Integrability and action-angle variables of post-Newtonian binary black holes

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Plan of the talk

- Introduction and theory

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- 1.5PN: action-angles

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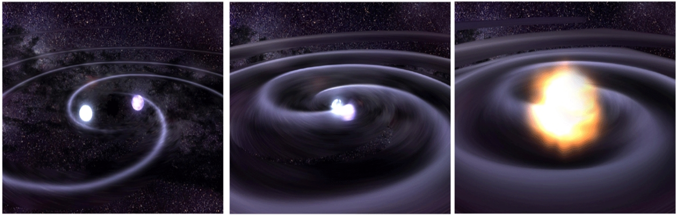
- 2PN: integrable or non-integrable?
- Conclusions

Introduction and theory

Gravitational waves from binary black holes



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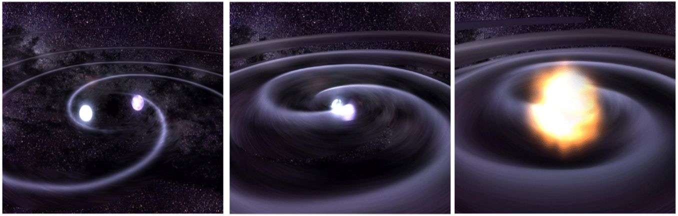

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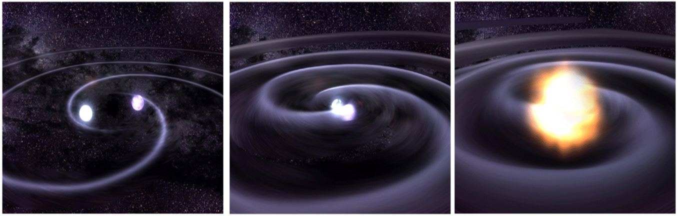

- **Stellar mass BBHs:** gravitational wave (GW) sources of LIGO and LISA.
- **Inspiral stage:** the longest-lived stage of BBH evolution.
- Quadrupole formula $\implies$ GWs are functions of black hole trajectories *(focus of the talk)*.

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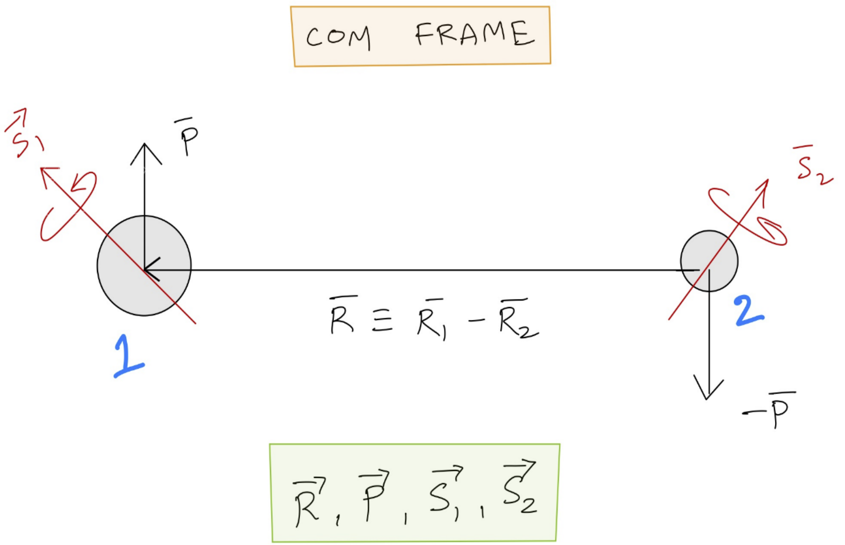
- **Starting point:** 2PN Hamiltonian due to [Barker, O'Connell-1975]

Phase space of spinning PN BBHs



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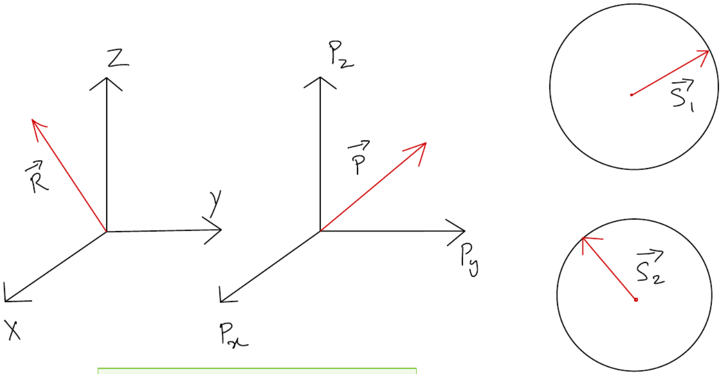

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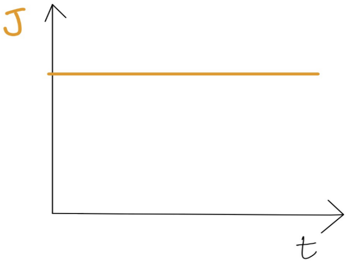

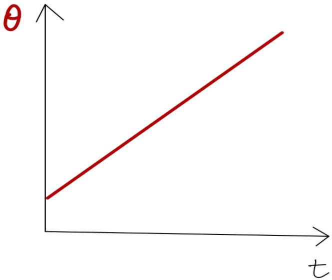

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It's nice to have integrable systems (they occur rarely), and extra nice to have action-angles.

1.5PN: action-angle variables

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RESULTS: action-angles & the solution at 1.5PN

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- **Other examples:** Radius of convergence of Taylor series, Fermat's last theorem.

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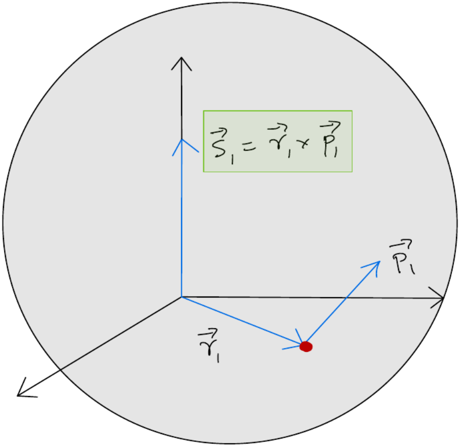
- **Numerical check:** matches with numerical solution [github.com/sashwattanay/BBH-PN-Toolkit].

A pictorial mnemonic for fictitious variables



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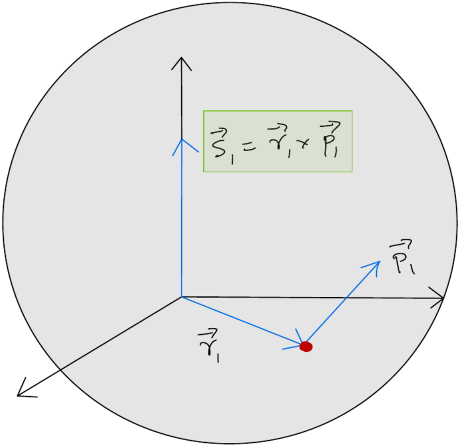

Essentially, trade the spherical manifold in favor of a Cartesian manifold.

2PN: Integrable or non-integrable?

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- See the Introduction of [gr-qc:0511009] and [2012.06586] for details.

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- The non-exact nature of integrability $\implies$ the tension b/w the two camps.

The fourth commuting constant of motion

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The 4th commuting constant is

The fourth commuting constant of motion

With the definitions:

$$
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\end{aligned}
$$

The 4th commuting constant is

$$
\boldsymbol{L^2} - \epsilon \left[\frac{(m_2 \ P^i S_{1i} + m_1 \ P^i S_{2i})^2}{m_1^2 \ m_2^2} + \frac{2G(m_2 \ R^i S_{1i} + m_1 \ R^i S_{2i})^2}{(m_1 + m_2)(R^i R_i)^{3/2}} \right.
$$
$$
\left. + \left(\frac{P^i P_i}{m_1 m_2} - \frac{2Gm_1 m_2}{(m_1 + m_2)\sqrt{R^i R_i}} \right) S_{1a} S_2^a \right] .
$$

And the 5th commuting constant is ...

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$$
\vec{S}_{\text{eff}} \cdot \vec{L} + \frac{1}{2}\left(S_1^a S_{2a}\right) + \frac{\epsilon\left(P^a S_{1a}\right)^2}{m_1^2} + \frac{3m_2\epsilon\left(P^a S_{1a}\right)^2}{4m_1^3} - \frac{2Gm_2^2\epsilon\left(R^a S_{1a}\right)^2}{\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}}
$$
$$
- \frac{3Gm_2^3\epsilon\left(R^a S_{1a}\right)^2}{2m_1\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} + \frac{3\epsilon\left(P^a S_{1a}\right)\left(P^a S_{2a}\right)}{4m_1^2}
$$
$$
+ \frac{3\epsilon\left(P^a S_{1a}\right)\left(P^a S_{2a}\right)}{4m_2^2} + \frac{2\epsilon\left(P^a S_{1a}\right)\left(P^a S_{2a}\right)}{m_1 m_2} + \frac{3m_1\epsilon\left(P^a S_{2a}\right)^2}{4m_2^3}
$$
$$
+ \frac{\epsilon\left(P^a S_{2a}\right)^2}{m_2^2} - \frac{3Gm_1^2\epsilon\left(R^a S_{1a}\right)\left(R^a S_{2a}\right)}{2\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}}
$$
$$
- \frac{4Gm_1 m_2\epsilon\left(R^a S_{1a}\right)\left(R^a S_{2a}\right)}{\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} - \frac{3Gm_2^2\epsilon\left(R^a S_{1a}\right)\left(R^a S_{2a}\right)}{2\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}}
$$
$$
- \frac{2Gm_1^2\epsilon\left(R^a S_{2a}\right)^2}{\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}} - \frac{3Gm_1^3\epsilon\left(R^a S_{2a}\right)^2}{2m_2\left(m_1+m_2\right)\left(R_a R^a\right)^{3/2}}.
$$

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- Redo EOB for spinning systems by matching PN and EMRI action angles.

Right Part

Thank you!

Questions?

Refs:

- Papers: 2012.06586, 2110.15351, 2210.01605, 2110.09608.
- Lecture notes: 2206.05799
- Mathematica package: github.com/sashwattanay/BBH-PN-Toolkit