List 6

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Exercise 1

Our Equation of motion is given by:

$$\ddot{\phi} - \frac{g + \ddot{h}}{l}\phi = 0 \tag{1}$$

We may also define $a(t) = \frac{g+\ddot{h}}{l}$.

(a) Lets define $a(t)$ in the following way:

$$a(t) = \begin{cases} w_1^2,\ t \in (0, \pi) \\ w_2^2,\ t \in (\pi, 2\pi) \end{cases} \tag{2}$$

where in both cases $|\ddot{h}(t)| < g$.

Lets calculate F_1. The equation of motion for $[0, \pi]$ is:

$$\ddot{\phi} + \omega_1^2\phi = 0 \tag{3}$$

The general solution for this equation is simply:

$$\phi(t) = c_1 e^{\omega_1 t} + c_2 e^{-\omega_1 t} \tag{4}$$

Therefore, $\dot{\phi(t)} = \omega_1 c_1 e^{\omega_1 t} - c_2 e^{-\omega_1 t}$. From that, one may deduce the following:

$$x = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} e^{\omega_1 t} & e^{-\omega_1 t} \\ \omega_1 e^{\omega_1 t} & -\omega_1 e^{-\omega_1 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{5}$$

In order to discover the columns of F_1, lets define some initial conditions. For F_1, lets fix an initial condition $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This implies:

$$\begin{cases} c_1 + c_2 = 1 \\ c_1\omega_1 - c_2\omega_1 = 0 \end{cases} \tag{6}$$

Therefore, $a_1 = a_2 = \frac{1}{2}$ in this case.

Analogously, for the second column of F_1 lets define $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This implies that $c_1 = -c_2 = \frac{1}{\omega_1}$.
This same analysis can be done for F_2. Therefore, F_1 and F_2 are given as follows:

$$F_1 = \begin{pmatrix} \cosh(\omega_1\pi) & \frac{1}{\omega_1}\sinh(\omega_1\pi) \\ \omega_1 \sinh(\omega_1\pi) & \cosh(\omega_1\pi) \end{pmatrix} \tag{7}$$

$$F_2 = \begin{pmatrix} \cosh(\omega_2\pi) & \frac{1}{\omega_2}\sinh(\omega_2\pi) \\ \omega_2\sinh(\omega_2\pi) & \cosh(\omega_2\pi) \end{pmatrix} \tag{8}$$

We know that $F = F_2F_1$. Therefore (Assuming $\omega_1 < \omega_2$):

$$|\mathrm{tr}(F)| = |2\cosh(\omega_1\pi)\cosh(\omega_2\pi) + (\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2})\sinh(\omega_1\pi)\sinh(\omega_2\pi)| \tag{9}$$

$$|\mathrm{tr}(F)| < |2\cosh(\omega_1\pi)\cosh(\omega_2\pi) + \sinh(\omega_1\pi)\sinh(\omega_2\pi)| \tag{10}$$

$$|\mathrm{tr}(F)| < |\cosh([\omega_1+\omega_2]\pi)| < \left|\cosh\left(\pi\sqrt{\frac{4g}{l}}\right)\right| \tag{11}$$

Notice now that if $l < \frac{4g\pi^2}{[\cosh^{-1}(2)]^2} \approx 227.6\,m$, then $|\mathrm{tr}(F)| < 2$. Therefore the system is stable except for very long handles.

(b) This problem is analogous to the previous one, except for some things. First in the period from $[\pi, 2\pi]$ we know that $a(t) = -\omega_2^2 < 0$, second we know that $|\ddot{h}(t)| > g$, $\forall t$. Therefore, F_1 and F_2 are:

$$F_1 = \begin{pmatrix} \cosh(\omega_1\pi) & \frac{1}{\omega_1}\sinh(\omega_1\pi) \\ \omega_1\sinh(\omega_1\pi) & \cosh(\omega_1\pi) \end{pmatrix} \tag{12}$$

F_2 deduction is slightly different this time, the solution of the original ODE (when studied inside $[\pi, 2\pi]$) gives the following:

$$\ddot{\phi} - \omega_1^2\phi = 0 \tag{13}$$

The solution for this equation is:

$$\phi(t) = c_1\sin(\omega_2 t) + c_2\cos(\omega_2 t) \tag{14}$$

Therefore,

$$x = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \sin(\omega_2 t) & \cos(\omega_2 t) \\ \omega_2\cos(\omega_2 t) & -\omega_2\sin(\omega_2 t) \end{pmatrix} \begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \tag{15}$$

setting $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives $c_3 = 0$, $c_4 = 1$. The other initial condition used on the first problem gives $c_3 = \frac{1}{\omega_2}$, $c_4 = 0$. Therefore, we know that:

$$F_2 = \begin{pmatrix} \cos(\omega_2\pi) & \frac{1}{\omega_2}\sin(\omega_2\pi) \\ -\omega_2\sin(\omega_2\pi) & \cos(\omega_2\pi) \end{pmatrix} \tag{16}$$

Notice that F contains products of hyperbolic functions with trigonometric ones. Therefore, a stability analysis without numerical values for each constant is improbable.

Exercise 2

(a) Our Lagrangian for each mass is given by $\mathcal{L} = T - U$, where $T_{2i+1} = \frac{m_1}{2}\dot{x}_{2i+1}$, $T_{2i} = \frac{m_2}{2}\dot{x}_{2i}$ and $U_i = -\frac{-k}{2}(2x_i - x_{i+1} - x_{i-1})^2$. Therefore, by Euler-Lagrange equation:

$$\begin{cases} m_1\ddot{x}_n + k(2x_n - y_{n-1} - y_n) & = 0 \\ m_2\ddot{y}_n + k(2y_n - x_n - x_{n+1}) & = 0 \end{cases} \tag{17}$$

Where we corrected the indices to $n \in \mathbb{Z}$ in order to stay consistent with the statement of the problem.

(b) First, lets assume a solution of the form:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix} e^{ikn} \tag{18}$$

Therefore, we have through the equation of motion:

$$\begin{cases} m_1\ddot{\zeta}(t)e^{ikn} + k(2\zeta(t)e^{ikn} - \eta(t)e^{ik(n-1)} - \eta(t)e^{ikn}) = 0 \\ m_2\ddot{\eta}(t)e^{ikn} + k(2\eta(t)e^{ikn} - \zeta(t)e^{ikn} - \zeta(t)e^{ik(n+1)}) = 0 \end{cases} \tag{19}$$

Dividing everything by e^{ikn} we have:

$$\begin{cases} m_1\ddot{\zeta}(t) + 2k\zeta(t) - k(e^{-ik}+1)\eta(t) = 0 \\ m_2\ddot{\eta}(t) + 2k\eta(t) - k(e^{ik}+1)\zeta(t) = 0 \end{cases} \tag{20}$$

In a matrix form, we have $\left(\text{where } \vec{q} = \begin{pmatrix} \zeta(t) \\ \eta(t) \end{pmatrix}\right)$:

$$\ddot{\vec{q}} + \underbrace{\begin{pmatrix} 2\omega_1^2 & -\omega_1^2(e^{-ik}+1) \\ -\omega_2^2(e^{ik}+1) & 2\omega_2^2 \end{pmatrix}}_{\text{K}} \vec{q} = 0 \tag{21}$$

or alternatively:

$$\ddot{\vec{q}} + K\vec{q} = 0 \tag{22}$$

Therefore, in order to find the general solution for this ODE, one need only to find the eigenvalue λ that solves:

$$\det\left[\lambda^2 \mathbb{1} + K\right] = 0 \tag{23}$$

$$\det\left[\lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2\omega_1^2 & -\omega_1^2(e^{-ik}+1) \\ -\omega_2^2(e^{ik}+1) & 2\omega_2^2 \end{pmatrix}\right] = 0 \tag{24}$$

$$\det\left[\begin{pmatrix} 2\omega_1^2+\lambda^2 & -\omega_1^2(e^{-ik}+1) \\ -\omega_2^2(e^{ik}+1) & 2\omega_2^2+\lambda^2 \end{pmatrix}\right] = 0 \tag{25}$$

$$(\lambda^2+2\omega_1^2)(\lambda^2+2\omega_2^2) - \omega_1^2\omega_2^2(e^{-ik}+1)(e^{ik}+1) = 0 \tag{26}$$

$$\tag{27}$$

Thus, the eigenvalues of this matrix are:

$$\left\{\lambda = \pm\sqrt{-\omega_1^2 - \omega_2^2 \pm \sqrt{\omega_1^4 + \omega_2^4 + 2\omega_1^2\omega_2^2\cos(k)}}\right. \tag{28}$$

After calculating the eigenvectors $\vec{u}_i$ of each eigenvalue λ_i, one may find the general solution as:

$$\vec{q}(t) = \sum_{i=1}^{4} e^{\lambda_i t}\vec{u}_i \tag{29}$$