Integrability and action-angle variables of post-Newtonian binary black holes

Sashwat Tanay
with Leo Stein, Gihyuk Cho*
& José T. Gálvez Ghersi†

Univ. of Mississippi, SNU (Seoul)*, UTEC (Lima)†

Northwestern University (Jul 2023)

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stanay@olemiss.edu

Plan of the talk

- Introduction and theory

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- 1.5PN: action-angles

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- 2PN: integrable or non-integrable?

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- 2PN: integrable or non-integrable?
- Conclusions

Introduction and theory

Gravitational waves from binary black holes

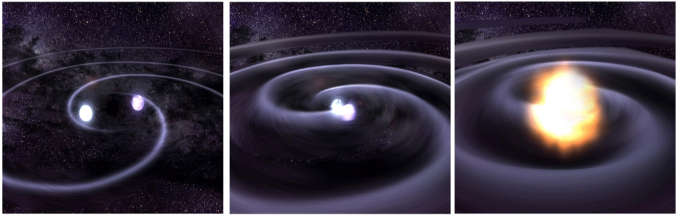

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Image credit: www.eoportal.org

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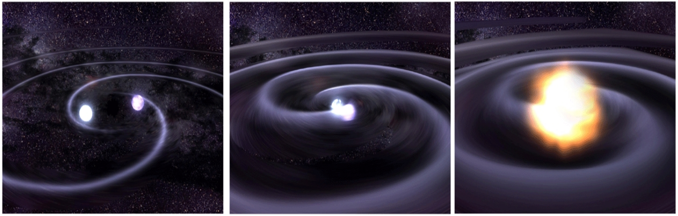

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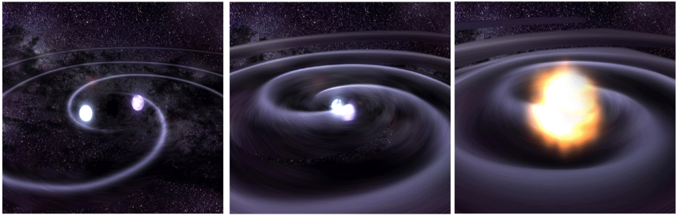

- **Stellar mass BBHs:** gravitational wave (GW) sources of LIGO and LISA.
- **Inspiral stage:** the longest-lived stage of BBH evolution.
- Quadrupole formula $\implies$ GWs are functions of black hole trajectories *(focus of the talk)*.

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- **Example:** schematic representation of Hamiltonian. Each factor of $1/c^2 \implies$ one PN order.

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- **Starting point:** 2PN Hamiltonian due to [Barker, O'Connell-1975]

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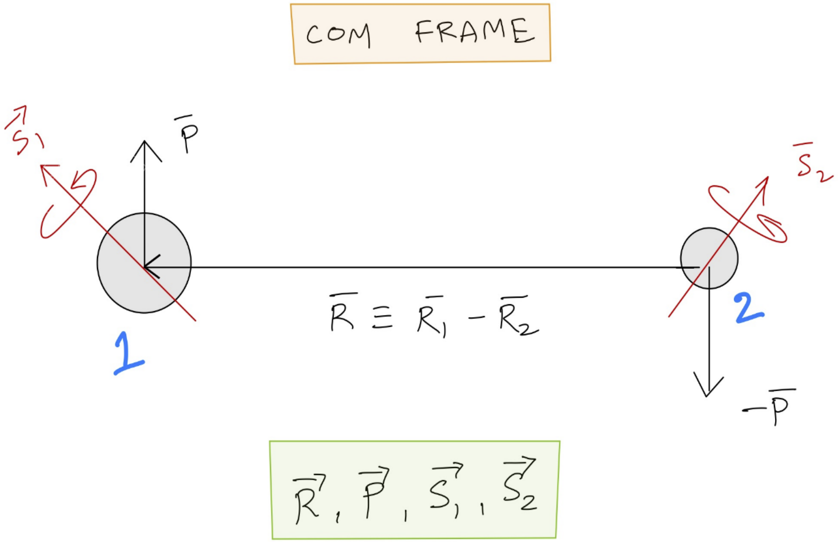

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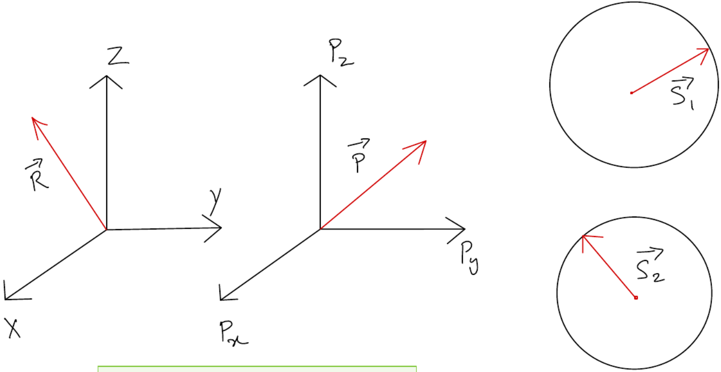

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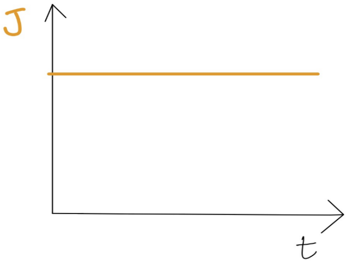

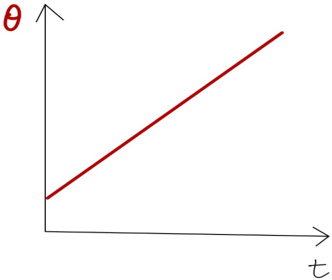

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It's nice to have integrable systems (they occur rarely), and extra nice to have action-angles.

1.5PN: action-angle variables

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RESULTS: action-angles & the solution at 1.5PN

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$$H = \underbrace{\left(\frac{P^2}{2\mu} - \frac{Gm_1 m_2}{R}\right)}_{\text{Newtonian}} + \frac{1}{c^2} F_1(\vec{R}, \vec{P}) + \frac{1}{c^3} F_2\left(\vec{R}, \vec{P}, \vec{S_1}, \vec{S_2}\right)$$

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- Necessary concepts laid out in Goldstein, Jose-Saletan, V. I. Arnold, Fasano-Marmi.

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- $\mathcal{J}_5$ is very lengthy.

$$J = JdJ^{1/2};\ L = LdL^{1/2};\ S1 = S1dS1^{1/2};\ S2 = S2dS2^{1/2};$$

$$\Delta 1 = \frac{\left(\frac{(J^2-L^2-S1^2-S2^2)}{2} - \frac{SeffdL}{\sigma 2}\right)}{\sigma 1 - \sigma 2};$$

$$\Delta 2 = \frac{\left(\frac{(J^2-L^2-S1^2-S2^2)}{2} - \frac{SeffdL}{\sigma 1}\right)}{\sigma 1 - \sigma 2};$$

$$\sigma 2 = 1 + \frac{9}{16 \times (-1 + \sigma 1)};$$

$$\sigma 1 = \left(1 + \frac{3}{4}\ \frac{m2}{m1}\right);\quad \sigma 2 = \left(1 + \frac{3}{4}\ \frac{m1}{m2}\right);$$

$$Seff = \delta 1\, S1 + \delta 2\, S2$$

$$cubic = L^2\, S1^2\, S2^2 - 2\, \sigma 1\, \sigma 2\, (\sigma 1 - \sigma 2)\, f\, (f - \Delta 1)\, (f - \Delta 2) - L^2\, (\sigma 1 - \sigma 2)^2\, f^2 - S2^2\, \sigma 2^2\, (f - \Delta 1)^2 - S1^2\, \sigma 1^2\, (f - \Delta 2)^2;$$

```
cubic = Collect[cubic // Expand // Simplify , f] ;
cubSol1 = ((Solve[cubic == 0, f])⟦1⟧);
cubSol2 = ((Solve[cubic == 0, f])⟦2⟧);
cubSol3 = ((Solve[cubic == 0, f])⟦3⟧);
f3 = f /. cubSol1 ;
f1 = f /. cubSol2 ;
f2 = f /. cubSol3 ;        (*The three roots of the cubic*)
```

$$A = 2\, \sigma 1\, \sigma 2\, (\sigma 2 - \sigma 1);$$

$$\beta = ((f2 - f1) / (f3 - f1))^{1/2};$$

$$B1 = \frac{1}{2}\left(\left(SeffdL + L^2\, (\sigma 1 + \sigma 2)\right) (J + L) + L\left(\sigma 1\, S1^2 + \sigma 2\, S2^2 + (\sigma 1 + \sigma 2)\, (\Delta 2\ \sigma 1 - \Delta 1\ \sigma 2)\right)\right);$$

$$B2 = \frac{1}{2}\left(\left(SeffdL + L^2\, (\sigma 1 + \sigma 2)\right) (J - L) - L\left(\sigma 1\, S1^2 + \sigma 2\, S2^2 + (\sigma 1 + \sigma 2)\, (\Delta 2\ \sigma 1 - \Delta 1\ \sigma 2)\right)\right);$$

$$D1 = L\, (L + J) + \Delta 2\ \sigma 1 - \Delta 1\ \sigma 2;$$

$$D2 = L\, (L - J) + \Delta 2\ \sigma 1 - \Delta 1\ \sigma 2;$$

$$B1S = \frac{1}{2}\left(-L^2\, S1\, \sigma 1 + J\, S1^2\, \sigma 1 - S1^3\, \sigma 1 - S1\, \Delta 2\, \sigma 1^2 + J^2\, S1\, \sigma 2 - 2\, J\, S1^2\, \sigma 2 + S1^3\, \sigma 2 - J\, \Delta 1\, \sigma 1\, \sigma 2 + 2\, S1\, \Delta 1\, \sigma 1\, \sigma 2 + J\, \Delta 1\, \sigma 2^2 - S1\, \Delta 1\, \sigma 2^2\right);$$

$$B2S = \frac{1}{2}\left(L^2\, S1\, \sigma 1 + J\, S1^2\, \sigma 1 + S1^3\, \sigma 1 + S1\, \Delta 2\, \sigma 1^2 - J^2\, S1\, \sigma 2 - 2\, J\, S1^2\, \sigma 2 - S1^3\, \sigma 2 - J\, \Delta 1\, \sigma 1\, \sigma 2 - 2\, S1\, \Delta 1\, \sigma 1\, \sigma 2 + J\, \Delta 1\, \sigma 2^2 + S1\, \Delta 1\, \sigma 2^2\right);$$

$$D1S = -J\, S1 + S1^2 - \Delta 1\ \sigma 2;$$

$$D2S = \left(J\, S1 + S1^2 - \Delta 1\ \sigma 2\right);$$

$$\alpha 1 = \frac{(\sigma 1 - \sigma 2)\, (f2 - f1)}{D1 - f1\, (\sigma 1 - \sigma 2)};$$

$$\alpha 2 = \frac{(\sigma 1 - \sigma 2)\,(f2 - f1)}{D2 - f1\,(\sigma 1 - \sigma 2)};$$

$$\alpha 1S = \frac{(-\sigma 1)\,(f2 - f1)}{D1S + f1\,\sigma 1};$$

$$\alpha 2S = \frac{(-\sigma 1)\,(f2 - f1)}{D2S + f1\,\sigma 1};$$

$$\Delta Y = \text{EllipticK}\left[\beta^2\right];$$

$$\Delta\lambda 1 = \frac{4\,\Delta Y}{(A\,(f3 - f1))^{1/2}};$$

$$\Delta\lambda 2 = \left(\frac{-1}{2\,JdJ^{1/2}}\right)\left(\frac{4}{(A\,(f3 - f1))^{1/2}}\right)\left(\frac{B1\ \text{EllipticPi}\left[\alpha 1,\ \beta^2\right]}{D1 - f1\,(\sigma 1 - \sigma 2)} + \frac{B2\ \text{EllipticPi}\left[\alpha 2,\ \beta^2\right]}{D2 - f1\,(\sigma 1 - \sigma 2)}\right);$$

$$\Delta\lambda 3 = \left(\frac{-1}{2\,LdL^{1/2}}\right)\left(\left(\frac{4}{(A\,(f3 - f1))^{1/2}}\right)\left(\frac{B1\ \text{EllipticPi}\left[\alpha 1,\ \beta^2\right]}{D1 - f1\,(\sigma 1 - \sigma 2)} - \frac{B2\ \text{EllipticPi}\left[\alpha 2,\ \beta^2\right]}{D2 - f1\,(\sigma 1 - \sigma 2)}\right) - (\text{SeffdL} + (\Delta 1 - \Delta 2)\,\sigma 1\,\sigma 2 + \text{LdL}\,(\sigma 1 + \sigma 2))\,\frac{\Delta\lambda 1}{\text{LdL}^{1/2}}\right);$$

$$\Delta\lambda 4 = \left(\frac{-1}{2\,S1dS1^{1/2}}\right)\left(\left(\frac{4}{(A\,(f3 - f1))^{1/2}}\right)\left(-\frac{B1S\ \text{EllipticPi}\left[\alpha 1S,\ \beta^2\right]}{D1S + f1\,\sigma 1} + \frac{B2S\ \text{EllipticPi}\left[\alpha 2S,\ \beta^2\right]}{D2S + f1\,\sigma 1}\right) + S1dS1^{1/2}\,(\sigma 2 - \sigma 1)\,\Delta\lambda 1\right);$$

$$\Delta\lambda 5 = \Delta\lambda 4\ /.\ \sigma 1 \to \sigma 2\ /.\ S1dS1 \to S1DS1\ /.\ S2dS2 \to S2DS2\ /.\ S1DS1 \to S2dS2\ /.\ S2DS2 \to S1dS1;$$

$$J5 = \frac{1}{\pi}\left(\text{SeffdL}\ \Delta\lambda 1 + J^2\,\Delta\lambda 2 + L^2\,\Delta\lambda 3 + S1^2\,\Delta\lambda 4 + S2\,\Delta\lambda 5\right);$$

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- **Other examples:** Radius of convergence of Taylor series, Fermat's last theorem.

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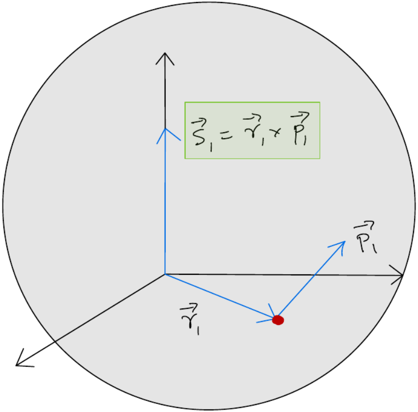
- **Numerical check:** matches with numerical solution [github.com/sashwattanay/BBH-PN-Toolkit].

A pictorial mnemonic for fictitious variables



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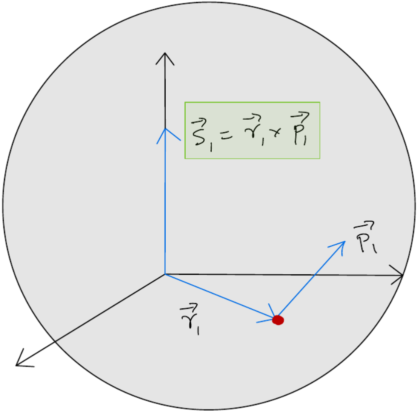

Essentially, trade the spherical manifold in favor of a Cartesian manifold.

2PN: Integrable or non-integrable?

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- See the Introduction of [gr-qc:0511009] and [2012.06586] for details.

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- The non-exact nature of integrability $\implies$ the tension b/w the two camps.

The fourth commuting constant of motion

With the definitions:

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$$\left. + \left(\frac{P^i P_i}{m_1 m_2} - \frac{2Gm_1 m_2}{(m_1 + m_2)\sqrt{R^i R_i}} \right) S_{1a} S_2^a \right].$$

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$$
\vec{S}_{\text{eff}} \cdot \vec{L} + \frac{1}{2}\left(S_1^a S_{2a}\right) + \frac{\epsilon\left(P^a S_{1a}\right)^2}{m_1^2} + \frac{3m_2\epsilon\left(P^a S_{1a}\right)^2}{4m_1^3} - \frac{2Gm_2^2\epsilon\left(R^a S_{1a}\right)^2}{\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}}
$$

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(2) Using classical mechanics to do GW research.

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- Redo EOB for spinning systems by matching PN and EMRI action angles.

Thank you!
Questions?

Refs:

- Papers: 2012.06586, 2110.15351, 2210.01605, 2110.09608.
- Lecture notes: 2206.05799
- Mathematica package: github.com/sashwattanay/BBH-PN-Toolkit