Closed-form solutions of spinning BBHs at 1.5PN (using action-angle variables)

Sashwat Tanay

References

- **RESEARCH PAPERS**
  - The standard way of computing the solution (without 1PN part): https://arxiv.org/abs/1908.02927
  - Action-angle-based solution: https://arxiv.org/abs/2012.06586, https://arxiv.org/abs/2110.15351

- **LECTURE NOTES**
  - Lecture notes (latest): https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/gh-action-result/pdflatex/lecture_notes/main.pdf
  - Lecture notes (for citation purposes): https://arxiv.org/abs/2206.05799

- **MATHEMATICA PACKAGE**
  - Mathematica package on GitHub: https://github.com/sashwattanay/BBH-PN-Toolkit

- **YOUTUBE VIDEO**
  - https://youtu.be/aoiCk5TtmvE

Lecture plan

Lecture style: standing on the shoulders of giants (due to time constraints)

- **Lecture 1**:
    - Theory
    - Strategy to compute solution from action-angles

- **Lecture 2**:
    - Construct the solution

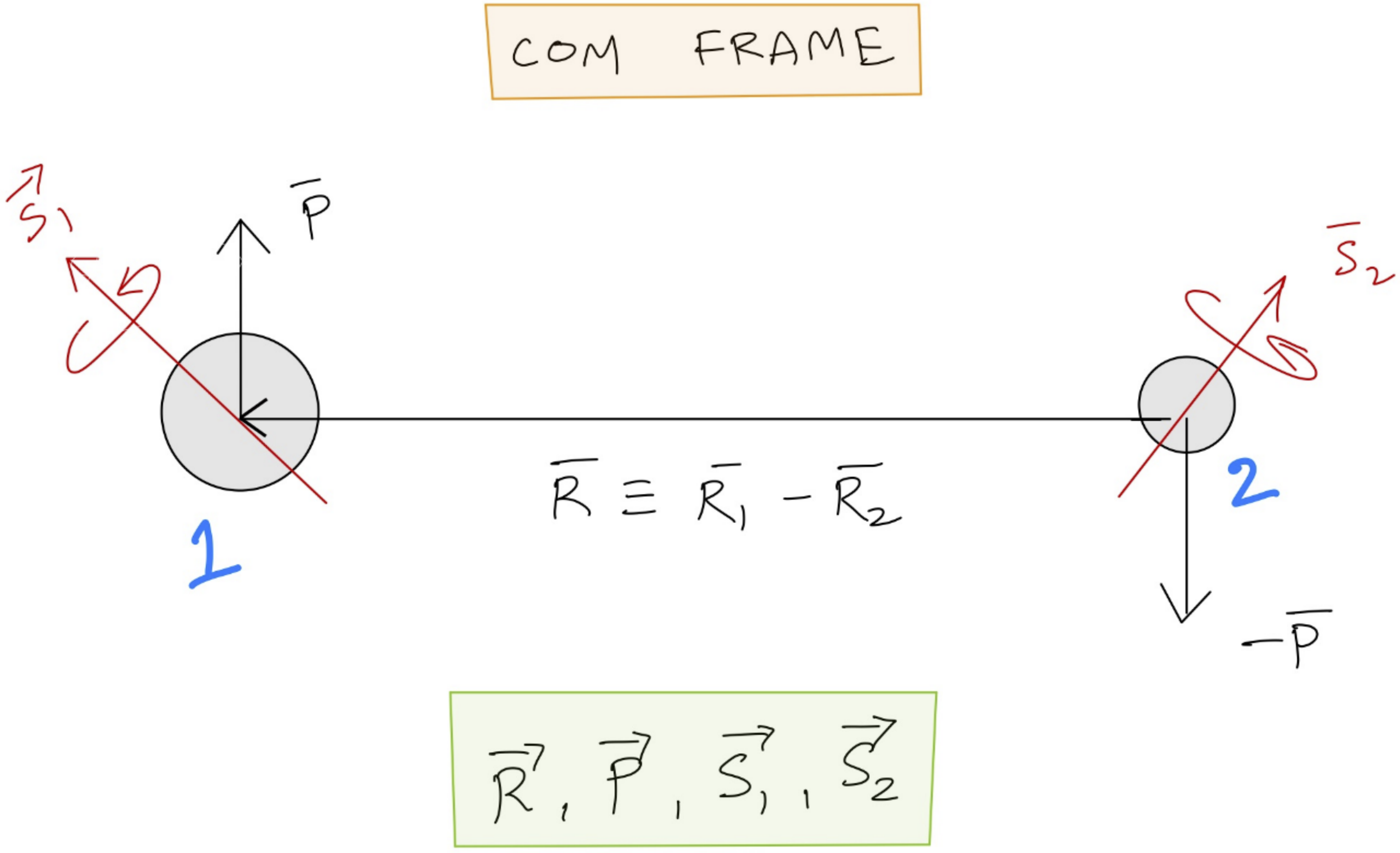
Lecture plan

- **Lecture 1**:
  - Theory
  - Strategy to compute solution from action-angles

- **Lecture 2**:
  - Construct the solution

Introduction to the system

Spinning 1.5PN BBH system



Phase space variables

$\vec{R}(t)$, $\vec{P}(t)$, $\vec{S}_1(t)$ and $\vec{S}_2(t)$

Statement of the problem

- The 1.5PN Hamiltonian is $H = H_{\mathrm{N}} + H_{1\mathrm{PN}} + H_{1.5\mathrm{PN}} + \mathcal{O}\left(c^{-4}\right)$ with
- $H_{\mathrm{N}} = \mu\left(\frac{p^2}{2} - \frac{1}{r}\right)$, $\qquad H_{1.5\mathrm{PN}} = \frac{2G}{c^2R^3}\vec{S}_{\mathrm{eff}} \cdot \vec{L}$.
- Hamilton's equations $\Longrightarrow$ $\frac{d\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)}{dt}$.
- **Problem**: Integrate Hamilton's eqns. to obtain $\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)$.

Historical context and the status quo

- The 1.5PN Hamiltonian is $H = H_{\mathrm{N}} + H_{1\mathrm{PN}} + H_{1.5\mathrm{PN}} + \mathcal{O}\left(c^{-4}\right)$.
- **1680s**: Issac Newton gave the Newtonian solution $R = a(1 - e\cos u)$.
- **1985**: Damour-Deruelle gave 1PN quasi-Keplerian solution.
- **2019**: Gihyuk Cho, H. M. Lee gave 1.5PN solution (1PN effects ignored for simplicity)
- **2021**: We worked out an equivalent action-angle based solution (subject of these lectures).
- Why action-angles? Extendible to 2PN via canonical perturbation theory (Goldstein).

Do we even the solution? YouTube link

- Show all three solutions
- YouTube video

EOMs with Poisson brackets

Standard approach

- Hamilton's eqns. are $\dot{q}_i = \dfrac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\dfrac{\partial H}{\partial q_i}$
- Leads to EOM $\dfrac{df}{dt} = \{f, H\}$ with $\{f, g\} \equiv \sum_{i=1}^{N} \left(\dfrac{\partial f}{\partial q_i}\dfrac{\partial g}{\partial p_i} - \dfrac{\partial f}{\partial p_i}\dfrac{\partial g}{\partial q_i} \right).$

EOMs with Poisson brackets for BBHs

Our approach

- Define EOMs: $\frac{df(t)}{dt} = \{f, H\}$ where $f = f\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)$.
- Define PBs: $\left\{R_i, P_j\right\} = \delta_{ji}$ $\left\{S_A^i, S_B^j\right\} = \delta_{AB}\epsilon_k^{ij} S_A^k$.

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\left\{f, g\left(v_i\right)\right\} = \{f, v_i\} \frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

PB Exercise 1

Prob: Compute $\left\{R_x, \sin P_x + P_x\right\}$.

Sol: Using the bilinearity and the chain rule (2nd and 4th rules) for PBs

$$
\begin{aligned}
& \left\{R_x, \sin P_x + P_x\right\} \\
= & \left\{R_x, \sin P_x\right\} + \left\{R_x, P_x\right\} \\
= & \left\{R_x, P_x\right\} \frac{\partial \sin P_x}{\partial P_x} + \left\{R_x, P_x\right\} \\
= & \cos P_x + 1.
\end{aligned}
$$

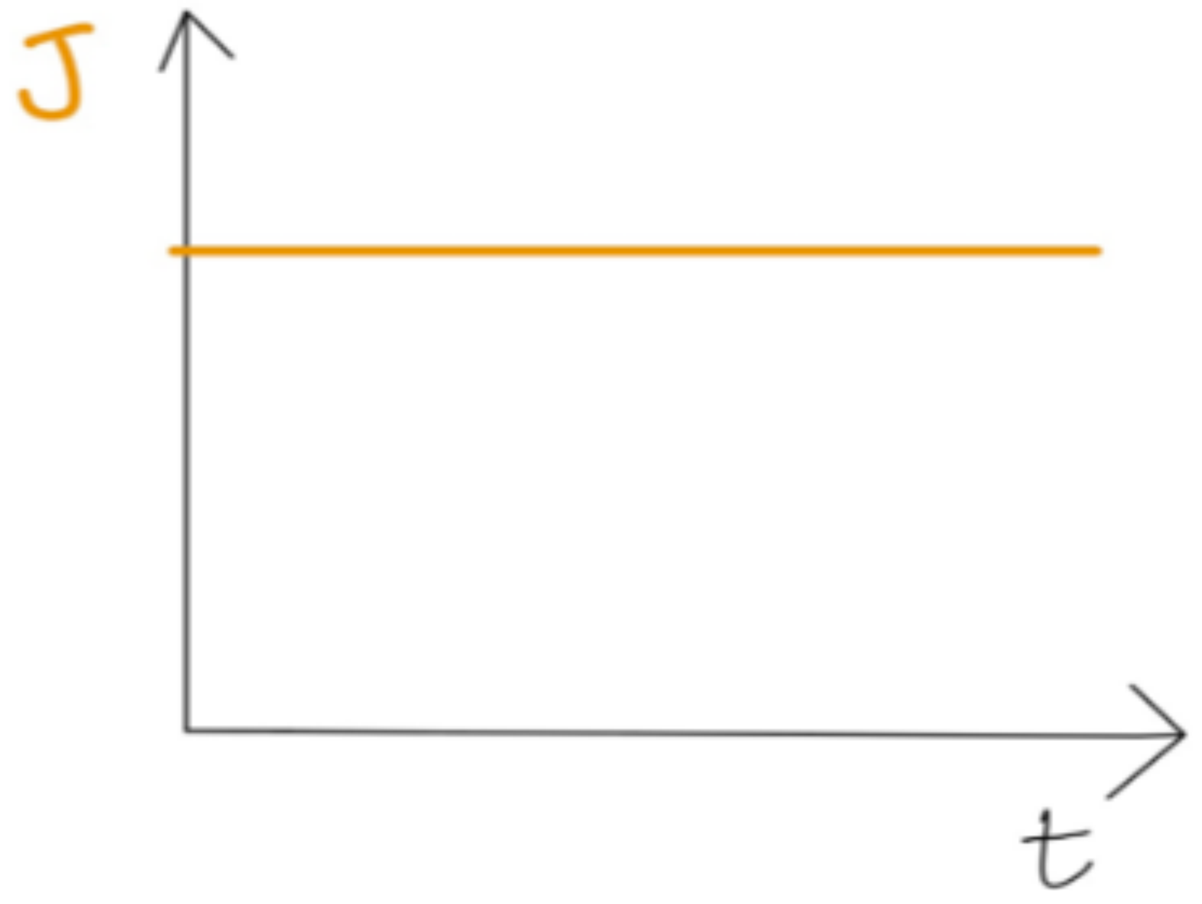
PB Exercise 2

Prob: Show that $\left\{\phi_A, S^z_B\right\} = \delta_{AB}$, where $\phi_A = \arctan\left(S^y_A/S^x_A\right)$ is the azimuthal angle of $\overrightarrow{S}_A$.

- Implies that $\phi \sim$ position; $S^z \sim$ momentum upon comparison with $\left\{R_i, P_j\right\} = \delta_{ji}$.
- **Lingo**: f and g commute if $\left\{f, g\right\} = 0$.
- How to evaluate PBs: See the YouTube video (...add hyperlink)

Integrable systems and action-angles

- **Integrable system**: canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}\left(\theta_i + 2\pi\right) = \{\vec{p}, \vec{q}\}\left(\theta_i\right)$.

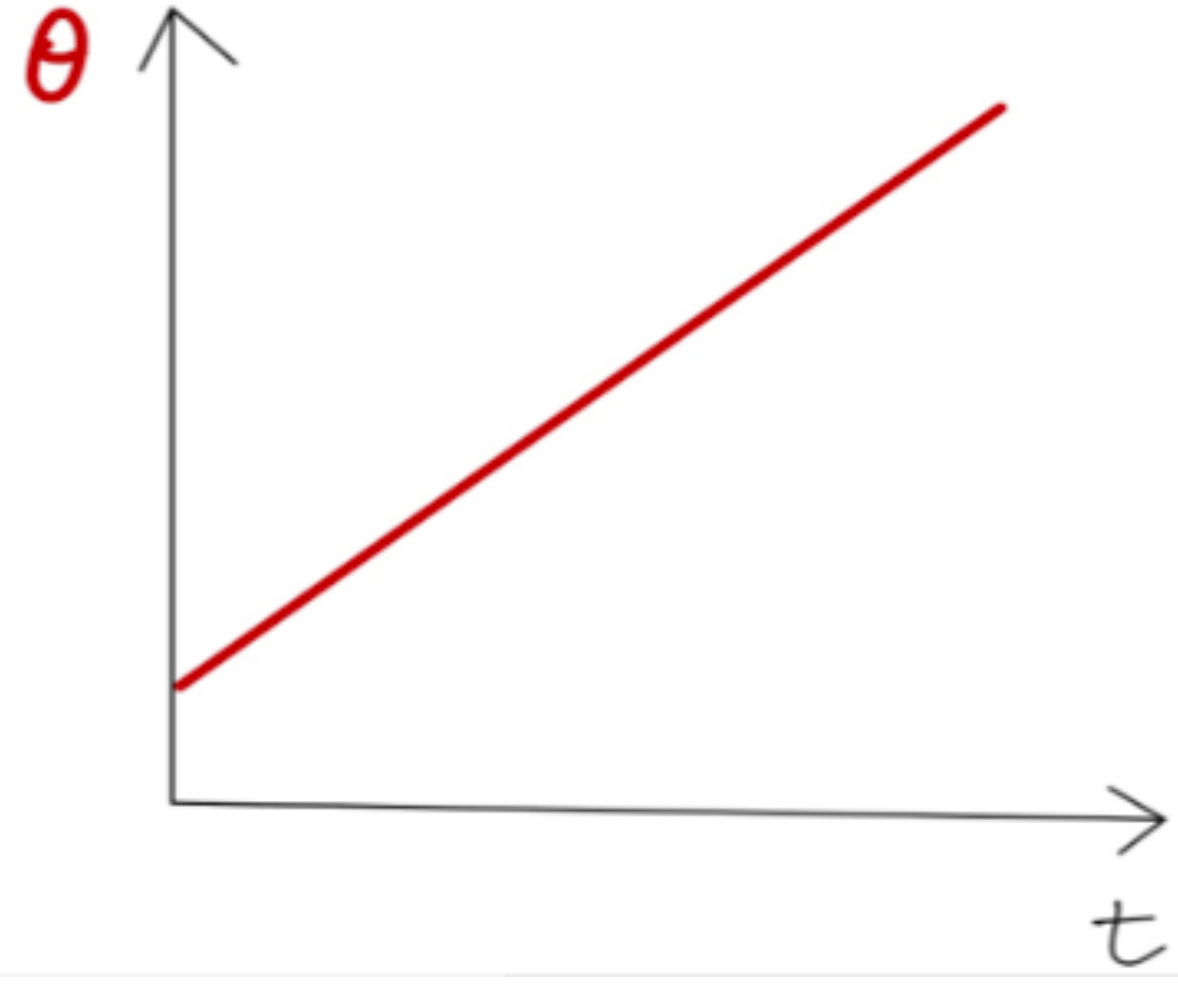


- Action $\mathcal{J}_i = \sim p$; angle $\theta_i = \sim q$.

- Hamilton's equations $\Longrightarrow$

$$\dot{\mathcal{J}}_i = -\,\partial H / \partial \theta_i = 0 \qquad \Longrightarrow \mathcal{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i(\vec{\mathcal{J}}) \qquad \Longrightarrow \theta_i = \omega_i(\vec{\mathcal{J}})t.$$



- Having action-angles ~ having closed-form solutions.

Liouville-Arnold theorem

- **Theorem**: $2n$ phase space variables & n commuting constants of motion $\Longrightarrow$ integrability.

- How to check if f is a constant of motion? Check if $\{f, H\} = 0$.

- For BBH phase space $(\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2)$, $2n \neq 12$. Positions-momenta delineation not clear for spins.

- Easy to check that $\left\{R_i, P_j\right\} = \delta_{ij}$ and $\left\{\phi_A, S_B^z\right\} = \delta_{AB}$ where $\phi_A = \arctan\left(S_A^y / S_A^x\right)$, the azimuthal angle for $\vec{S}_A$.

- $\left(\phi_A, S_A^z\right)$ are the positions, momenta of $\vec{S}_A$. Only 2 variables needed for $\vec{S}_A$ since $dS_A/dt = \left\{S_A, H\right\} = 0$.

- Hence $2n = 3 + 3 + 2 + 2 = 10 \implies 10/2 = 5$ commuting constants needed for integrability.

Commuting constants for BBHs

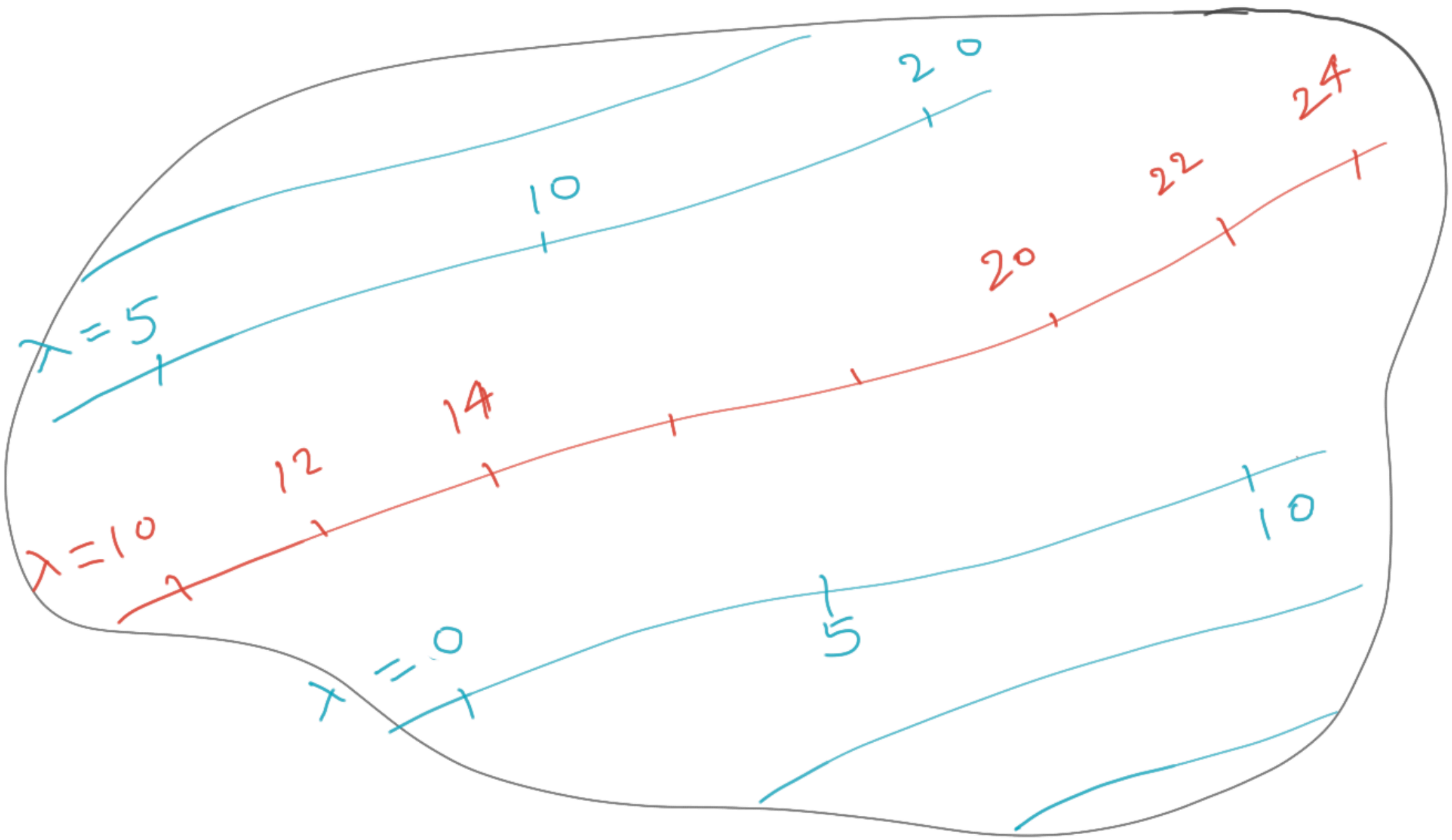
- With $m \equiv m_1 + m_2, \quad \mu \equiv m_1 m_2/m, \quad \nu \equiv \mu/m, \quad \overrightarrow{L} \equiv \overrightarrow{R} \times \overrightarrow{P}$
  $\sigma_1 \equiv \left(2 + 3m_2/m_1\right), \quad \sigma_2 \equiv \left(2 + 3m_1/m_2\right), \quad \overrightarrow{S}_{\text{eff}} \equiv \sigma_1 \overrightarrow{S}_1 + \sigma_2 \overrightarrow{S}_2$
  $\overrightarrow{J} = \overrightarrow{L} + \overrightarrow{S}_1 + \overrightarrow{S}_2.$

- The 5 commuting constants are long known: $H, J^2, L^2, J_z, \overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}.$

- Hence the 1.5PN BBH is integrable and has action-angles.

Curves, vectors, vector field and flow

- **Pictorial definition**: the vector is $d/d\lambda$ (a derivative operator).



Hamiltonian flow of a function $f(\vec{V})$

- $\vec{V} \equiv \left\{ \vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2 \right\}$, unless states otherwise.
- Hamiltonian flow of $f(\vec{V})$: $\dfrac{d\vec{V}}{d\lambda} = \{ \vec{V}, f\}$. f need not be the Hamiltonian.
- Solution of the flow given in the form $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda)$.
- Flow $\equiv$ Hamiltonian flow (for brevity).

Flow exercise 1

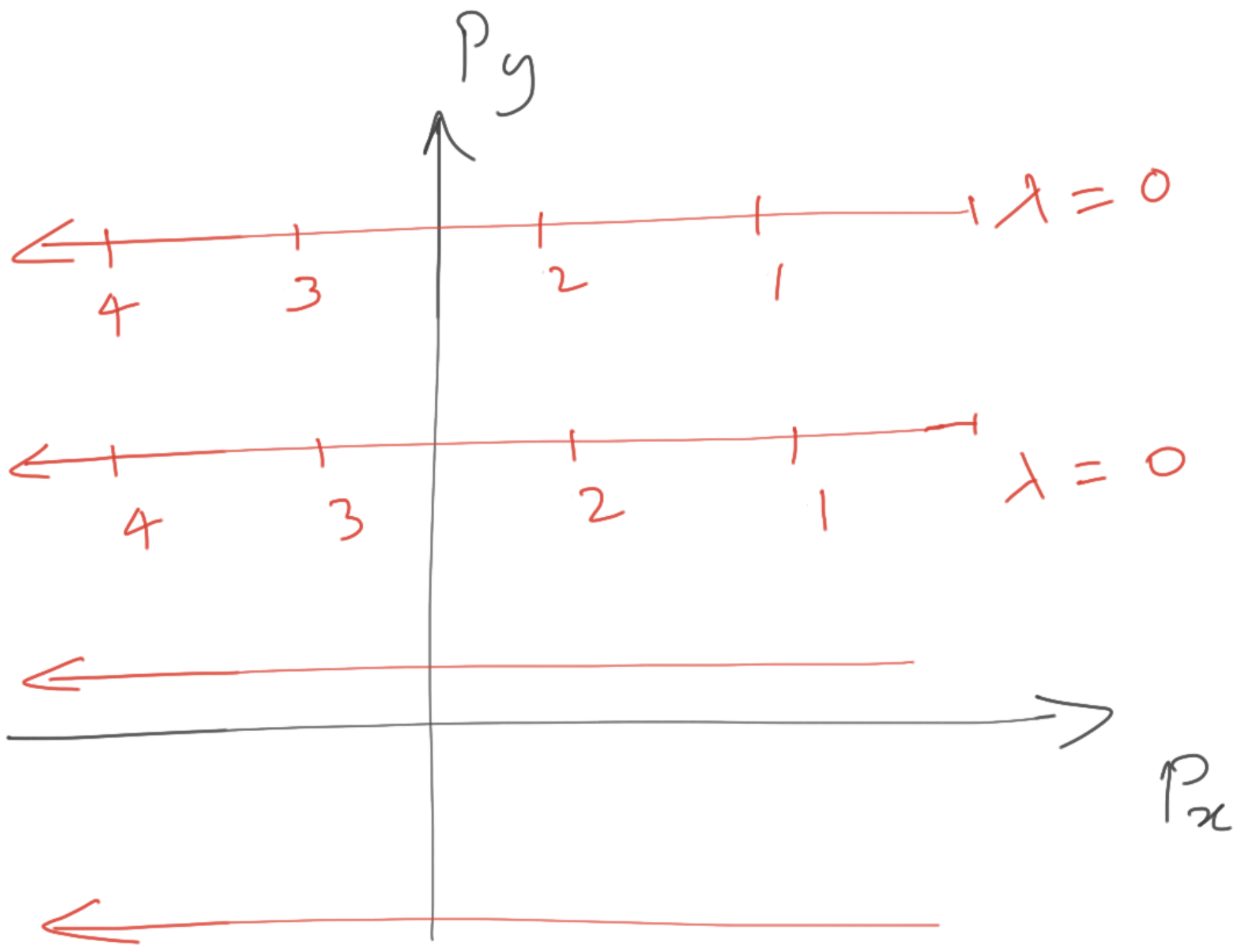
Prob: Solve the flow under R_x and draw pictures.

Sol: Under the R_x flow:

$$\frac{dP_x}{d\lambda} = \{P_x, R_x\} = -1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i\text{'s}.$$

$$\implies P_x - P_x(\lambda_0) = (\lambda_0 - \lambda)\,.$$



Flow exercise 2

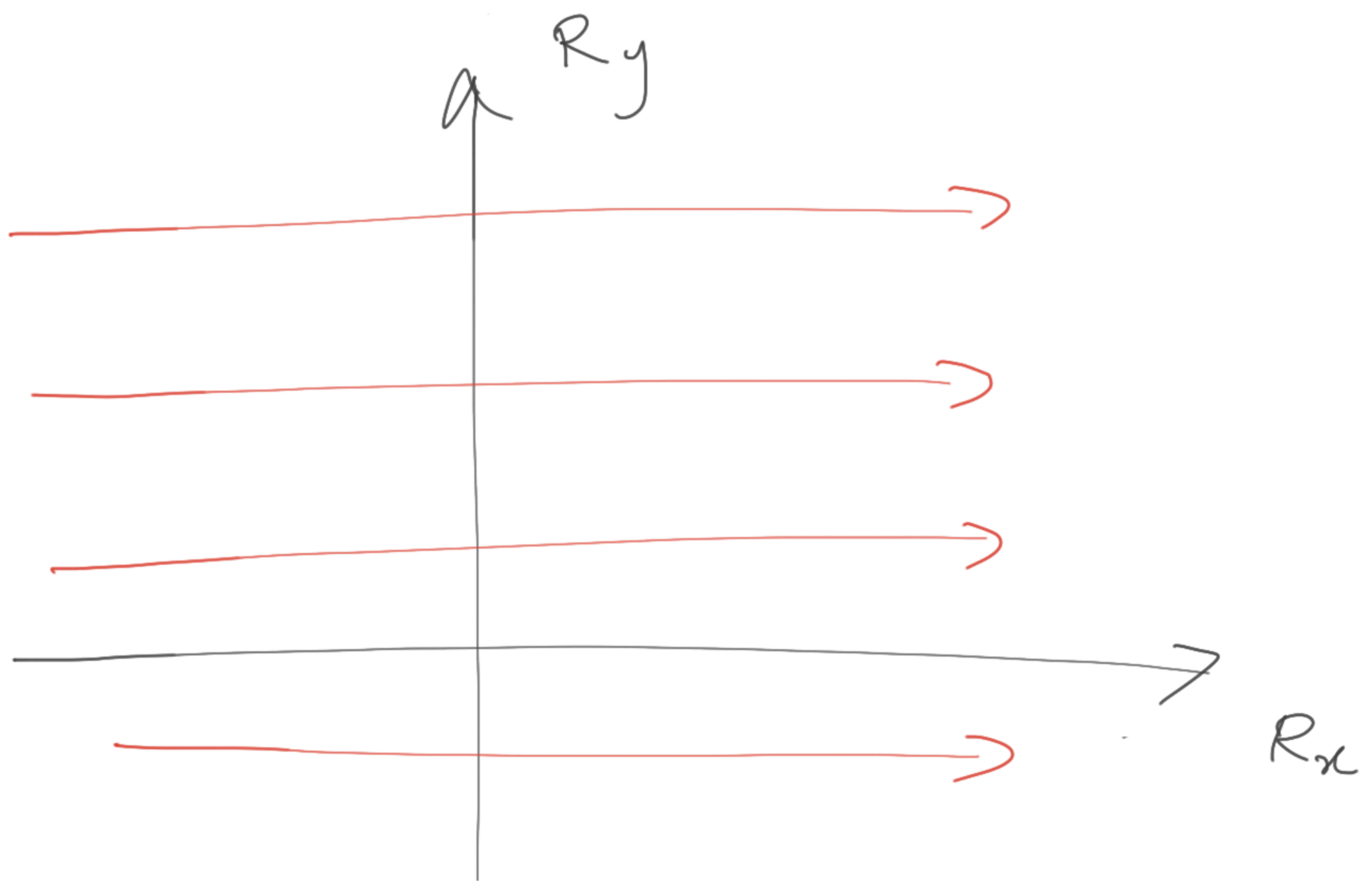
Prob: Solve the flow under P_x and draw pictures.

Sol: Under the R_x flow:

$$\frac{dR_x}{d\lambda} = \{R_x, P_x\} = 1.$$

$$\frac{dV^i}{d\lambda} = 0 \text{ for other } V^i\text{'s.}$$

$$\implies R_x - R_x(\lambda_0) = (\lambda - \lambda_0)\,.$$



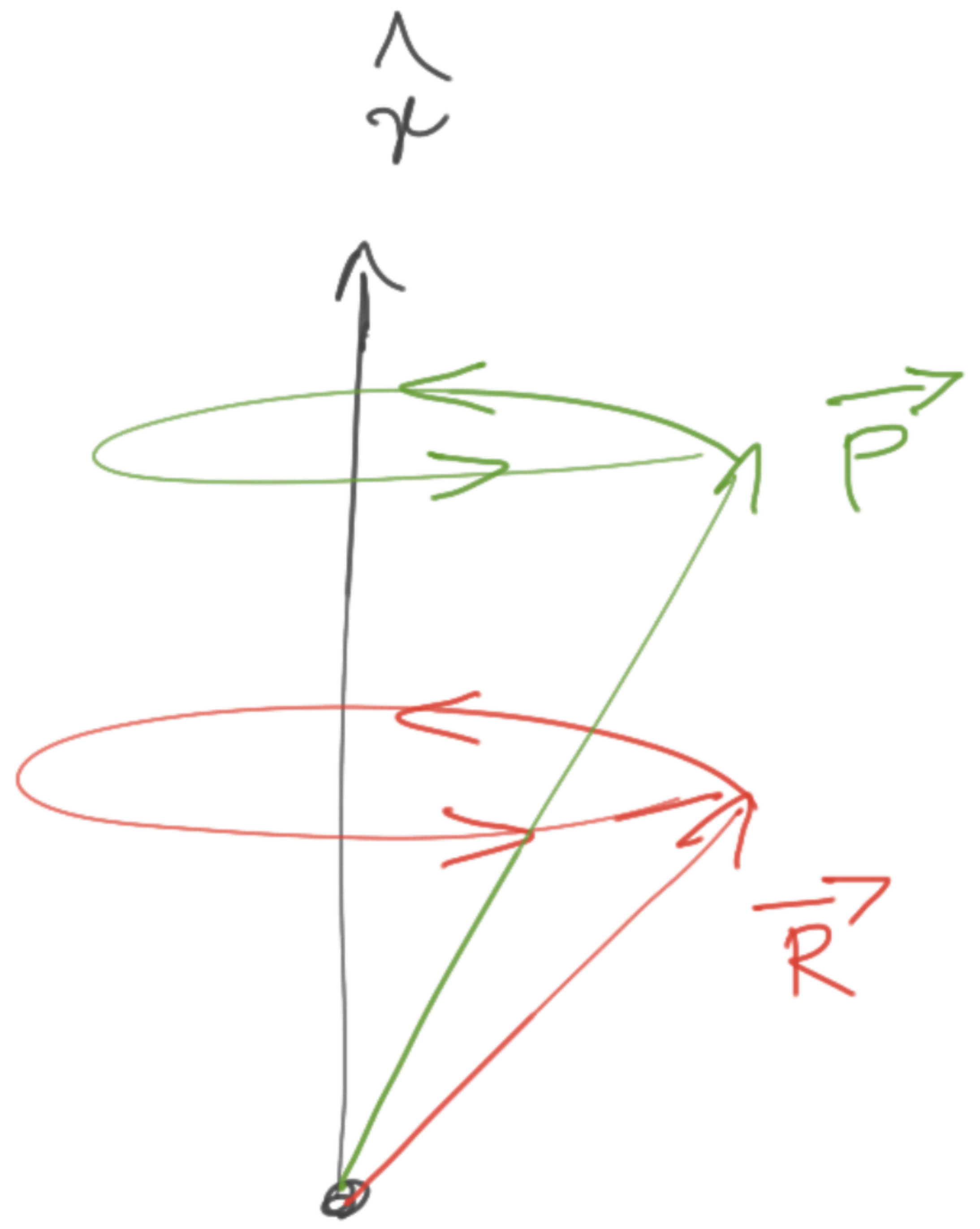
Flow exercise 3

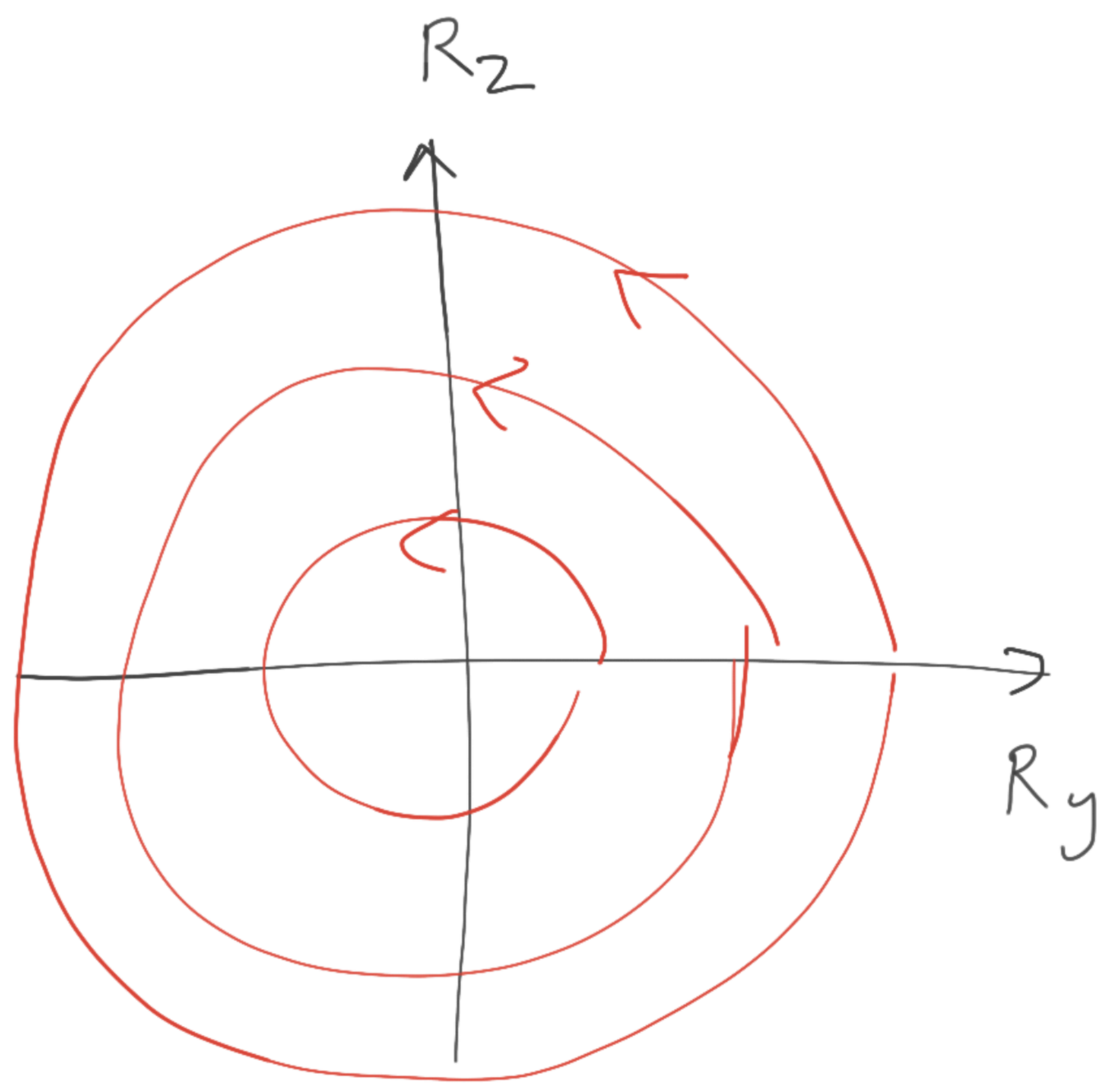
Prob: Draw pictures for L_x flow where $\overrightarrow{L} = \overrightarrow{R} \times \overrightarrow{P}$.

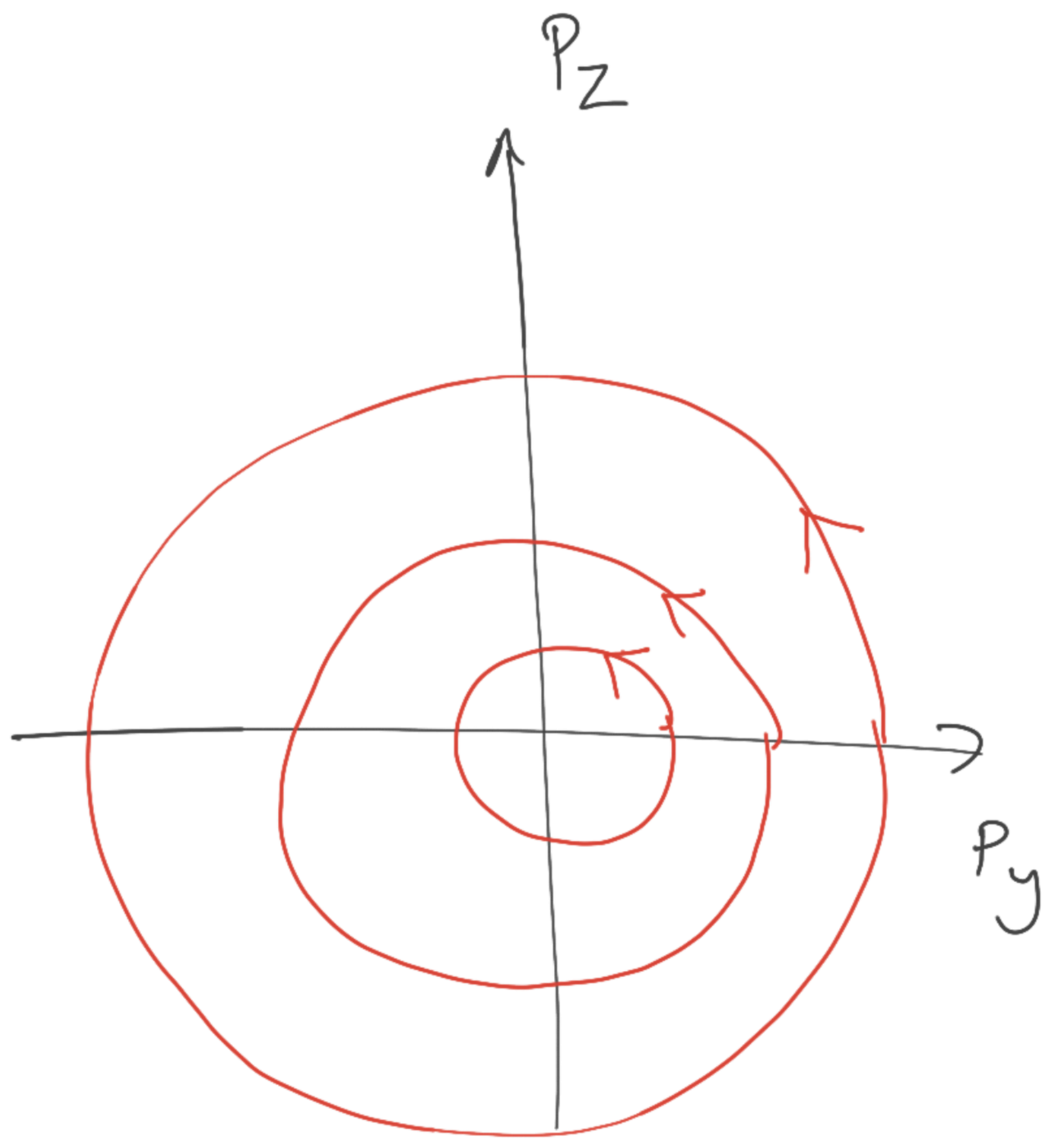
Sol: Under the L_x flow:

$$\frac{d\overrightarrow{R}}{d\lambda} = \hat{x} \times \overrightarrow{V}$$

$$\frac{d\overrightarrow{P}}{d\lambda} = \hat{x} \times \overrightarrow{V}$$





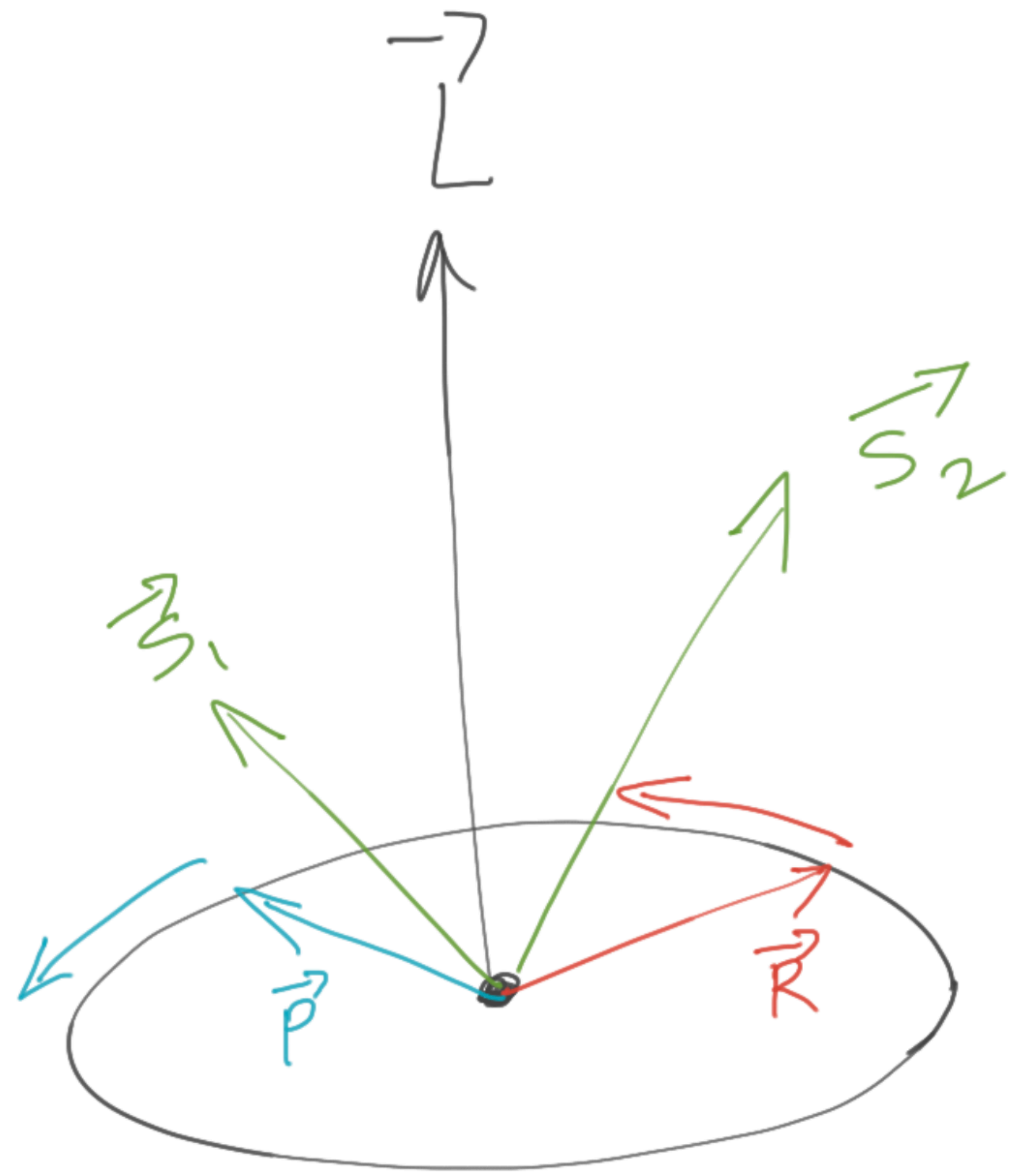


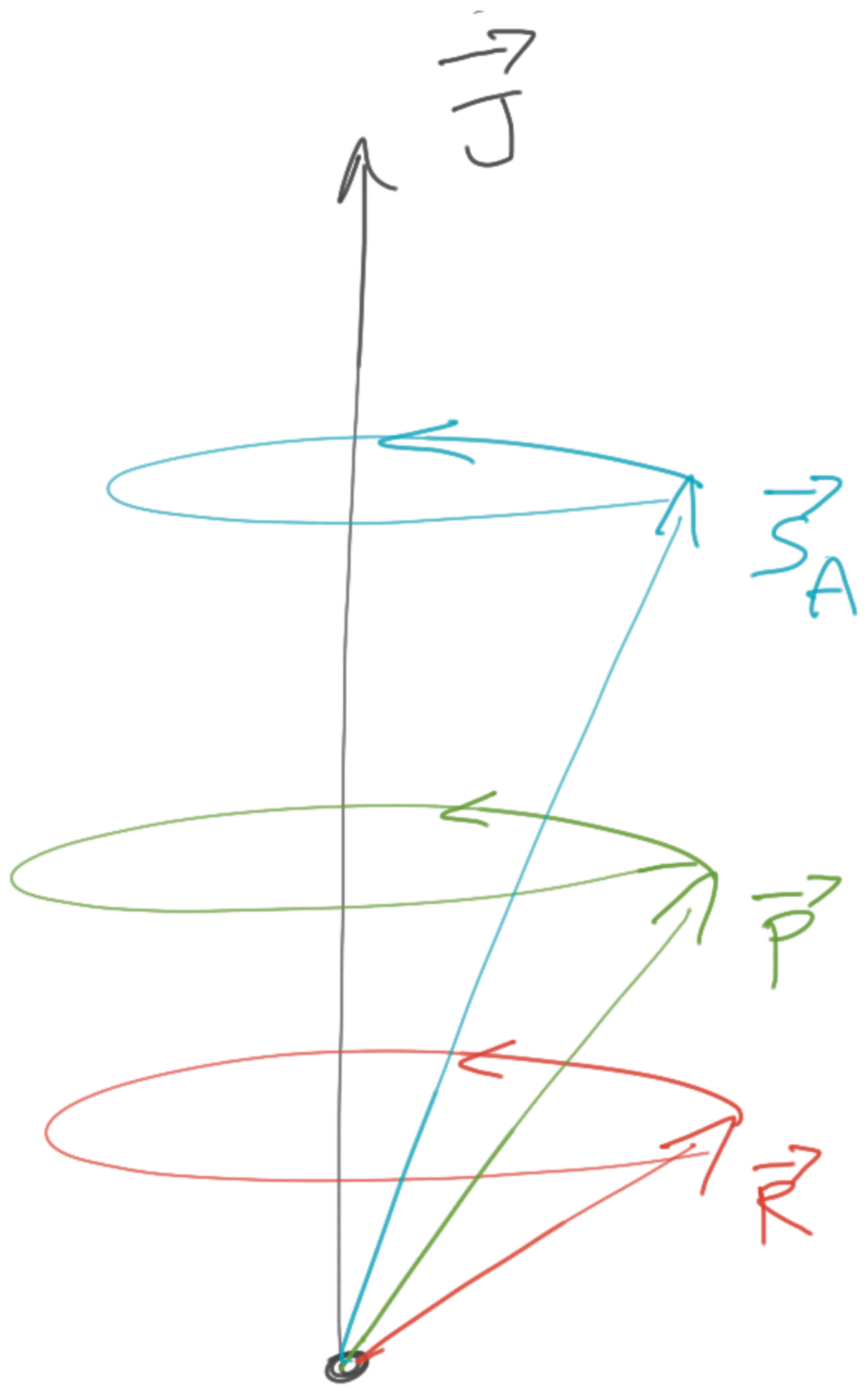
Flow exercise 4

Prob: Draw pictures for L^2 and J^2 flow where $\vec{L} = \vec{R} \times \vec{P}$ and $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$.

Sol: L^2 flow $\Longrightarrow \left\{ \vec{L}, L^2 \right\} = \left\{ \vec{S}_A, L^2 \right\} = 0 \Longrightarrow \vec{L}, \vec{S}_A$ remain fixed.

J^2 flow $\Longrightarrow \left\{ \vec{J}, J^2 \right\} = 0 \Longrightarrow \vec{J}$ remains fixed





Hamiltonian flow of the Hamiltonian H

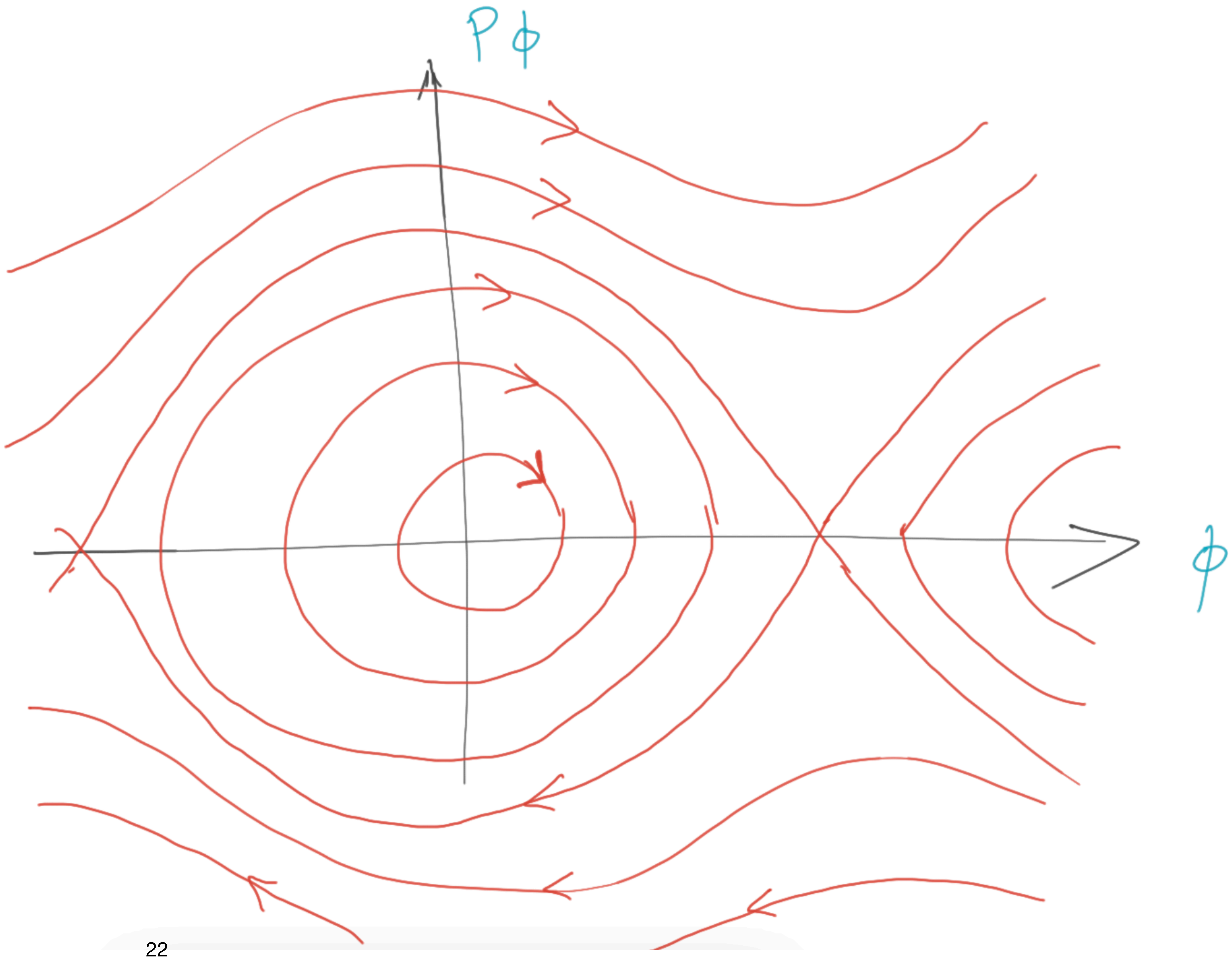
- With $\vec{V} \equiv \left\{\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2\right\}$, flow eqn. is $\dfrac{d\vec{V}}{d\lambda} = \{\vec{V}, f\}$.

- Flow under $H \implies$ $\dfrac{d\vec{V}}{d\lambda} = \{\vec{V}, H\}$.

- This is the **EOM.** Gives the **real-time evolution**, unlike other flows.

- Hamiltonian flow of the Hamiltonian is special.

- **Example**: flow under H for a pendulum



5 minute break

Coffee, questions?

Lecture plan

- **Lecture 1**:
  - Theory
  - Strategy to compute solution from action-angles
- **Lecture 2**:
  - Construct the solution

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions (subject of the next lecture).

- How to combine $\vec{C}$ flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \dfrac{d\theta_i}{dt}$.
- How to flow along the actions $\mathcal{J}_i$?
- Solution via action-angles.

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions (subject of the next lecture).

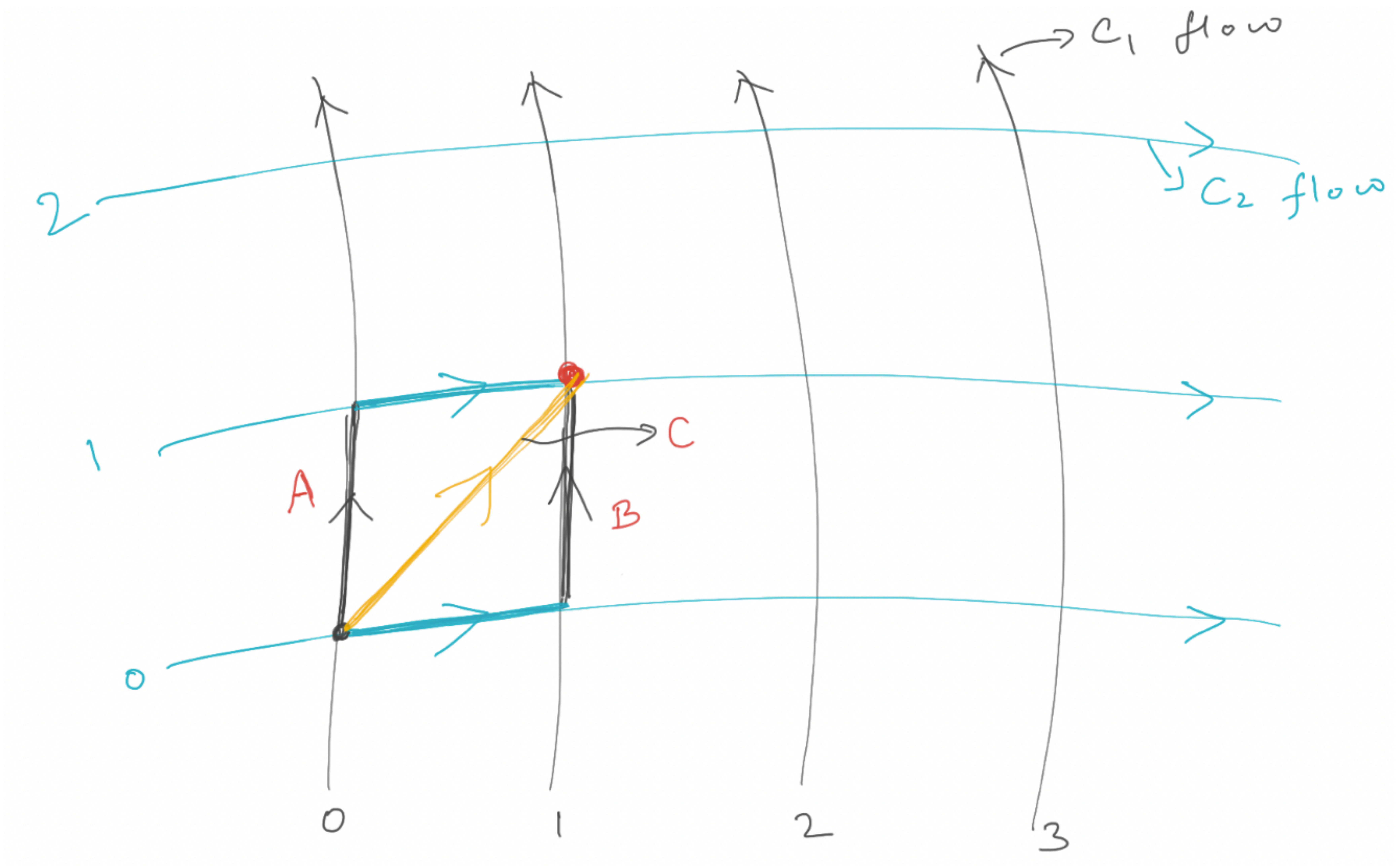
- How to combine $\vec{C}$ flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \dfrac{d\theta_i}{dt}$.
- How to flow along the actions $\mathcal{J}_i$?
- Solution via action-angles.

How to combine $\vec{C}$ flows?

- Assume C_i's are commuting quantities (don't have to be constants).
- **Notation**: Output of C_i flow $\dfrac{d\vec{V}}{d\lambda_i} = \{\vec{V}, C_i\}$ denoted by $\vec{V} = \vec{V}(\vec{V}_0, \Delta\lambda_i)$.
- **Result**: Order of flow does not matter, i.e. $(\quad (\vec{V}_0, \Delta\lambda_1) \quad , \Delta\lambda_2) = (\quad (\vec{V}_0, \Delta\lambda_2), \quad \Delta\lambda_1)$.
- **Result**: Simultaneous flows can be made sequential: $\dfrac{d\vec{V}}{d\lambda} = \{\vec{V}, C_1 + C_2\}$ by $\Delta\lambda$ is C_1 flow followed by C_2 flow (both by $\Delta\lambda$). Or in the reverse order.

How to combine $\overrightarrow{C}$ flows?

Pictorial depiction of the two flow rules



Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\rm eff} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions (subject of the next lecture).

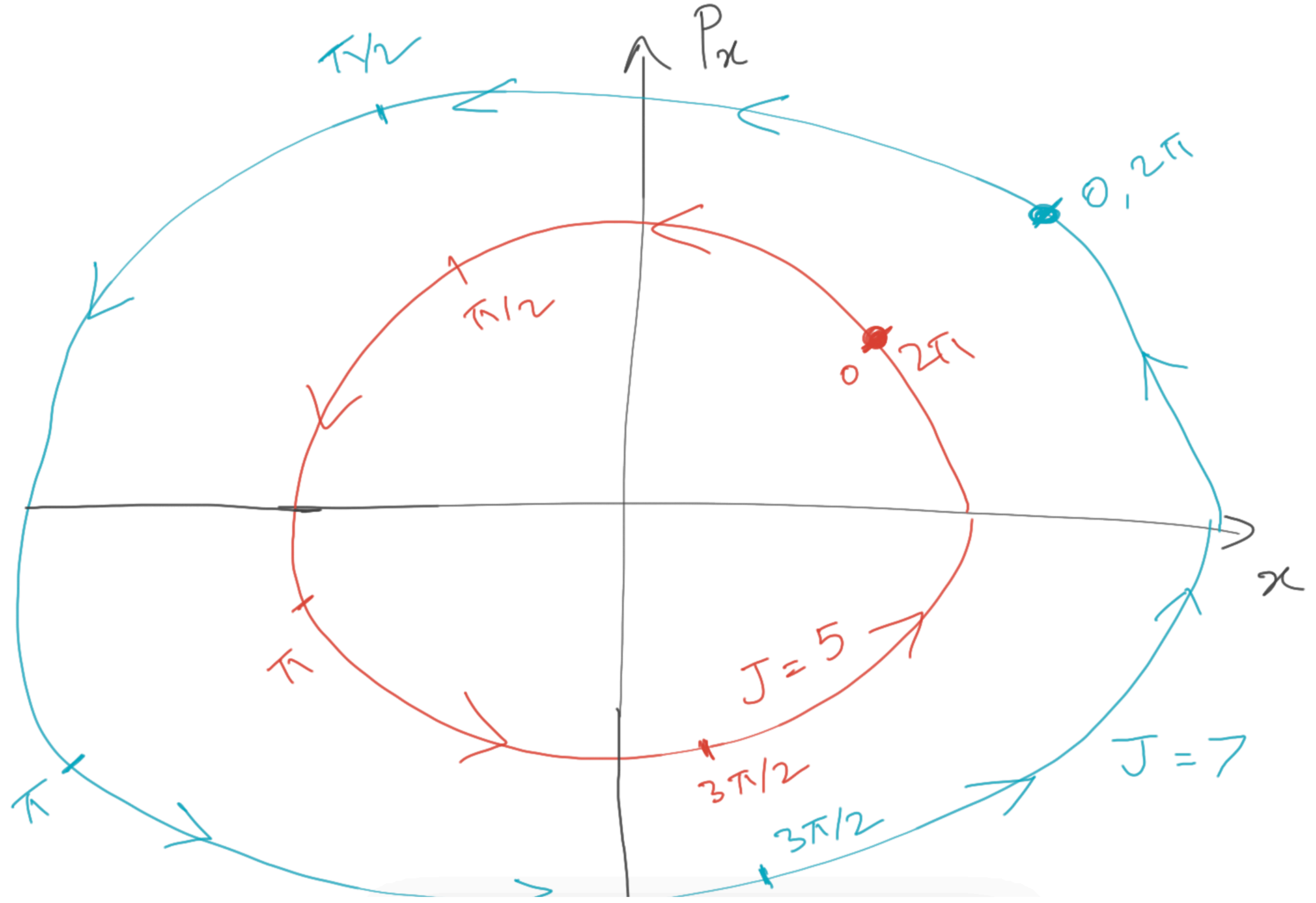
- How to combine $\vec{C}$ flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \dfrac{d\theta_i}{dt}$.
- How to flow along the actions $\mathcal{J}_i$?
- Solution via action-angles.

Construct action-angles

- $\mathcal{J}_i = \dfrac{1}{2\pi} \oint_{\gamma_i} \overrightarrow{P} \cdot d\overrightarrow{Q}$ with $\left\{R_i, P_j\right\} = \delta_{ij}$ and $\left\{\phi_A, S^z_B\right\} = \delta_{AB}$.
- Loop γ_i on the $\overrightarrow{C}$ = constant submanifold.
- No. of independent $\mathcal{J}_i's$ $= n$ despite infinite no. of loops.
- $\mathcal{J}_i$ flow by $2\pi \rightarrow$ loop (different from γ_i).
- Angle $\theta_i \equiv \lambda_i$ along the $\mathcal{J}_i$ flow.

Construct action-angles

Pictorial depiction of the construction



Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions (subject of the next lecture).

- How to combine $\vec{C}$ flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \dfrac{d\theta_i}{dt}$.
- How to flow along the actions $\mathcal{J}_i$?
- Solution via action-angles.

Integrable systems and action-angles

- **Integrable system**: canonical transformation $(\overrightarrow{p}, \overrightarrow{q}) \leftrightarrow (\overrightarrow{\mathscr{J}}, \overrightarrow{\theta})$ exists such that $H = H(\overrightarrow{\mathscr{J}})$ and $\{\overrightarrow{p}, \overrightarrow{q}\}\big(\theta_i + 2\pi\big) = \{\overrightarrow{p}, \overrightarrow{q}\}\big(\theta_i\big)$.

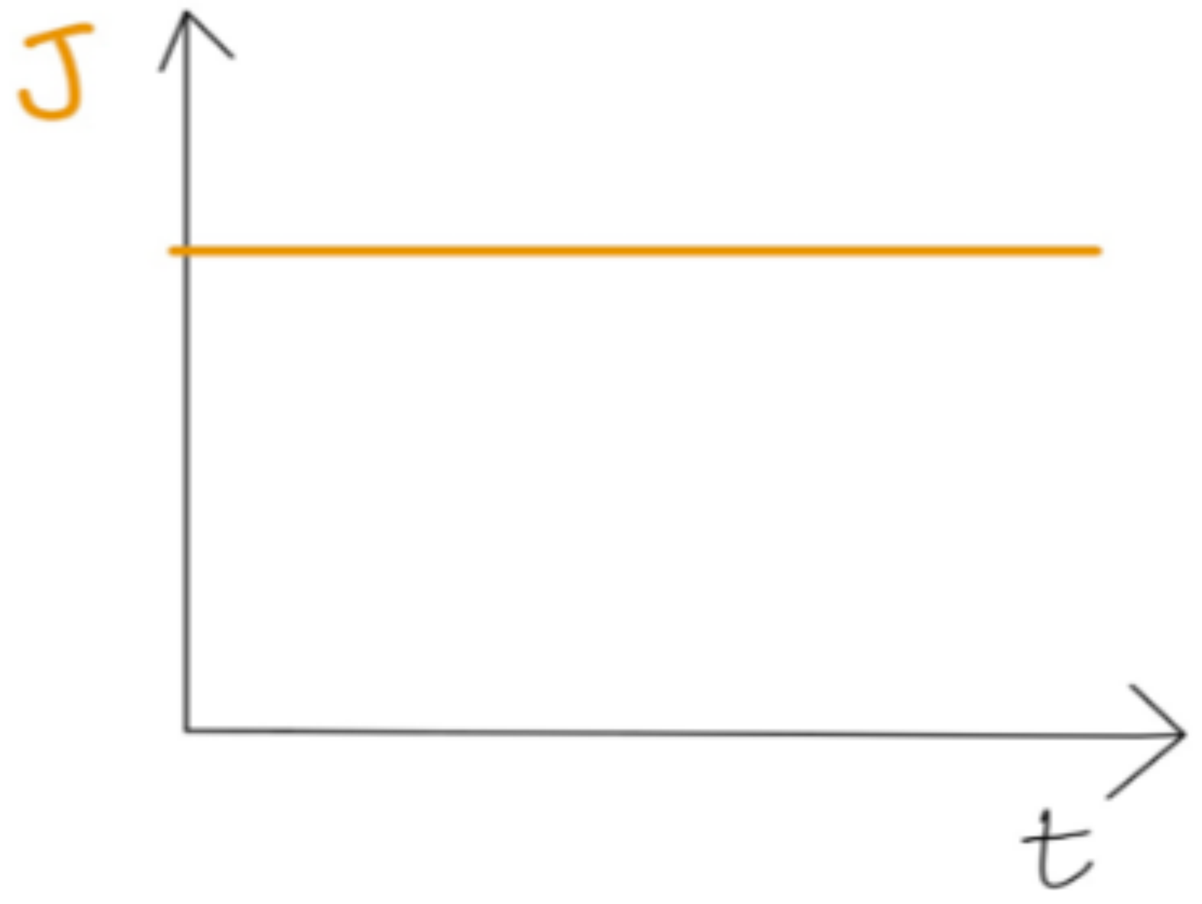
- Action $\mathscr{J}_i = \sim p$;   angle $\theta_i = \sim q$.

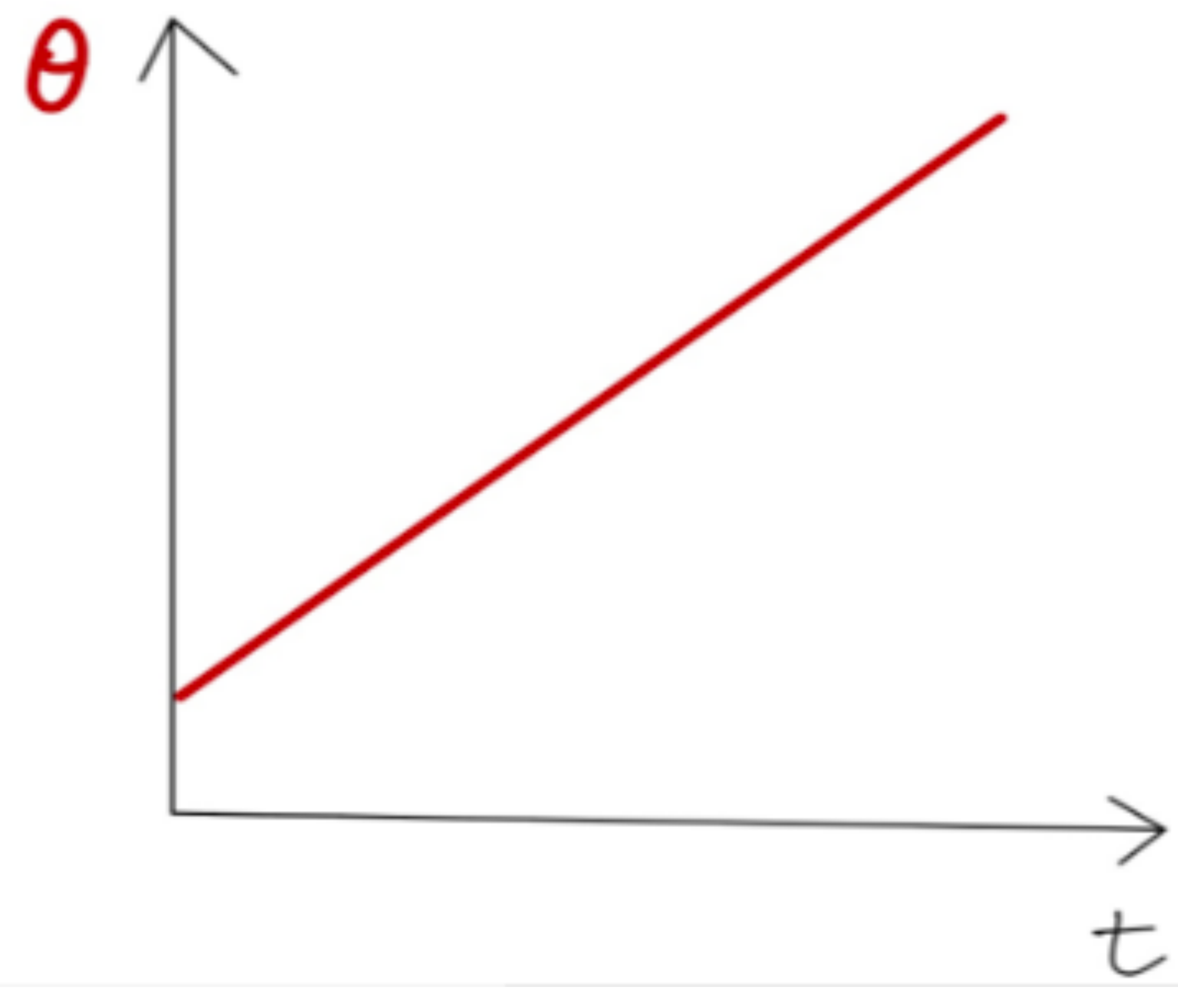
- Hamilton's equations $\Longrightarrow$

$$\dot{\mathscr{J}}_i = -\,\partial H/\partial\theta_i = 0 \qquad \Longrightarrow \mathscr{J}_i \text{ stay constant}$$

$$\dot{\theta}_i = \partial H/\partial \mathscr{J}_i \equiv \omega_i(\overrightarrow{\mathscr{J}}) \qquad \Longrightarrow \theta_i = \omega_i(\overrightarrow{\mathscr{J}})t.$$

- Having action-angles ~ having closed-form solutions.





Compute frequencies $\omega_i \equiv d\theta_i/dt$

- Recall $\dot{\theta}_i = \partial H/\partial \mathcal{J}_i \equiv \omega_i$.
- With $\overrightarrow{C} = \{H, J^2, L^2, J_z, \overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}\}$, assume we have $\mathcal{J}_i(\overrightarrow{C})$ (next lecture's subject).
- Compute the Jacobian $M_{ij}(\overrightarrow{C}) \equiv \dfrac{\partial \mathcal{J}_i}{\partial C_j}$ (consists of numeric constants).
- **Inverse function theorem**: If $N_{ij} \equiv \dfrac{\partial C_i}{\partial \mathcal{J}_j}$, then $N = M^{-1}$.
- The first row of N corresponding to $(C_1 = H)$ contains $\dot{\theta}_i = \partial H/\partial \mathcal{J}_i \equiv \omega_i$.

Action-angle-based solution: strategy

With $\overrightarrow{C} = \{H, J^2, L^2, J_z, \overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}\}$, assume we have *(i)* $\mathcal{J}_i(\overrightarrow{C})$ *(ii)* $\overrightarrow{C}$ flow solutions (subject of the next lecture).

- How to combine $\overrightarrow{C}$ flows?
- Construct action-angles.
- ~~Compute frequencies~~ $\cancel{\omega_i \equiv \frac{d\theta_i}{dt}}$.
- How to flow along the actions $\mathcal{J}_i$?
- Solution via action-angles.

EOMs with Poisson brackets for BBHs

Our approach

- Define EOMs: $\dfrac{df(t)}{dt} = \{f, H\}$ where $f = f\left(\vec{R}(t), \vec{P}(t), \vec{S}_1(t), \vec{S}_2(t)\right)$.
- Define PBs: $\left\{R_i, P_j\right\} = \delta_{ji}$ $\left\{S_A^i, S_B^j\right\} = \delta_{AB}\epsilon_k^{ij}S_A^k$.

$$\{f, g\} = -\{g, f\}$$

$$\{af + bg, h\} = a\{f, h\} + b\{g, h\}, \quad \{h, af + bg\} = a\{h, f\} + b\{h, g\}, a, b \in \mathbb{R},$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\},$$

$$\left\{f, g\left(v_i\right)\right\} = \{f, v_i\}\frac{\partial g}{\partial v_i},$$

- **How to define the system?** (i) specify the Hamiltonian (ii) define PBs (iii) define the EOMs (via PBs).

How to flow along the actions $\mathcal{J}_i$?

- With $\overrightarrow{C} = \{H, J^2, L^2, J_z, \overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}\}$, assume we have *(i)* $\mathcal{J}_i(\overrightarrow{C})$ *(ii)* $\overrightarrow{C}$ flow solutions (next lecture's subject).

- Using chain rule for PBs, $\dfrac{d\overrightarrow{V}}{d\lambda} = \left\{\overrightarrow{V}, \mathcal{J}_i\right\} = \left\{\overrightarrow{V}, C_j\right\}\left(\dfrac{\partial \mathcal{J}_i}{\partial C_j}\right) = 2.5\{\overrightarrow{V}, C_1\} + 5.1\{\overrightarrow{V}, C_2\}$ $= \{\overrightarrow{V},\ 2.5C_1 + 5.1C_2\}$.

- $\mathcal{J}_i$ flow by $\Delta\lambda$ $=$ (C_1 flow by $2.5\Delta\lambda$, then C_2 flow by $5.1\Delta\lambda$) , reverse the order.

Action-angle-based solution: strategy

With $\overrightarrow{C} = \{H, J^2, L^2, J_z, \overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}\}$, assume we have *(i)* $\mathcal{J}_i(\overrightarrow{C})$ *(ii)* $\overrightarrow{C}$ flow solutions (subject of the next lecture).

- How to combine $\overrightarrow{C}$ flows?
- Construct action-angles.
- ~~Compute frequencies~~ $\cancel{\omega_i \equiv \frac{d\theta_i}{dt}}$~~.~~
- ~~How to flow along the actions~~ $\cancel{\mathcal{J}_i}$~~?~~
- Solution via action-angles.

Solution via action-angles.

- Start with an initial $\vec{V}_0 = \{\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2\}$. Assign it $\vec{\theta} = \vec{0}$.
- We want $\vec{V} = \vec{V}(\vec{V_0}, t)$.
- Recall $\dot{\theta}_i = \partial H / \partial \mathcal{J}_i \equiv \omega_i$ and $\Delta\theta_i = \Delta\lambda_i$.
- After time t, $\theta_i(t) = \omega_i\, t$.
- How to increase the angles? Action flows increase the angles.
- We need to flow under $\mathcal{J}_i$'s by an amount $\lambda_i = \theta_i(t) = \omega_i\, t$.

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\rm eff} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions (subject of the next lecture).

- How to combine $\vec{C}$ flows?
- Construct action-angles.
- ~~Compute frequencies~~ $\cancel{\omega_i \equiv \frac{d\theta_i}{dt}}$.
- ~~How to flow along the actions~~ $\cancel{\mathcal{J}_i}$~~?~~
- ~~Solution via action-angles.~~

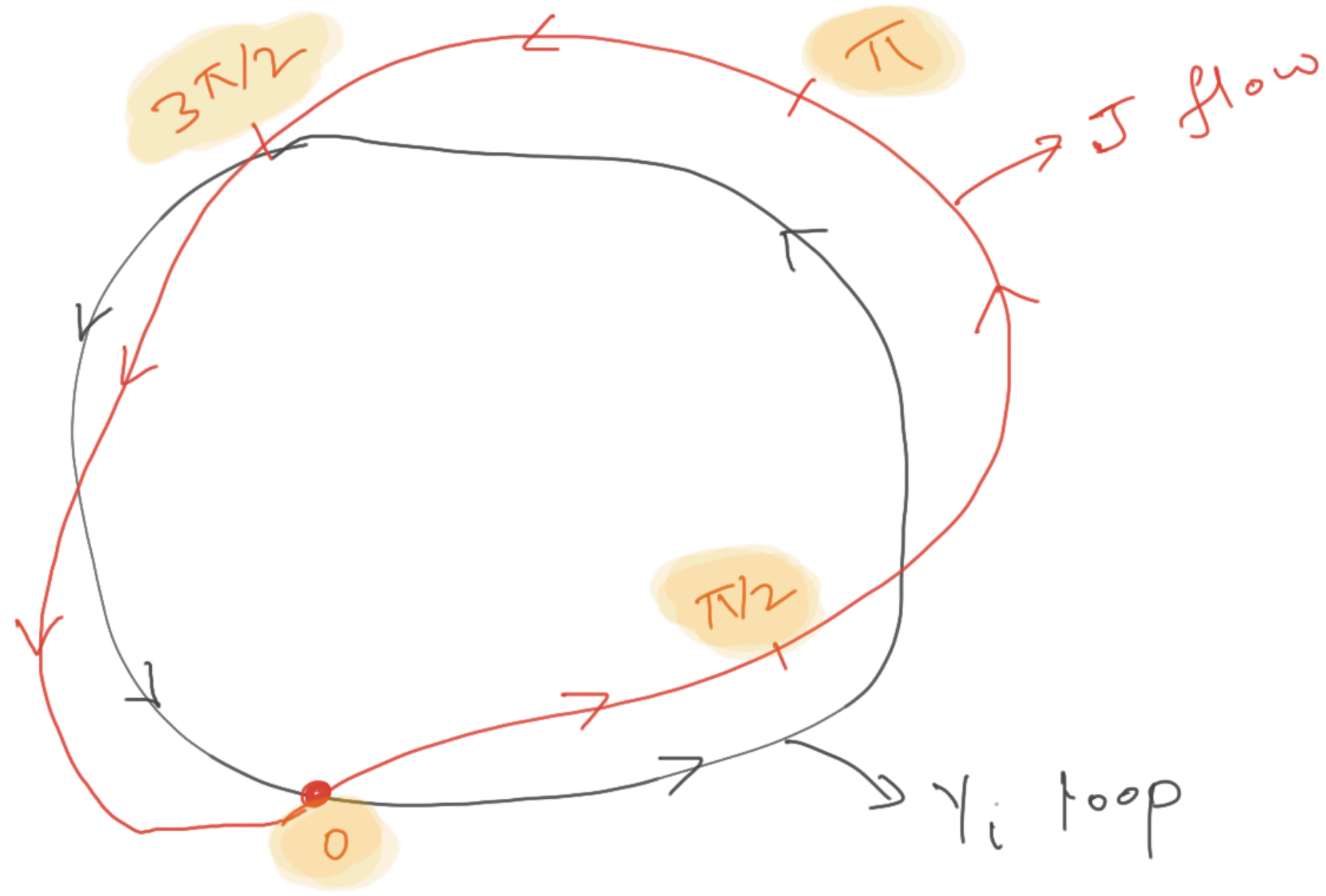
Construct action-angles

- $\mathscr{J}_i = \dfrac{1}{2\pi}\oint_{\gamma_i} \overrightarrow{P} \cdot d\overrightarrow{Q}$. Loop γ_i on the $\overrightarrow{C}$ = constant submanifold.

- $\mathscr{J}_i$ flow by $2\pi \to$ loop (different from γ_i).

- Angle $\theta_i \equiv \lambda_i$ along the $\mathscr{J}_i$ flow.

- **To show:** $\{\theta_i, J_k\} = \delta_{ij}$, $\{J_i, J_k\} = 0$, $\{\theta_i, \theta_k\} = 0$.



Construct action-angles

- Using $\theta_i = \lambda_i$, $\dfrac{d\theta_i}{d\lambda_i} = 1$ and $\dfrac{d\theta_i}{d\lambda_i} = \left\{\theta_i, \mathcal{J}_i\right\} \implies \left\{\theta_i, \mathcal{J}_i\right\} = 1$.

- From definition $\mathcal{J}_i$ and chain rule for PBs, $\mathcal{J}_i = \mathcal{J}_i(\vec{C}) \implies \left\{J_i, J_k\right\}$
$= \dfrac{\partial J_i}{\partial C_l}\dfrac{\partial J_k}{\partial C_m}\left\{C_l, C_m\right\} = 0.$

- $\left\{\theta_i, \theta_j\right\} = 0$ involves changing $\mathcal{J}_i$, which does not happen with real evolution. Hence ignore.

- "**Integrable system**: canonical transformation $(\vec{p}, \vec{q}) \leftrightarrow (\vec{\mathcal{J}}, \vec{\theta})$ exists such that $H = H(\vec{\mathcal{J}})$ and $\{\vec{p}, \vec{q}\}(\theta_i + 2\pi) = \{\vec{p}, \vec{q}\}(\theta_i)$." that lead to $\dot{\mathcal{J}}_i = 0;\ \theta_i = \omega_i\, t$ is satisfied because action flow makes a loop after 2π .

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\mathrm{eff}} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions **(subject of the next lecture).**

- How to combine $\vec{C}$ flows?
- ~~Construct action-angles.~~
- ~~Compute frequencies~~ $\cancel{\omega_i \equiv \frac{d\theta_i}{dt}}$~~.~~
- ~~How to flow along the actions~~ $\cancel{\mathcal{J}_i}$~~?~~
- ~~Solution via action-angles.~~

Lecture plan

- **Lecture 1**:
  - Theory
  - Strategy to compute solution from action-angles
- **Lecture 2**:
  - Construct the solution

Action-angle-based solution: strategy

With $\vec{C} = \{H, J^2, L^2, J_z, \vec{S}_{\text{eff}} \cdot \vec{L}\}$, assume we have *(i)* $\mathcal{J}_i(\vec{C})$ *(ii)* $\vec{C}$ flow solutions

(subject of this lecture).

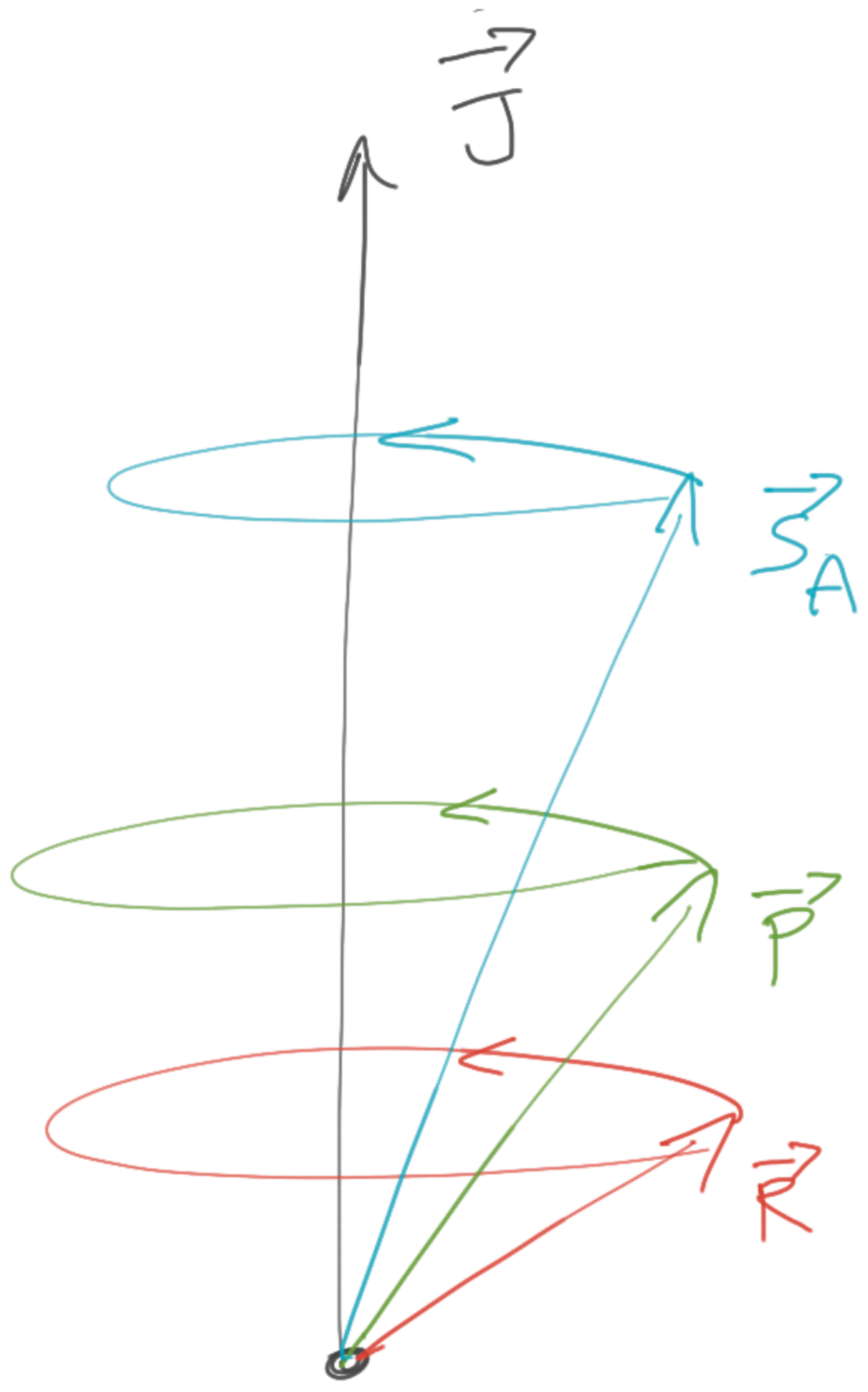
- How to combine $\vec{C}$ flows?
- Construct action-angles.
- Compute frequencies $\omega_i \equiv \dfrac{d\theta_i}{dt}$.
- How to flow along the actions $\mathcal{J}_i$?
- Solution via action-angles.

Computing actions: strategy

- $\mathcal{J}_i = \dfrac{1}{2\pi} \oint_{\gamma_i} \vec{P} \cdot d\vec{Q}$; loop γ_i is on the surface of constant $\vec{C}$.
- $\left\{R_i, P_j\right\} = \delta_{ij}$ and $\left\{\phi_A, S^z_B\right\} = \delta_{AB}$
- How to be on the surface of constant $\vec{C}$? Flow along C_i's: $\dfrac{dC_i}{d\lambda} = \left\{C_i, C_j\right\} = 0.$
- $\mathcal{J} = \mathcal{J}^{\mathrm{orb}} + \mathcal{J}^{\mathrm{spin}}$
- $\mathcal{J}^{\mathrm{orb}} = \dfrac{1}{2\pi} \oint_{\mathscr{C}} \sum_i P_i dR^i$ $\qquad \mathcal{J}^{\mathrm{spin}}_A = \dfrac{1}{2\pi} \oint_A^z d\phi_A\,.$

Computing $\mathscr{J}_1$

- With $\vec{V} = \{\vec{R}, \vec{P}, \vec{L}, \vec{S}_1, \vec{S}_2\}$, J^2 flow $\Longrightarrow \dfrac{d\vec{V}}{d\lambda} = 2\vec{J} \times \vec{V} \equiv \vec{n} \times \vec{V}$.
- $\left\{\vec{J}, J^2\right\} = 0.$
- **Solution**: $\phi(\lambda) = n\ \lambda + \phi_0$.
- Loop closes after flowing by $\Delta\lambda = 2\pi/n = 2\pi/(2J) = \pi/J$.
- $\mathscr{J}^{\text{orb}} = \dfrac{1}{2\pi}\displaystyle\int_0^{\Delta\lambda} P_i \dfrac{dR^i}{d\lambda} d\lambda = \dfrac{1}{2\pi}\int_0^{\Delta\lambda} \vec{P}\cdot(\vec{n}\times\vec{R})d\lambda = \dfrac{1}{2\pi}\int_0^{\Delta\lambda} \vec{n}\cdot\vec{L}\,d\lambda = \hat{n}\cdot\vec{L}$.
- $\mathscr{J}^{\text{spin}}_A = \dfrac{1}{2\pi}\displaystyle\oint S^z_A d\phi_A = S^z_A = \hat{n}\cdot\vec{S}_A$ (with $\vec{n}$ along z-axis)
- The spin integral is rotationally invariant, but not manifestly so.
- $\mathscr{J}_1 = \hat{n}\cdot\left(\vec{L} + \vec{S}_1 + \vec{S}_2\right) = \hat{n}\cdot\vec{J} = J.$
- **Summary**: We have computed $\mathscr{J}_1$ and also computed the solution to $C_1 = J^2$.



Computing $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$

- **For flows under** J^2, J_z, **and** L^2: $\dfrac{d\vec{V}}{d\lambda} = \vec{n} \times \vec{V}$. $\vec{n} = 2\vec{J}, \hat{z}$, and $2\vec{L}$ (with $\vec{n}$ being fixed)
- **Exception**: Under L^2 flow, spins don't move.
- **Solution**: $\phi(\lambda) = n\ \lambda + \phi_0$. Doesn't apply to spins under the L^2 flow.
- Loop closes after flowing by $\Delta\lambda = 2\pi/n$.
- $\mathcal{J}^{\text{orb}} = \dfrac{1}{2\pi}\displaystyle\int_0^{\Delta\lambda} P_i \frac{dR^i}{d\lambda} d\lambda = \frac{1}{2\pi}\int_0^{\Delta\lambda} \vec{P} \cdot (\vec{n} \times \vec{R}) d\lambda = \frac{1}{2\pi}\int_0^{\Delta\lambda} \vec{n} \cdot \vec{L}\, d\lambda = \hat{n} \cdot \vec{L}$.
- $\mathcal{J}_A^{\text{spin}} = \dfrac{1}{2\pi}\displaystyle\oint S_A^z d\phi_A = S_A^z = \hat{n} \cdot \vec{S}_A$ (with $\vec{n}$ along z-axis)
- $\mathcal{J}_1 = J,\ \mathcal{J}_2 = J_z, \mathcal{J}_3 = L$.
- **Summary**: We have computed $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\}$ and also computed the solution to $C_i = \{J^2, J_z, L^2\}$.

Computing $\mathscr{J}_4$

- We won't compute it here.

- $\mathscr{J}_4$ has a Newtonian version (Eq. (10.139) of Goldstein).

- 1PN version given in Eq. (3.10) of Damour-Schafer.

- 1.5PN version in Eq. (38) of [arXiv: 2012.06586].

Taking stock

- We solved the flows under $C_i = \{J^2, J_z, L^2\}$.
- Finding the solution $\overrightarrow{V}(\overrightarrow{V}_0, \Delta\lambda)$ of a flow under C_i: $\dfrac{d\overrightarrow{V}}{d\lambda} = \{\overrightarrow{V}, C_i\}$ is basically solving an ODE.
- Solution of flow under $\overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}$ given in [arXiv:2110.15351].
- Solution of flow under H given in [arXiv:1908.02927]. This is the standard solution of the system. Omit 1PN terms for simplicity.
- Quite lengthy but nothing esoteric.
- **Future focus**: compute $\mathcal{J}_5$.

$\mathscr{J}_5$ computation

For $\overrightarrow{S}_{\text{eff}} \cdot \overrightarrow{L}$ flow:

$$\frac{d\overrightarrow{R}}{d\lambda} = \overrightarrow{S}_{\text{eff}} \times \overrightarrow{R}$$

$$\frac{d\overrightarrow{P}}{d\lambda} = \overrightarrow{S}_{\text{eff}} \times \overrightarrow{P}$$

$$\frac{d\overrightarrow{S}_a}{d\lambda} = \sigma_a \left(\overrightarrow{L} \times \overrightarrow{S}_a \right)$$

$$\frac{d\overrightarrow{L}}{d\lambda} = \overrightarrow{S}_{\text{eff}} \times \overrightarrow{L}$$

-
- **Important**: $\overrightarrow{n}$ not fixed

$\mathscr{J}_5$ computation

$$2\pi\mathscr{J} = 2\pi\left(\mathscr{J}^{\text{orb}} + \mathscr{J}^{\text{spin}}\right)$$

$$= \int_{\lambda_i}^{\lambda_f}\left(P_i dR^i + S_1^z d\phi_1^z + S_2^z d\phi_2^z\right)$$

- $$= \int_{\lambda_i}^{\lambda_f}\left(P_i\frac{dR^i}{d\lambda} + S_1^z\frac{d\phi_1^z}{d\lambda} + S_2^z\frac{d\phi_2^z}{d\lambda}\right)d\lambda$$

- $$2\pi\mathscr{J}^{\text{orb}} = \int_{\lambda_i}^{\lambda_f}\overrightarrow{P}\cdot\left(\overrightarrow{S}_{\text{eff}}\times\overrightarrow{R}\right)d\lambda = \int_{\lambda_i}^{\lambda_f}\left(S_{\text{eff}}\cdot L\right)d\lambda = \left(S_{\text{eff}}\cdot L\right)\Delta\lambda$$

- Can't do spin sector integral because $\overrightarrow{S}_A \neq \overrightarrow{R}_A \times \overrightarrow{P}_A$. A refers to the BHs.

$\mathcal{J}_5$ computation: enter fictitious variables

- Define $\vec{R}_a, \vec{P}_a$ (fictitious variables) such that $\vec{S}_a \equiv \vec{R}_a \times \vec{P}_a$.
- **Hamiltonian**: Now a function of $\vec{R}, \vec{P}, \vec{R}_{1/2}, \vec{P}_{1/2}$ and not $\vec{R}, \vec{P}, \vec{S}_1, \vec{S}_2$.
- **PBs and EOMs**: $\left\{R_i, P_j\right\} = \delta_{ij}, \quad \left\{R_{ai}, P_{bj}\right\} = \delta_{ab}\delta_{ji}; \qquad \dfrac{df}{dt} = \{f, H\}$.
- $\left\{R_i, P_j\right\} = \delta_{ij}, \quad \left\{R_{ai}, P_{bj}\right\} = \delta_{ab}\delta_{ji} \implies \left\{R_i, P_j\right\} = \delta_{ij}, \quad \{\phi_A, S_B^z\} = \delta_{AB}$
- PBs $\rightarrow$ EOMs $\implies$ The standard phase space (**SPS**) is equivalent to the extended phase space (**EPS**).
- **Integrability equivalency:** EPS needs $n = 2n/2 = 18/2 = 9 = (5+4)$ C_i's. The next 4 C_i's are S_a^2 and $\vec{R}_a \cdot \vec{P}_a$.

$\mathscr{J}_5$ computation: sanity checks

- **Check 1**: Final $\mathscr{J}_5$ depends on $\vec{R}$, $\vec{P}$, $\vec{S}_1$ and $\vec{S}_2$.

- **Check 2**: Numerical flow by 2π under $\mathscr{J}_5$ closes a loop in the SPS picture.

- We have all seen fictitious variables before (in spirit)!

- Inventing complex numbers to do real integrals.

542 Chapter 11 Complex Variable Theory

11.8.19 Prove that $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \pi \ln 2$.

11.8.20 Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,dx = \frac{\pi a}{\sin \pi a},$$

where $-1 < a < 1$.

Hint. Use the contour shown in Fig. 11.26, noting that $z = 0$ is a branch point and the positive x-axis can be chosen to be a cut line.

11.8.21 Show that

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 11.8.16 is a special case of this result.

11.8.22 Show that

$$\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = 2\pi/n$.

11.8.23 (a) Show that

$$f(z) = z^4 - 2z^2\cos 2\theta + 1$$

has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin\theta} = \frac{\pi}{2^{1/2}(1-\cos 2\theta)^{1/2}}.$$

Exercise 11.8.22 ($n = 4$) is a special case of this result.

$\mathscr{J}_5$ computation using fictitious variables

$$\mathscr{J}_k = \frac{1}{2\pi}\oint_{\mathscr{C}_k}\left(\vec{P}\cdot d\vec{R} + \vec{P}_1\cdot d\vec{R}_1 + \vec{P}_2\cdot d\vec{R}_2\right)$$

EOMs for $\vec{S}_{\text{eff}}\cdot\vec{L}$ flow are

$$\frac{d\vec{R}}{d\lambda} = \vec{S}_{\text{eff}}\times\vec{R}$$

$$\frac{d\vec{P}}{d\lambda} = \vec{S}_{\text{eff}}\times\vec{P}$$

$$\frac{d\vec{R}_a}{d\lambda} = \sigma_a\left(\vec{L}\times\vec{R}_a\right)$$

$$\frac{d\vec{P}_a}{d\lambda} = \sigma_a\left(\vec{L}\times\vec{P}_a\right)$$

$$2\pi\mathscr{J}_{S_{\text{eff}}\cdot L} = 2\pi\left(\mathscr{J}^{\text{orb}} + \mathscr{J}^{\text{spin}}\right)$$

$$= \int_{\lambda_i}^{\lambda_f}\left(P_i\frac{dR^i}{d\lambda} + P_{1i}\frac{dR_1^i}{d\lambda} + P_{2i}\frac{dR_2^i}{d\lambda}\right)d\lambda$$

$$= \int_{\lambda_i}^{\lambda_f}\left(\vec{P}\cdot\left(\vec{S}_{\text{eff}}\times\vec{R}\right) + \vec{P}_1\cdot\left(\sigma_1\vec{L}\times\vec{R}_1\right)\right.$$

$$\left. + \vec{P}_2\cdot\left(\sigma_2\vec{L}\times\vec{R}_2\right)\right)d\lambda$$

$$= 2\int_{\lambda_i}^{\lambda_f}\left(S_{\text{eff}}\cdot L\right)d\lambda = 2\left(S_{\text{eff}}\cdot L\right)\Delta\lambda_{S_{\text{eff}}\cdot L}$$

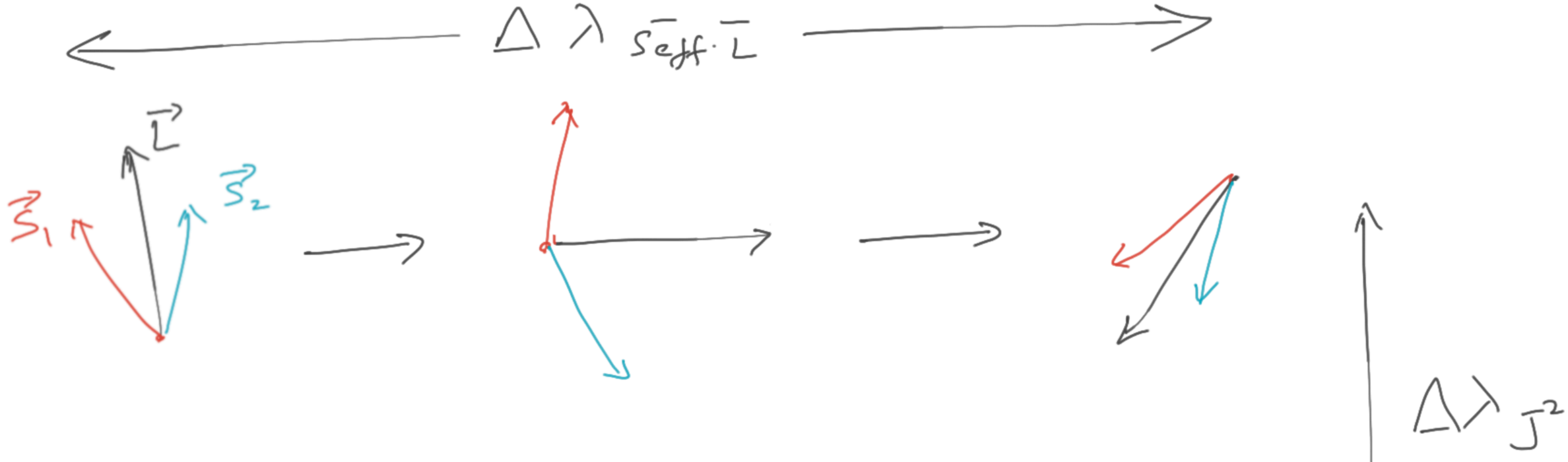
$$\mathscr{J}_{S_{\text{eff}}\cdot L} = \frac{\left(S_{\text{eff}}\cdot L\right)\Delta\lambda_{S_{\text{eff}}\cdot L}}{\pi}$$

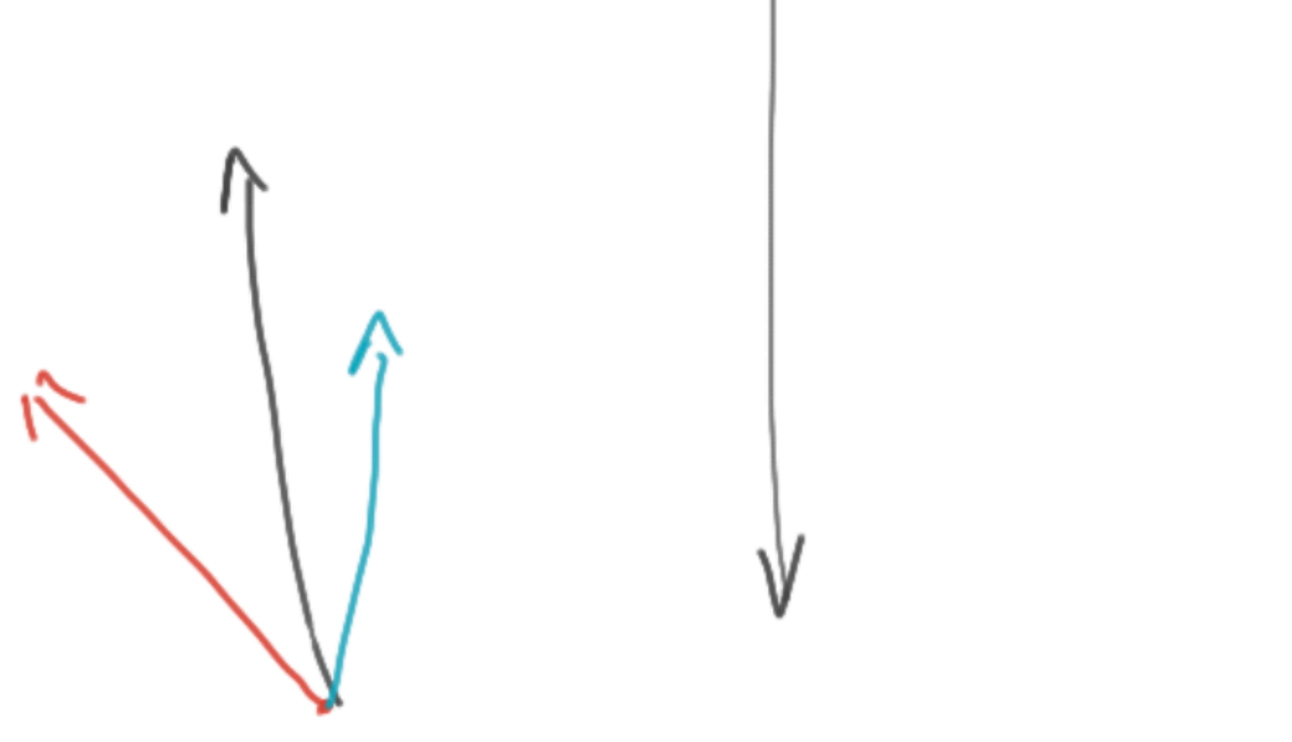
$\mathcal{J}_5$ computation using fictitious variables

- $$\begin{aligned} \mathcal{J}_{S_{\text{eff}}\cdot L} &= \frac{\left(S_{\text{eff}}\cdot L\right)\Delta\lambda_{S_{\text{eff}}\cdot L}}{\pi}, \\ \mathcal{J}_{J^2} &= \frac{J^2\Delta\lambda_{J^2}}{\pi}, \\ \mathcal{J}_{L^2} &= \frac{L^2\Delta\lambda_{L^2}}{\pi}, \\ \mathcal{J}_{S_1^2} &= \frac{S_1^2\Delta\lambda_{S_1^2}}{\pi}, \\ \mathcal{J}_{S_2^2} &= \frac{S_2^2\Delta\lambda_{S_2^2}}{\pi}. \end{aligned} \implies \mathcal{J}_5 = \frac{1}{\pi}\left\{\left(S_{\text{eff}}\cdot L\right)\Delta\lambda_{S_{\text{eff}}\cdot L} + J^2\Delta\lambda_{J^2} + L^2\Delta\lambda_{L^2} + S_1^2\Delta\lambda_{S_1^2} + S_2^2\Delta\lambda_{S_2^2}\right\}.$$

- Loop for $\mathcal{J}_5$ is closed by flowing under 5 C_i's (not one).
- Flow amounts $\Delta\lambda_i$'s give $\mathcal{J}_5$

$\mathcal{J}_5$ computation: flow overview





- **Flows needed:** $\vec{S}_{\text{eff}} \cdot \vec{L}, J^2$ $\quad$ ($\vec{L}$, $\vec{S}_a$ restored).
- **Flows needed**: L^2, S_a^2 $\quad$ ($\vec{R}$, $\vec{P}$, $\vec{R}_a$, $\vec{P}_a$ restored).
- $\mathcal{J}_5 = \dfrac{1}{\pi}\left\{ \left(S_{\text{eff}} \cdot L\right) \Delta\lambda_{S_{\text{eff}}\cdot L} + J^2 \Delta\lambda_{J^2} + L^2 \Delta\lambda_{L^2} + S_1^2 \Delta\lambda_{S_1^2} + S_2^2 \Delta\lambda_{S_2^2} \right\}$

References

- **RESEARCH PAPERS**
  - The standard way of computing the solution (without 1PN part): https://arxiv.org/abs/1908.02927
  - Action-angle-based solution: https://arxiv.org/abs/2012.06586, https://arxiv.org/abs/2110.15351

- **LECTURE NOTES**
  - Lecture notes (latest): https://github.com/sashwattanay/lectures_integrability_action-angles_PN_BBH/blob/gh-action-result/pdflatex/lecture_notes/main.pdf
  - Lecture notes (for citation purposes): https://arxiv.org/abs/2206.05799

- **MATHEMATICA PACKAGE**
  - Mathematica package on GitHub: https://github.com/sashwattanay/BBH-PN-Toolkit

- **YOUTUBE VIDEO**
  - https://youtu.be/aoiCk5TtmvE

THE END

Please send comments on the lecture notes and the presentation _/_

Thank you!