Complex Analysis

Samir Salmen
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Outline

1| Lecture I: Introduction to Complex Analysis

Complex Analysis is the natural analogy of Real Analysis to the field of real numbers. like real numbers that are represented on a line, complex numbers (of the form $a + bi \mid a, b \in \mathbb{R}$ and $i^2 = -1$) are represented on a plane.
Complex analysis is more close to geometric than algebraic thinking, this is justified by the concept that complex analysis can be defined as following:

1 Definition. *Complex analysis is the local relevant for the study of complex manifolds.*

This phrase contains much of the motivation about complex analysis. Lets develop this thought:

1. Important in mathematics and physics.
   (a) String theory (with complex geometry)
   (b) Analytic functions appear often on Fourier analysis, Harmonic analysis, number theory...

   .1 Example (Riemman zeta function)**.**

   $$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\, dx.\, s \in$$

   Are all non trivial zeros of this function on the $Re[x] = \frac{1}{2}$ line?

   I

Lets define the goals of this course now:

- Analytic functions and power series.
- the Riemann sphere and Mobius transformation.
- Complex integration.
- some "bonus topics".

Now, lets extend the usual real concepts to the complex plane.

- Trigonometric: $e^{ix} = \cos(x) + i\sin(x) \implies \cos(x) = \frac{\exp(ix)+\exp(-ix)}{2}$
- Differentiation: Different from Real Functions, there is only two classes of complex functions. C^∞ and discontinuous ones.
- Integration: Like differentiation, integration is also different. When you integrate a function on the complex plane, the answer will depend on the path taken. However, **Cauchy Theorems** states that the integral of $f(x)$ *almost* does not depend on th path taken (the set of functions that depend on it are called **non-holomorphic**).
- *Difficult* real operations: We can learn to realize some difficult calculations more easily using complex analysis (like $\int_0^\infty \frac{\sin(x)}{x}\, dx$).
- Analytic continuation: A function that is *complex differentiable* on $[a, b]$ has a *unique* differentiable analytic continuation.

I Construction of $\mathbb{C}$

Classical analysis: Works over $(\mathbb{R}, +, \cdot)$ an **ordered field**, i.e.

- $\forall x, y \in \mathbb{R}$, either $x < y$, $x > y$, $x = y$ holds true.

- $x, y > 0 \implies x + y > 0$ and $xy > 0$.

However, $\mathbb{R}$ is ***not* algebraic closed**. ($x^2 + 1$ has no roots $/\mathbb{R}$).

Question: Can $(\mathbb{R}^2, +)$ (Abelian group) be made into a field? [what about $(\mathbb{R}^n, +)$?].

1 Theorem. If this is possible, then $\exists z \in (\mathbb{R}^2, +, \cdot)$ such that $z^2 = -1$.

Demonstração. Chose a basis $\{1, e\}$ of $\mathbb{R}^2$

$$\begin{aligned} z &= x \cdot 1 + y \cdot e \\ z^2 &= x^2 \cdot 1 + 2xy \cdot e + y^2 \cdot e^2 \\ e &= a \cdot 1 + b \cdot e, a, b \in \mathbb{R} \\ z^2 &= (x + ay^2) \cdot 1 + (2xy + by^2) \cdot e \end{aligned}$$

We want to find $(x, y) \in \mathbb{R}$ s.t $z^2 = -1$. Then we must have $2x + by^2 = 0$. So either $y = 0$ or $x = -\frac{b}{2}y$, lets discard the first one because it implies that $z \in \mathbb{R}$. Therefore, if we go after the second possibility a straight foward calculation gives:

$$\begin{aligned} z^2 &= ((-\frac{b}{2}y)^2 + ay^2) \cdot 1 + (0) \cdot e \\ z^2 &= (a + \frac{b^2}{4})y \cdot 1 \end{aligned}$$

.

One can prove that $y = -\frac{1}{(a + \frac{b^2}{4})}$. Therefore $z^2 = -1$. This proof was first introduced by Gauss in his thesis, 1799. $\square$

2| Lecture II: Complex Differentiation

$\mathbb{C}$ is a metric space with $d(z, w) = |z - w|$ isometric to $\mathbb{R}^2$.
Triangle ineq: $|z + w| \leq |z| + |w|$.
Topology: Open disks $\{z \in \mathbb{C} : |z - w| < R\} = D(w, R)$ (analogous with closed disks)

2 Definition (open sets in $\mathbb{C}$)**.** *$A \subset \mathbb{C}$ is open if $\forall a \in A$ has a disk $D(a, \epsilon) \leq A$ for some $\epsilon > 0$.*