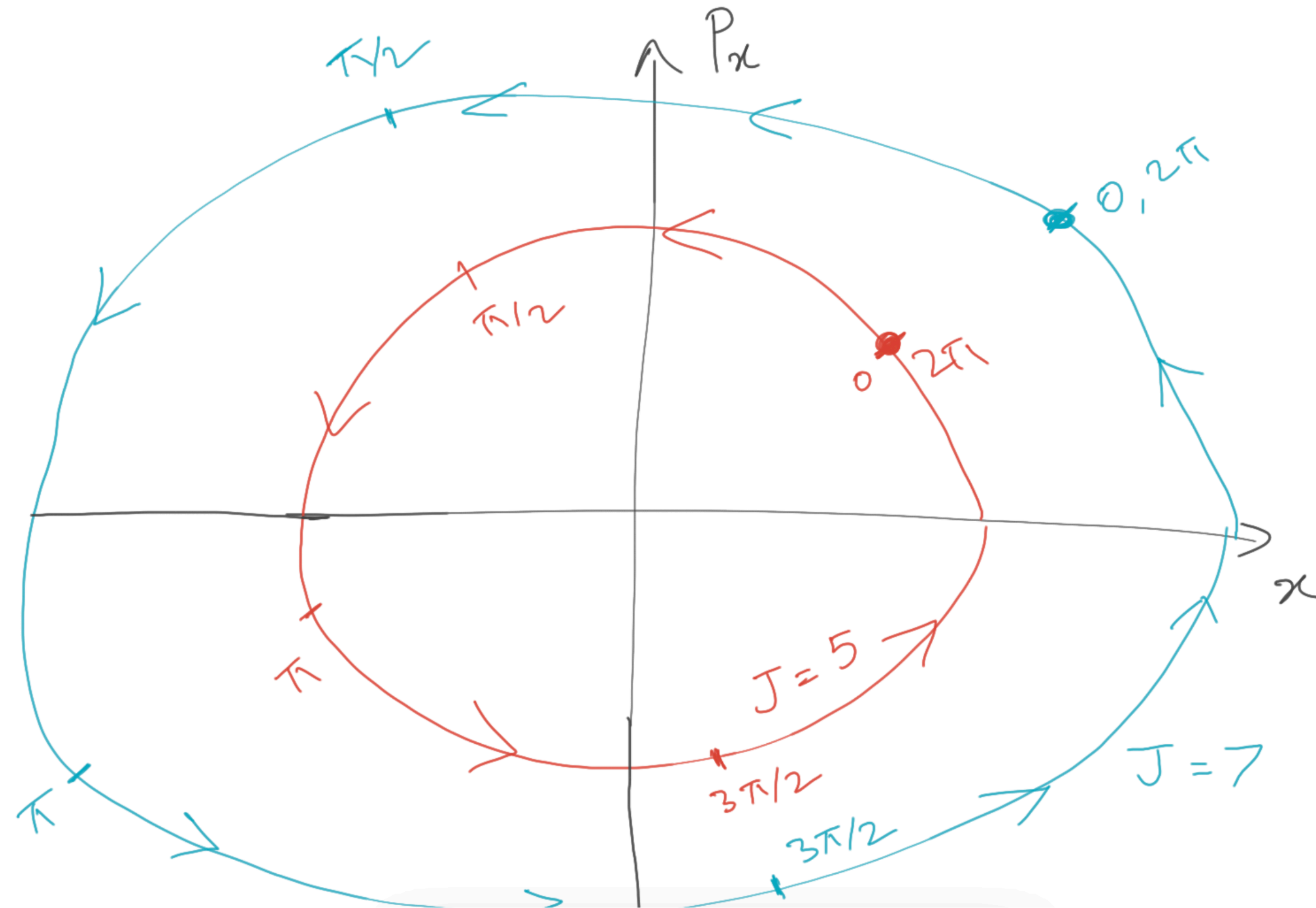
# Construct action-angles

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$  with  $\{R_i, P_j\} = \delta_{ij}$  and  $\{\phi_A, S_B^z\} = \delta_{AB}$ .
- Loop  $\gamma_i$  on the  $\vec{C} = \text{constant}$  submanifold.
- No. of independent  $\mathcal{J}_i$ 's  $= n$ , despite infinite no. of loops.
- $\mathcal{J}_i$  flow by  $2\pi \rightarrow$  loop (different from  $\gamma_i$ ).
- Angle  $\theta_i \equiv \lambda_i$  along the  $\mathcal{J}_i$  flow.



# Construct action-angles

Pictorial depiction of the construction



# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
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- How to combine  $\vec{C}$  flows?
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- Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .
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# Integrable systems and action-angles

- **Integrable system:** canonical transformation  $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$  exists such that  $H = H(\vec{\mathcal{J}})$  and  $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$ .

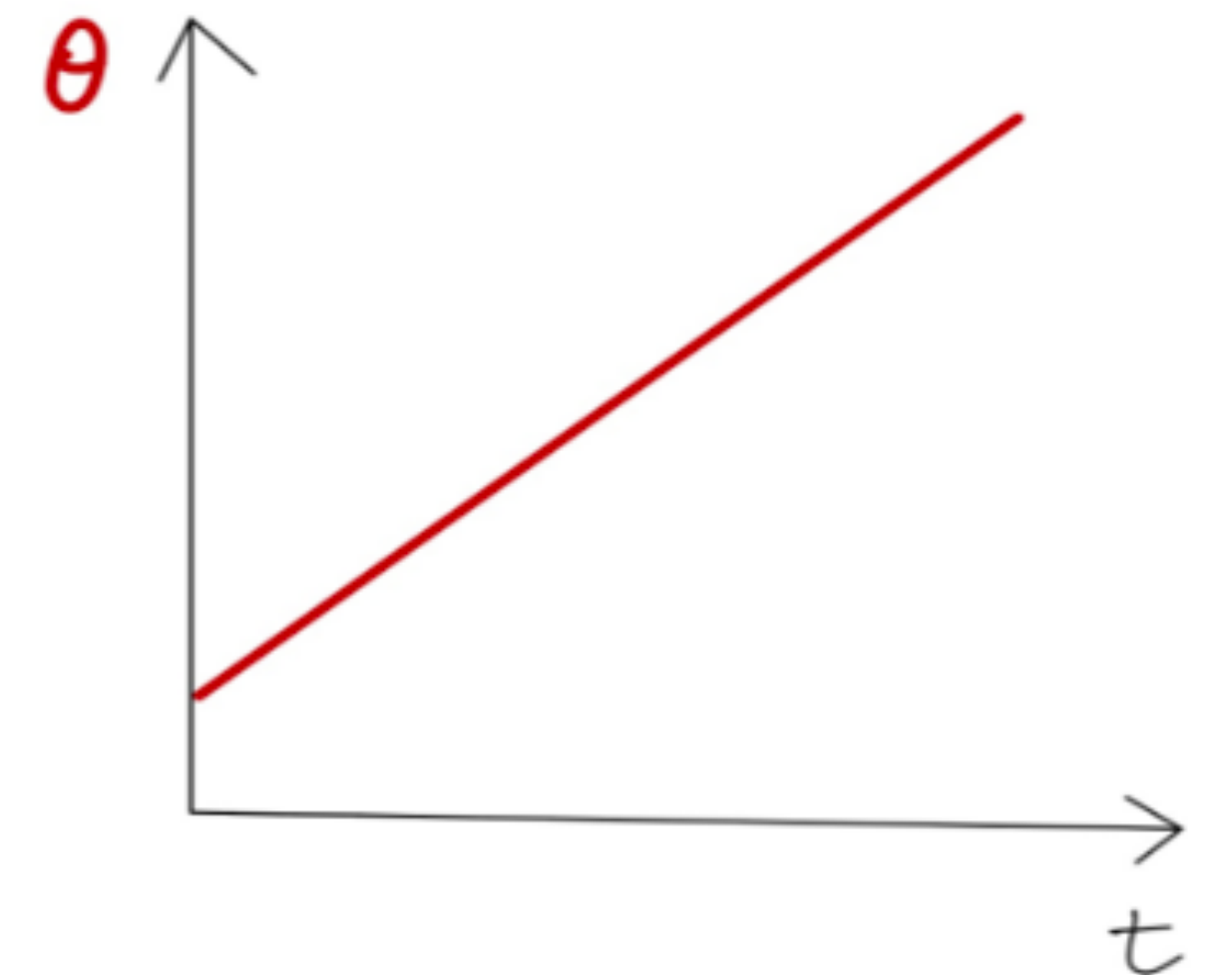
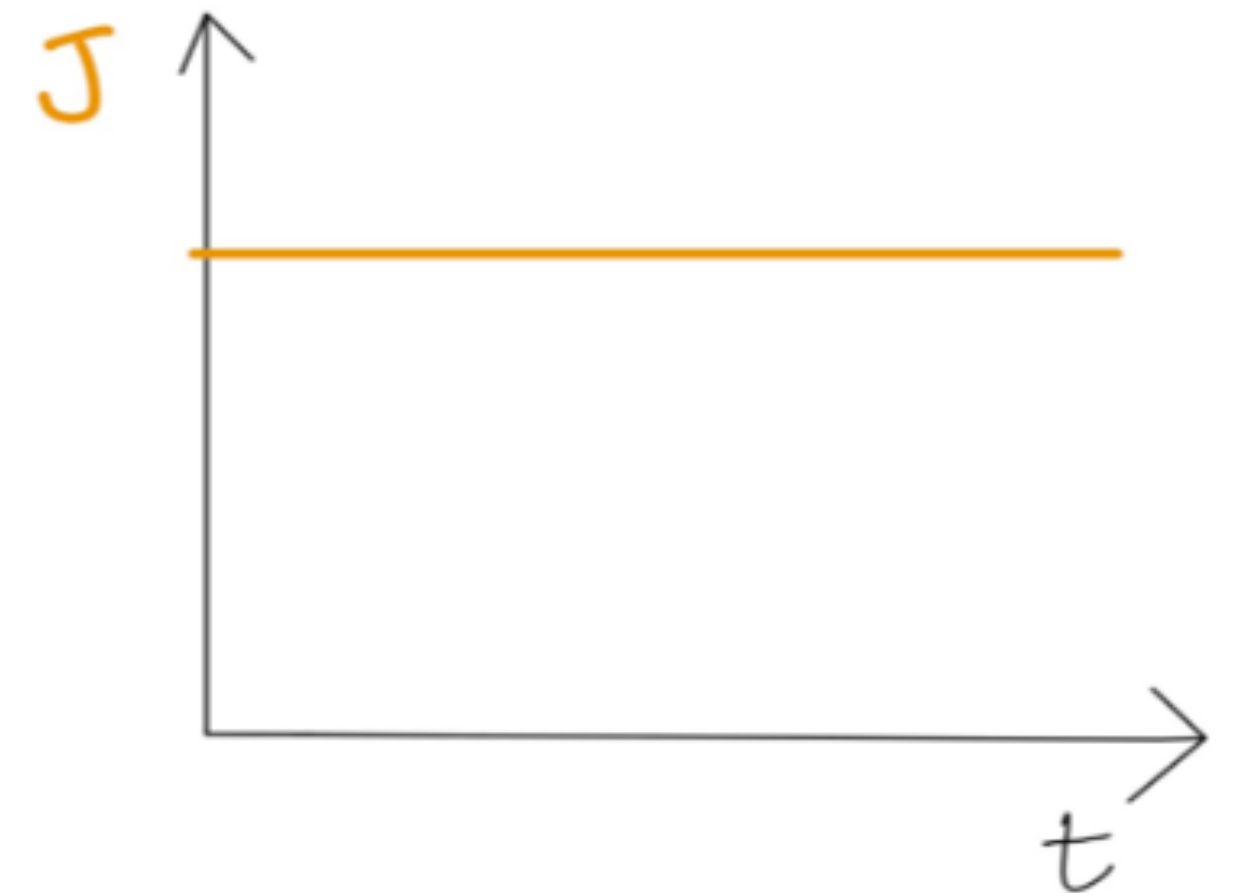
- Action  $\mathcal{J}_i = \sim p$ ; angle  $\theta_i = \sim q$ .

- Hamilton's equations  $\implies$

$$\dot{\mathcal{J}}_i = -\partial H / \partial \theta_i = 0 \quad \implies \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \quad \implies \theta_i = \omega_i(\vec{\mathcal{J}})t.$$

- Having action-angles  $\sim$  having closed-form solutions.





# Compute frequencies $\omega_i \equiv d\theta_i/dt$

- Recall  $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$ .
- With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have  $\mathcal{J}_i(\vec{C})$  (next lecture's subject).
- Compute the Jacobian  $M_{ij}(\vec{C}) \equiv \frac{\partial \mathcal{J}_i}{\partial C_j}$  (consists of numeric constants).
- **Inverse function theorem:** If  $N_{ij} \equiv \frac{\partial C_i}{\partial \mathcal{J}_j}$ , then  $N = M^{-1}$ .
- The first row of  $N$  corresponding to  $(C_1 = H)$  contains  $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$ .

# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
(subject of the next lecture).

- How to combine  $\vec{C}$  flows?
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- ~~Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .~~
- How to flow along the actions  $\mathcal{J}_i$ ?
- Solution via action-angles.

# EOMs with Poisson brackets for BBHs

## Our approach

- Define EOMs:  $\frac{df(t)}{dt} = \{f, H\}$  where  $f = f\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)$ .
- Define PBs:  $\left\{R_i, P_j\right\} = \delta_{ji}$   $\left\{S_A^i, S_B^j\right\} = \delta_{AB} \epsilon_k^{ij} S_A^k$ .

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, \quad a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\left\{f, g(v_i)\right\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

# How to flow along the actions $\mathcal{J}_i$ ?

- With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions (next lecture's subject).
- Using chain rule for PBs, 
$$\frac{d\vec{V}}{d\lambda} = \left\{ \vec{V}, \mathcal{J}_i \right\} = \left\{ \vec{V}, C_j \right\} \left( \frac{\partial \mathcal{J}_i}{\partial C_j} \right) = 2.5 \{ \vec{V}, C_1 \} + 5.1 \{ \vec{V}, C_2 \}$$
  
 $= \{ \vec{V}, 2.5C_1 + 5.1C_2 \}.$
- $\mathcal{J}_i$  flow by  $\Delta\lambda$  =  $(C_1$  flow by  $2.5\Delta\lambda$ , then  $C_2$  flow by  $5.1\Delta\lambda$ ). Or reverse the order.

# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
(subject of the next lecture).

- How to combine  $\vec{C}$  flows?
- Construct action-angles.
- ~~Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .~~
- ~~How to flow along the actions  $\mathcal{J}_i$ ?~~
- Solution via action-angles.

# Solution via action-angles.

- Start with an initial  $\overrightarrow{V}_0 = \{ \overrightarrow{R}, \overrightarrow{P}, \overrightarrow{S}_1, \overrightarrow{S}_2 \}$ . Assign it  $\overrightarrow{\theta} = \overrightarrow{0}$ .
- We want  $\overrightarrow{V} = \overrightarrow{V}(\overrightarrow{V}_0, t)$ .
- Recall  $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$  and  $\Delta \theta_i = \Delta \lambda_i$ .
- After time  $t$ ,  $\theta_i(t) = \omega_i t$ .
- How to increase the angles? Action flows increase the angles.
- We need to flow under  $\mathcal{J}_i$ 's by an amount  $\lambda_i = \theta_i(t) = \omega_i t$ .

# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
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- How to combine  $\vec{C}$  flows?
- Construct action-angles.
- ~~Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .~~
- ~~How to flow along the actions  $\mathcal{J}_i$ ?~~
- ~~Solution via action-angles.~~



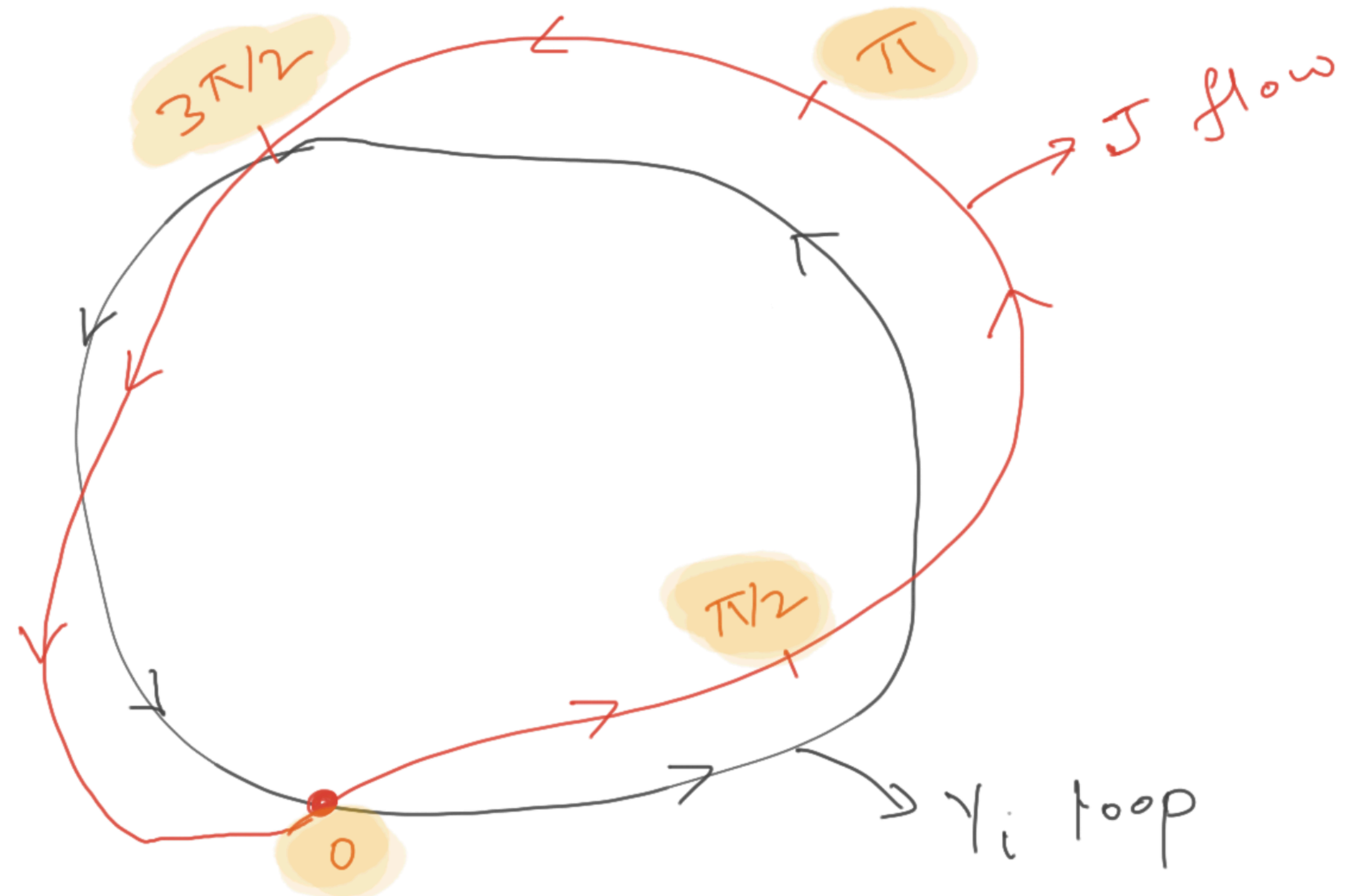
# Construct action-angles

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$ . Loop  $\gamma_i$  on the  $\vec{C} = \text{constant}$  submanifold.

- $\mathcal{J}_i$  flow by  $2\pi \rightarrow$  loop (different from  $\gamma_i$ ).

- Angle  $\theta_i \equiv \lambda_i$  along the  $\mathcal{J}_i$  flow.

- **To show:**  $\{\theta_i, J_k\} = \delta_{ij}$ ,  $\{J_i, J_k\} = 0$ ,  $\{\theta_i, \theta_k\} = 0$ .





# Construct action-angles

- Using  $\theta_i = \lambda_i$ ,  $\frac{d\theta_i}{d\lambda_i} = 1$  and  $\frac{d\theta_i}{d\lambda_i} = \{\theta_i, \mathcal{J}_i\} \implies \{\theta_i, \mathcal{J}_i\} = 1$ .
- From definition  $\mathcal{J}_i$  and chain rule for PBs,  $\mathcal{J}_i = \mathcal{J}_i(\vec{C}) \implies \{J_i, J_k\} = \frac{\partial J_i}{\partial C_l} \frac{\partial J_k}{\partial C_m} \{C_l, C_m\} = 0$ .
- $\{\theta_i, \theta_j\} = 0$  involves changing  $\mathcal{J}_i$ , which does not happen with real evolution.  
Hence ignore.
- “**Integrable system:** canonical transformation  $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$  exists such that  $H = H(\vec{\mathcal{J}})$  and  $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$ .” that lead to  $\dot{\mathcal{J}}_i = 0$ ;  $\theta_i = \omega_i t$  is satisfied because action flow makes a loop after  $2\pi$ .

# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
(subject of the next lecture).

- How to combine  $\vec{C}$  flows?
- ~~Construct action-angles.~~
- ~~Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .~~
- ~~How to flow along the actions  $\mathcal{J}_i$ ?~~
- ~~Solution via action-angles.~~

# Lecture plan

- **Lecture 1:**
  - Theory
  - Strategy to compute solution from action-angles
- **Lecture 2:**
  - Construct the solution

**THE END**

Please send comments on the lecture notes and  
the presentation 

Thank you!

# Lecture plan

- **Lecture 1:**
  - Theory
  - Strategy to compute solution from action-angles
- **Lecture 2:**
  - Construct the solution

# Action-angle-based solution: strategy

With  $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$ , assume we have (i)  $\mathcal{J}_i(\vec{C})$  (ii)  $\vec{C}$  flow solutions  
(subject of this lecture).

- How to combine  $\vec{C}$  flows?
- Construct action-angles.
- Compute frequencies  $\omega_i \equiv \frac{d\theta_i}{dt}$ .
- How to flow along the actions  $\mathcal{J}_i$ ?
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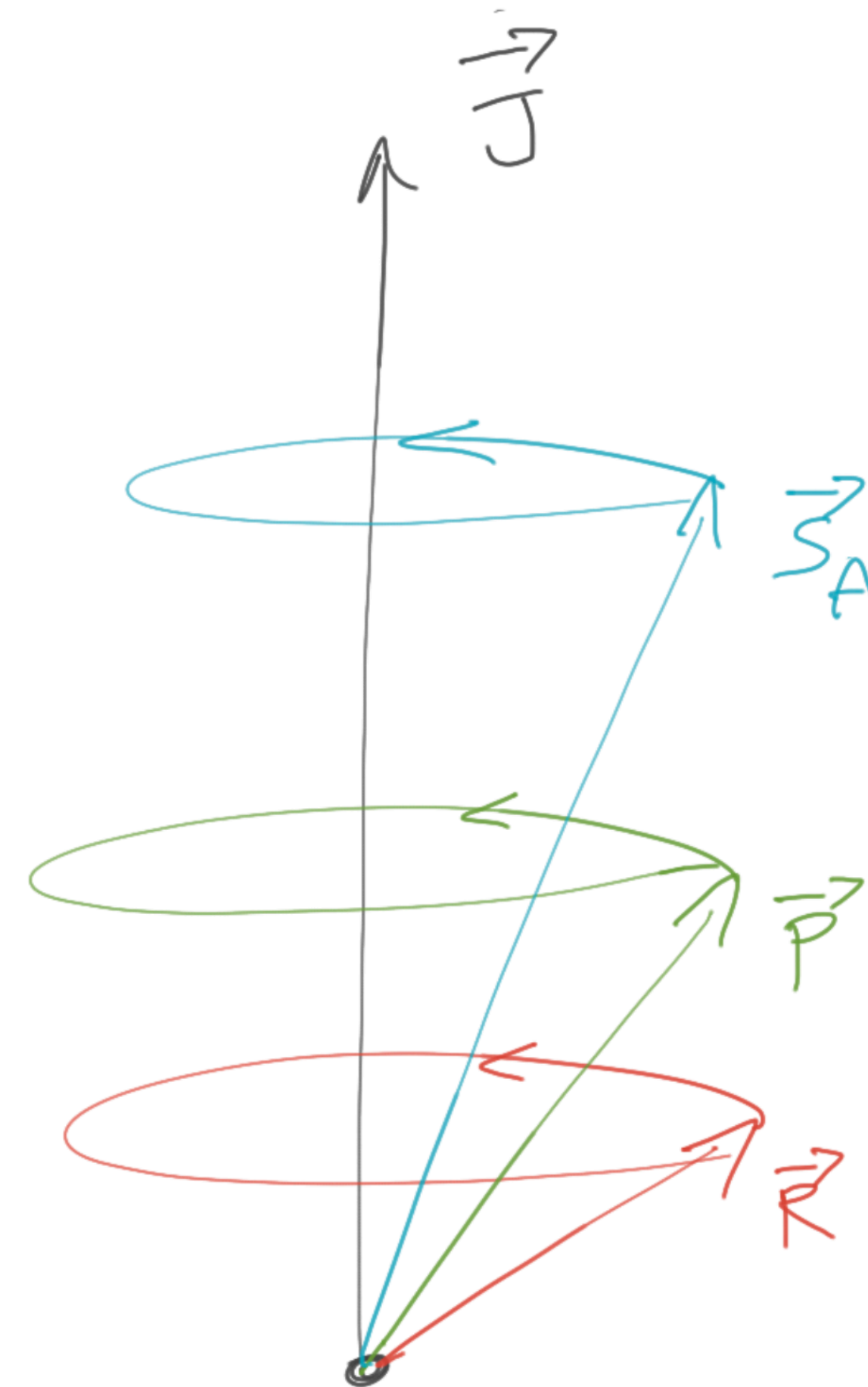
# Computing actions: strategy

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$ ; loop  $\gamma_i$  is on the surface of constant  $\vec{C}$ .
- $\{R_i, P_j\} = \delta_{ij}$  and  $\{\phi_A, S_B^z\} = \delta_{AB}$
- How to be on the surface of constant  $\vec{C}$ ? Flow along  $C_i$ 's:  $\frac{dC_i}{d\lambda} = \{C_i, C_j\} = 0$ .
- $\mathcal{J} = \mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}$
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \sum_i P_i dR^i$        $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint_A^z S_A^z d\phi_A$ .



# Computing $\mathcal{I}_1$

- With  $\vec{V} = \{\vec{R}, \vec{P}, \vec{L}, \vec{S}_1, \vec{S}_2\}$ ,  $J^2$  flow  $\implies \frac{d\vec{V}}{d\lambda} = 2\vec{J} \times \vec{V} \equiv \vec{n} \times \vec{V}$ .
- $\{\vec{J}, J^2\} = 0$ .
- **Solution:**  $\phi(\lambda) = n \lambda + \phi_0$ .
- Loop closes after flowing by  $\Delta\lambda = 2\pi/n = 2\pi/(2J) = \pi/J$ .
- $\mathcal{I}^{\text{orb}} = \frac{1}{2\pi} \int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{n} \cdot \vec{L} d\lambda = \hat{n} \cdot \vec{L}$ .
- $\mathcal{I}_A^{\text{spin}} = \frac{1}{2\pi} \oint S_A^z d\phi_A = S_A^z = \hat{n} \cdot \vec{S}_A$  (with  $\vec{n}$  along z-axis)
- The spin integral is **rotationally invariant**, but not manifestly so.
- $\oint S_z d\phi = \int dS_z \wedge d\phi = S \int d(\cos \theta) \wedge d\phi = -S \int \sin \theta d\theta \wedge d\phi = -\text{Area} / S$
- $\mathcal{I}_1 = \hat{n} \cdot (\vec{L} + \vec{S}_1 + \vec{S}_2) = \hat{n} \cdot \vec{J} = J$ .
- **Summary:** We have computed  $\mathcal{I}_1$  and also computed the solution to  $C_1 = J^2$ .



# Computing $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$

- **For flows under  $J^2, J_z$ , and  $L^2$ :**  $\frac{d\vec{V}}{d\lambda} = \vec{n} \times \vec{V}$ .  $\vec{n} = 2\vec{J}, \hat{z}$ , and  $2\vec{L}$  (with  $\vec{n}$  being fixed)
- **Exception:** Under  $L^2$  flow, spins don't move.
- **Solution:**  $\phi(\lambda) = n\lambda + \phi_0$ . Doesn't apply to spins under the  $L^2$  flow.
- Loop closes after flowing by  $\Delta\lambda = 2\pi/n$ .
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{n} \cdot \vec{L} d\lambda = \hat{n} \cdot \vec{L}$ .
- $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint S_A^z d\phi_A = S_A^z = \hat{n} \cdot \vec{S}_A$  (with  $\vec{n}$  along  $z$ -axis)
- $\mathcal{J}_1 = J, \mathcal{J}_2 = J_z, \mathcal{J}_3 = L$ .
- **Summary:** We have computed  $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$  and also computed the solution to  $C_i = \{J^2, J_z, L^2\}$ .

# Computing $\mathcal{I}_4$

- We won't compute it here.
- $\mathcal{I}_4$  has a Newtonian version (Eq. (10.139) of Goldstein).
- 1PN version given in Eq. (3.10) of Damour-Schafer.
- 1.5PN version in Eq. (38) of [arXiv: 2012.06586].

# Taking stock

- We solved the flows under  $C_i = \{J^2, J_z, L^2\}$ .
- Finding the solution  $\vec{V}(\vec{V}_0, \Delta\lambda)$  of a flow under  $C_i$ :  $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, C_i\}$  is basically solving an ODE.
- Solution of flow under  $\vec{S}_{\text{eff}} \cdot \vec{L}$  in [arXiv:2110.15351].
- Solution of flow under  $H$  in [arXiv:1908.02927]. They omit 1PN terms for simplicity. Call it the **standard solution**.
- Above solutions: quite lengthy but not esoteric.
- **Future focus:** compute  $\mathcal{I}_5$ .

**5 minute break**

**Coffee, questions?**

# $\mathcal{I}_5$ computation

For  $\vec{S}_{\text{eff}} \cdot \vec{L}$  flow:

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{P}$$

$$\frac{d\vec{S}_a}{d\lambda} = \sigma_a \left( \vec{L} \times \vec{S}_a \right)$$

$$\frac{d\vec{L}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{L}$$

•

- **Important:**  $\vec{n}$  not fixed:  $\{ \vec{n}, S_{\text{eff}} \cdot L \} \neq 0$ .

# $\mathcal{I}_5$ computation

$$2\pi\mathcal{I} = 2\pi \left( \mathcal{I}^{\text{orb}} + \mathcal{I}^{\text{spin}} \right)$$

$$= \int_{\lambda_i}^{\lambda_f} \left( P_i dR^i + S_1^z d\phi_1^z + S_2^z d\phi_2^z \right)$$

- $$= \int_{\lambda_i}^{\lambda_f} \left( P_i \frac{dR^i}{d\lambda} + S_1^z \frac{d\phi_1^z}{d\lambda} + S_2^z \frac{d\phi_2^z}{d\lambda} \right) d\lambda$$

- $$2\pi\mathcal{I}^{\text{orb}} = \int_{\lambda_i}^{\lambda_f} \vec{P} \cdot \left( \vec{S}_{\text{eff}} \times \vec{R} \right) d\lambda = \int_{\lambda_i}^{\lambda_f} (S_{\text{eff}} \cdot L) d\lambda = (S_{\text{eff}} \cdot L) \Delta\lambda$$

- Can't do spin sector integral because  $\vec{S}_A \neq \vec{R}_A \times \vec{P}_A$ . “A” is BH index.



# $\mathcal{I}_5$ computation: enter fictitious variables

- Define  $\vec{R}_a, \vec{P}_a$  (fictitious variables) such that  $\vec{S}_a \equiv \vec{R}_a \times \vec{P}_a$ .
- **Hamiltonian:** Now a function of  $\vec{R}, \vec{P}, \vec{R}_{1/2}, \vec{P}_{1/2}$  and not  $\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$ .
- **PBs and EOMs:**  $\left\{ R_i, P_j \right\} = \delta_{ij}, \quad \left\{ R_{ai}, P_{bj} \right\} = \delta_{ab} \delta_{ji}; \quad \frac{df}{dt} = \{f, H\}.$
- $\left\{ R_i, P_j \right\} = \delta_{ij}, \quad \left\{ R_{ai}, P_{bj} \right\} = \delta_{ab} \delta_{ji} \implies \left\{ R_i, P_j \right\} = \delta_{ij}, \quad \left\{ \phi_A, S_B^z \right\} = \delta_{AB}$
- PBs  $\rightarrow$  EOMs  $\implies$  The standard phase space (**SPS**) is equivalent to the extended phase space (**EPS**).
- **Integrability equivalency:** EPS needs  $n = 2n/2 = 18/2 = 9 = (5 + 4)$   $C_i$ 's. The next 4  $C_i$ 's are  $S_a^2$  and  $\vec{R}_a \cdot \vec{P}_a$ .

# $\mathcal{I}_5$ computation: sanity checks

- **Check 1:** Final  $\mathcal{I}_5$  depends on  $\overrightarrow{R}$ ,  $\overrightarrow{P}$ ,  $\overrightarrow{S}_1$  and  $\overrightarrow{S}_2$ .
- **Check 2:** Numerical flow by  $2\pi$  under  $\mathcal{I}_5$  closes a loop in the SPS picture.
- We have all seen fictitious variables before (in spirit)!
- Inventing complex numbers to do real integrals.

11.8.19 Prove that  $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx = \pi \ln 2$ .

11.8.20 Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2} dx = \frac{\pi a}{\sin \pi a},$$

where  $-1 < a < 1$ .

*Hint.* Use the contour shown in Fig. 11.26, noting that  $z = 0$  is a branch point and the positive  $x$ -axis can be chosen to be a cut line.

11.8.21 Show that

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.16 is a special case of this result.

11.8.22 Show that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}.$$

*Hint.* Try the contour shown in Fig. 11.30, with  $\theta = 2\pi/n$ .

11.8.23 (a) Show that

$$f(z) = z^4 - 2z^2 \cos 2\theta + 1$$

has zeros at  $e^{i\theta}$ ,  $e^{-i\theta}$ ,  $-e^{i\theta}$ , and  $-e^{-i\theta}$ .

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.22 ( $n = 4$ ) is a special case of this result.

# $\mathcal{J}_5$ computation using fictitious variables

$$\mathcal{J}_k = \frac{1}{2\pi} \oint_{\mathcal{C}_k} \left( \vec{P} \cdot d\vec{R} + \vec{P}_1 \cdot d\vec{R}_1 + \vec{P}_2 \cdot d\vec{R}_2 \right)$$

EOMs for  $\vec{S}_{\text{eff}} \cdot \vec{L}$  flow are

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{P}$$

$$\frac{d\vec{R}_a}{d\lambda} = \sigma_a \left( \vec{L} \times \vec{R}_a \right)$$

$$\frac{d\vec{P}_a}{d\lambda} = \sigma_a \left( \vec{L} \times \vec{P}_a \right)$$

$$\begin{aligned} 2\pi \mathcal{J}_{S_{\text{eff}} \cdot L} &= 2\pi \left( \mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}} \right) \\ &= \int_{\lambda_i}^{\lambda_f} \left( P_i \frac{dR^i}{d\lambda} + P_{1i} \frac{dR_1^i}{d\lambda} + P_{2i} \frac{dR_2^i}{d\lambda} \right) d\lambda \\ &= \int_{\lambda_i}^{\lambda_f} \left( \vec{P} \cdot \left( \vec{S}_{\text{eff}} \times \vec{R} \right) + \vec{P}_1 \cdot \left( \sigma_1 \vec{L} \times \vec{R}_1 \right) \right. \\ &\quad \left. + \vec{P}_2 \cdot \left( \sigma_2 \vec{L} \times \vec{R}_2 \right) \right) d\lambda \\ &= 2 \int_{\lambda_i}^{\lambda_f} (S_{\text{eff}} \cdot L) d\lambda = 2 (S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L} \\ \mathcal{J}_{S_{\text{eff}} \cdot L} &= \frac{(S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L}}{\pi} \end{aligned}$$

# $\mathcal{J}_5$ computation using fictitious variables

$$\mathcal{J}_{S_{\text{eff}} \cdot L} = \frac{(S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L}}{\pi},$$

$$\mathcal{J}_{J^2} = \frac{J^2 \Delta\lambda_{J^2}}{\pi},$$

$$\mathcal{J}_{L^2} = \frac{L^2 \Delta\lambda_{L^2}}{\pi},$$

$$\mathcal{J}_{S_1^2} = \frac{S_1^2 \Delta\lambda_{S_1^2}}{\pi},$$

$$\mathcal{J}_{S_2^2} = \frac{S_2^2 \Delta\lambda_{S_2^2}}{\pi}.$$

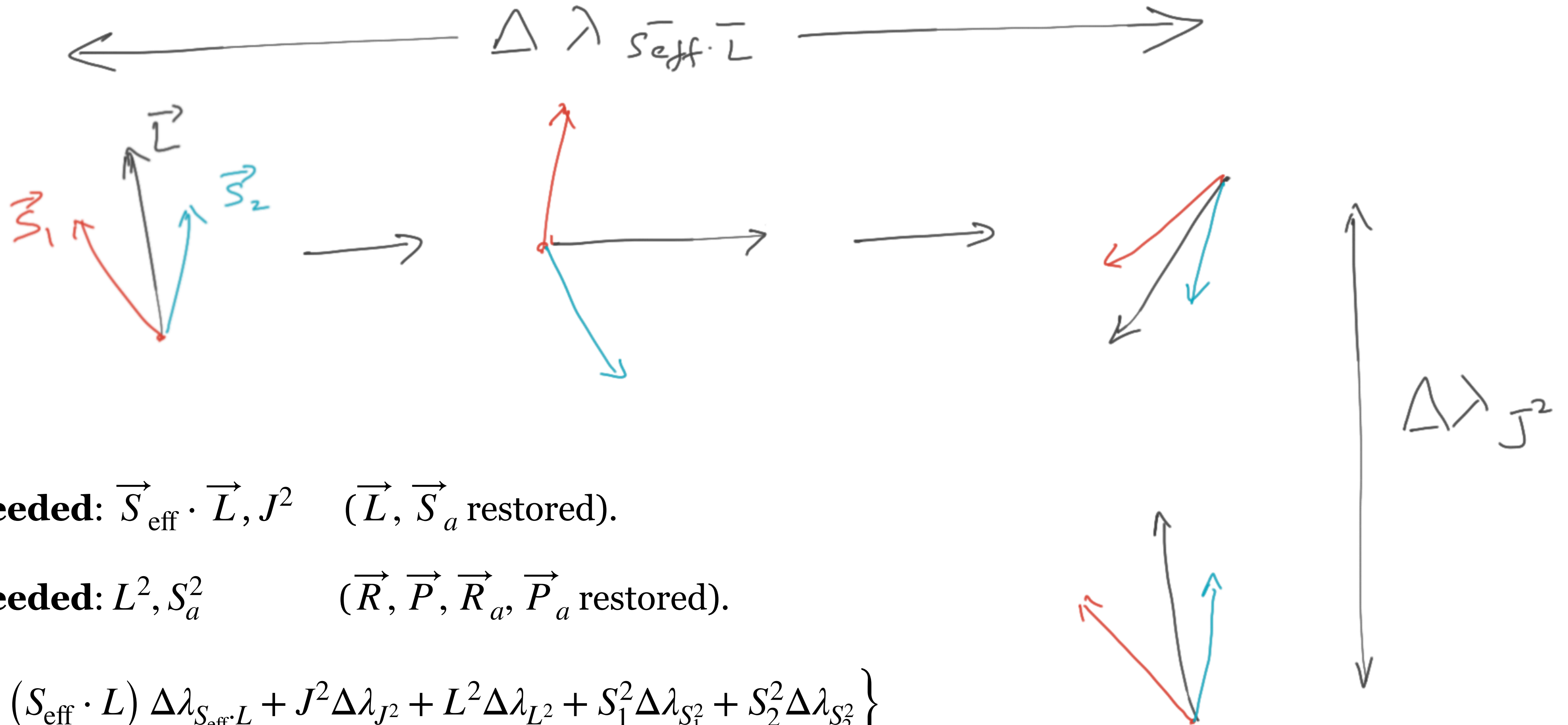
$$\Rightarrow \quad \mathcal{J}_5 = \frac{1}{\pi} \left\{ (S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L} + J^2 \Delta\lambda_{J^2} + L^2 \Delta\lambda_{L^2} + S_1^2 \Delta\lambda_{S_1^2} + S_2^2 \Delta\lambda_{S_2^2} \right\}.$$

•

• Loop for  $\mathcal{J}_5$  is closed by flowing under 5  $C_i$ 's (not one).

• Flow amounts  $\Delta\lambda_i$ 's give  $\mathcal{J}_5$

# $\mathcal{I}_5$ computation: flow overview



**THE END**

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the presentation 

Thank you!