C*-Algebra

Samir Salmen
February 2021

Outline

1| Order vector spaces and positivity

I Basic Notions

a Cones

Let V be a vector space over a field. We say that $P \subset V$ is a **cone** if: $\forall \lambda \geq 0$ and $v \in P, \lambda v \in P$.
A given cone is **convex** if it is closed under vector addiction.
Note that a convex cone is also a convex set.

b Preorders

Any convex cone $P \subset V$ naturally defines a pre-order relation in V, denoted here by $\succeq^P$:

$$v \succeq^P v' \iff v - v' \in P,\ v, v' \in V$$

from that follows:

- For all $v, v', w, w' \in P$ with $v \succeq^P v'$ and $w \succeq^P w'$, one has $v + w \succeq^P v' + w'$.
- For all $\lambda \geq 0$ and $v, v' \in P$ with $v \succeq^P v'$, one has $\lambda v \succeq^P \lambda v'$

Conversely, if $\succeq$ is a pre-order in V with the above two proprieties, then it tis the preorder associeted with the convex cone

$$V^+ := \{v \in V : v \succeq 0\}..$$

1 Definition. *The pair $(V, \succeq)$ is called "preordered vector space", while the elements that are greater in this order than zero are called the "positive elements".*