

Closed-form solutions of spinning BBHs at 1.5PN (using action-angle variables)

Lecture Workshop (Univ. of Illinois Urbana-Champaign)

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References

- **RESEARCH PAPERS**
 - The standard way of computing the solution (without 1PN part): <https://arxiv.org/abs/1908.02927>
 - Action-angle-based solution: <https://arxiv.org/abs/2012.06586>, <https://arxiv.org/abs/2110.15351>
- **LECTURE NOTES**
 - Lecture notes (latest): https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/github-action-result/pdflatex/lecture_notes/main.pdf
 - Lecture notes (for citation purposes): <https://arxiv.org/abs/2206.05799>
- **MATHEMATICA PACKAGE**
 - <https://github.com/sashwattanay/BBH-PN-Toolkit>
- **YOUTUBE VIDEO**
 - <https://youtu.be/aoiCk5TtmvE>
- **THE PRESENTATION**
 - https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/main/UIUC_workshop_presentation/uiuc_workshop_presentation.pdf

Lecture plan

Lecture style: standing on the shoulders of giants (due to time constraints)

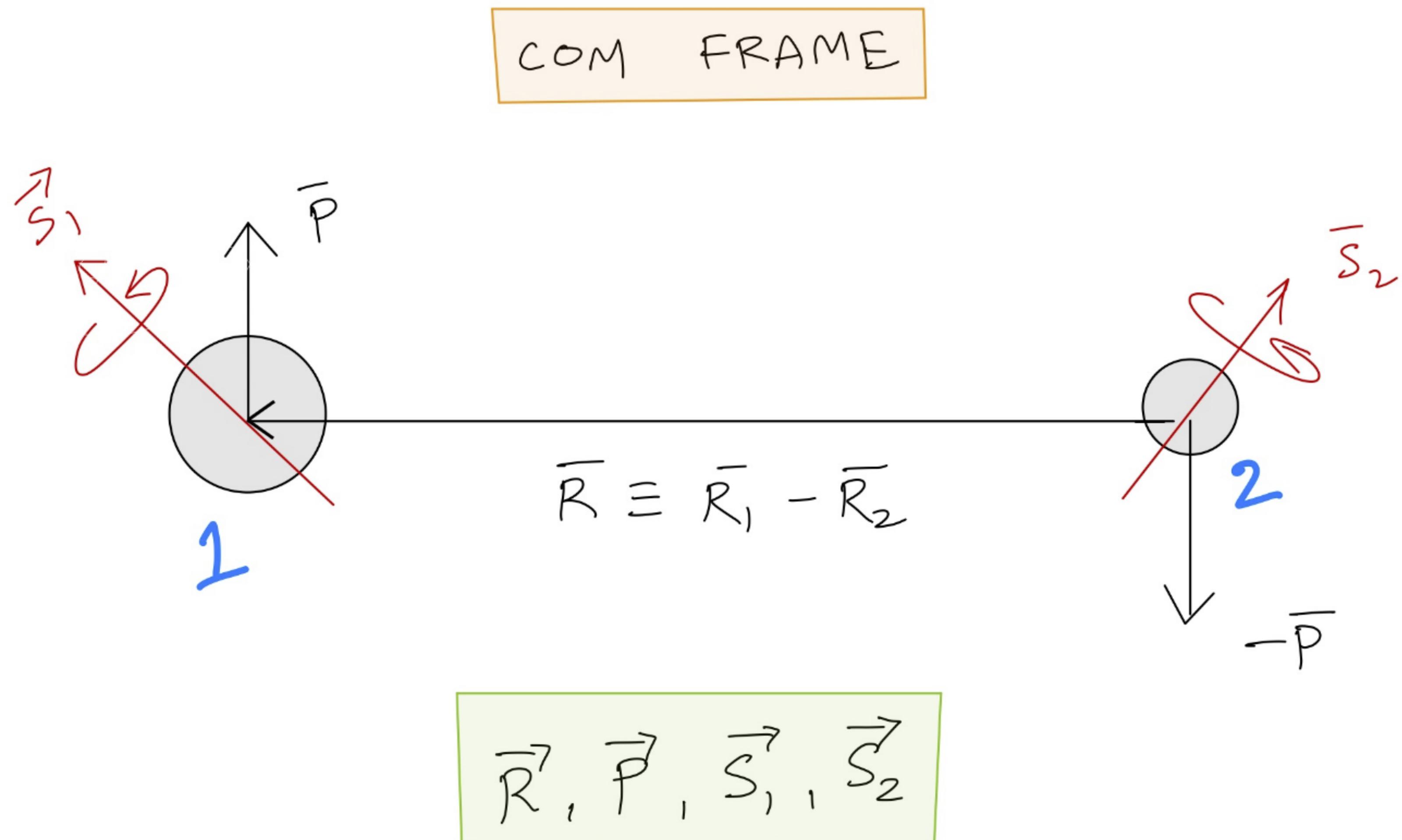
- **Lecture 1:**
 - Theory
 - Strategy to compute solution from action-angles
- **Lecture 2:**
 - Finish the solution (first 4 actions)
 - Finish the solution (last action)

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Introduction to the system

Spinning 1.5PN BBH system



Phase space variables

$\vec{R}(t), \vec{P}(t), \vec{S}_1(t)$ and $\vec{S}_2(t)$

Statement of the problem

- The 1.5PN Hamiltonian is $H = H_N + H_{1\text{PN}} + H_{1.5\text{PN}} + \mathcal{O}(c^{-4})$ with
- $H_N = \mu \left(\frac{p^2}{2} - \frac{1}{r} \right)$, $H_{1.5\text{PN}} = \frac{2G}{c^2 R^3} \vec{S}_{\text{eff}} \cdot \vec{L}$.
- Hamilton's equations $\Rightarrow \frac{d(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t))}{dt}$.
- **Problem:** Integrate Hamilton's eqns. to obtain $\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)$.

Historical context and the status quo

- The 1.5PN Hamiltonian is $H = H_N + H_{1\text{PN}} + H_{1.5\text{PN}} + \mathcal{O}(c^{-4})$.
- **1680s:** Issac Newton gave the Newtonian solution $R = a(1 - e \cos u)$.
- **1985:** Damour-Deruelle gave 1PN quasi-Keplerian solution.
- **2019:** Gihyuk Cho, H. M. Lee gave 1.5PN solution (1PN effects ignored for simplicity)
- **2020 & 2021:** We worked out an equivalent action-angle based solution ([subject of these lectures](#)).
- **Why action-angles?** Extendible to 2PN via canonical perturbation theory ([Goldstein](#)).
- **Do we even have the solutions?** See the plot of analytical and numerical solutions (via a Mathematica package) in the [YouTube video](#) @10:33 (in References)

EOMs with Poisson brackets

Standard approach

- Hamilton's eqns. are $\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = - \frac{\partial H}{\partial q_i}$
- Leads to EOM $\frac{df}{dt} = \{f, H\}$ with $\{f, g\} \equiv \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$.

EOMs with Poisson brackets for BBHs

Our approach

- Define EOMs: $\frac{df(t)}{dt} = \{f, H\}$ where $f = f(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t))$.
- Define PBs: $\{R_i, P_j\} = \delta_{ji}$ $\{S_A^i, S_B^j\} = \delta_{AB}\epsilon_k^{ij}S_A^k$.

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\{f, g(v_i)\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

PB Exercise 1

Prob: Compute $\{R_x, \sin P_x + P_x\}$.

Sol: Using the bilinearity and the chain rule (2nd and 4th rules) for PBs

$$\begin{aligned} & \{R_x, \sin P_x + P_x\} \\ &= \{R_x, \sin P_x\} + \{R_x, P_x\} \\ &= \{R_x, P_x\} \frac{\partial \sin P_x}{\partial P_x} + \{R_x, P_x\} \\ &= \cos P_x + 1. \end{aligned}$$

PB Exercise 2

Prob: Show that $\{\phi_A, S_B^z\} = \delta_{AB}$, where $\phi_A = \arctan(S_A^y/S_A^x)$ is the azimuthal angle of \vec{S}_A .

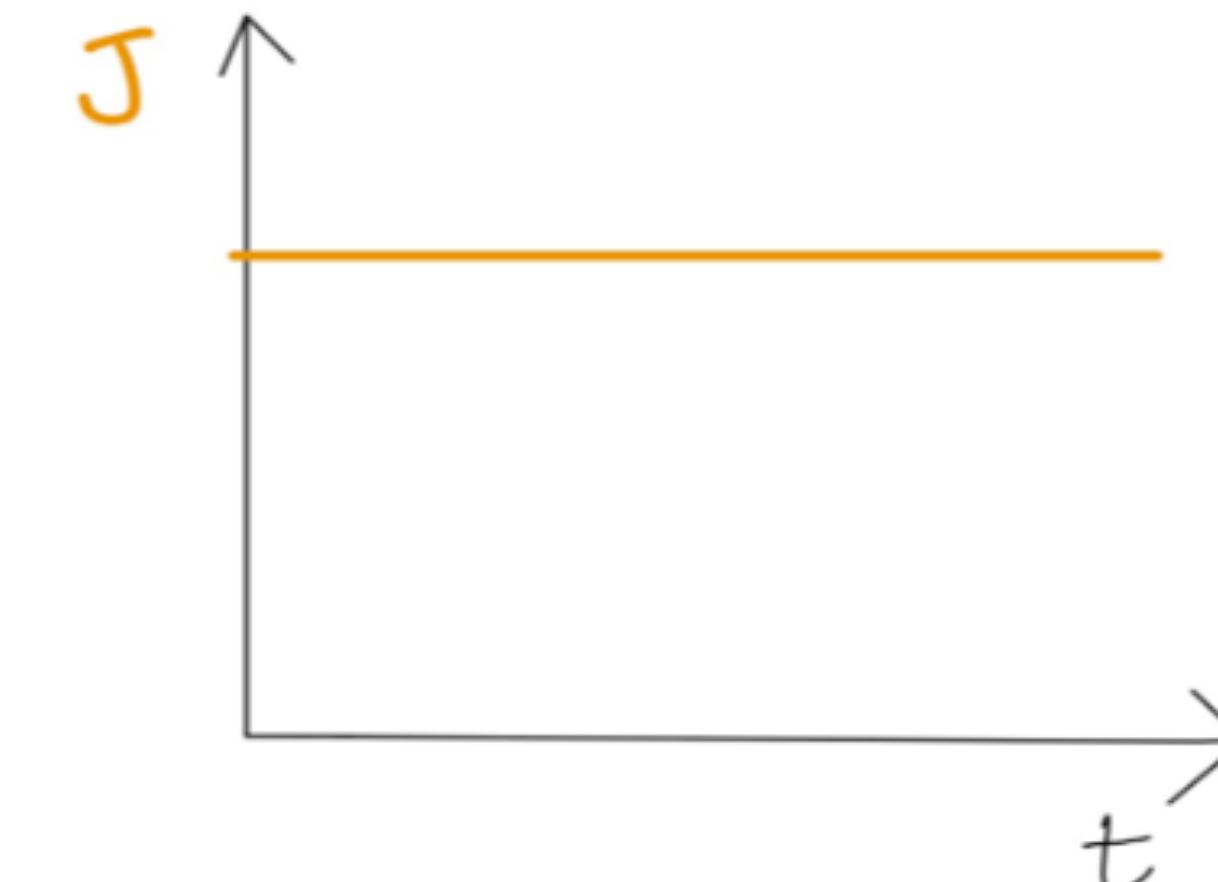
- Implies that $\phi \sim$ position; $S^z \sim$ momentum upon comparison with $\{R_i, P_j\} = \delta_{ji}$.
- **Lingo:** f and g commute if $\{f, g\} = 0$.
- **How to evaluate general PBs quickly?:** Use the Mathematica notebook. See the YouTube video @14:22 (in References)

Spin canonical coordinates

- $\phi \equiv$ azimuthal angle of \vec{S} .
- **Standard approach:** (ϕ, S_z, S) are functions of (S_x, S_y, S_z) .
- **Symplectic geometric approach:** Symplectic manifolds are always **even-dimensional** (position-momentum pairing); (S_x, S_y, S_z) doesn't fit in.
- Spin sphere of constant radius S . S is not a dynamical variable; just like m_1, m_2 .
- On this sphere we have coordinates $(S_z, \phi) \equiv$ (momentum, position); i.e. $\{\phi, S_z\} = 1$. Hence **even-dimensional**.
- $S_x = \sqrt{S^2 - S_z^2} \cos \phi, \quad S_y = \sqrt{S^2 - S_z^2} \sin \phi.$

Integrable systems and action-angles

- **Integrable system:** canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$.



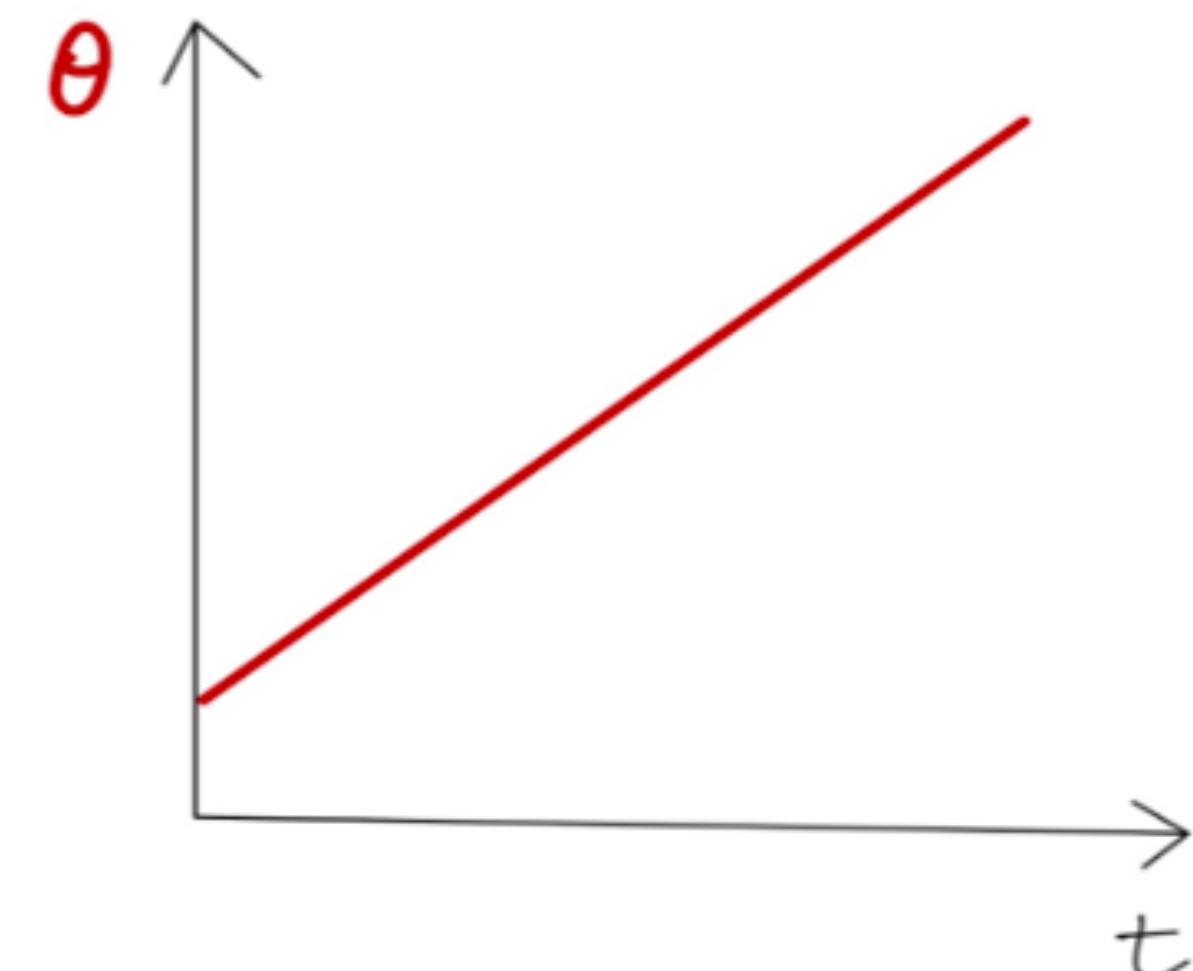
- Action $\mathcal{J}_i \sim p$; angle $\theta_i \sim q$. (Definition 1)

- Hamilton's equations \Rightarrow

$$\dot{\mathcal{J}}_i = -\partial H/\partial\theta_i = 0 \quad \Rightarrow \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H/\partial\mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \quad \Rightarrow \theta_i = \omega_i(\vec{\mathcal{J}})t.$$

- Having action-angles \sim having closed-form solutions.



Liouville-Arnold theorem

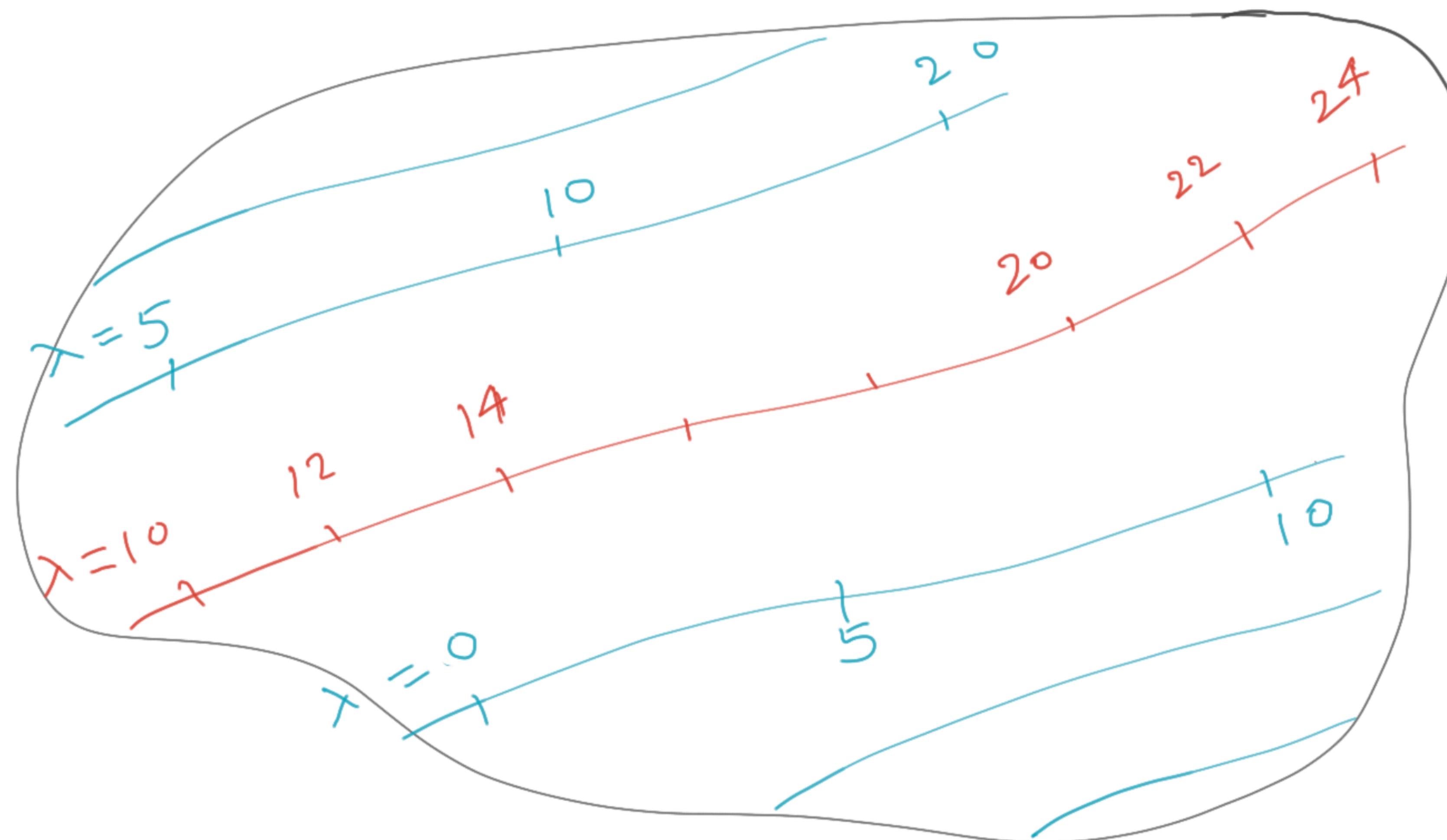
- **Theorem:** $2n$ phase space variables & n commuting constants of motion \implies integrability.
- How to check if f is a constant of motion? Check if $\dot{f} = \{f, H\} = 0$.
- For BBH phase space $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$, $2n \neq 12$. Positions-momenta delineation not clear for spins.
- Easy to check that $\{R_i, P_j\} = \delta_{ij}$ and $\{\phi_A, S_B^z\} = \delta_{AB}$ where $\phi_A = \arctan(S_A^y/S_A^x)$, the azimuthal angle for \vec{S}_A .
- (ϕ_A, S_A^z) are the positions, momenta of \vec{S}_A . Only 2 variables needed for \vec{S}_A since $dS_A/dt = \{S_A, H\} = 0$.
- Hence $2n = 3 + 3 + 2 + 2 = 10 \implies 10/2 = 5$ commuting constants needed for integrability.

Commuting constants for BBHs

- With $m \equiv m_1 + m_2$, $\mu \equiv m_1 m_2 / m$, $\nu \equiv \mu/m$, $\vec{L} \equiv \vec{R} \times \vec{P}$,
 $\frac{\sigma_1}{J} \equiv \frac{\vec{L}}{\vec{L} + \vec{S}_1 + \vec{S}_2} = \frac{2 + 3m_2/m_1}{2 + 3m_1/m_2}$, $\sigma_2 \equiv (2 + 3m_1/m_2)$, $\vec{S}_{\text{eff}} \equiv \sigma_1 \vec{S}_1 + \sigma_2 \vec{S}_2$.
- The 5 commuting constants are long known: $H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}$.
- Hence the 1.5PN BBH is integrable and has action-angles.

Curves, vectors, vector fields and flows

- **Pictorial definition:** the vector is $d/d\lambda$ (a derivative operator).



Flow of a function $f(\vec{V})$

- $\vec{V} \equiv \{\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2\}$, unless states otherwise.
- Flow of $f(\vec{V})$: $\frac{dV^i}{d\lambda} = \{V^i, f\}$ or $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, f\}.$
- Solution of the flow given in the form $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda).$
- Under f flow, g changes as $\frac{dg}{d\lambda} = \frac{\partial g}{\partial V_k} \frac{\partial V_k}{\partial \lambda} = \frac{\partial g}{\partial V_k} \{V_k, f\} = \{g, f\}.$

Flow exercise 1

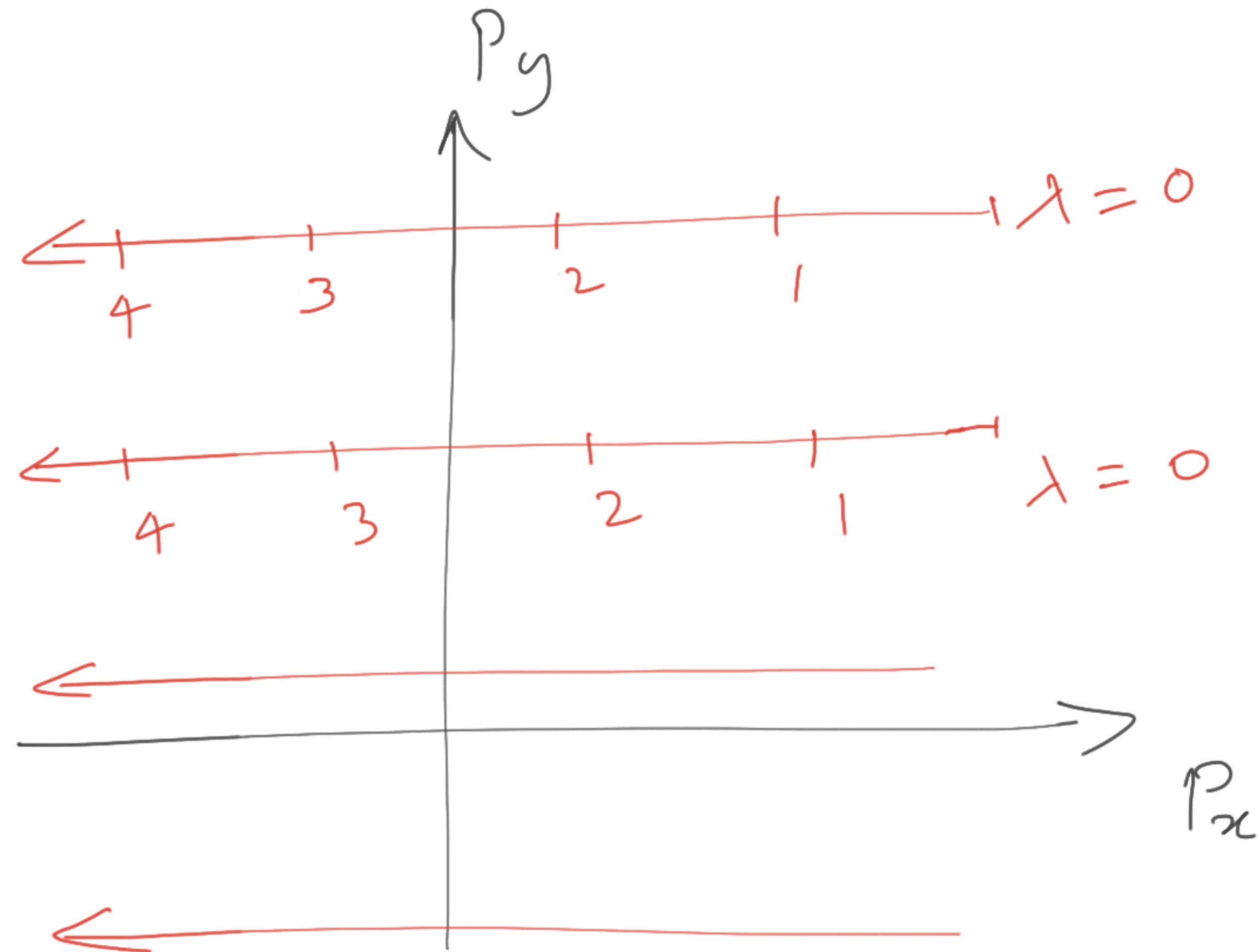
Prob: Solve the flow under R_x and draw pictures.

Sol: Under the R_x flow:

$$\frac{dP_x}{d\lambda} = \{P_x, R_x\} = -1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i \text{'s.}$$

$$\implies P_x - P_x(\lambda_0) = (\lambda_0 - \lambda).$$



Flow exercise 2

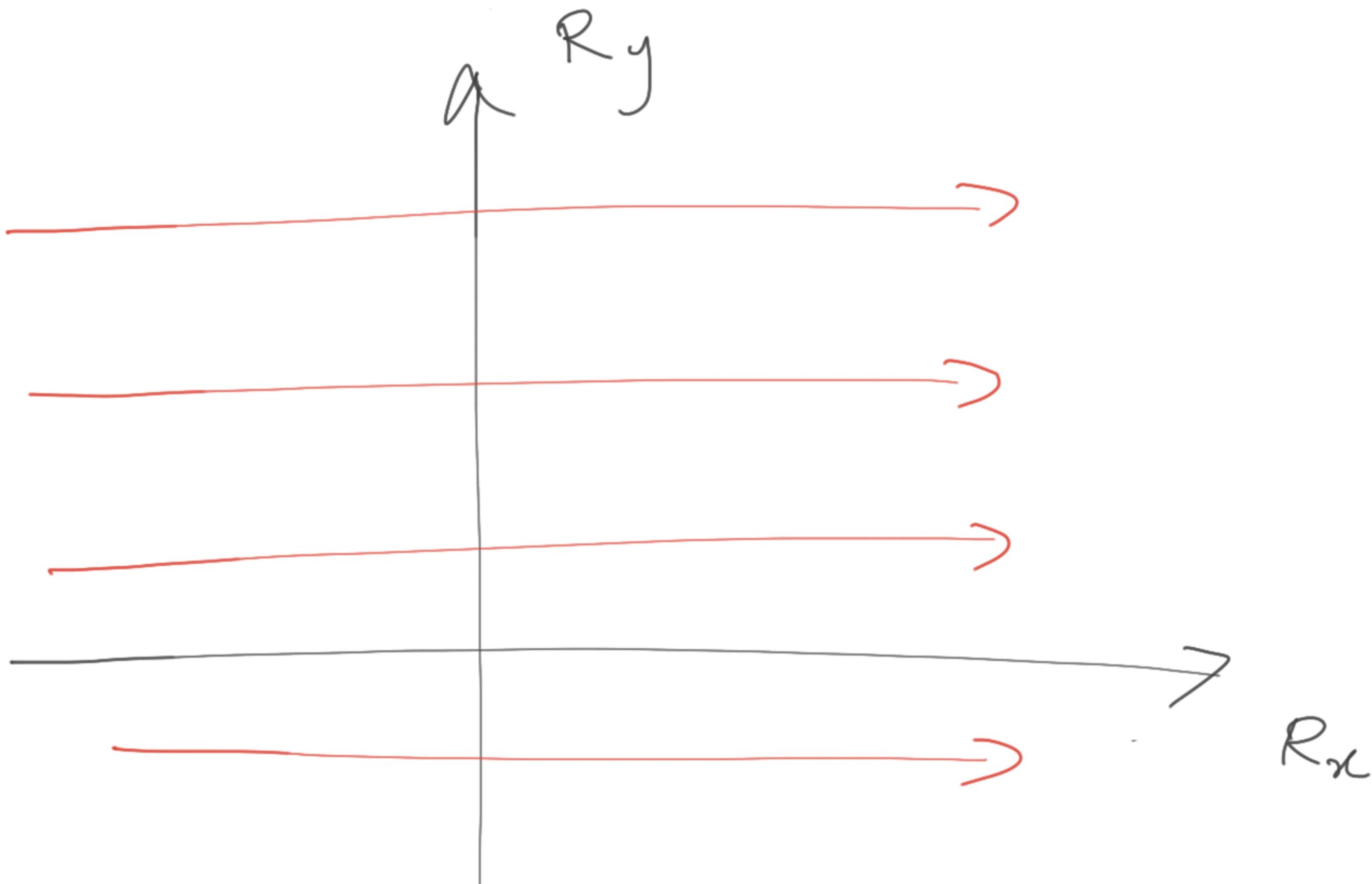
Prob: Solve the flow under P_x and draw pictures.

Sol: Under the R_x flow:

$$\frac{dR_x}{d\lambda} = \{R_x, P_x\} = 1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i \text{'s.}$$

$$\implies R_x - R_x(\lambda_0) = (\lambda - \lambda_0).$$



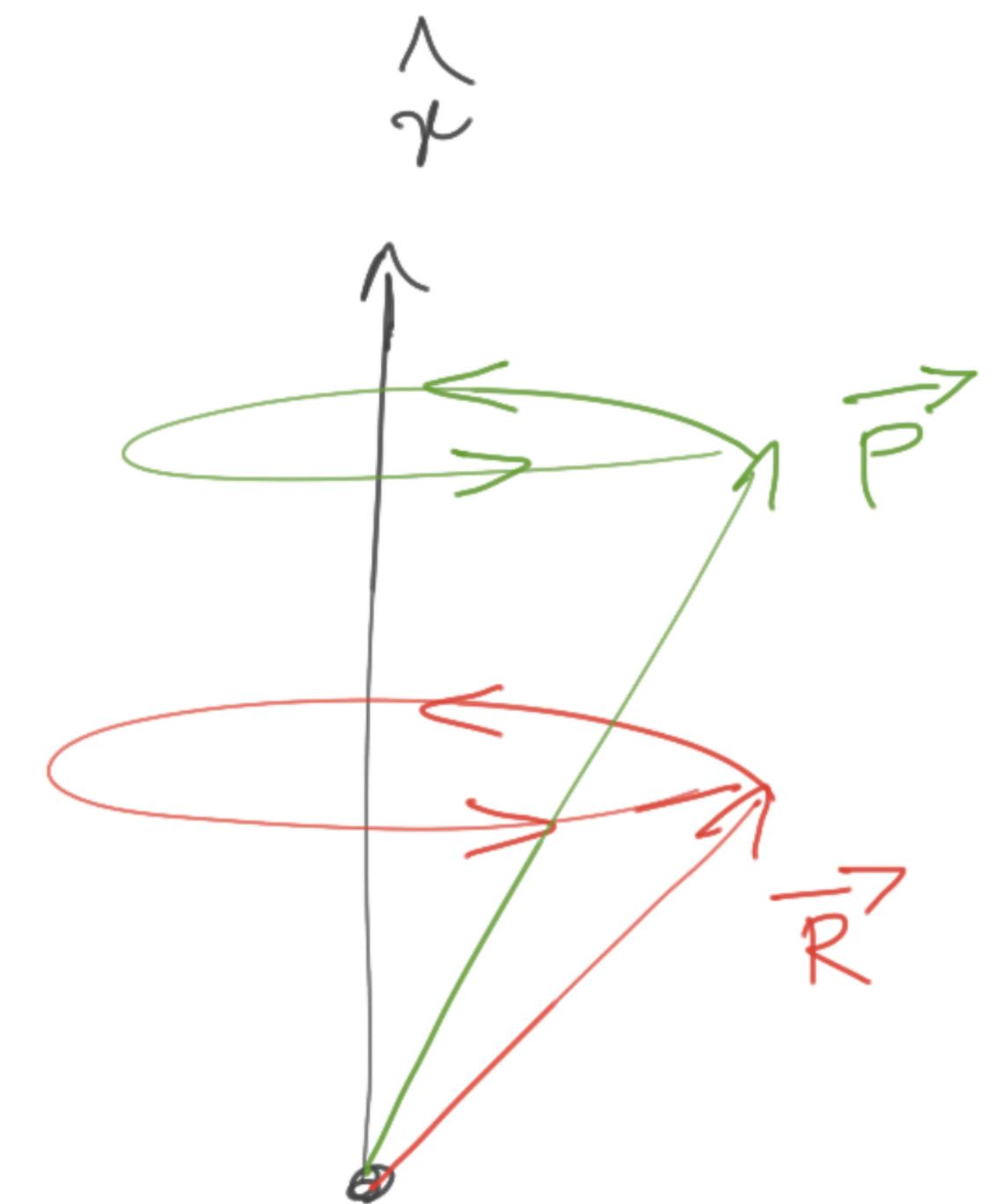
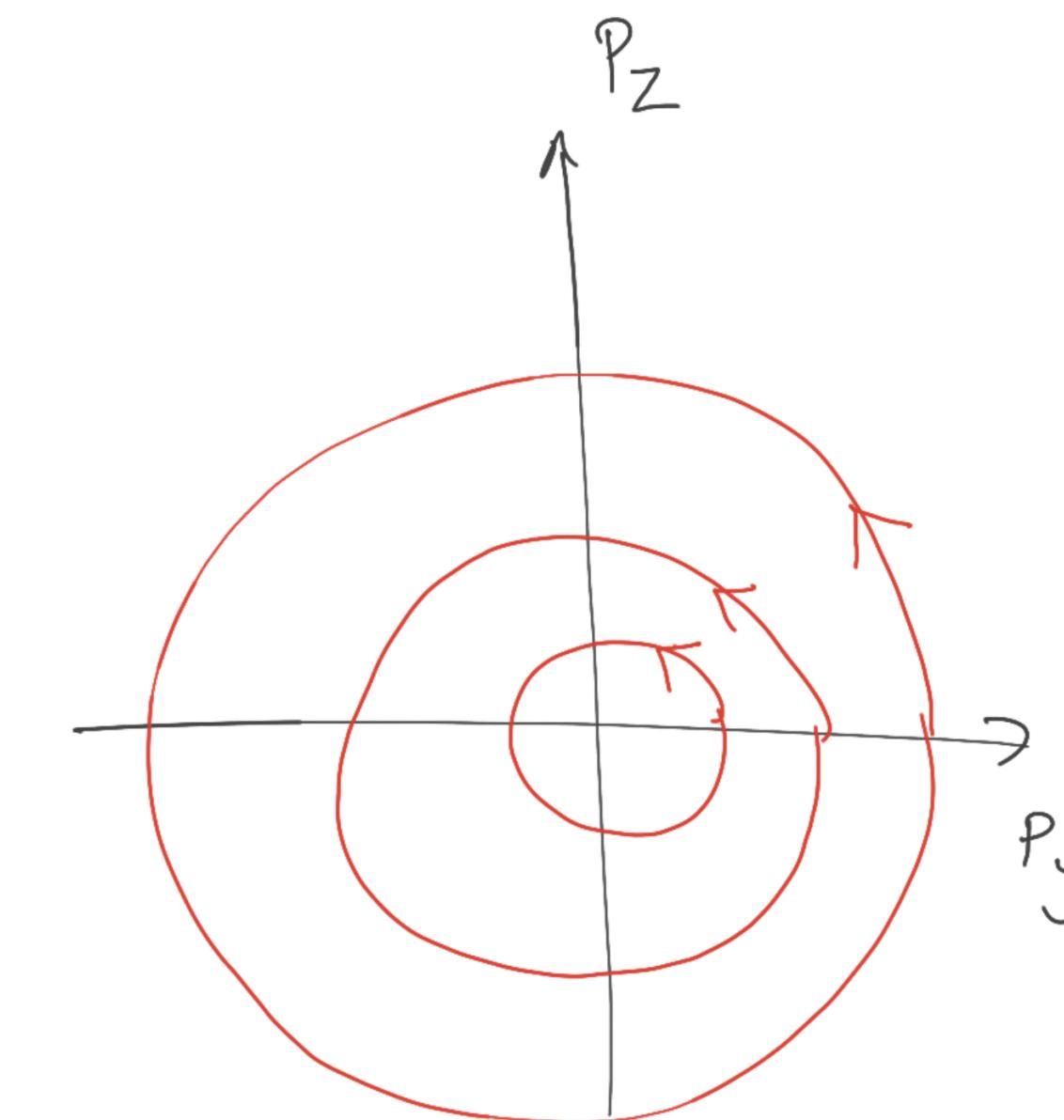
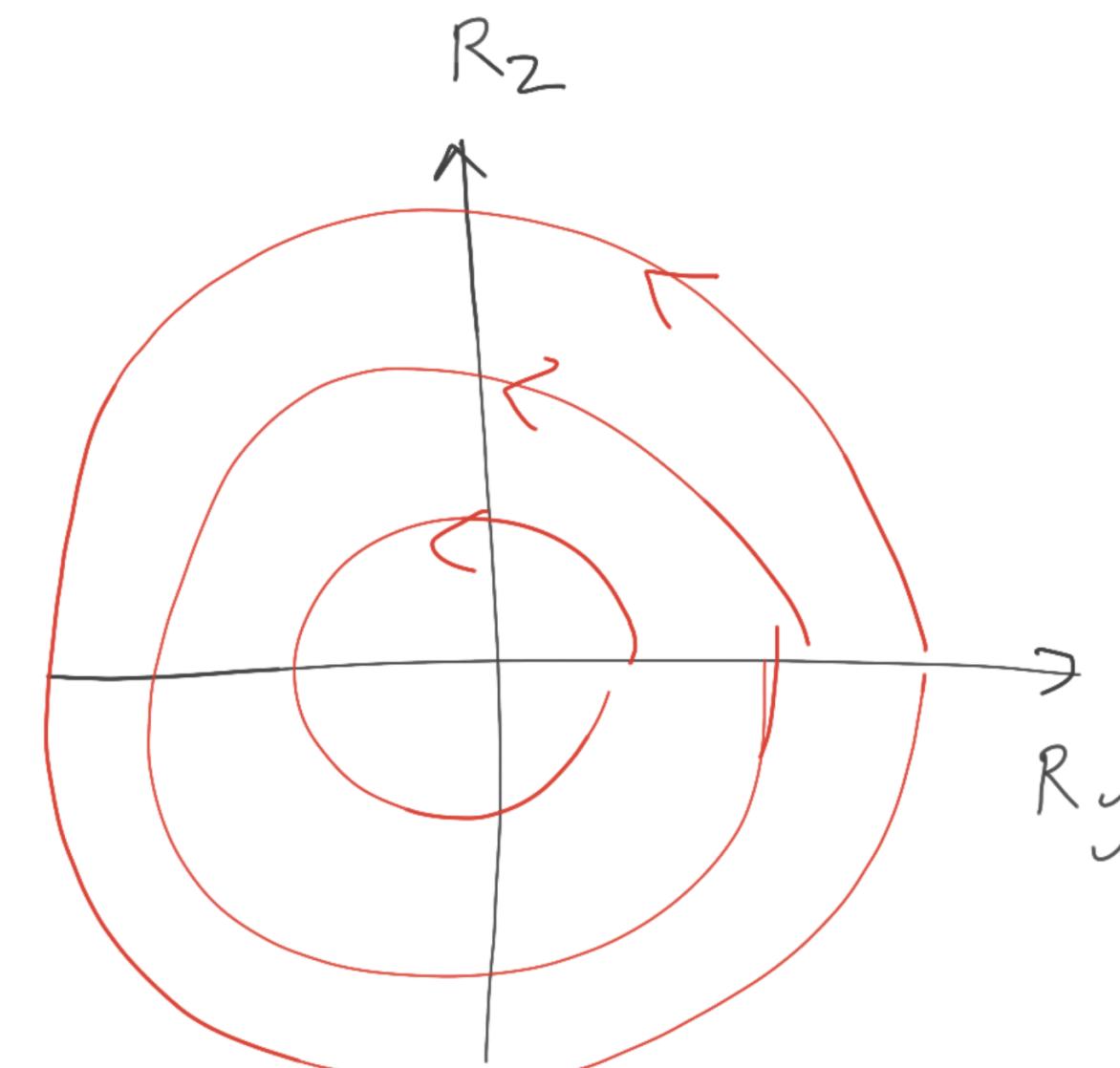
Flow exercise 3

Prob: Draw pictures for L_x flow where $\vec{L} = \vec{R} \times \vec{P}$.

Sol: Under the L_x flow:

$$\frac{d\vec{R}}{d\lambda} = \hat{x} \times \vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \hat{x} \times \vec{P}$$

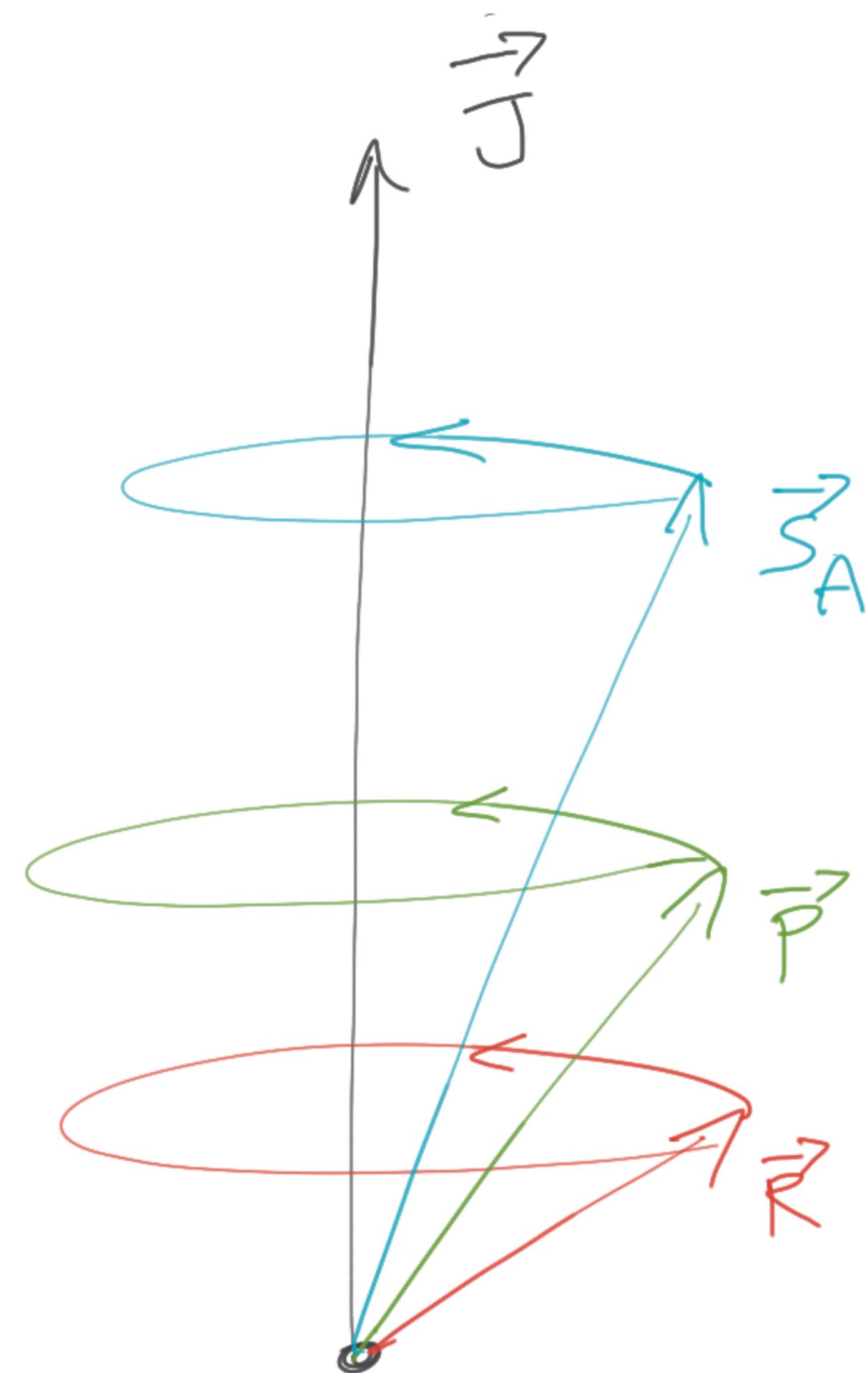
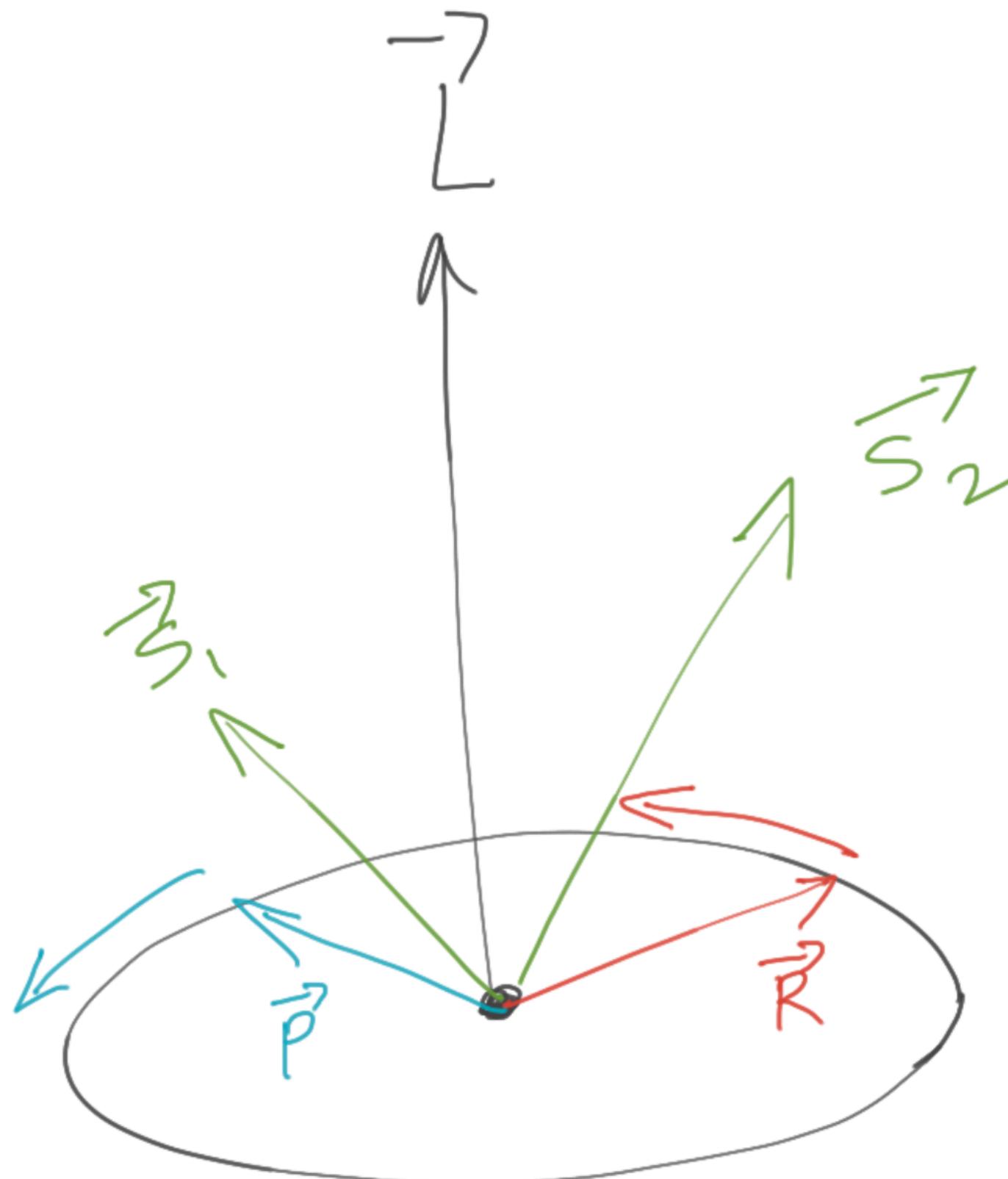


Flow exercise 4

Prob: Draw pictures for L^2 and J^2 flow where $\vec{L} = \vec{R} \times \vec{P}$ and $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$.

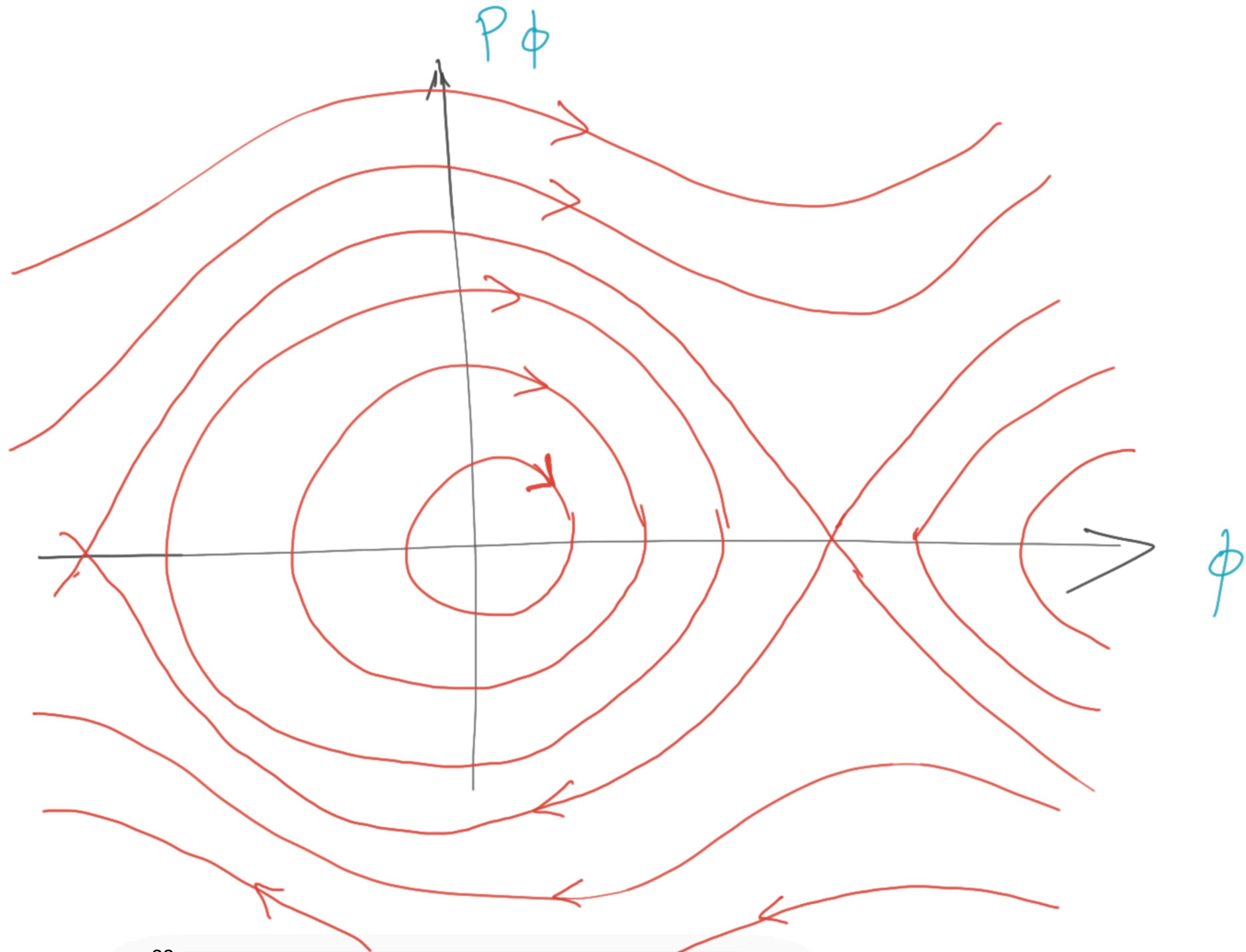
Sol: L^2 flow $\Rightarrow \{\vec{L}, L^2\} = \{\vec{S}_A, L^2\} = 0 \Rightarrow \vec{L}, \vec{S}_A$ remain fixed.

J^2 flow $\Rightarrow \{\vec{J}, J^2\} = 0 \Rightarrow \vec{J}$ remains fixed



Flow under the Hamiltonian H

- With $\vec{V} \equiv \{\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2\}$, flow eqn. is $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, f\}$.
- Flow under $H \implies \frac{d\vec{V}}{d\lambda} = \{\vec{V}, H\}$.
- This is the **EOM**. Gives the **real-time evolution**, unlike other flows.
- Flow of the Hamiltonian is special!
- Example:** flow under H for a pendulum



5 minute break

Coffee, questions?

Lecture plan

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Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions (subject of the next lecture).

- How to combine \vec{C} flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
- Solution via action-angles (the holy grail)

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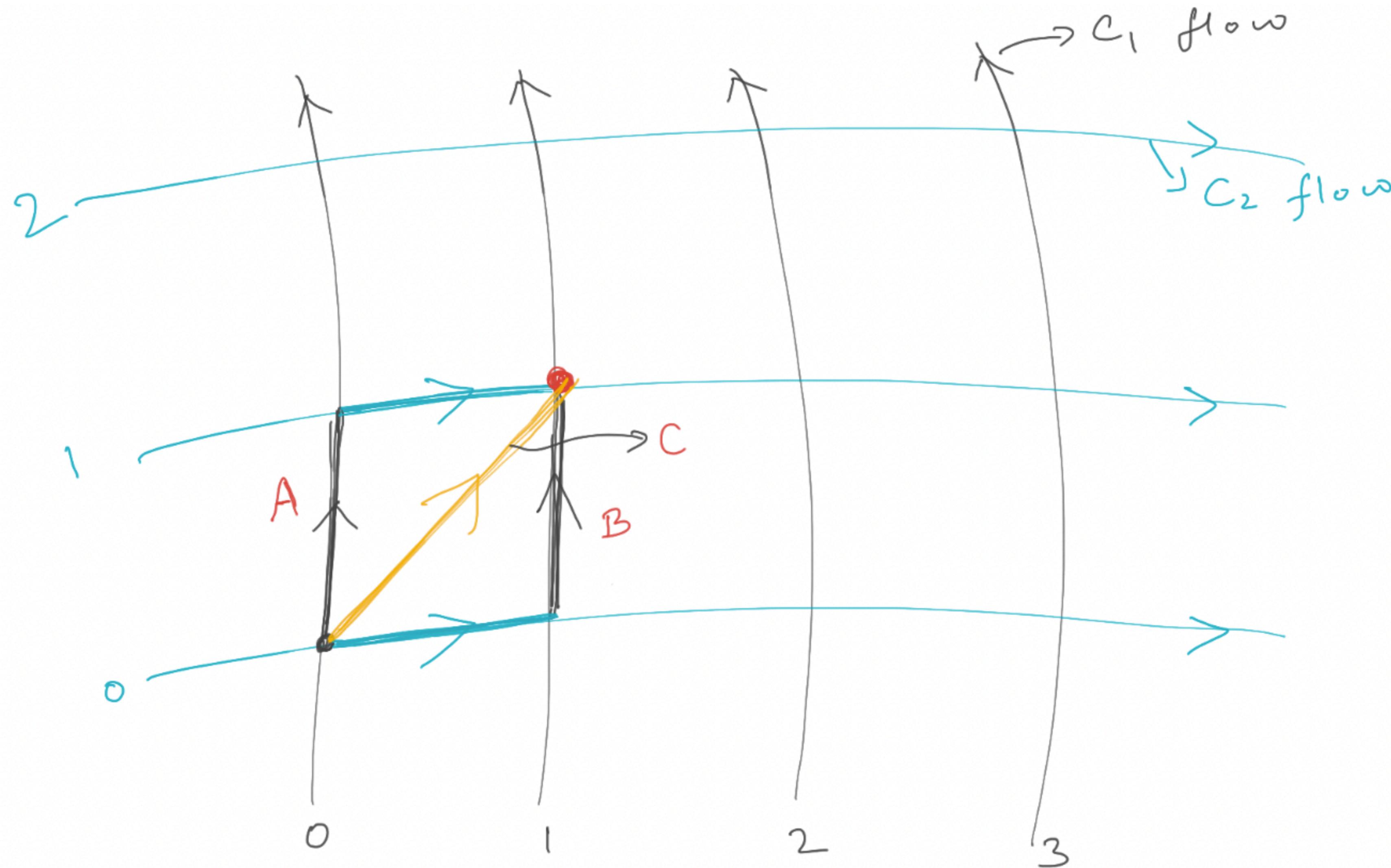
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How to combine \vec{C} flows?

- Assume C_i 's are commuting quantities (don't have to be constants).
- **Notation:** Output of C_i flow $\frac{d\vec{V}}{d\lambda_i} = \{\vec{V}, C_i\}$ denoted by $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda_i)$.
- **Result:** $\frac{d\vec{V}}{d\lambda_i} = \{\vec{V}, \alpha C_i\} \implies \frac{d\vec{V}}{d(\alpha\lambda_i)} = \{\vec{V}, C_i\}$; Flow under $10C_1$ by $\Delta\lambda \sim$ flow under C_1 by $10\Delta\lambda$.
- **Result:** Order of flow does not matter, i.e. $\vec{V}(\vec{V}_0, \Delta\lambda_1, \Delta\lambda_2) = \vec{V}(\vec{V}_0, \Delta\lambda_2, \Delta\lambda_1)$.
- **Result:** Simultaneous flows can be made sequential: $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, C_1 + C_2\}$ by $\Delta\lambda$ is C_1 flow followed by C_2 flow (both by $\Delta\lambda$). Or in the reverse order.

How to combine \vec{C} flows?

Pictorial depiction of the two flow rules



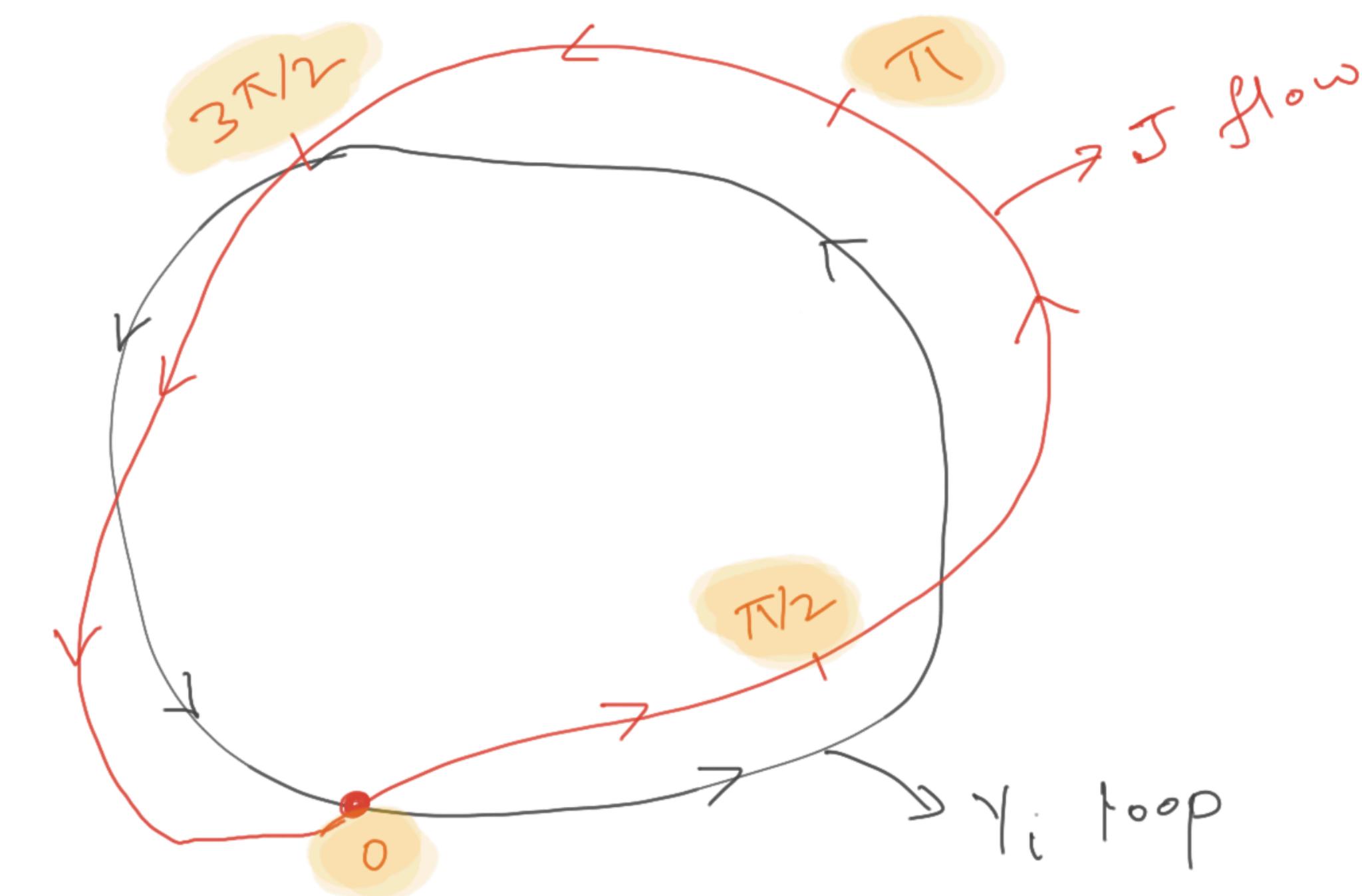
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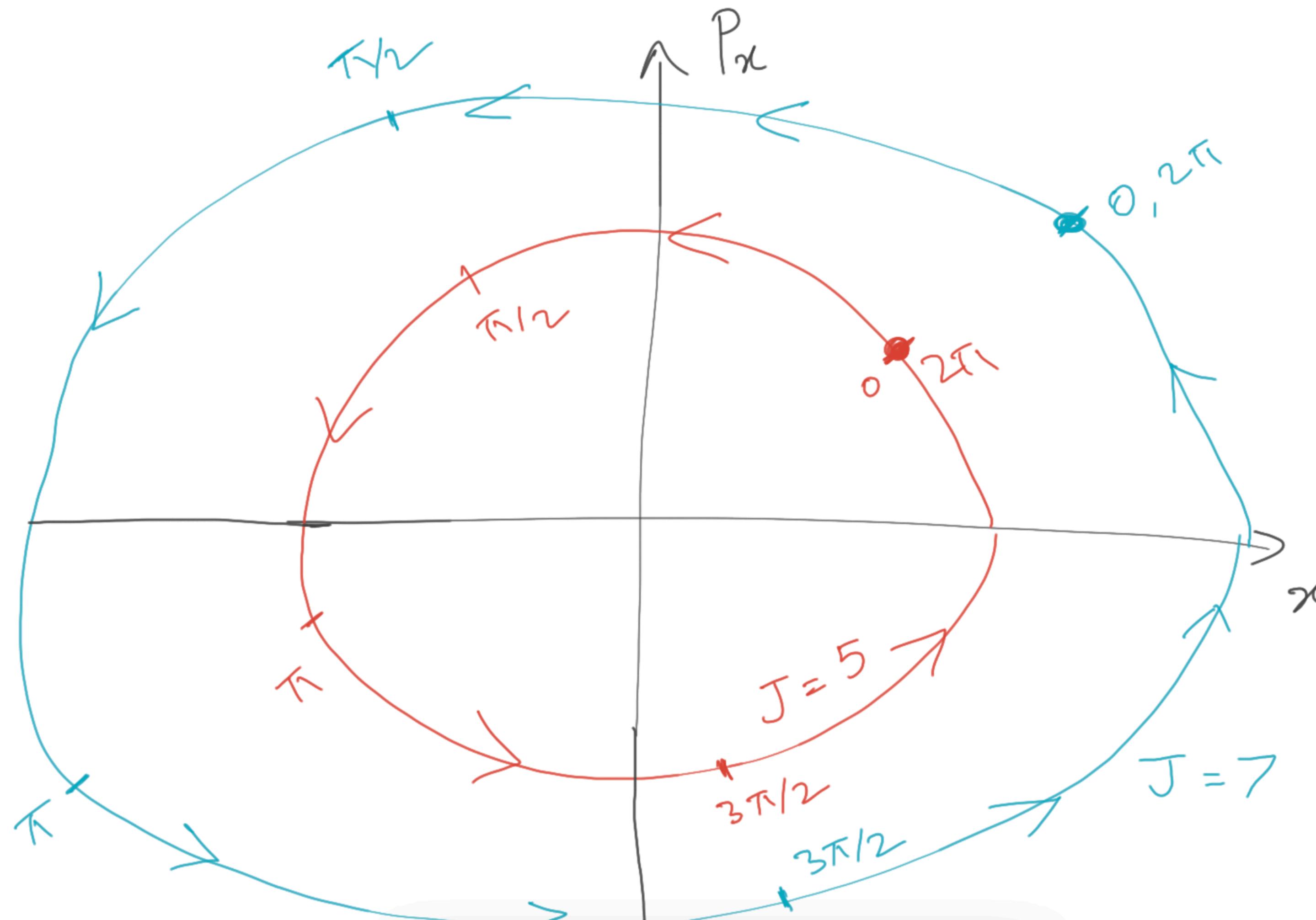
Construct action-angles (Definition 2)

- Loop \sim closed path in the phase space.
- Motion happens on the $2n - n = n$ dimensional $\vec{C} = \text{constant}$ submanifold
- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$, with loop γ_i on the $\vec{C} = \text{constant}$ submanifold with $\{R_i, P_j\} = \delta_{ij}; \{\phi_A, S_B^z\} = \delta_{AB}$.
- \mathcal{J}_i flow by $2\pi \rightarrow$ loop (different from γ_i).
- How to increase only 1 angle? $\theta_i \equiv \lambda_i$ along the \mathcal{J}_i flow.



Construct action-angles

Pictorial depiction of the construction (SHO)



Construct action-angles

- To show: $\{\theta_i, \mathcal{J}_k\} = \delta_{ij}$, $\{\mathcal{J}_i, \mathcal{J}_k\} = 0$, $\{\theta_i, \theta_k\} = 0$.
- Using $\theta_i = \lambda_i$, $\frac{d\theta_i}{d\lambda_i} = 1$ and $\frac{d\theta_i}{d\lambda_i} = \{\theta_i, \mathcal{J}_i\} \implies \{\theta_i, \mathcal{J}_i\} = 1$.
- From the definition of \mathcal{J}_i and chain rule for PBs, $\mathcal{J}_i = \mathcal{J}_i(\vec{C})$ (Stokes' theorem)
 $\implies \{\mathcal{J}_i, \mathcal{J}_k\} = \frac{\partial \mathcal{J}_i}{\partial C_l} \frac{\partial \mathcal{J}_k}{\partial C_m} \{C_l, C_m\} = 0$.
- $\{\theta_i, \theta_j\} = 0$ involves changing \mathcal{J}_i , which does not happen with real evolution.
Hence ignore.
- “**Integrable system:** canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$.” that lead to $\dot{\mathcal{J}}_i = 0$; $\theta_i = \omega_i t$ is satisfied because action flow makes a loop after 2π . Hence, **Definition 1 ~ Definition 2**.

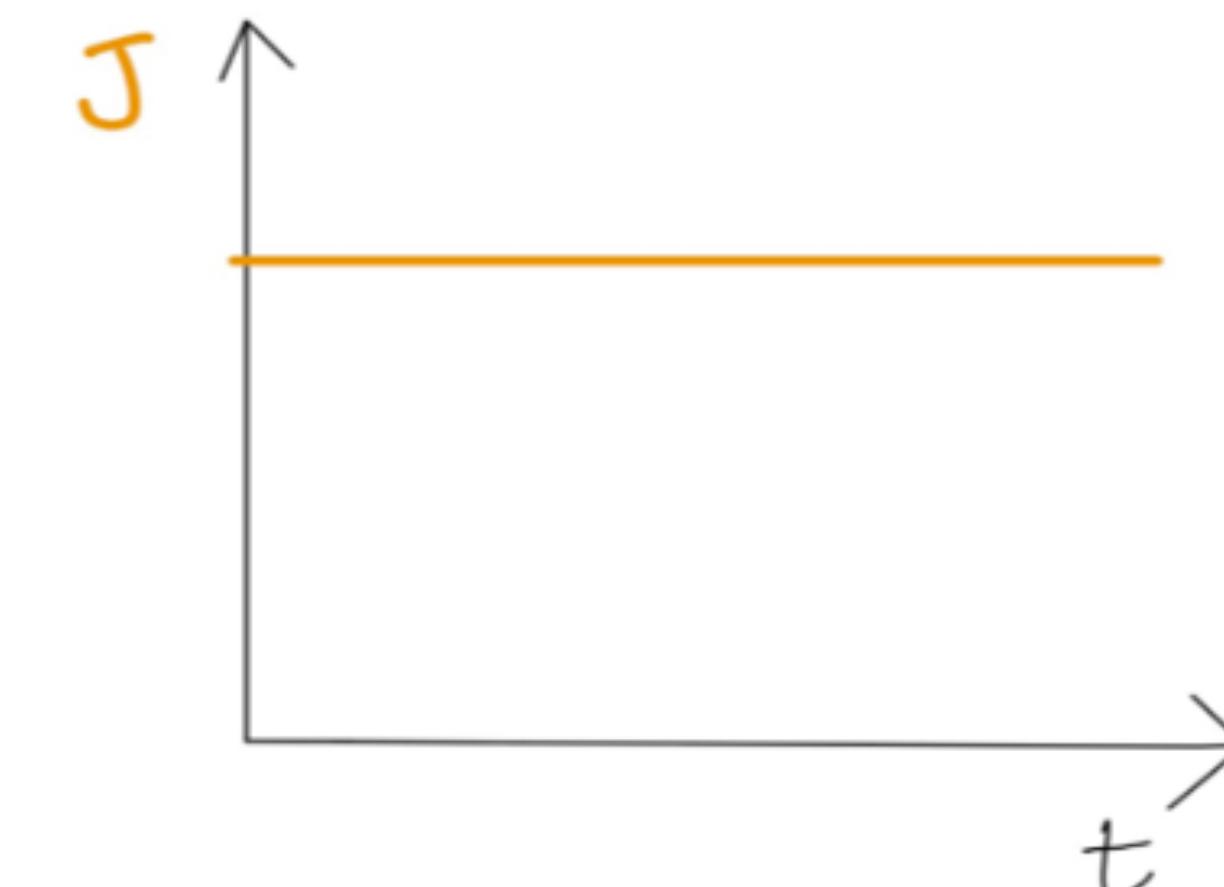
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Integrable systems and action-angles

- **Integrable system:** canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$.



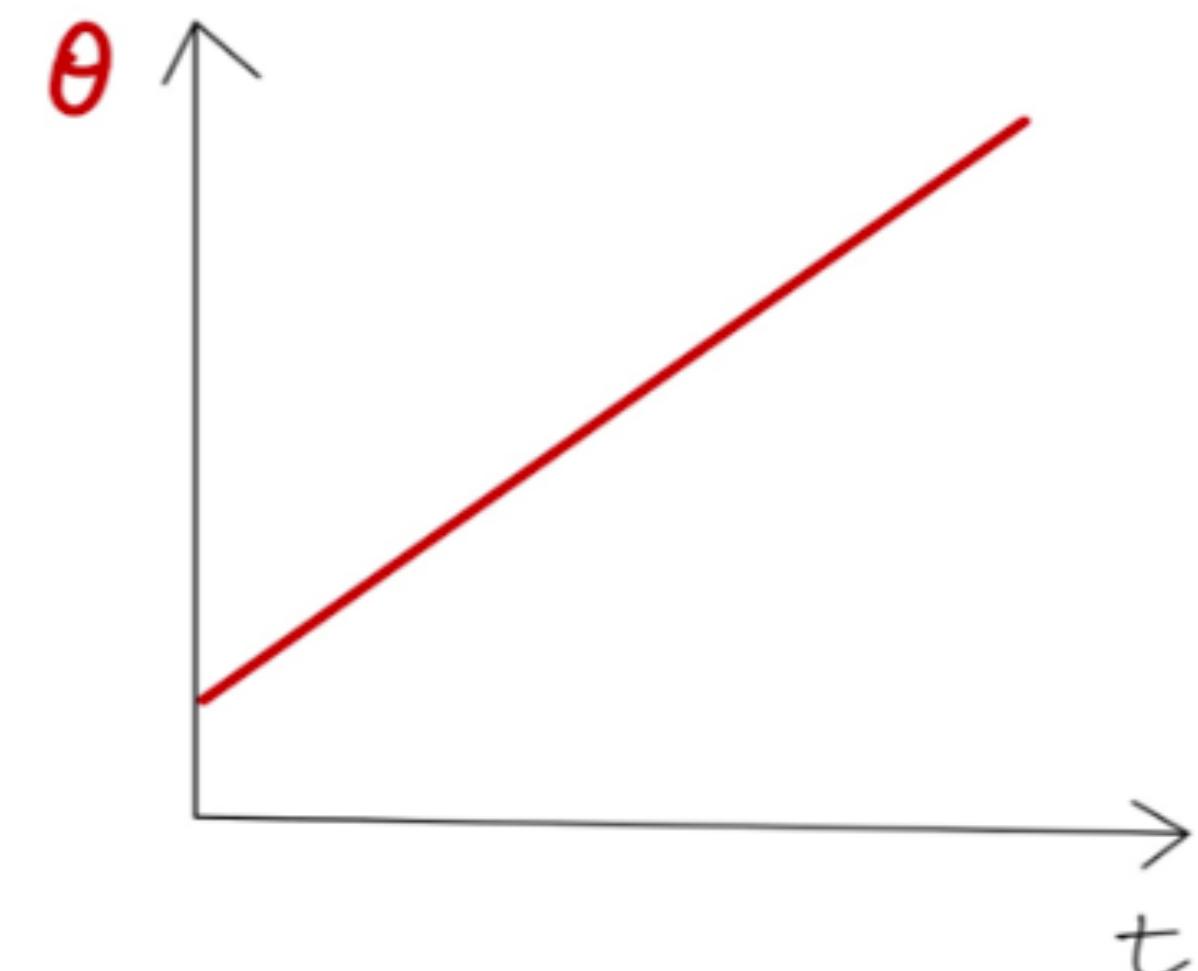
- Action $\mathcal{J}_i = \sim p$; angle $\theta_i = \sim q$.

- Hamilton's equations \Rightarrow

$$\dot{\mathcal{J}}_i = - \partial H / \partial \theta_i = 0 \quad \Rightarrow \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \quad \Rightarrow \theta_i = \omega_i(\vec{\mathcal{J}})t.$$

- Having action-angles \sim having closed-form solutions.



Compute frequencies $\omega_i \equiv d\theta_i/dt$

- Recall $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$.
- With $\vec{C} = \{J^2, L^2, J_z, H, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have $\mathcal{J}_i(\vec{C})$ (next lecture's subject).
- Compute the Jacobian $M_{ij}(\vec{C}) = \frac{\partial \mathcal{J}_i}{\partial C_j}$
(consists of numeric constants).
- **Inverse function theorem:** If $N_{ij} \equiv \frac{\partial C_i}{\partial \mathcal{J}_j}$, then $N = M^{-1}$.
- The fourth row of N corresponding to $(C_4 = H)$ contains $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$.

$$M = \begin{pmatrix} \frac{\delta \mathcal{J}_1}{\delta C_1} & \frac{\delta \mathcal{J}_1}{\delta C_2} & \frac{\delta \mathcal{J}_1}{\delta C_3} & \frac{\delta \mathcal{J}_1}{\delta C_4} & \frac{\delta \mathcal{J}_1}{\delta C_5} \\ \frac{\delta \mathcal{J}_2}{\delta C_1} & \frac{\delta \mathcal{J}_2}{\delta C_2} & \frac{\delta \mathcal{J}_2}{\delta C_3} & \frac{\delta \mathcal{J}_2}{\delta C_4} & \frac{\delta \mathcal{J}_2}{\delta C_5} \\ \frac{\delta \mathcal{J}_3}{\delta C_1} & \frac{\delta \mathcal{J}_3}{\delta C_2} & \frac{\delta \mathcal{J}_3}{\delta C_3} & \frac{\delta \mathcal{J}_3}{\delta C_4} & \frac{\delta \mathcal{J}_3}{\delta C_5} \\ \frac{\delta \mathcal{J}_4}{\delta C_1} & \frac{\delta \mathcal{J}_4}{\delta C_2} & \frac{\delta \mathcal{J}_4}{\delta C_3} & \frac{\delta \mathcal{J}_4}{\delta C_4} & \frac{\delta \mathcal{J}_4}{\delta C_5} \\ \frac{\delta \mathcal{J}_5}{\delta C_1} & \frac{\delta \mathcal{J}_5}{\delta C_2} & \frac{\delta \mathcal{J}_5}{\delta C_3} & \frac{\delta \mathcal{J}_5}{\delta C_4} & \frac{\delta \mathcal{J}_5}{\delta C_5} \end{pmatrix}$$

$$N = \begin{pmatrix} \frac{\delta C_1}{\delta \mathcal{J}_1} & \frac{\delta C_1}{\delta \mathcal{J}_2} & \frac{\delta C_1}{\delta \mathcal{J}_3} & \frac{\delta C_1}{\delta \mathcal{J}_4} & \frac{\delta C_1}{\delta \mathcal{J}_5} \\ \frac{\delta C_2}{\delta \mathcal{J}_1} & \frac{\delta C_2}{\delta \mathcal{J}_2} & \frac{\delta C_2}{\delta \mathcal{J}_3} & \frac{\delta C_2}{\delta \mathcal{J}_4} & \frac{\delta C_2}{\delta \mathcal{J}_5} \\ \frac{\delta C_3}{\delta \mathcal{J}_1} & \frac{\delta C_3}{\delta \mathcal{J}_2} & \frac{\delta C_3}{\delta \mathcal{J}_3} & \frac{\delta C_3}{\delta \mathcal{J}_4} & \frac{\delta C_3}{\delta \mathcal{J}_5} \\ \frac{\delta C_4}{\delta \mathcal{J}_1} & \frac{\delta C_4}{\delta \mathcal{J}_2} & \frac{\delta C_4}{\delta \mathcal{J}_3} & \frac{\delta C_4}{\delta \mathcal{J}_4} & \frac{\delta C_4}{\delta \mathcal{J}_5} \\ \frac{\delta C_5}{\delta \mathcal{J}_1} & \frac{\delta C_5}{\delta \mathcal{J}_2} & \frac{\delta C_5}{\delta \mathcal{J}_3} & \frac{\delta C_5}{\delta \mathcal{J}_4} & \frac{\delta C_5}{\delta \mathcal{J}_5} \end{pmatrix}$$

Action-angle-based solution: strategy

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- How to combine \vec{C} flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
- Solution via action-angles (the holy grail)

EOMs with Poisson brackets for BBHs

Our approach

- Define EOMs: $\frac{df(t)}{dt} = \{f, H\}$ where $f = f(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t))$.
- Define PBs: $\{R_i, P_j\} = \delta_{ji}$ $\{S_A^i, S_B^j\} = \delta_{AB}\epsilon_k^{ij}S_A^k$.

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\{f, g(v_i)\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

How to flow along the actions \mathcal{J}_i ?

- With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions (next lecture's subject).
- Using chain rule for PBs, $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, \mathcal{J}_i\} = \{\vec{V}, C_j\} \left(\frac{\partial \mathcal{J}_i}{\partial C_j} \right) = 2.5\{\vec{V}, C_1\} + 5.1\{\vec{V}, C_2\} = \{\vec{V}, 2.5C_1 + 5.1C_2\}$.
- \mathcal{J}_i flow by $\Delta\lambda = (C_1 \text{ flow by } 2.5\Delta\lambda, \text{ then } C_2 \text{ flow by } 5.1\Delta\lambda)$. Or reverse the order.

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions (subject of the next lecture).

- How to combine \vec{C} flows?
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- Solution via action-angles (the holy grail)

Solution via action-angles (the holy grail)

- Start with an initial $\vec{V}_0 = \{\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2\}$. Assign it $\vec{\theta} = \vec{0}$.
- We want $\vec{V} = \vec{V}(\vec{V}_0, t)$.
- Recall $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$ and $\Delta\theta_i = \Delta\lambda_i$.
- After time t , $\theta_i(t) = \omega_i t$.
- How to increase the angles? Action flows increase the angles.
- We need to flow under \mathcal{J}_i 's by an amount $\lambda_i = \theta_i(t) = \omega_i t$.

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have (i) $\mathcal{J}_i(\vec{C})$ (ii) \vec{C} flow solutions
(subject of the next lecture).

- How to combine \vec{C} flows?
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Lecture plan

- **Lecture 1:**
 - Theory
 - Strategy to compute solution from action-angles
- **Lecture 2:**
 - Finish the solution (first 4 actions)
 - Finish the solution (last action)

THE END

Please send comments on the lecture notes and
the presentation _/_

Thank you!

Lecture plan

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Action-angle-based solution: strategy

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(subject of this lecture).

- How to combine \vec{C} flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \frac{d\theta_i}{dt}$.
- How to flow along the actions \mathcal{J}_i ?
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Spin canonical coordinates

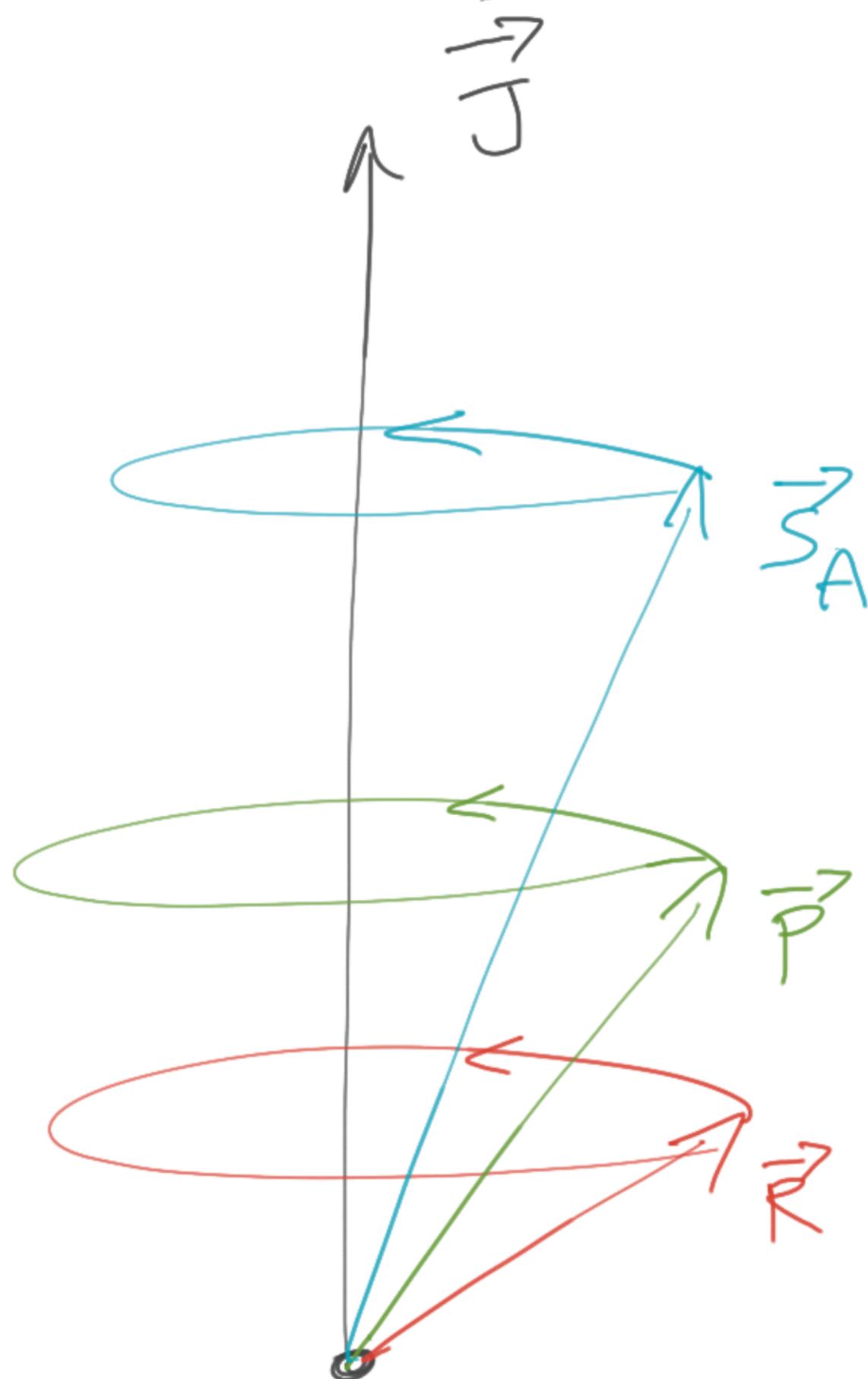
- $\phi \equiv$ azimuthal angle of \vec{S} .
- **Standard approach:** (ϕ, S_z, S) are functions of (S_x, S_y, S_z) .
- **Symplectic geometric approach:** Symplectic manifolds are always **even-dimensional** (position-momentum pairing); (S_x, S_y, S_z) doesn't fit in.
- Spin sphere of constant radius S . S is not a dynamical variable; just like m_1, m_2 .
- On this sphere we have coordinates $(S_z, \phi) \equiv$ (momentum, position); i.e. $\{\phi, S_z\} = 1$. Hence **even-dimensional**.
- $S_x = \sqrt{S^2 - S_z^2} \cos \phi, \quad S_y = \sqrt{S^2 - S_z^2} \sin \phi.$

Computing actions: strategy

- $\mathcal{J}_i = \frac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q};$ loop γ_i is on the surface of constant $\vec{C}.$
- $\{R_i, P_j\} = \delta_{ij}$ and $\{\phi_A, S_B^z\} = \delta_{AB}$
- How to be on the surface of constant $\vec{C}?$ Flow along C'_i s: $\frac{dC_i}{d\lambda} = \{C_i, C_j\} = 0.$
- $\mathcal{J} = \mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}$
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \oint_{\mathcal{C}} \sum_i P_i dR^i$ $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint_A S_A^z d\phi_A.$

Computing \mathcal{J}_1

- With $\vec{V} = \{\vec{R}, \vec{P}, \vec{L}, \vec{S}_1, \vec{S}_2\}$, J^2 flow $\Rightarrow \frac{d\vec{V}}{d\lambda} = 2\vec{J} \times \vec{V} \equiv \vec{n} \times \vec{V}$.
- $\{\vec{J}, J^2\} = 0$.
- Solution:** $\phi(\lambda) = n \lambda + \phi_0$.
- Loop closes after flowing by $\Delta\lambda = 2\pi/n = 2\pi/(2J) = \pi/J$.
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{n} \cdot \vec{L} d\lambda = \hat{n} \cdot \vec{L}$.
- $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint S_A^z d\phi_A = S_A^z = \hat{n} \cdot \vec{S}_A$ (with \vec{n} along z -axis)
- The spin integral is **rotationally invariant**, but not manifestly so.
- $\oint S_z d\phi = \int dS_z \wedge d\phi = S \int d(\cos \theta) \wedge d\phi = -S \int \sin \theta d\theta \wedge d\phi = -\text{Area}/S$
- $\mathcal{J}_1 = \hat{n} \cdot (\vec{L} + \vec{S}_1 + \vec{S}_2) = \hat{n} \cdot \vec{J} = J$.
- Summary:** We have computed \mathcal{J}_1 and also computed the solution to $C_1 = J^2$.



Computing $\mathcal{J}_1, \mathcal{J}_2$ and \mathcal{J}_3

- **For flows under J^2, J_z , and L^2 :** $\frac{d\vec{V}}{d\lambda} = \vec{n} \times \vec{V}$. $\vec{n} = 2\vec{J}, \hat{\vec{z}}$, and $2\vec{L}$ (with \vec{n} being fixed)
- **Exception:** Under L^2 flow, spins don't move.
- **Solution:** $\phi(\lambda) = n \lambda + \phi_0$. Doesn't apply to spins under the L^2 flow.
- Loop closes after flowing by $\Delta\lambda = 2\pi/n$.
- $\mathcal{J}^{\text{orb}} = \frac{1}{2\pi} \int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi} \int_0^{\Delta\lambda} \vec{n} \cdot \vec{L} d\lambda = \hat{\vec{n}} \cdot \vec{L}$.
- $\mathcal{J}_A^{\text{spin}} = \frac{1}{2\pi} \oint S_A^z d\phi_A = S_A^z = \hat{\vec{n}} \cdot \vec{S}_A$ (with \vec{n} along z -axis)
- $\mathcal{J}_1 = J, \mathcal{J}_2 = J_z, \mathcal{J}_3 = L$.
- **Summary:** We have computed $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ and also computed the solution to $C_i = \{J^2, J_z, L^2\}$.

Computing \mathcal{J}_4

- We won't compute it here.
- \mathcal{J}_4 has a Newtonian version (Eq. (10.139) of Goldstein).
- 1PN version given in Eq. (3.10) of Damour-Schafer.
- 1.5PN version in Eq. (38) of [arXiv: 2012.06586].
- $$\mathcal{J}_4 = -L + \frac{GM\mu^{3/2}}{\sqrt{-2H}} + \frac{GM}{c^2} \left[\frac{3GM\mu^2}{L} + \frac{\sqrt{-H}\mu^{1/2}(\nu - 15)}{\sqrt{32}} - \frac{2G\mu^3}{L^3} \vec{S}_{\text{eff}} \cdot \vec{L} \right] + \mathcal{O}(c^{-4})$$

Taking stock

- We solved the flows under $C_i = \{J^2, J_z, L^2\}$.
- Finding the solution $\vec{V}(\vec{V}_0, \Delta\lambda)$ of a flow under C_i : $\frac{d\vec{V}}{d\lambda} = \{\vec{V}, C_i\}$ is basically solving an ODE.
- Solution of flow under $\vec{S}_{\text{eff}} \cdot \vec{L}$ in [arXiv:2110.15351].
- Solution of flow under H in [arXiv:1908.02927]. They omit 1PN terms for simplicity. Call it the **standard solution**.
- Above solutions: quite lengthy but not esoteric.
- **Future focus:** compute \mathcal{J}_5 and a peek into $S_{\text{eff}} \cdot L$ flow

Lecture plan

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5 minute break
Coffee, questions?

\mathcal{J}_5 computation

For $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow:

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{P}$$

$$\frac{d\vec{S}_a}{d\lambda} = \sigma_a \left(\vec{L} \times \vec{S}_a \right)$$

$$\frac{d\vec{L}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{L}$$

- **Important:** \vec{n} not fixed: $\{\vec{n}, S_{\text{eff}} \cdot L\} \neq 0$.

\mathcal{J}_5 computation

$$2\pi \mathcal{J} = 2\pi (\mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}})$$

$$= \int_{\lambda_i}^{\lambda_f} \left(P_i dR^i + S_1^z d\phi_1^z + S_2^z d\phi_2^z \right)$$

- $= \int_{\lambda_i}^{\lambda_f} \left(P_i \frac{dR^i}{d\lambda} + S_1^z \frac{d\phi_1^z}{d\lambda} + S_2^z \frac{d\phi_2^z}{d\lambda} \right) d\lambda$

- $2\pi \mathcal{J}^{\text{orb}} = \int_{\lambda_i}^{\lambda_f} \overrightarrow{P} \cdot \left(\overrightarrow{S}_{\text{eff}} \times \overrightarrow{R} \right) d\lambda = \int_{\lambda_i}^{\lambda_f} (S_{\text{eff}} \cdot L) d\lambda = (S_{\text{eff}} \cdot L) \Delta\lambda$

- Can't do spin sector integral because $\overrightarrow{S}_A \neq \overrightarrow{R}_A \times \overrightarrow{P}_A$. “A” is BH index.

\mathcal{J}_5 computation: enter fictitious variables

- Define \vec{R}_a, \vec{P}_a (fictitious variables) such that $\vec{S}_a \equiv \vec{R}_a \times \vec{P}_a$.
- **Hamiltonian:** Now a function of $\vec{R}, \vec{P}, \vec{R}_{1/2}, \vec{P}_{1/2}$ and not $\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$.
- **PBs and EOMs:** $\{R_i, P_j\} = \delta_{ij}, \quad \{R_{ai}, P_{bj}\} = \delta_{ab}\delta_{ji}; \quad \frac{df}{dt} = \{f, H\}$.
- $\{R_i, P_j\} = \delta_{ij}, \quad \{R_{ai}, P_{bj}\} = \delta_{ab}\delta_{ji} \implies \{R_i, P_j\} = \delta_{ij}, \quad \{\phi_A, S_B^z\} = \delta_{AB}$
- PBs \rightarrow EOMs \implies The standard phase space (**SPS**) is equivalent to the extended phase space (**EPS**).
- **Integrability equivalency:** EPS needs $n = 2n/2 = 18/2 = 9 = (5 + 4)$ C_i 's. The next 4 C_i 's are S_a^2 and $\vec{R}_a \cdot \vec{P}_a$.

\mathcal{J}_5 computation: sanity checks

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- **Check 1:** Final \mathcal{J}_5 depends on \vec{R} , \vec{P} , \vec{S}_1 and \vec{S}_2 .
- **Check 2:** Numerical flow by 2π under \mathcal{J}_5 closes a loop in the SPS picture.
- We have all seen fictitious variables before (in spirit)!
- Inventing complex numbers to do real integrals.

11.8.19 Prove that $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx = \pi \ln 2$.

11.8.20 Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2} dx = \frac{\pi a}{\sin \pi a},$$

where $-1 < a < 1$.

Hint. Use the contour shown in Fig. 11.26, noting that $z = 0$ is a branch point and the positive x -axis can be chosen to be a cut line.

11.8.21 Show that

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.16 is a special case of this result.

11.8.22 Show that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = 2\pi/n$.

11.8.23 (a) Show that

$$f(z) = z^4 - 2z^2 \cos 2\theta + 1$$

has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.22 ($n = 4$) is a special case of this result.

\mathcal{J}_5 computation using fictitious variables

$$\mathcal{J}_k = \frac{1}{2\pi} \oint_{\mathcal{C}_k} \left(\vec{P} \cdot d\vec{R} + \vec{P}_1 \cdot d\vec{R}_1 + \vec{P}_2 \cdot d\vec{R}_2 \right)$$

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\text{eff}} \times \vec{P}$$

$$\frac{d\vec{R}_a}{d\lambda} = \sigma_a (\vec{L} \times \vec{R}_a)$$

$$\frac{d\vec{P}_a}{d\lambda} = \sigma_a (\vec{L} \times \vec{P}_a)$$

EOMs for $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow are

$$\begin{aligned} 2\pi \mathcal{J}_{S_{\text{eff}} \cdot L} &= 2\pi (\mathcal{J}^{\text{orb}} + \mathcal{J}^{\text{spin}}) \\ &= \int_{\lambda_i}^{\lambda_f} \left(P_i \frac{dR^i}{d\lambda} + P_{1i} \frac{dR_1^i}{d\lambda} + P_{2i} \frac{dR_2^i}{d\lambda} \right) d\lambda \\ &= \int_{\lambda_i}^{\lambda_f} \left(\vec{P} \cdot (\vec{S}_{\text{eff}} \times \vec{R}) + \vec{P}_1 \cdot (\sigma_1 \vec{L} \times \vec{R}_1) \right. \\ &\quad \left. + \vec{P}_2 \cdot (\sigma_2 \vec{L} \times \vec{R}_2) \right) d\lambda \\ &= 2 \int_{\lambda_i}^{\lambda_f} (S_{\text{eff}} \cdot L) d\lambda = 2 (S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L} \end{aligned}$$

$$\mathcal{J}_{S_{\text{eff}} \cdot L} = \frac{(S_{\text{eff}} \cdot L) \Delta\lambda_{S_{\text{eff}} \cdot L}}{\pi}$$

\mathcal{J}_5 computation using fictitious variables

$$\mathcal{J}_{S_{\text{eff}} \cdot L} = \frac{(S_{\text{eff}} \cdot L) \Delta \lambda_{S_{\text{eff}} \cdot L}}{\pi},$$

$$\mathcal{J}_{J^2} = \frac{J^2 \Delta \lambda_{J^2}}{\pi},$$

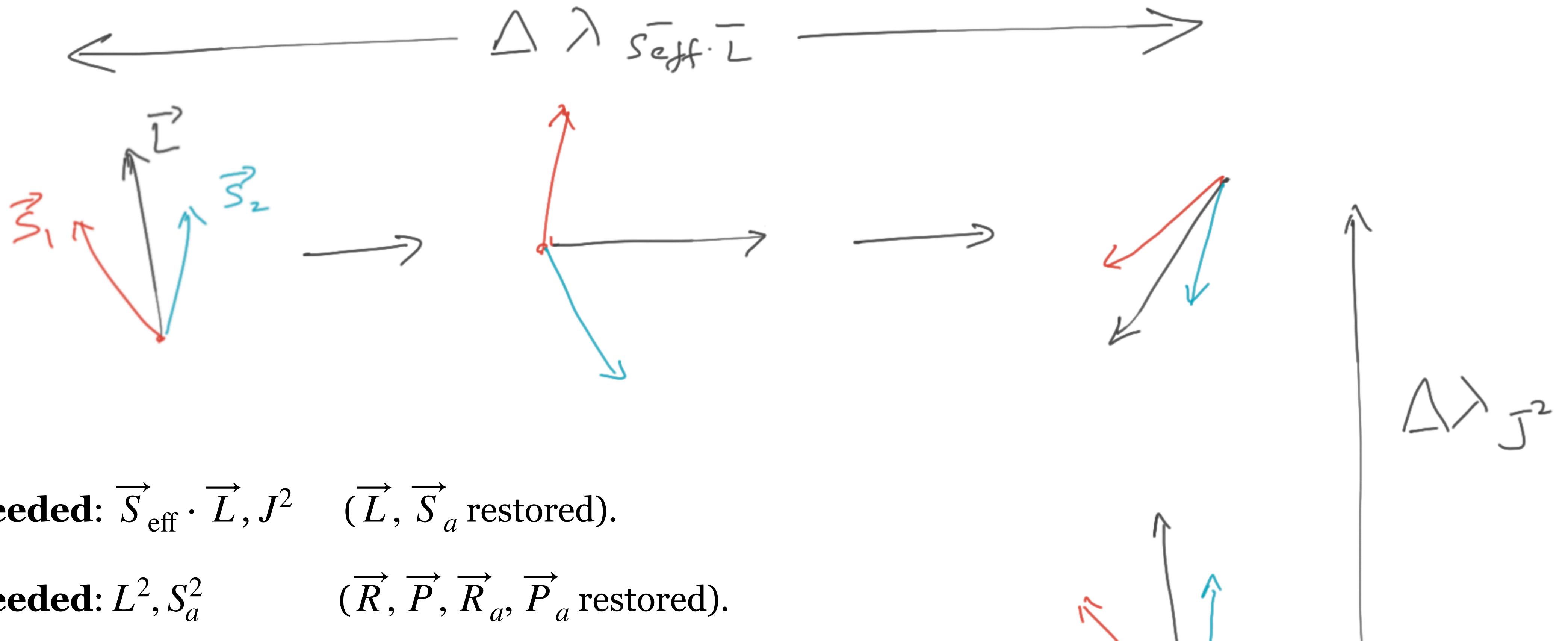
$$\mathcal{J}_{L^2} = \frac{L^2 \Delta \lambda_{L^2}}{\pi}, \quad \Rightarrow \quad \mathcal{J}_5 = \frac{1}{\pi} \left\{ (S_{\text{eff}} \cdot L) \Delta \lambda_{S_{\text{eff}} \cdot L} + J^2 \Delta \lambda_{J^2} + L^2 \Delta \lambda_{L^2} + S_1^2 \Delta \lambda_{S_1^2} + S_2^2 \Delta \lambda_{S_2^2} \right\}.$$

- $\mathcal{J}_{S_1^2} = \frac{S_1^2 \Delta \lambda_{S_1^2}}{\pi},$

$$\mathcal{J}_{S_2^2} = \frac{S_2^2 \Delta \lambda_{S_2^2}}{\pi}.$$

- Loop for \mathcal{J}_5 is closed by flowing under 5 C_i 's (not one).
- Flow amounts $\Delta \lambda_i$'s give \mathcal{J}_5

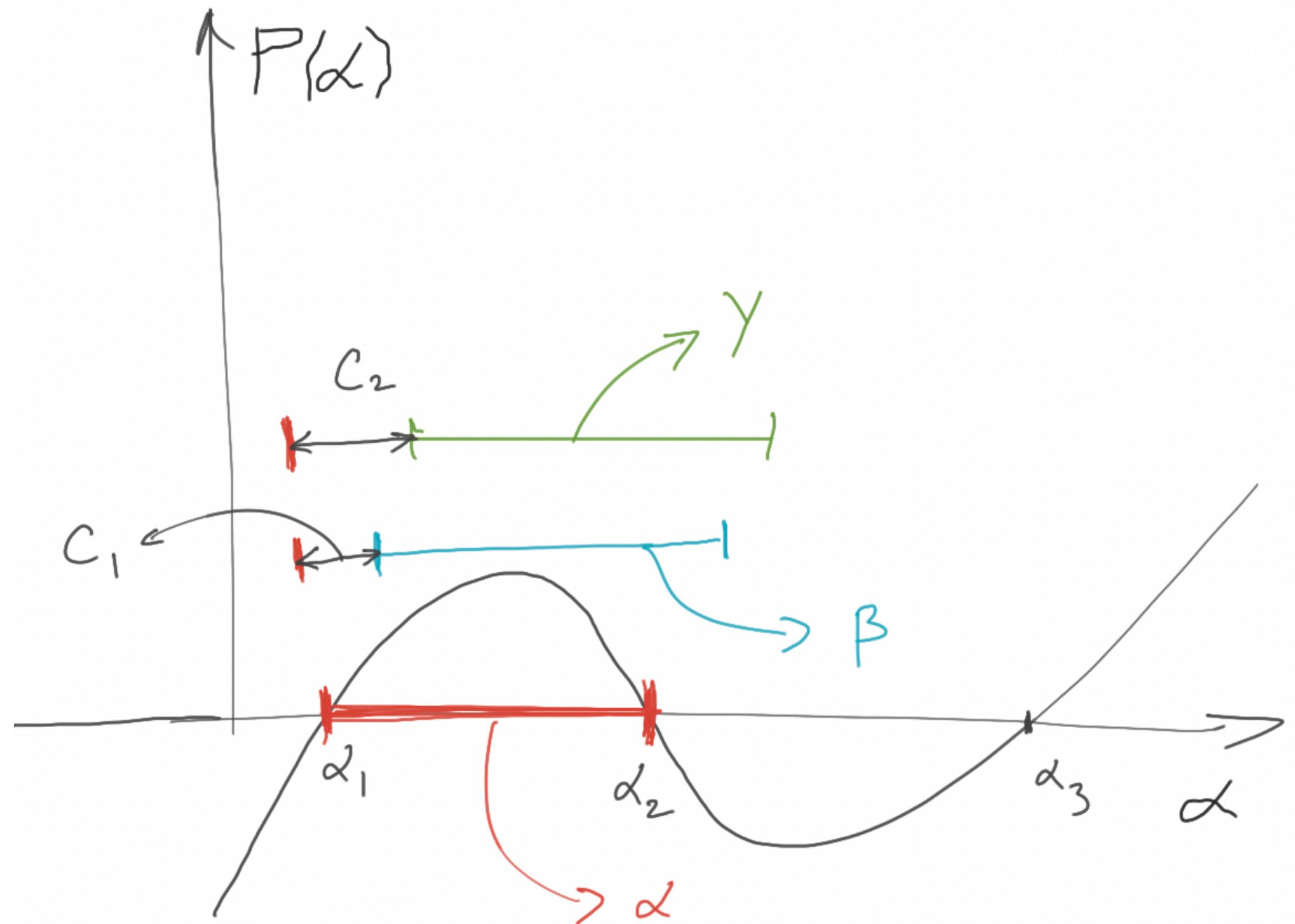
\mathcal{J}_5 computation: flow overview



- **Flows needed:** $\vec{S}_{\text{eff}} \cdot \vec{L}, J^2$ (\vec{L}, \vec{S}_a restored).
- **Flows needed:** L^2, S_a^2 ($\vec{R}, \vec{P}, \vec{R}_a, \vec{P}_a$ restored).
- $$\mathcal{J}_5 = \frac{1}{\pi} \left\{ (S_{\text{eff}} \cdot L) \Delta \lambda_{S_{\text{eff}} \cdot L} + J^2 \Delta \lambda_{J^2} + L^2 \Delta \lambda_{L^2} + S_1^2 \Delta \lambda_{S_1^2} + S_2^2 \Delta \lambda_{S_2^2} \right\}$$

\mathcal{J}_5 computation: $\{\vec{L}, \vec{S}_1, \vec{S}_2\}$ triad acts like a lung under $\vec{S}_{\text{eff}} \cdot \vec{L}$ flow

- I will be sloppy; drop some constants (L, S_1, S_2) and ignore - signs
- $\alpha, \beta, \gamma = \cosine$ of the mutual angles of the $\{\vec{L}, \vec{S}_1, \vec{S}_2\}$ triad.
- **Key point:** $d\alpha/d\lambda \equiv \alpha' = \beta' = \gamma' = \vec{L} \cdot (\vec{S}_1 \times \vec{S}_2) = \pm \sqrt{1 + 2\alpha\beta\gamma - \alpha^2 - \beta^2 - \gamma^2}$.
- $\alpha = \beta - C_1 = \gamma - C_2 \quad \& \quad \alpha' = \pm \sqrt{P(\alpha)}$; $P(\alpha)$ = cubic in α .
- α at extrema $\implies \beta, \gamma$ also at extrema \implies breathing lung.
- $a(\lambda) = a_1 + (a_2 - a_1)\operatorname{sn}^2(u, k)$; $u = \frac{1}{2}\sqrt{A(a_3 - a_1)}(C_4 + (\lambda - \lambda_0))$
- $k = \sqrt{\frac{a_2 - a_1}{a_3 - a_1}}$; $C_4 = \frac{2}{\sqrt{A(a_3 - a_1)}} F\{\arcsin \sqrt{\frac{a(\lambda_0) - a_1}{a_2 - a_1}}\}$;
- $A = A(m_1, m_2)$
- Flow amount $\Delta\lambda = \text{period} = \frac{4K(k)}{\sqrt{A(a_3 - a_1)}}$.



THE END

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Thank you!