

Closed-form solutions of spinning BBHs at $1.5PN$ (using action-angle variables)

Sashwat Tanay

References

- **RESEARCH PAPERS**

- The standard way of computing the solution (without 1PN part): <https://arxiv.org/abs/1908.02927>
- Action-angle-based solution: <https://arxiv.org/abs/2012.06586>, <https://arxiv.org/abs/2110.15351>

- **LECTURE NOTES**

- Lecture notes (latest): https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/gh-action-result/pdflatex/lecture_notes/main.pdf
- Lecture notes (for citation purposes): <https://arxiv.org/abs/2206.05799>

- **MATHEMATICA PACKAGE**

- Mathematica package on GitHub: <https://github.com/sashwattanay/BBH-PN-Toolkit>

- **YOUTUBE VIDEO**

- <https://youtu.be/aoiCk5TtmvE>

- **THIS PRESENTATION**

- https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/main/UIUC_workshop_presentation/uiuc_workshop_presentation.pdf

Lecture plan

Lecture style: standing on the shoulders of giants (due to time constraints)

- **Lecture 1:**
 - Theory
 - Strategy to compute solution from action-angles
- **Lecture 2:**
 - Construct the solution

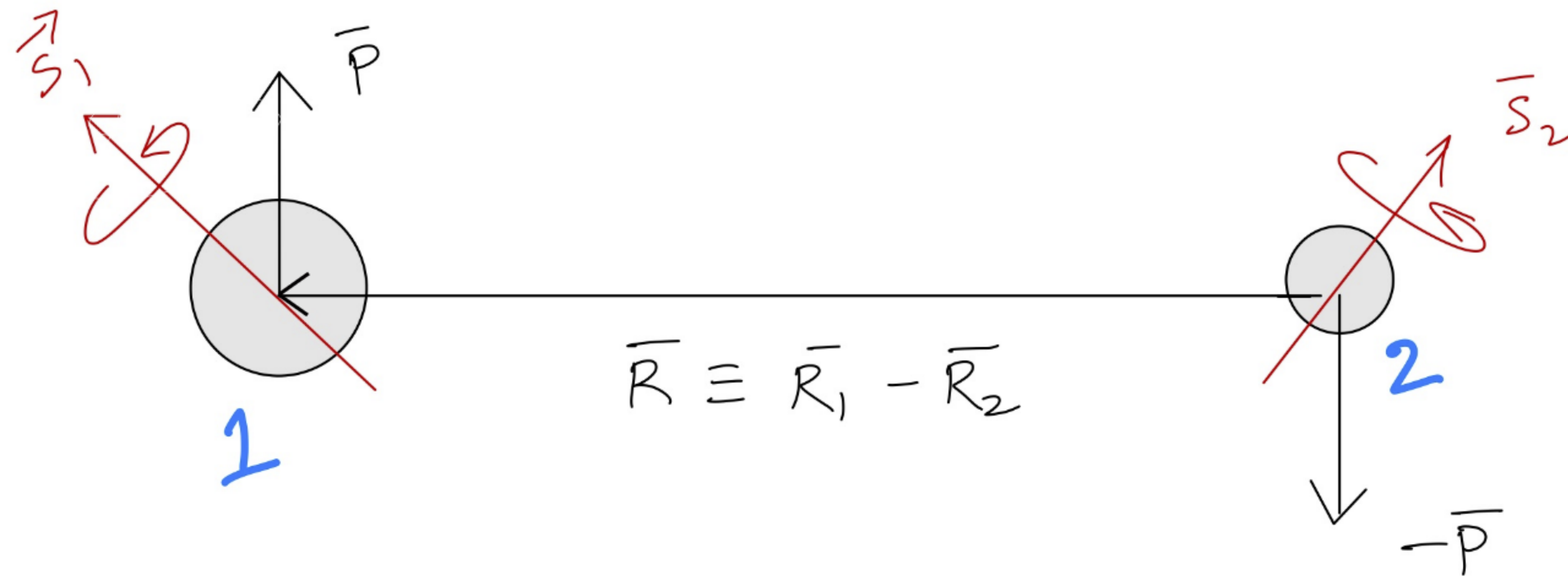
Lecture plan

- **Lecture 1:**
 - Theory
 - Strategy to compute solution from action-angles
- **Lecture 2:**
 - Construct the solution

Introduction to the system

Spinning 1.5PN BBH system

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Phase space variables

$\vec{R}(t)$, $\vec{P}(t)$, $\vec{S}_1(t)$ and $\vec{S}_2(t)$

$\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$

Statement of the problem

- The 1.5PN Hamiltonian is $H = H_{\text{N}} + H_{1\text{PN}} + H_{1.5\text{PN}} + \mathcal{O}(c^{-4})$ with
- $H_{\text{N}} = \mu \left(\frac{p^2}{2} - \frac{1}{r} \right), \quad H_{1.5\text{PN}} = \frac{2G}{c^2 R^3} \vec{S}_{\text{eff}} \cdot \vec{L}.$
- Hamilton's equations $\implies \frac{d \left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t) \right)}{dt}.$
- **Problem:** Integrate Hamilton's eqns. to obtain $\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t).$

Historical context and the status quo

- The 1.5PN Hamiltonian is $H = H_N + H_{1\text{PN}} + H_{1.5\text{PN}} + \mathcal{O}(c^{-4})$.
- **1680s**: Issac Newton gave the Newtonian solution $R = a(1 - e \cos u)$.
- **1985**: Damour-Deruelle gave 1PN quasi-Keplerian solution.
- **2019**: Gihyuk Cho, H. M. Lee gave 1.5PN solution (1PN effects ignored for simplicity)
- **2021**: We worked out an equivalent action-angle based solution (subject of these lectures).
- Why action-angles? Extendible to 2PN via canonical perturbation theory (Goldstein).

Do we even the solution? YouTube link

- Show all three solutions
- YouTube video

EOMs with Poisson brackets

Standard approach

- Hamilton's eqns. are $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$
- Leads to EOM $\frac{df}{dt} = \{f, H\}$ with $\{f, g\} \equiv \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$

EOMs with Poisson brackets for BBHs

Our approach

- Define EOMs: $\frac{df(t)}{dt} = \{f, H\}$ where $f = f\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)$.
- Define PBs: $\left\{R_i, P_j\right\} = \delta_{ji}$ $\left\{S_A^i, S_B^j\right\} = \delta_{AB} \epsilon_k^{ij} S_A^k$.

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, \quad a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\left\{f, g(v_i)\right\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

PB Exercise 1

Prob: Compute $\{R_x, \sin P_x + P_x\}$.

Sol: Using the bilinearity and the chain rule (2nd and 4th rules) for PBs

$$\begin{aligned} & \{R_x, \sin P_x + P_x\} \\ &= \{R_x, \sin P_x\} + \{R_x, P_x\} \\ &= \{R_x, P_x\} \frac{\partial \sin P_x}{\partial P_x} + \{R_x, P_x\} \\ &= \cos P_x + 1. \end{aligned}$$

PB Exercise 2

Prob: Show that $\{\phi_A, S_B^z\} = \delta_{AB}$, where $\phi_A = \arctan(S_A^y/S_A^x)$ is the azimuthal angle of \vec{S}_A .

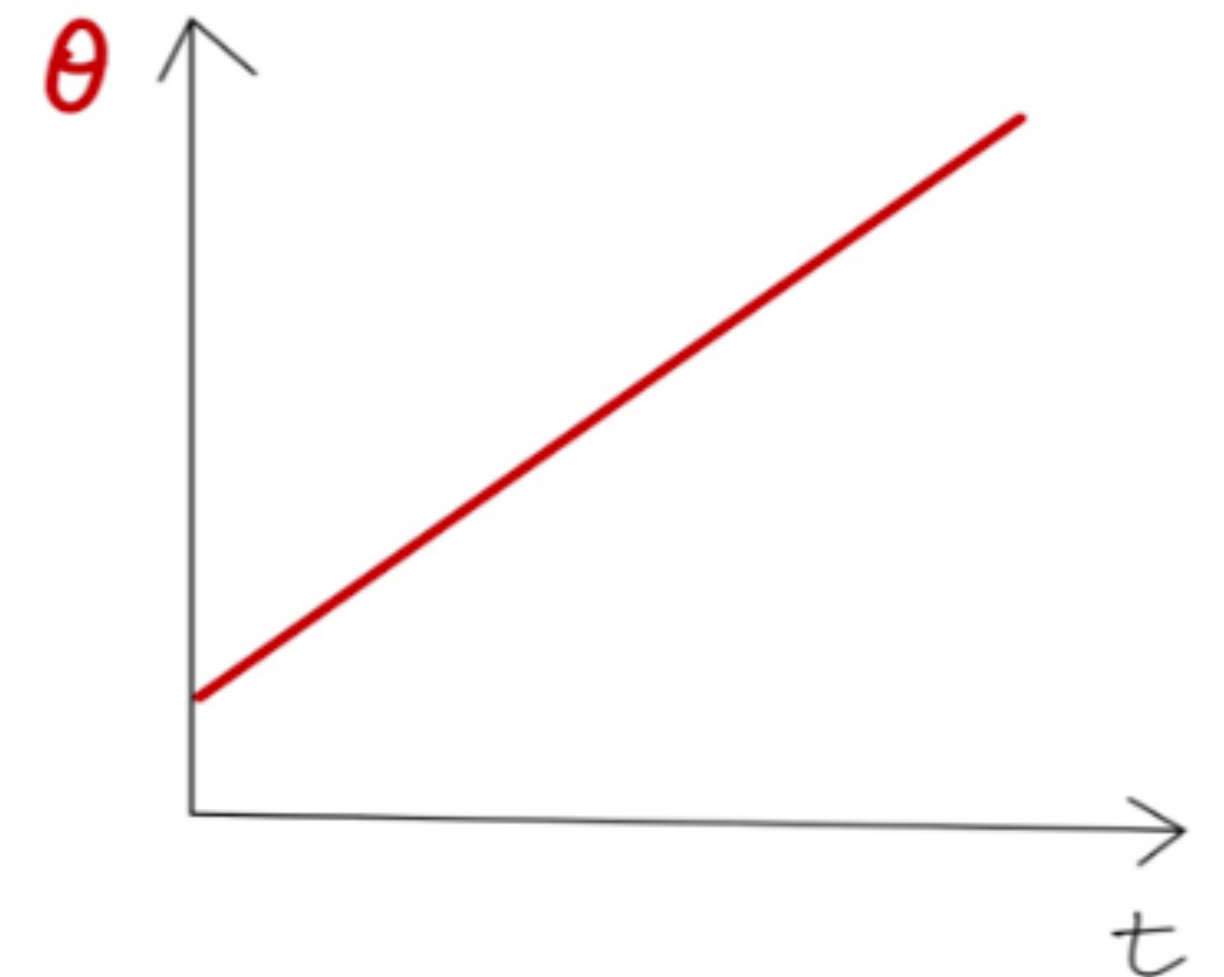
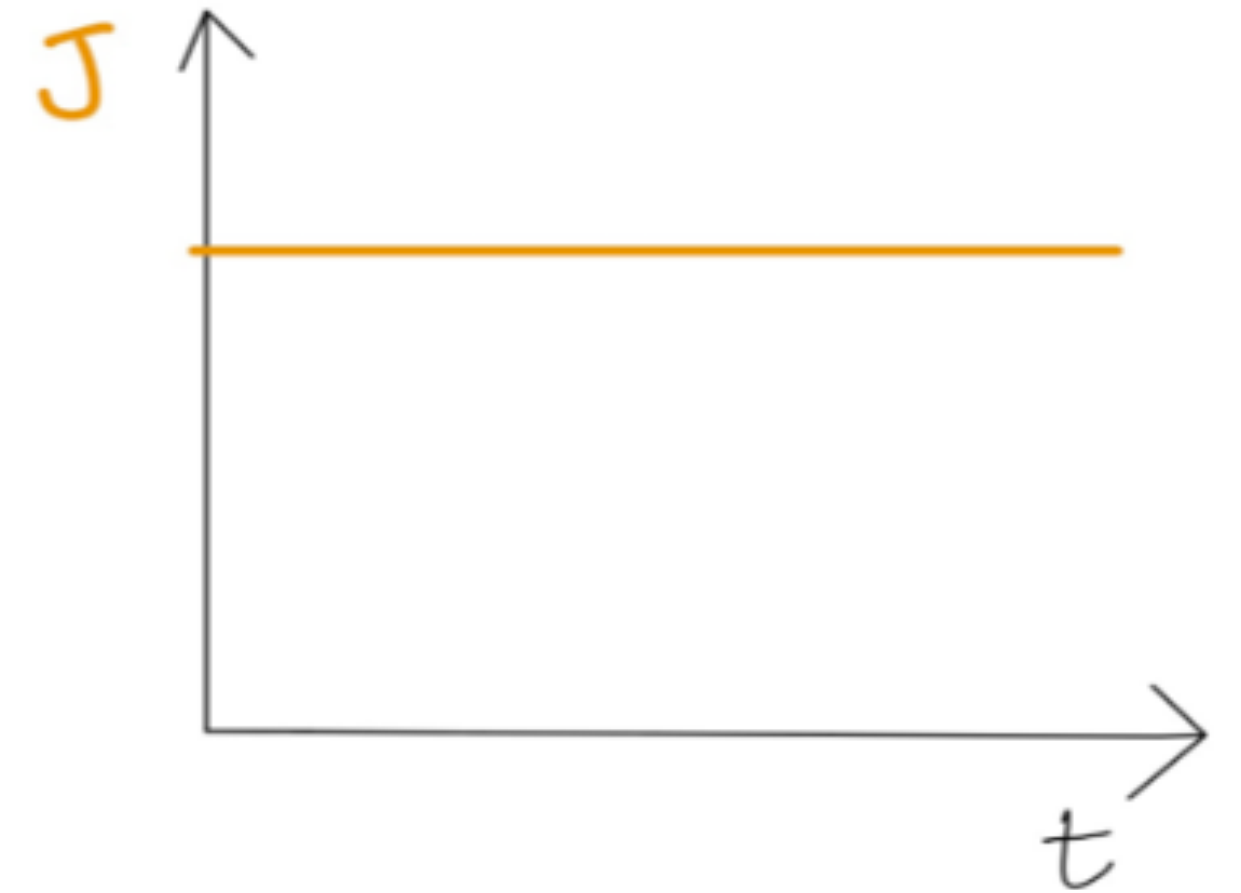
- Implies that $\phi \sim$ position; $S^z \sim$ momentum upon comparison with $\{R_i, P_j\} = \delta_{ji}$.
- **Lingo:** f and g commute if $\{f, g\} = 0$.
- How to evaluate PBs: See the YouTube video (...add hyperlink)

Integrable systems and action-angles

- **Integrable system:** canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$.
- Action $\mathcal{J}_i = \sim p$; angle $\theta_i = \sim q$.
- Hamilton's equations \implies

$$\dot{\mathcal{J}}_i = -\partial H / \partial \theta_i = 0 \quad \implies \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \quad \implies \theta_i = \omega_i(\vec{\mathcal{J}})t.$$
- Having action-angles \sim having closed-form solutions.



Liouville-Arnold theorem

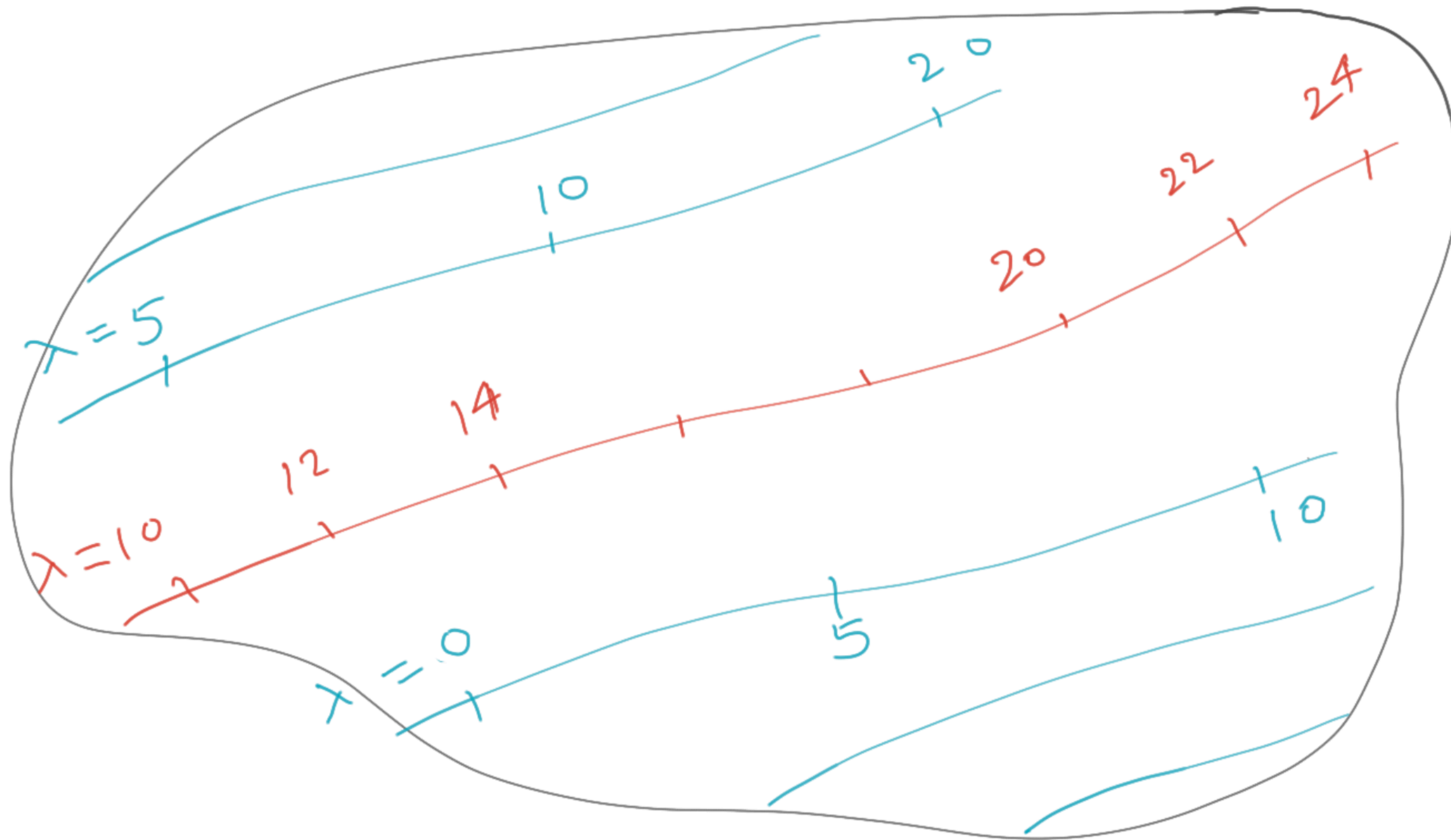
- **Theorem:** $2n$ phase space variables & n commuting constants of motion \implies integrability.
- How to check if f is a constant of motion? Check if $\{f, H\} = 0$.
- For BBH phase space $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$, $2n \neq 12$. Positions-momenta delineation not clear for spins.
- Easy to check that $\{R_i, P_j\} = \delta_{ij}$ and $\{\phi_A, S_B^z\} = \delta_{AB}$ where $\phi_A = \arctan(S_A^y/S_A^x)$, the azimuthal angle for \vec{S}_A .
- (ϕ_A, S_A^z) are the **positions, momenta** of \vec{S}_A . Only 2 variables needed for \vec{S}_A since $dS_A/dt = \{S_A, H\} = 0$.
- Hence $2n = 3 + 3 + 2 + 2 = 10 \implies 10/2 = 5$ commuting constants needed for integrability.

Commuting constants for BBHs

- With $m \equiv m_1 + m_2$, $\mu \equiv m_1 m_2 / m$, $\nu \equiv \mu / m$, $\vec{L} \equiv \vec{R} \times \vec{P}$
 $\sigma_1 \equiv (2 + 3m_2/m_1)$, $\sigma_2 \equiv (2 + 3m_1/m_2)$, $\vec{S}_{\text{eff}} \equiv \sigma_1 \vec{S}_1 + \sigma_2 \vec{S}_2$
 $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$.
- The 5 commuting constants are long known: $H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}$.
- Hence the 1.5PN BBH is integrable and has action-angles.

Curves, vectors, vector field and flow

- **Pictorial definition:** the vector is $d/d\lambda$ (a derivative operator).



Hamiltonian flow of a function $f(\vec{V})$

- $\vec{V} \equiv \{ \vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2 \}$, unless states otherwise.
- Hamiltonian flow of $f(\vec{V})$: $\frac{d\vec{V}}{d\lambda} = \{ \vec{V}, f \}$. f need not be the Hamiltonian.
- Solution of the flow given in the form $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda)$.
- Flow \equiv Hamiltonian flow (for brevity).

Flow exercise 1

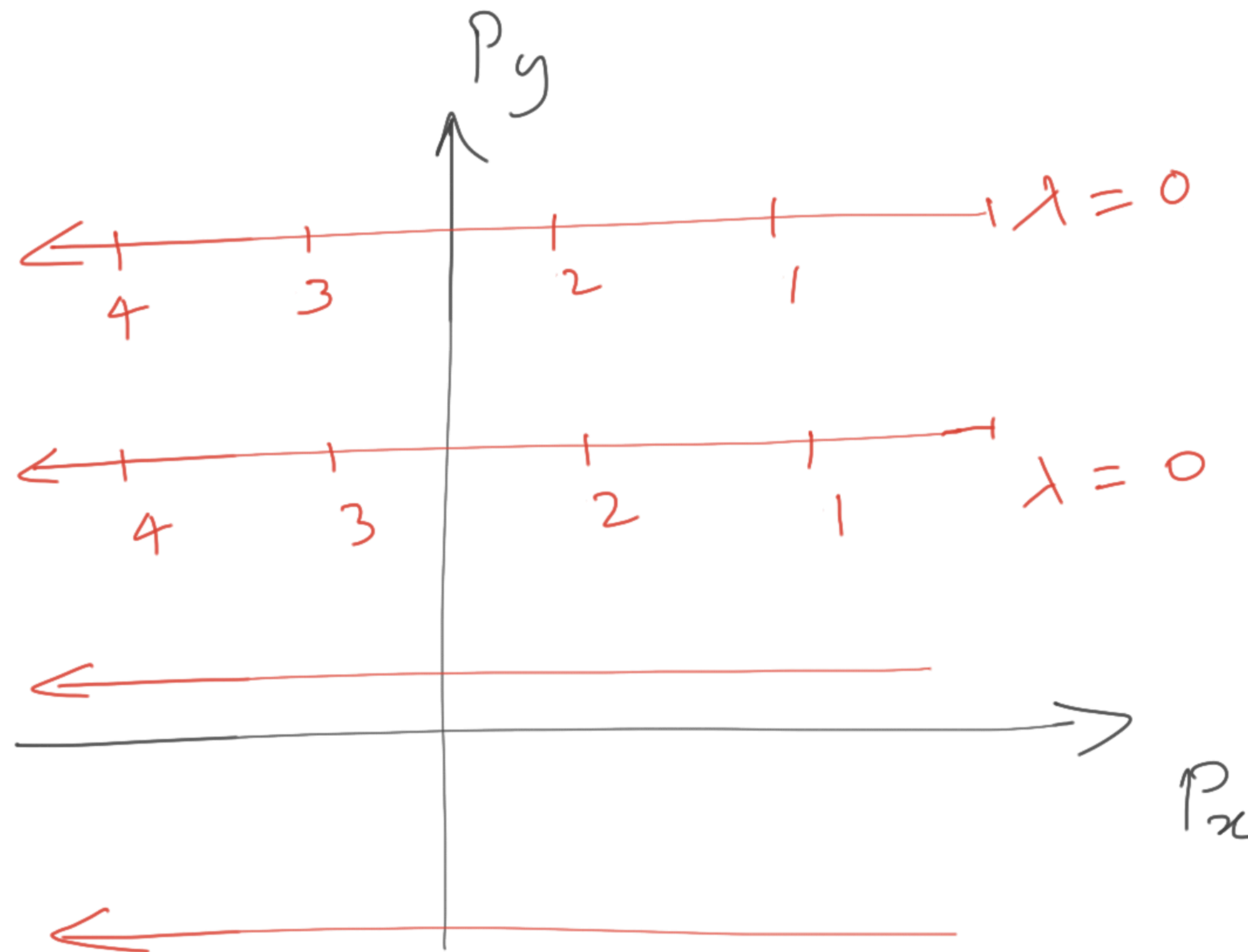
Prob: Solve the flow under R_x and draw pictures.

Sol: Under the R_x flow:

$$\frac{dP_x}{d\lambda} = \{P_x, R_x\} = -1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i\text{'s.}$$

$$\Rightarrow P_x - P_x(\lambda_0) = (\lambda_0 - \lambda).$$



Flow exercise 2

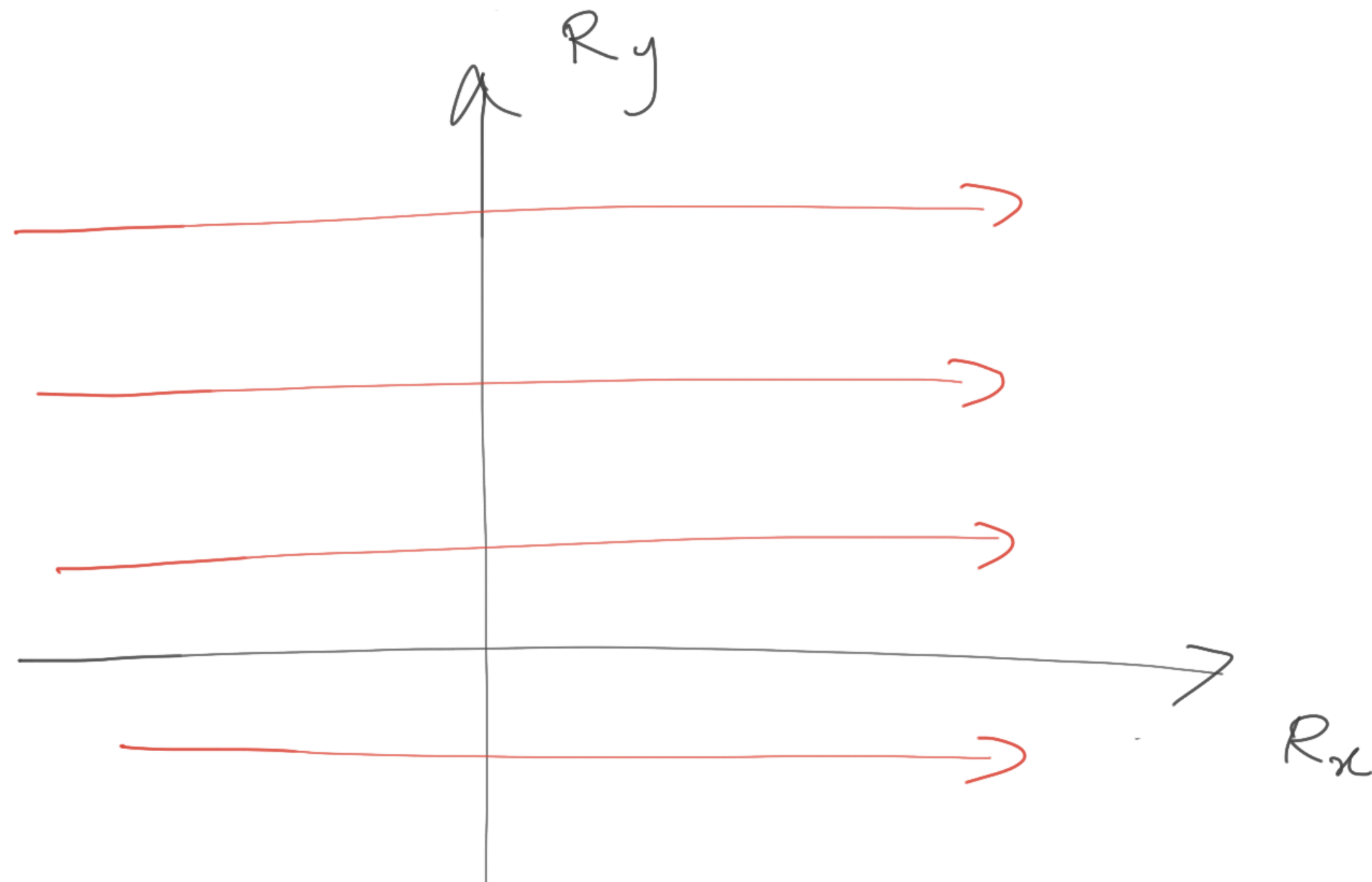
Prob: Solve the flow under P_x and draw pictures.

Sol: Under the R_x flow:

$$\frac{dR_x}{d\lambda} = \{R_x, P_x\} = 1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i\text{'s.}$$

$$\implies R_x - R_x(\lambda_0) = (\lambda - \lambda_0).$$



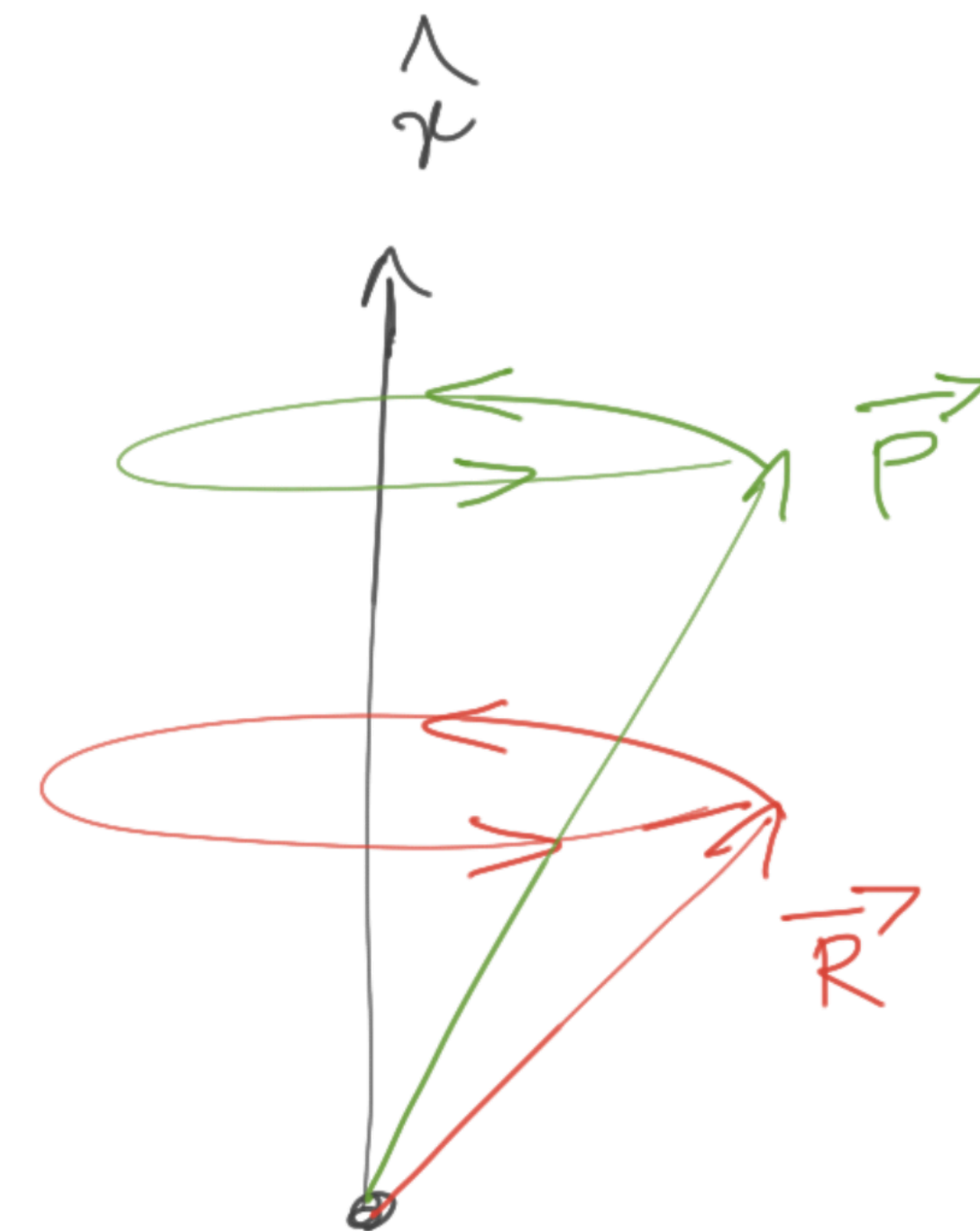
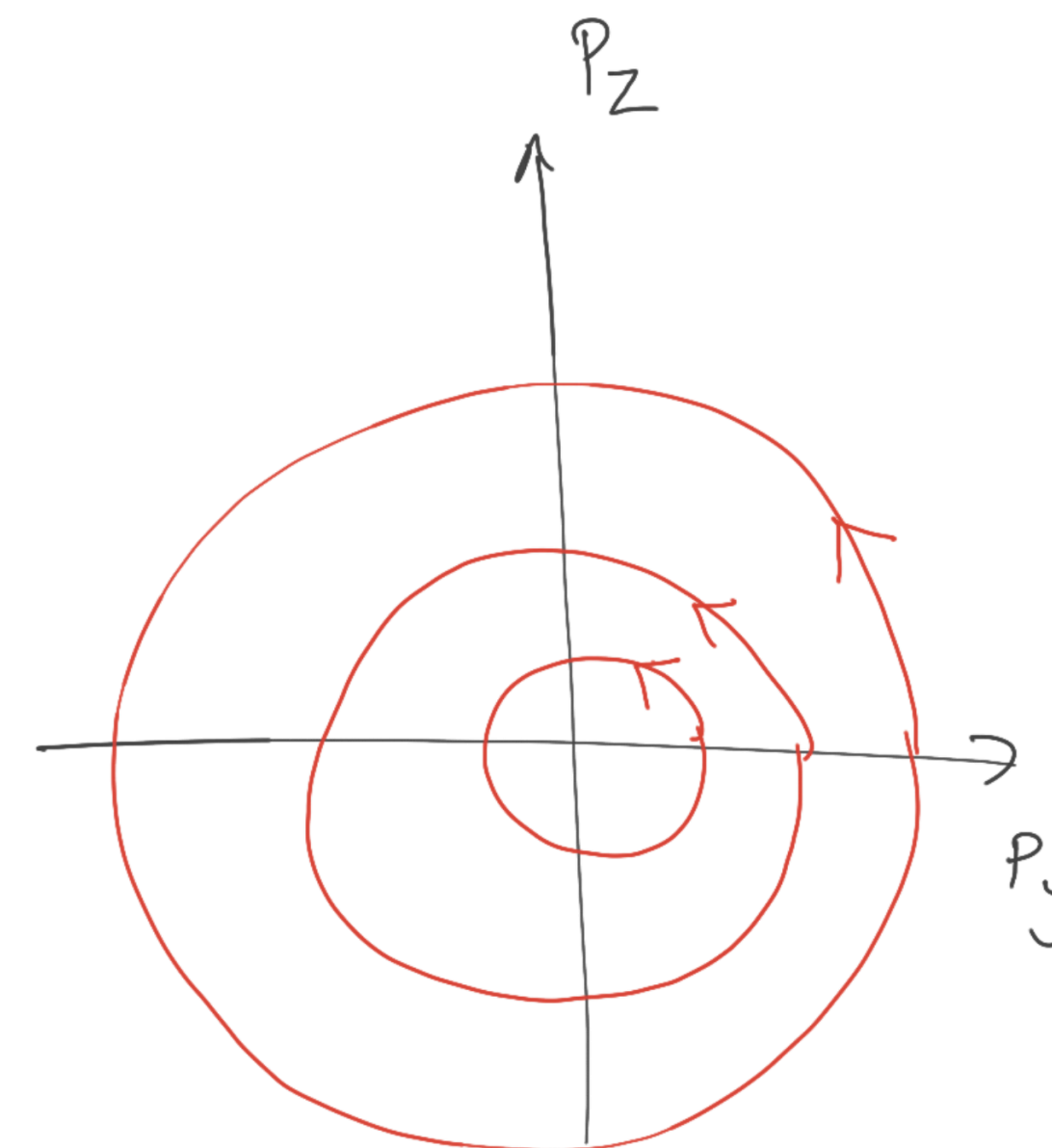
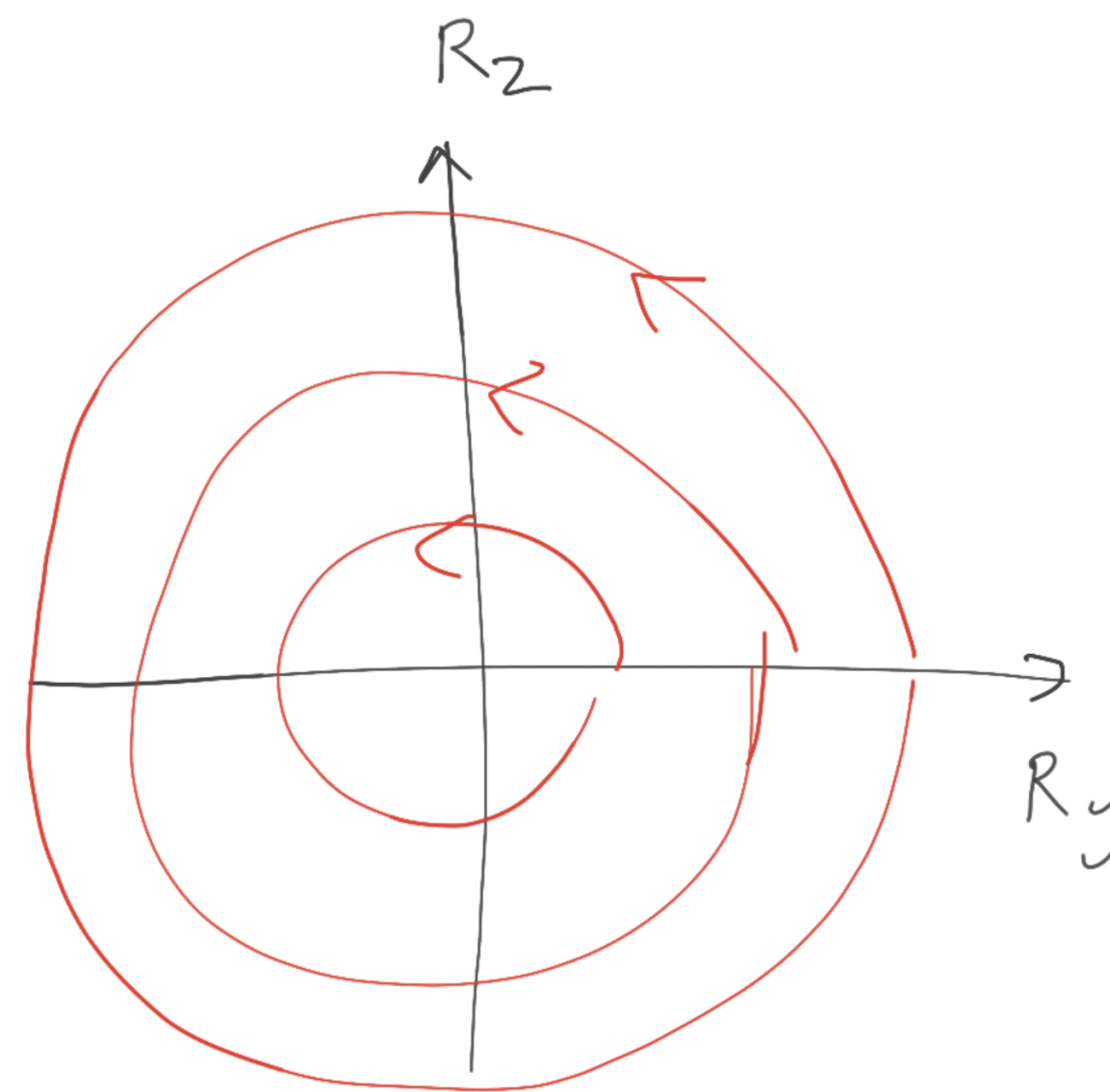
Flow exercise 3

Prob: Draw pictures for L_x flow where $\vec{L} = \vec{R} \times \vec{P}$.

Sol: Under the L_x flow:

$$\frac{d\vec{R}}{d\lambda} = \hat{x} \times \vec{V}$$

$$\frac{d\vec{P}}{d\lambda} = \hat{x} \times \vec{V}$$

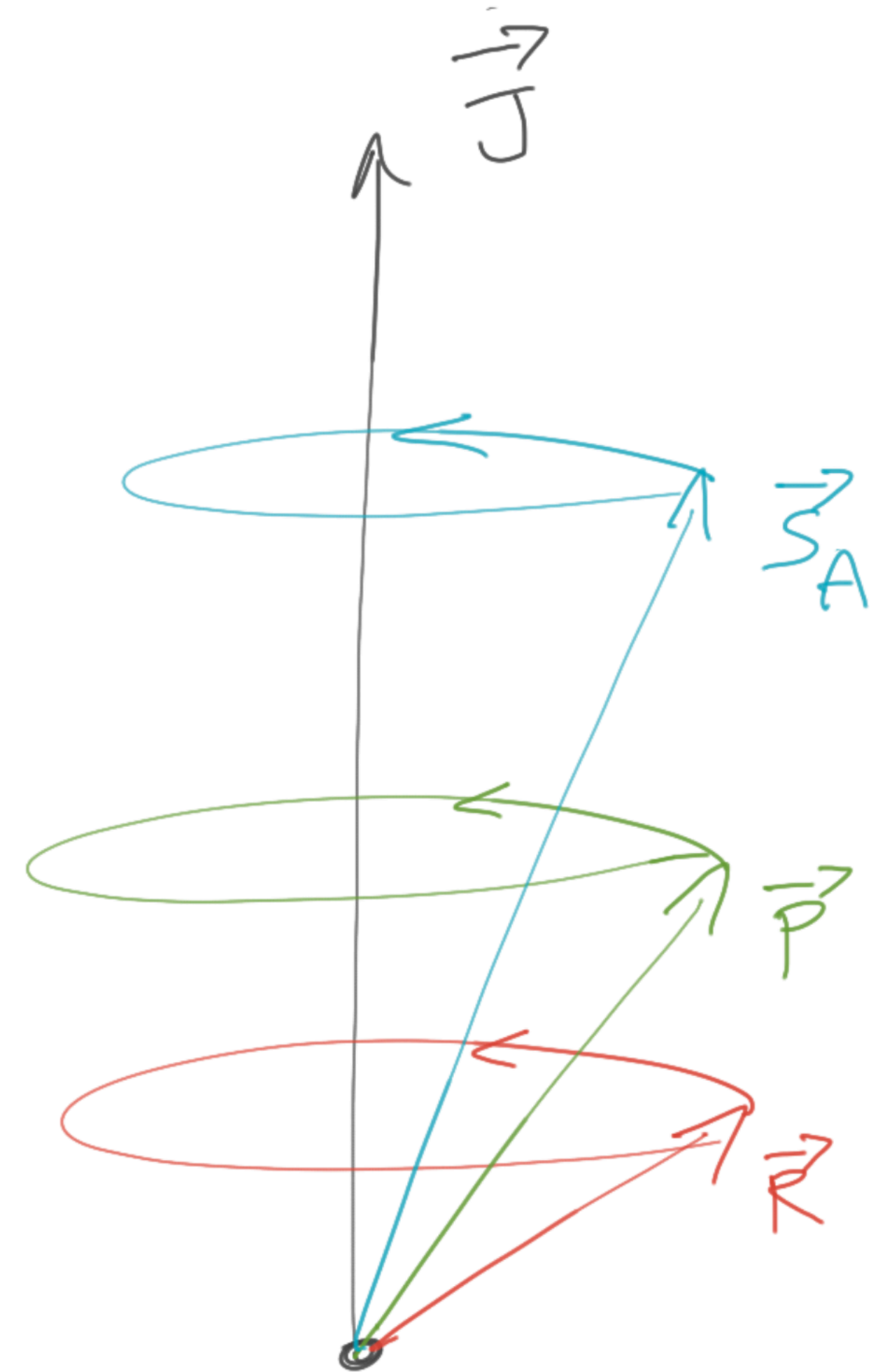
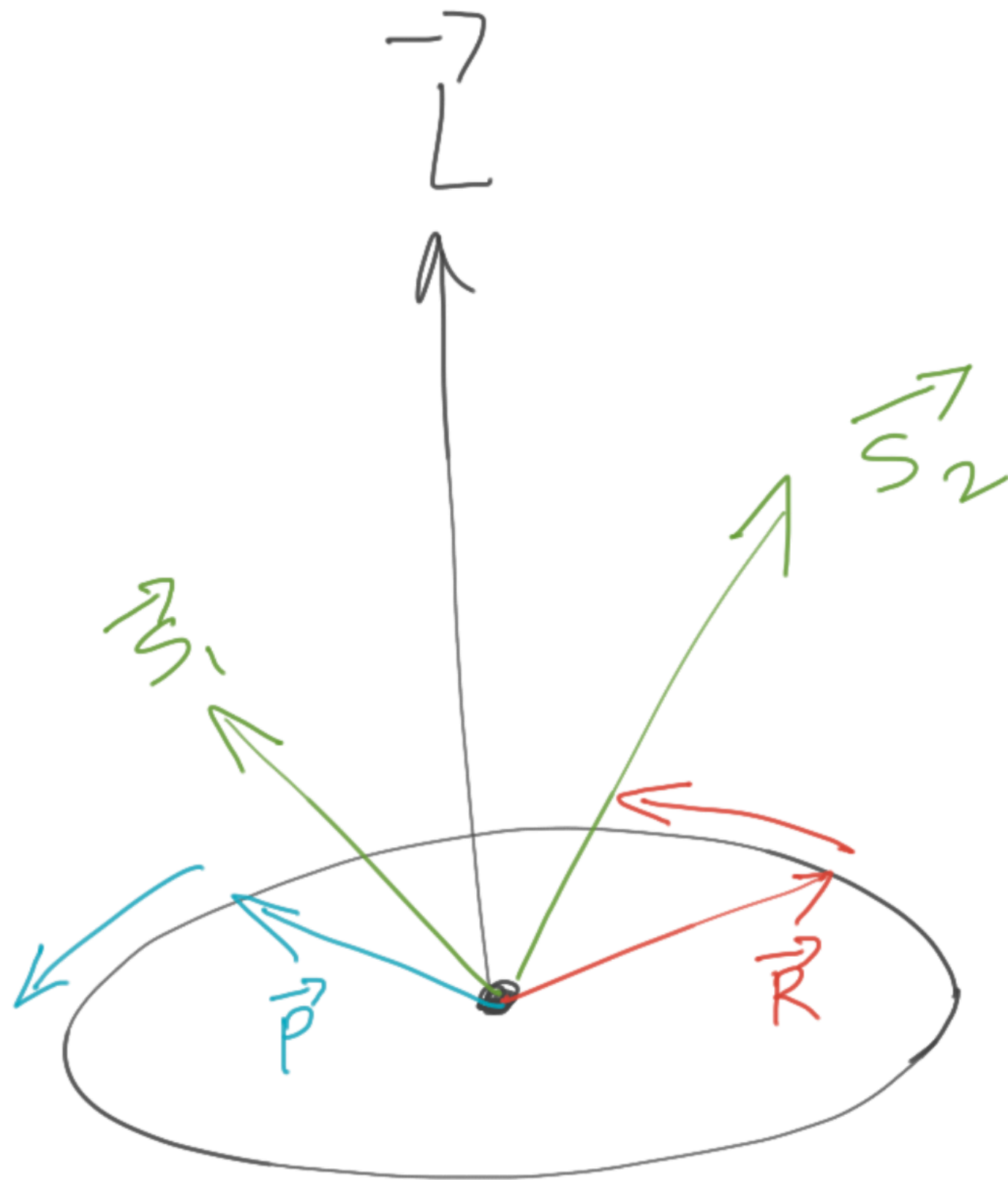


Flow exercise 4

Prob: Draw pictures for L^2 and J^2 flow where $\vec{L} = \vec{R} \times \vec{P}$ and $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$.

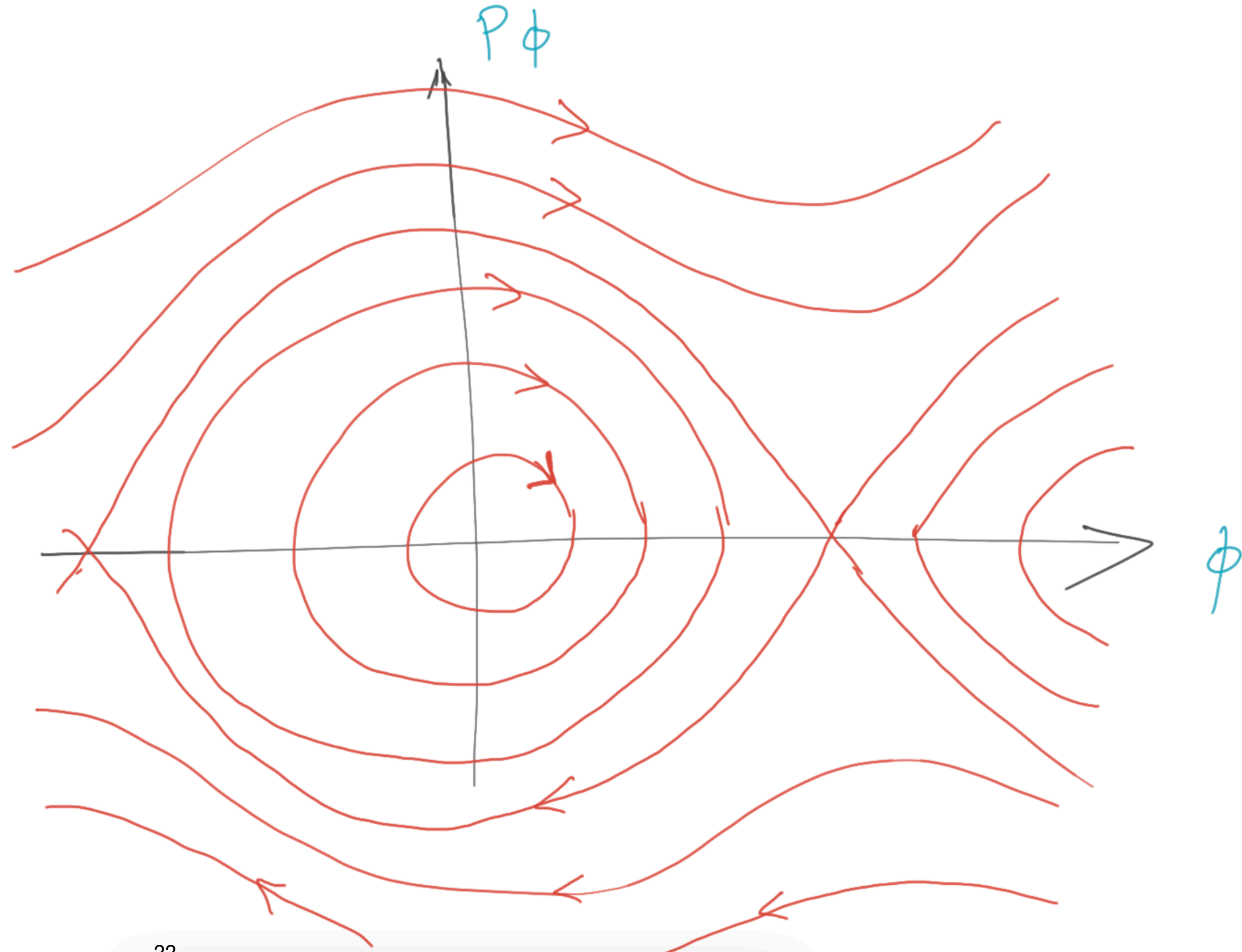
Sol: L^2 flow $\implies \{\vec{L}, L^2\} = \{\vec{S}_A, L^2\} = 0 \implies \vec{L}, \vec{S}_A$ remain fixed.

J^2 flow $\implies \{\vec{J}, J^2\} = 0 \implies \vec{J}$ remains fixed



Hamiltonian flow of the Hamiltonian H

- With $\vec{V} \equiv \{ \vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2 \}$, flow eqn. is $\frac{d\vec{V}}{d\lambda} = \{ \vec{V}, f \}$.
- Flow under $H \implies \frac{d\vec{V}}{d\lambda} = \{ \vec{V}, H \}$.
- This is the **EOM**. Gives the **real-time evolution**, unlike other flows.
- Hamiltonian flow of the Hamiltonian is special.
- **Example:** flow under H for a pendulum



5 minute break

Coffee, questions?

Lecture plan

- **Lecture 1:**
 - Theory
 - Strategy to compute solution from action-angles
- **Lecture 2:**
 - Construct the solution

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of the next lecture).

- How to combine \vec{C} flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
- Solution via action-angles.

Action-angle-based solution: strategy

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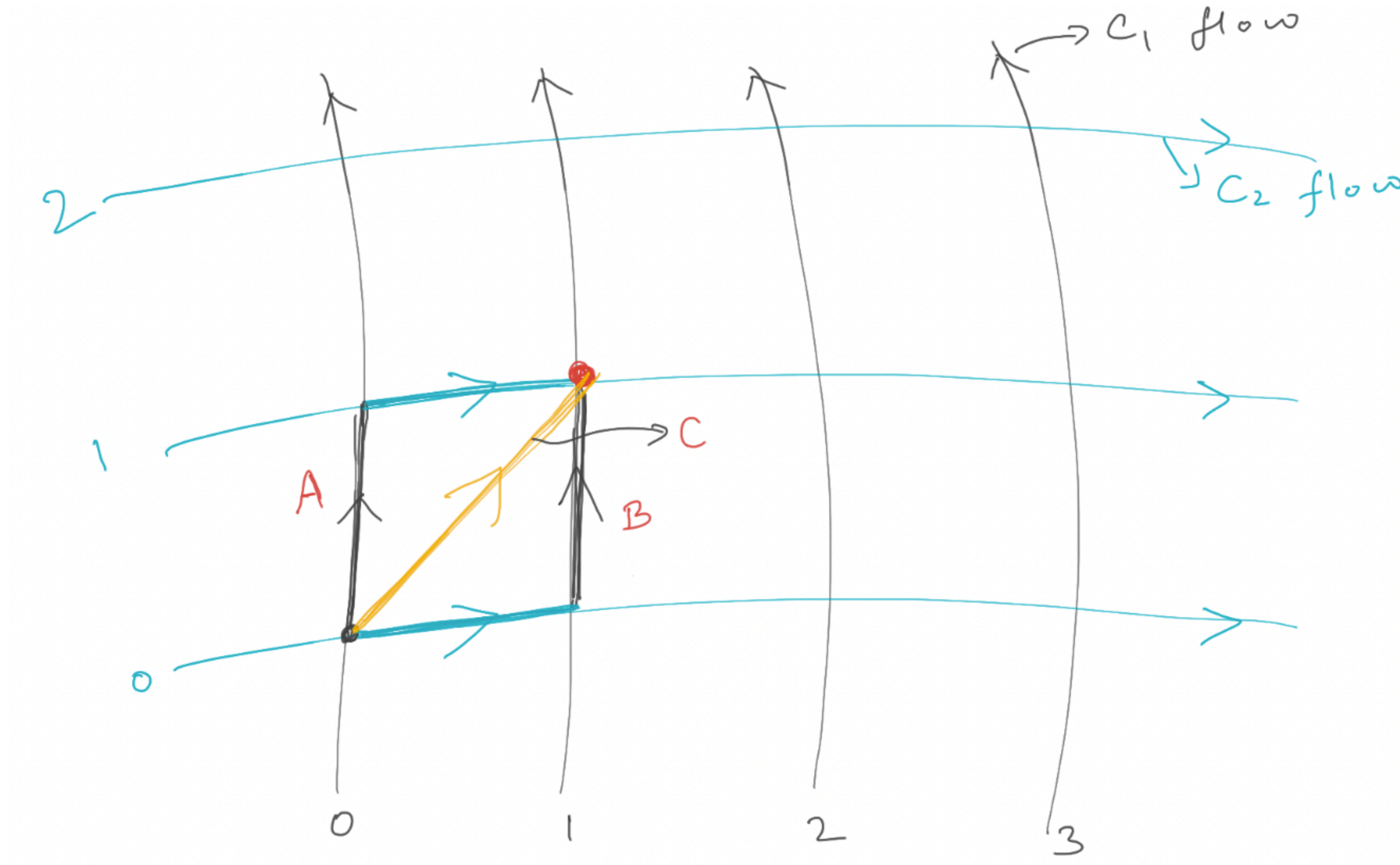
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- Solution via action-angles.

How to combine \vec{C} flows?

- Assume C_i 's are commuting quantities (don't have to be constants).
- **Notation:** Output of C_i flow $\frac{d\vec{V}}{d\lambda_i} = \{ \vec{V}, C_i \}$ denoted by $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda_i)$.
- **Result:** Order of flow does not matter, i.e. $((\vec{V}_0, \Delta\lambda_1), \Delta\lambda_2) = ((\vec{V}_0, \Delta\lambda_2), \Delta\lambda_1)$.
- **Result:** Simultaneous flows can be made sequential: $\frac{d\vec{V}}{d\lambda} = \{ \vec{V}, C_1 + C_2 \}$ by $\Delta\lambda$ is C_1 flow followed by C_2 flow (both by $\Delta\lambda$). Or in the reverse order.

How to combine \vec{C} flows?

Pictorial depiction of the two flow rules



Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
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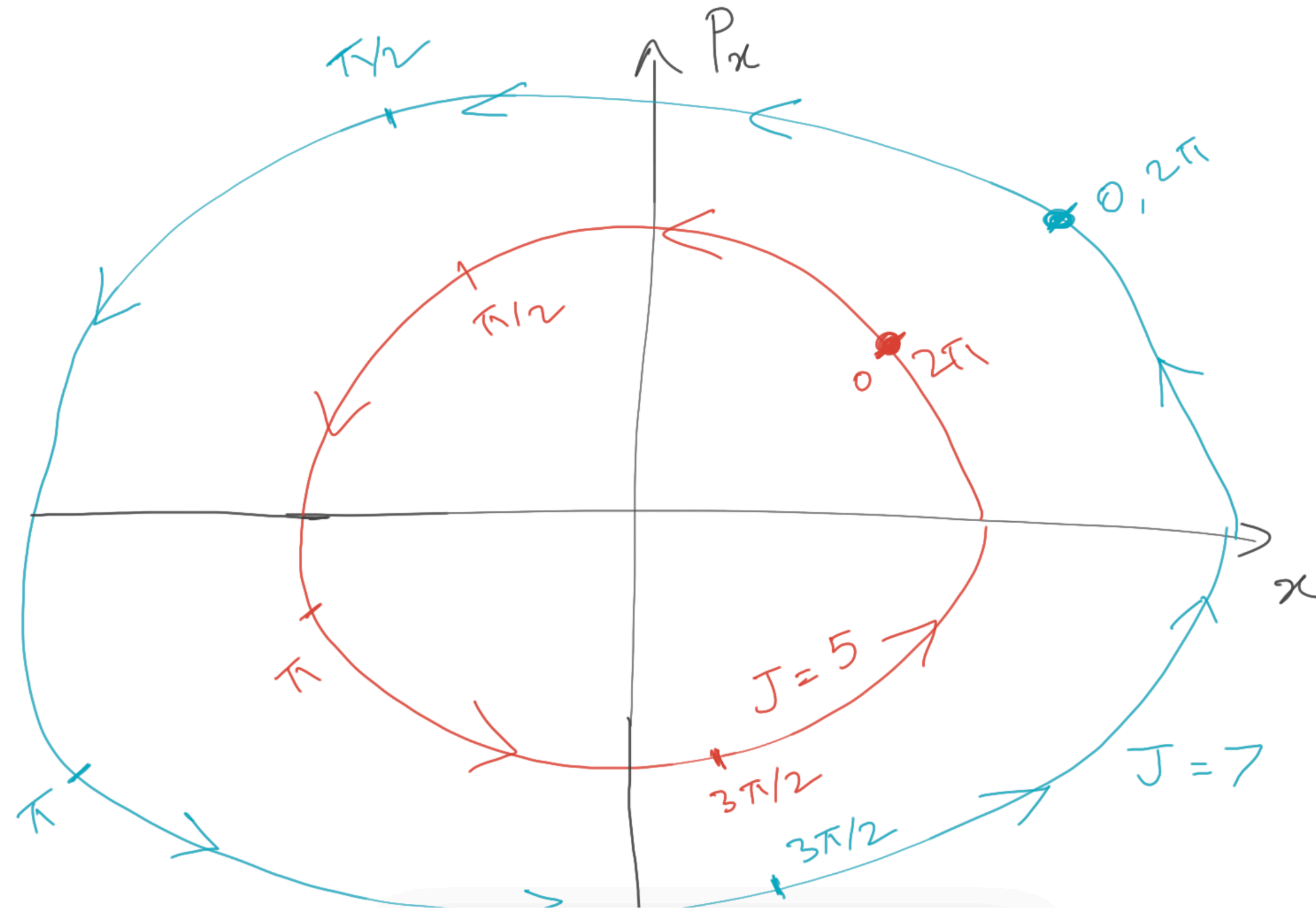
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- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
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Construct action-angles

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$ with $\{R_i, P_j\} = \delta_{ij}$ and $\{\phi_A, S_B^z\} = \delta_{AB}$.
- Loop γ_i on the $\vec{C} = \text{constant}$ submanifold.
- No. of independent \mathcal{J}_i 's = n despite infinite no. of loops.
- \mathcal{J}_i flow by $2\pi \rightarrow$ loop (different from γ_i).
- Angle $\theta_i \equiv \lambda_i$ along the \mathcal{J}_i flow.

Construct action-angles

Pictorial depiction of the construction



Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of the next lecture).

- How to combine \vec{C} flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
- Solution via action-angles.

Integrable systems and action-angles

- **Integrable system:** canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$.

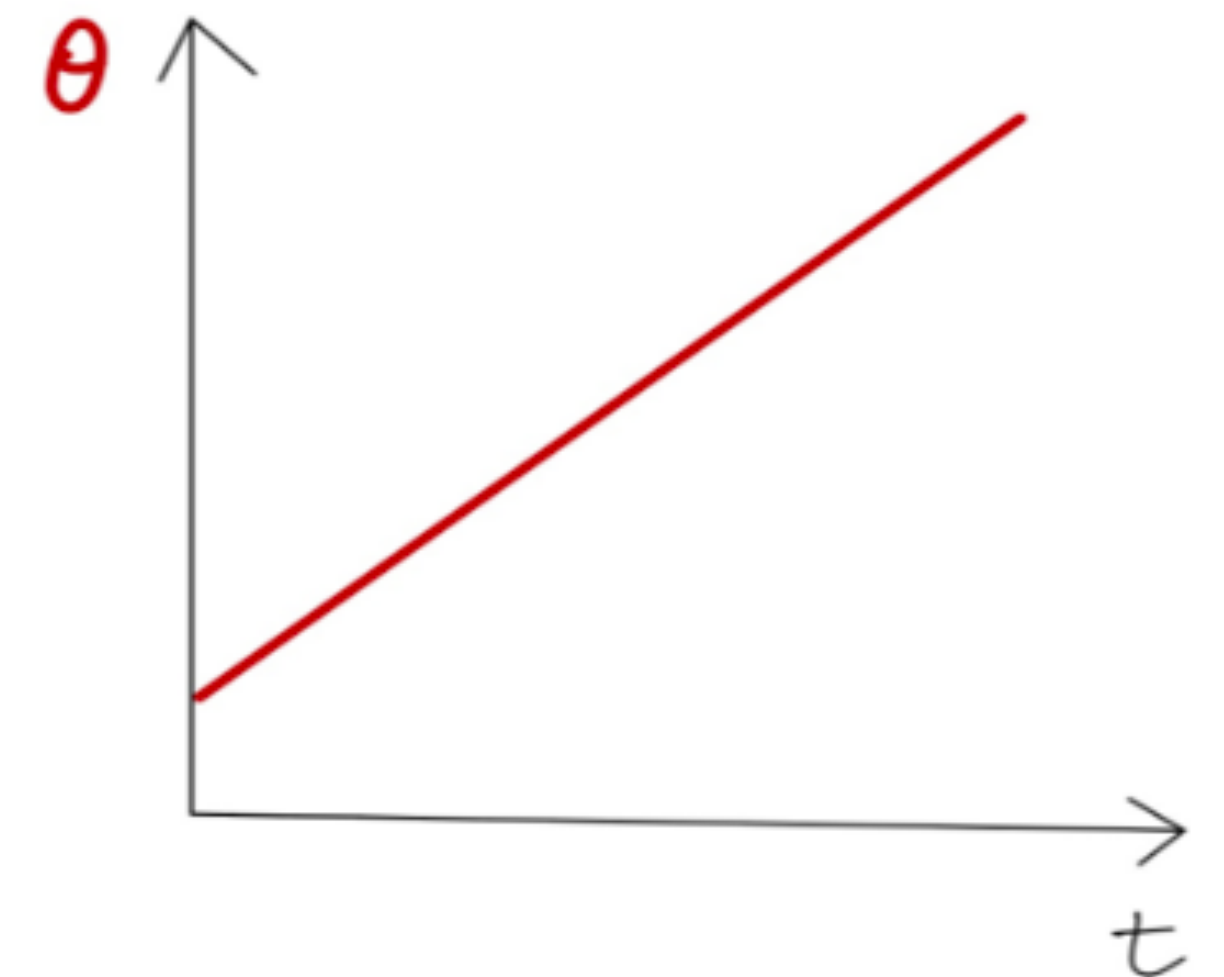
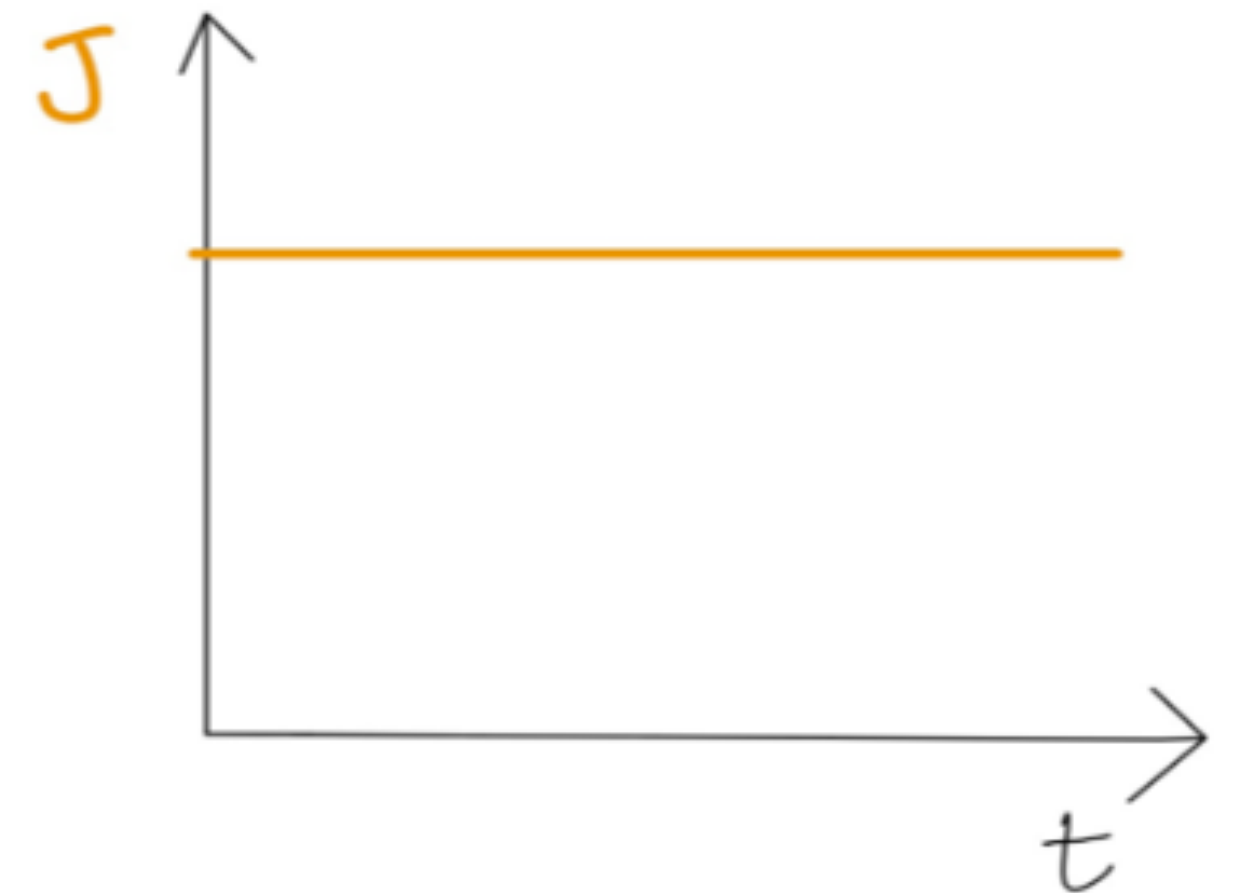
- Action $\mathcal{J}_i = \sim p$; angle $\theta_i = \sim q$.

- Hamilton's equations \implies

$$\dot{\mathcal{J}}_i = -\partial H / \partial \theta_i = 0 \quad \implies \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \quad \implies \theta_i = \omega_i(\vec{\mathcal{J}})t.$$

- Having action-angles \sim having closed-form solutions.



Compute frequencies $\omega_i \equiv d\theta_i/dt$

- Recall $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$.
- With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have $\mathcal{J}_i(\vec{C})$ (next lecture's subject).
- Compute the Jacobian $M_{ij}(\vec{C}) \equiv \frac{\partial \mathcal{J}_i}{\partial C_j}$ (consists of numeric constants).
- **Inverse function theorem:** If $N_{ij} \equiv \frac{\partial C_i}{\partial \mathcal{J}_j}$, then $N = M^{-1}$.
- The first row of N corresponding to $(C_1 = H)$ contains $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$.

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of the next lecture).

- How to combine \vec{C} flows?
- Construct action-angles.
- ~~Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.~~
- How to flow along the actions \mathcal{J}_i ?
- Solution via action-angles.

EOMs with Poisson brackets for BBHs

Our approach

- Define EOMs: $\frac{df(t)}{dt} = \{f, H\}$ where $f = f\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)$.
- Define PBs: $\left\{R_i, P_j\right\} = \delta_{ji}$ $\left\{S_A^i, S_B^j\right\} = \delta_{AB} \epsilon_k^{ij} S_A^k$.

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, \quad a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\left\{f, g(v_i)\right\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

How to flow along the actions \mathcal{J}_i ?

- With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions (next lecture's subject).
- Using chain rule for PBs,
$$\frac{d\vec{V}}{d\lambda} = \left\{ \vec{V}, \mathcal{J}_i \right\} = \left\{ \vec{V}, C_j \right\} \left(\frac{\partial \mathcal{J}_i}{\partial C_j} \right) = 2.5 \{ \vec{V}, C_1 \} + 5.1 \{ \vec{V}, C_2 \}$$

$$= \{ \vec{V}, 2.5C_1 + 5.1C_2 \}.$$
- \mathcal{J}_i flow by $\Delta\lambda$ = $(C_1$ flow by $2.5\Delta\lambda$, then C_2 flow by $5.1\Delta\lambda$), reverse the order.

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of the next lecture).

- How to combine \vec{C} flows?
- Construct action-angles.
- ~~Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.~~
- ~~How to flow along the actions \mathcal{J}_i ?~~
- Solution via action-angles.

Solution via action-angles.

- Start with an initial $\overrightarrow{V}_0 = \{ \overrightarrow{R}, \overrightarrow{P}, \overrightarrow{S}_1, \overrightarrow{S}_2 \}$. Assign it $\overrightarrow{\theta} = \overrightarrow{0}$.
- We want $\overrightarrow{V} = \overrightarrow{V}(\overrightarrow{V}_0, t)$.
- Recall $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$ and $\Delta \theta_i = \Delta \lambda_i$.
- After time t , $\theta_i(t) = \omega_i t$.
- How to increase the angles? Action flows increase the angles.
- We need to flow under \mathcal{J}_i 's by an amount $\lambda_i = \theta_i(t) = \omega_i t$.

Action-angle-based solution: strategy

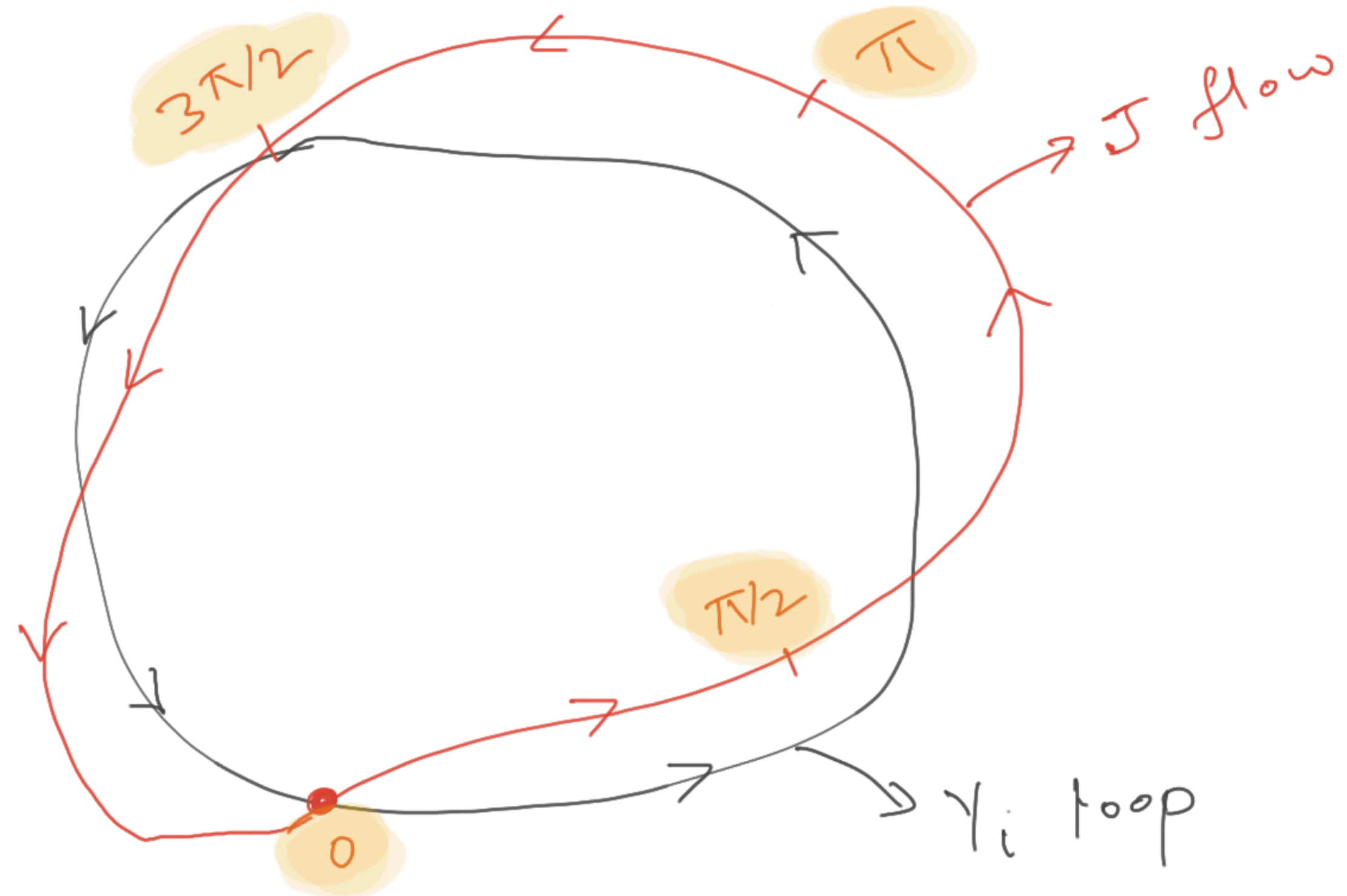
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- How to combine \vec{C} flows?
- Construct action-angles.
- ~~Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.~~
- ~~How to flow along the actions \mathcal{J}_i ?~~
- ~~Solution via action-angles.~~

Construct action-angles

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$. Loop γ_i on the $\vec{C} = \text{constant}$ submanifold.

- \mathcal{J}_i flow by $2\pi \rightarrow$ loop (different from γ_i).
- Angle $\theta_i \equiv \lambda_i$ along the \mathcal{J}_i flow.
- **To show:** $\{\theta_i, J_k\} = \delta_{ij}$, $\{J_i, J_k\} = 0$, $\{\theta_i, \theta_k\} = 0$.



Construct action-angles

- Using $\theta_i = \lambda_i$, $\frac{d\theta_i}{d\lambda_i} = 1$ and $\frac{d\theta_i}{d\lambda_i} = \{\theta_i, \mathcal{J}_i\} \implies \{\theta_i, \mathcal{J}_i\} = 1$.
- From definition \mathcal{J}_i and chain rule for PBs, $\mathcal{J}_i = \mathcal{J}_i(\vec{C}) \implies \{J_i, J_k\} = \frac{\partial J_i}{\partial C_l} \frac{\partial J_k}{\partial C_m} \{C_l, C_m\} = 0$.
- $\{\theta_i, \theta_j\} = 0$ involves changing \mathcal{J}_i , which does not happen with real evolution.
Hence ignore.
- “**Integrable system:** canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$.” that lead to $\dot{\mathcal{J}}_i = 0$; $\theta_i = \omega_i t$ is satisfied because action flow makes a loop after 2π .

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of the next lecture).

- How to combine \vec{C} flows?
- ~~Construct action-angles.~~
- ~~Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.~~
- ~~How to flow along the actions \mathcal{J}_i ?~~
- ~~Solution via action-angles.~~

Lecture plan

- **Lecture 1:**
 - Theory
 - Strategy to compute solution from action-angles
- **Lecture 2:**
 - Construct the solution

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of this lecture).

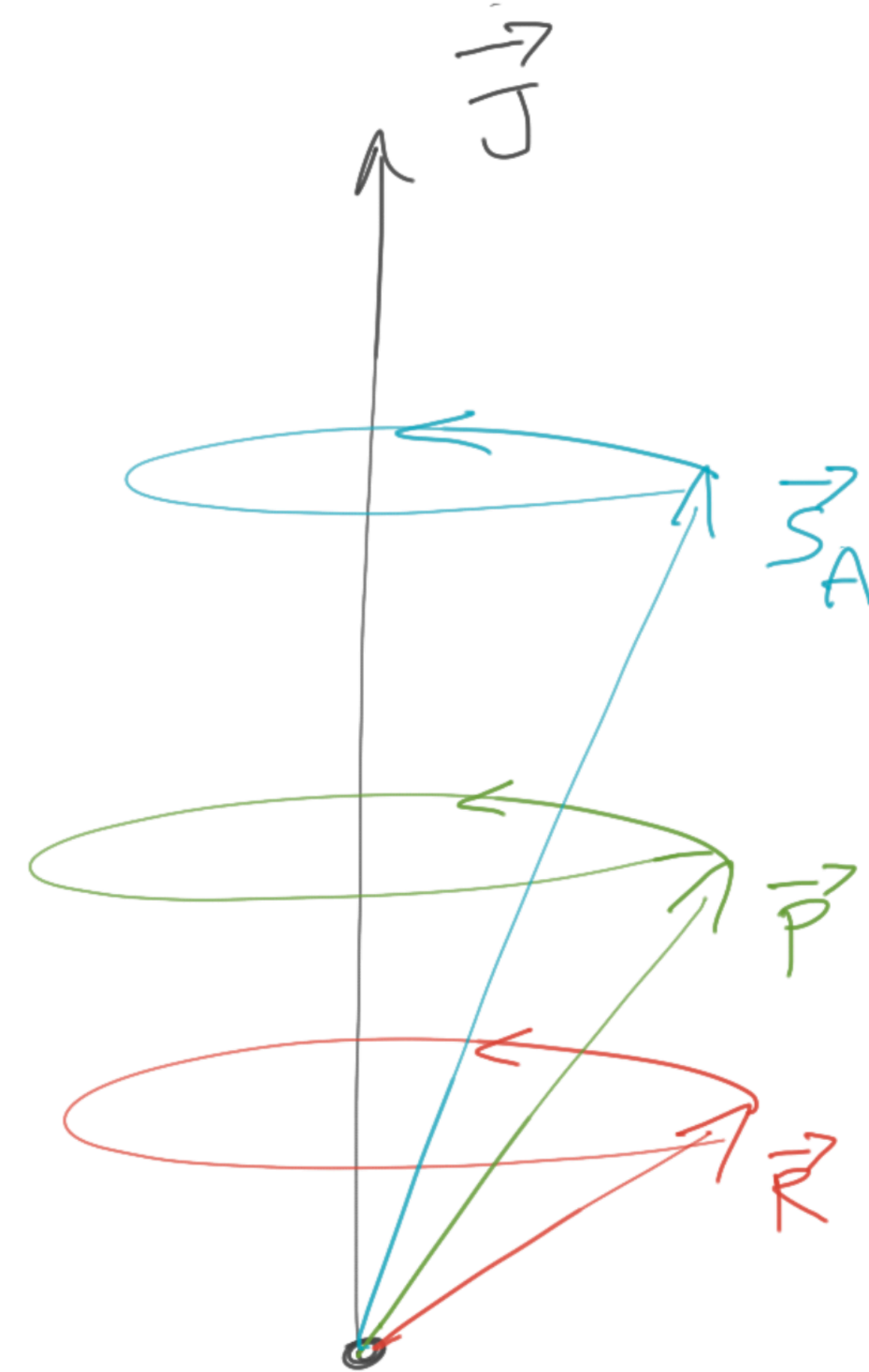
- How to combine \vec{C} flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
- Solution via action-angles.

Computing actions: strategy

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$; loop γ_i is on the surface of constant \vec{C} .
- $\{R_i, P_j\} = \delta_{ij}$ and $\{\phi_A, S_B^z\} = \delta_{AB}$
- How to be on the surface of constant \vec{C} ? Flow along C_i 's: $\frac{dC_i}{d\lambda} = \{C_i, C_j\} = 0$.
- $\mathcal{J} = \mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}$
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \sum_i P_i dR^i$ $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint_A^z d\phi_A.$

Computing \mathcal{J}_1

- With $\vec{V} = \{\vec{R}, \vec{P}, \vec{L}, \vec{S}_1, \vec{S}_2\}$, J^2 flow $\implies \frac{d\vec{V}}{d\lambda} = 2\vec{J} \times \vec{V} \equiv \vec{n} \times \vec{V}$.
- $\{\vec{J}, J^2\} = 0$.
- **Solution:** $\phi(\lambda) = n \lambda + \phi_0$.
- Loop closes after flowing by $\Delta\lambda = 2\pi/n = 2\pi/(2J) = \pi/J$.
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{n} \cdot \vec{L} d\lambda = \hat{n} \cdot \vec{L}$.
- $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint S_A^z d\phi_A = S_A^z = \hat{n} \cdot \vec{S}_A$ (with \vec{n} along z -axis)
- The spin integral is **rotationally invariant**, but not manifestly so.
- $\mathcal{J}_1 = \hat{n} \cdot (\vec{L} + \vec{S}_1 + \vec{S}_2) = \hat{n} \cdot \vec{J} = J$.
- **Summary:** We have computed \mathcal{J}_1 and also computed the solution to $C_1 = J^2$.



Computing $\mathcal{J}_1, \mathcal{J}_2$ and \mathcal{J}_3

- **For flows under J^2, J_z , and L^2 :** $\frac{d\vec{V}}{d\lambda} = \vec{n} \times \vec{V}$. $\vec{n} = 2\vec{J}, \hat{z}$, and $2\vec{L}$ (with \vec{n} being fixed)
- **Exception:** Under L^2 flow, spins don't move.
- **Solution:** $\phi(\lambda) = n\lambda + \phi_0$. Doesn't apply to spins under the L^2 flow.
- Loop closes after flowing by $\Delta\lambda = 2\pi/n$.
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{n} \cdot \vec{L} d\lambda = \hat{n} \cdot \vec{L}$.
- $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint S_A^z d\phi_A = S_A^z = \hat{n} \cdot \vec{S}_A$ (with \vec{n} along z-axis)
- $\mathcal{J}_1 = J, \mathcal{J}_2 = J_z, \mathcal{J}_3 = L$.
- **Summary:** We have computed $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ and also computed the solution to $C_i = \{J^2, J_z, L^2\}$.

Computing \mathcal{I}_4

- We won't compute it here.
- \mathcal{I}_4 has a Newtonian version (Eq. (10.139) of Goldstein).
- 1PN version given in Eq. (3.10) of Damour-Schafer.
- 1.5PN version in Eq. (38) of [arXiv: 2012.06586].

Taking stock

- We solved the flows under $C_i = \{J^2, J_z, L^2\}$.
- Finding the solution $\vec{V}(\vec{V}_0, \Delta\lambda)$ of a flow under C_i : $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, C_i\}$ is basically solving an ODE.
- Solution of flow under $\vec{S}_{\text{eff}} \cdot \vec{L}$ given in [arXiv:2110.15351].
- Solution of flow under H given in [arXiv:1908.02927]. This is the standard solution of the system. Omit 1PN terms for simplicity.
- Quite lengthy but nothing esoteric.
- **Future focus:** compute \mathcal{I}_5 .

\mathcal{I}_5 computation

For $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow:

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{P}$$

$$\frac{d\vec{S}_a}{d\lambda} = \sigma_a \left(\vec{L} \times \vec{S}_a \right)$$

$$\frac{d\vec{L}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{L}$$

-
- **Important:** \vec{n} not fixed

\mathcal{I}_5 computation

$$2\pi\mathcal{I} = 2\pi \left(\mathcal{I}^{\text{orb}} + \mathcal{I}^{\text{spin}} \right)$$

$$= \int_{\lambda_i}^{\lambda_f} \left(P_i dR^i + S_1^z d\phi_1^z + S_2^z d\phi_2^z \right)$$

- $$= \int_{\lambda_i}^{\lambda_f} \left(P_i \frac{dR^i}{d\lambda} + S_1^z \frac{d\phi_1^z}{d\lambda} + S_2^z \frac{d\phi_2^z}{d\lambda} \right) d\lambda$$

- $$2\pi\mathcal{I}^{\text{orb}} = \int_{\lambda_i}^{\lambda_f} \vec{P} \cdot \left(\vec{S}_{\text{eff}} \times \vec{R} \right) d\lambda = \int_{\lambda_i}^{\lambda_f} (S_{\text{eff}} \cdot L) d\lambda = (S_{\text{eff}} \cdot L) \Delta\lambda$$

- Can't do spin sector integral because $\vec{S}_A \neq \vec{R}_A \times \vec{P}_A$. A refers to the BHs.

\mathcal{I}_5 computation: enter fictitious variables

- Define \vec{R}_a, \vec{P}_a (fictitious variables) such that $\vec{S}_a \equiv \vec{R}_a \times \vec{P}_a$.
- **Hamiltonian:** Now a function of $\vec{R}, \vec{P}, \vec{R}_{1/2}, \vec{P}_{1/2}$ and not $\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$.
- **PBs and EOMs:** $\left\{ R_i, P_j \right\} = \delta_{ij}, \quad \left\{ R_{ai}, P_{bj} \right\} = \delta_{ab} \delta_{ji}; \quad \frac{df}{dt} = \{f, H\}.$
- $\left\{ R_i, P_j \right\} = \delta_{ij}, \quad \left\{ R_{ai}, P_{bj} \right\} = \delta_{ab} \delta_{ji} \implies \left\{ R_i, P_j \right\} = \delta_{ij}, \quad \left\{ \phi_A, S_B^z \right\} = \delta_{AB}$
- PBs \rightarrow EOMs \implies The standard phase space (**SPS**) is equivalent to the extended phase space (**EPS**).
- **Integrability equivalency:** EPS needs $n = 2n/2 = 18/2 = 9 = (5 + 4)$ C_i 's. The next 4 C_i 's are S_a^2 and $\vec{R}_a \cdot \vec{P}_a$.

\mathcal{I}_5 computation: sanity checks

- **Check 1:** Final \mathcal{I}_5 depends on \overrightarrow{R} , \overrightarrow{P} , \overrightarrow{S}_1 and \overrightarrow{S}_2 .
- **Check 2:** Numerical flow by 2π under \mathcal{I}_5 closes a loop in the SPS picture.
- We have all seen fictitious variables before (in spirit)!
- Inventing complex numbers to do real integrals.

11.8.19 Prove that $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx = \pi \ln 2$.

11.8.20 Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2} dx = \frac{\pi a}{\sin \pi a},$$

where $-1 < a < 1$.

Hint. Use the contour shown in Fig. 11.26, noting that $z = 0$ is a branch point and the positive x -axis can be chosen to be a cut line.

11.8.21 Show that

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.16 is a special case of this result.

11.8.22 Show that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = 2\pi/n$.

11.8.23 (a) Show that

$$f(z) = z^4 - 2z^2 \cos 2\theta + 1$$

has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.22 ($n = 4$) is a special case of this result.

\mathcal{J}_5 computation using fictitious variables

$$\mathcal{J}_k = \frac{1}{2\pi} \oint_{\mathcal{C}_k} \left(\vec{P} \cdot d\vec{R} + \vec{P}_1 \cdot d\vec{R}_1 + \vec{P}_2 \cdot d\vec{R}_2 \right)$$

EOMs for $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow are

$$\begin{aligned} \frac{d\vec{R}}{d\lambda} &= \vec{S}_{\text{eff}} \times \vec{R} \\ \frac{d\vec{P}}{d\lambda} &= \vec{S}_{\text{eff}} \times \vec{P} \\ \frac{d\vec{R}_a}{d\lambda} &= \sigma_a \left(\vec{L} \times \vec{R}_a \right) \\ \frac{d\vec{P}_a}{d\lambda} &= \sigma_a \left(\vec{L} \times \vec{P}_a \right) \end{aligned}$$

$$\begin{aligned} 2\pi \mathcal{J}_{S_{\text{eff}} \cdot L} &= 2\pi \left(\mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}} \right) \\ &= \int_{\lambda_i}^{\lambda_f} \left(P_i \frac{dR^i}{d\lambda} + P_{1i} \frac{dR_1^i}{d\lambda} + P_{2i} \frac{dR_2^i}{d\lambda} \right) d\lambda \\ &= \int_{\lambda_i}^{\lambda_f} \left(\vec{P} \cdot \left(\vec{S}_{\text{eff}} \times \vec{R} \right) + \vec{P}_1 \cdot \left(\sigma_1 \vec{L} \times \vec{R}_1 \right) \right. \\ &\quad \left. + \vec{P}_2 \cdot \left(\sigma_2 \vec{L} \times \vec{R}_2 \right) \right) d\lambda \\ &= 2 \int_{\lambda_i}^{\lambda_f} (S_{\text{eff}} \cdot L) d\lambda = 2 (S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L} \\ \mathcal{J}_{S_{\text{eff}} \cdot L} &= \frac{(S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L}}{\pi} \end{aligned}$$

\mathcal{J}_5 computation using fictitious variables

$$\mathcal{J}_{S_{\text{eff}} \cdot L} = \frac{(S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L}}{\pi},$$

$$\mathcal{J}_{J^2} = \frac{J^2 \Delta\lambda_{J^2}}{\pi},$$

$$\mathcal{J}_{L^2} = \frac{L^2 \Delta\lambda_{L^2}}{\pi},$$

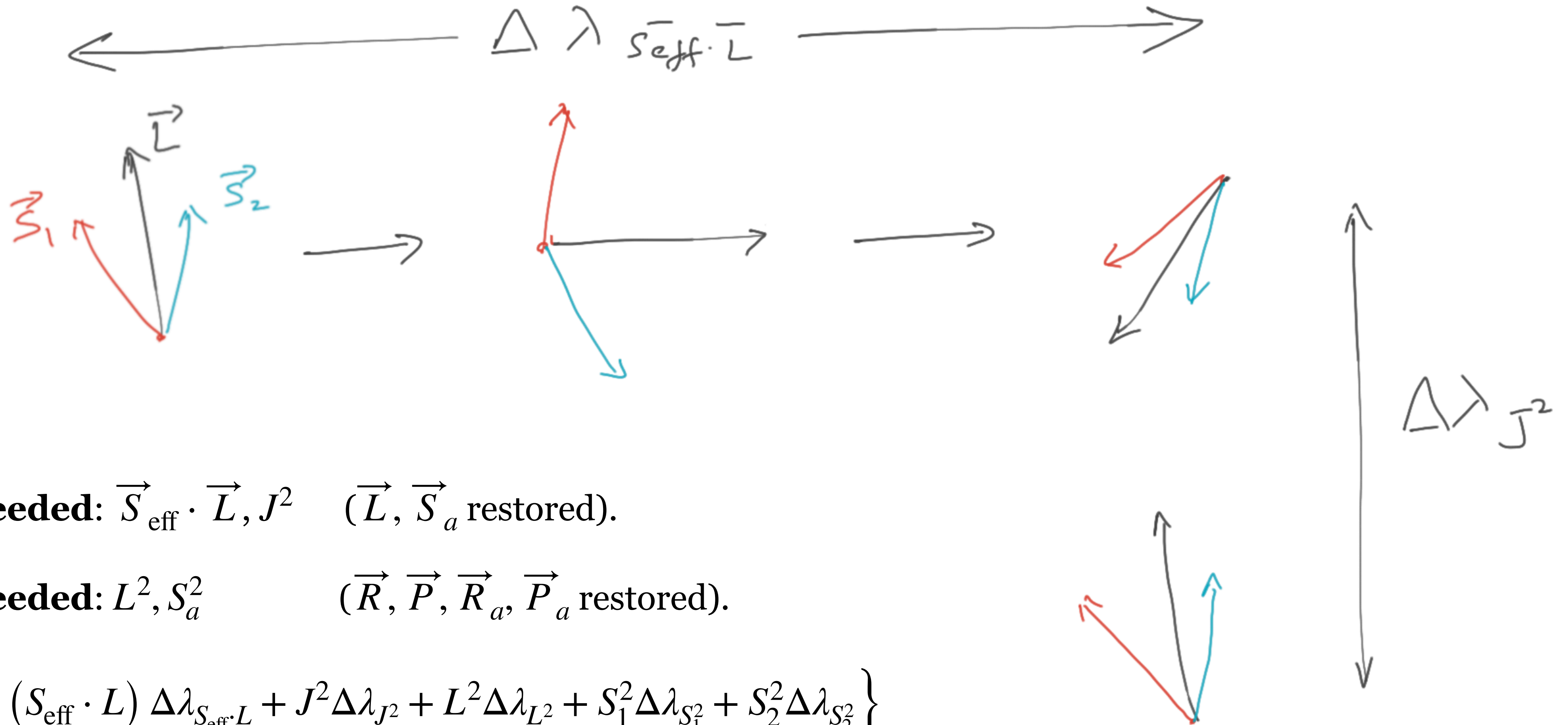
$$\Rightarrow \quad \mathcal{J}_5 = \frac{1}{\pi} \left\{ (S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L} + J^2 \Delta\lambda_{J^2} + L^2 \Delta\lambda_{L^2} + S_1^2 \Delta\lambda_{S_1^2} + S_2^2 \Delta\lambda_{S_2^2} \right\}.$$

• $\mathcal{J}_{S_1^2} = \frac{S_1^2 \Delta\lambda_{S_1^2}}{\pi},$

$$\mathcal{J}_{S_2^2} = \frac{S_2^2 \Delta\lambda_{S_2^2}}{\pi}.$$

- Loop for \mathcal{J}_5 is closed by flowing under 5 C_i 's (not one).
- Flow amounts $\Delta\lambda_i$'s give \mathcal{J}_5

\mathcal{I}_5 computation: flow overview



THE END

Please send comments on the lecture notes and
the presentation 

Thank you!