11 December 2021 11:03

V is a vector space over F.

$\{v_1, v_2, \ldots v_n\}$ orthogonal if $\langle v_i, v_j \rangle = 0$ $i \neq j$

$\langle v_i, v_j \rangle = v_i^T v_j = 0$ $i \neq j$

(eg) 1. $V = \mathbb{R}^2$ $\quad \left\{ \underset{v_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$

$v_1^T v_2 = (1, 0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.0 + 0.1 = 0$

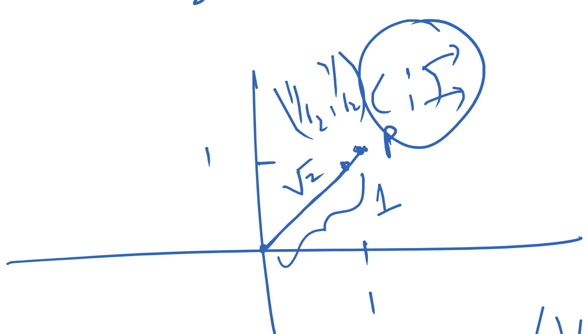
Yes, v_1 & v_2 are orthogonal

2. $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ Are they orthogonal

$(1, -1) \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2.1 + 0.-1 = 2 \neq 0$

$\therefore$ They are not orthogonal.

3.



$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{1^2+1^2} \\ 1/\sqrt{1^2+1^2} \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 1/\sqrt{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \to \begin{pmatrix} 1/\sqrt{1^2+3^2} \\ 3/\sqrt{1^2+3^2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$

$\left\|\begin{pmatrix}1\\3\end{pmatrix}\right\| \neq 1$!

$\left\|\begin{pmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{pmatrix}\right\| = 1 \rightarrow$ normalizing

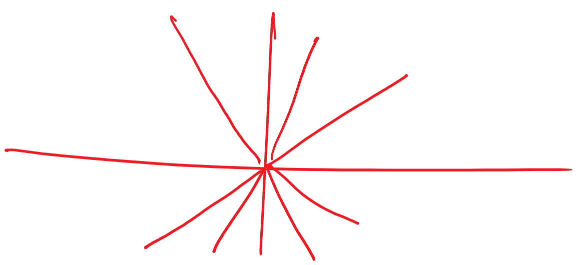
$v_1 \begin{pmatrix}1\\3\end{pmatrix} \quad v_2 \begin{pmatrix}3\\-1\end{pmatrix}$ orthogonal ?

orthonormality.

$v_1' \begin{pmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{pmatrix} \cdot v_2' \begin{pmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{pmatrix}$ $\left\{\begin{array}{l} v_1' \cdot v_2' = 0 \\ \|v_1'\| = 1, \quad \|v_2'\| = 1 \end{array}\right.$

(eg) $V = \mathbb{R}^2$: $\left\{ \underbrace{\alpha \begin{pmatrix}1\\1\end{pmatrix} + \beta \begin{pmatrix}1\\-1\end{pmatrix}}, \begin{pmatrix}2\\3\end{pmatrix} \right\}$

Basis for $\mathbb{R}^2$, only two vectors



$A_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} * & * & & * \\ C_1 & C_2 & \cdots & C_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$

$C_1, C_2, \ldots C_n$ are LI.

$\rightarrow$ G.S process

$\underline{\underline{Q}} = \{q_1, q_2, \ldots, q_n\}$ that are LI & orthogonal!

$$A = QR$$

$$Q^T A = \underbrace{Q^T \cdot Q}_{} \cdot R = I \cdot R = R$$

$$\begin{array}{cc} & n \\ n & \begin{bmatrix} | & | & | & \cdots & | \\ | & | & | & & | \\ | & | & | & & | \end{bmatrix} \end{array}$$

A

n^2 elements

$\begin{bmatrix} | \\ | \\ | \\ | \end{bmatrix} [| \; | \; | \; | \; \cdots \; |]$ $2n$

$\left[\begin{array}{l} A_{4\times3} \to \text{deg. of freedom} = 12. \\ Q_{m\times n} = Q_{4\times3} \end{array}\right.$

$R_{3\times3}$

$$R_{3\times3}. \quad \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \to 6$$

$$Q_{4\times3} : \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \\ q_{41} & q_{42} & q_{43} \end{bmatrix}$$

✓ $q_{11}^2 + q_{21}^2 + q_{31}^2 + q_{41}^2 = 1$

✓ $q_{11} q_{12} + q_{21} * q_{22} + q_{31} \cdot q_{32} + q_{41} \cdot q_{42} = 0$

$\cdot$ ~~$q_{11} \; q_{12}$~~ $+ q_{21} \cdot q_{23} + q_{31} \cdot q_{33} + q_{41} \cdot q_{43} = 0$

For the first col. 1 deg. of freedom only

For the second col. 4 $\rightarrow$ 3 $\rightarrow$ 2

For the third col. $\rightarrow$ 4 $\rightarrow$ 3

$Q \rightarrow$ total deg $= 1+2+3 = 6$

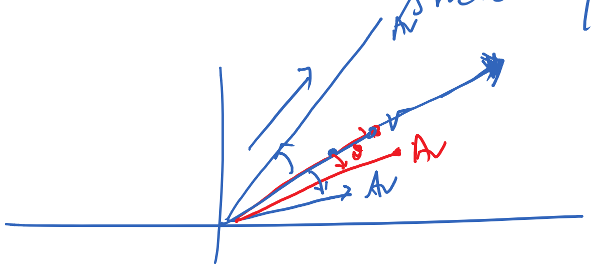
$R \rightarrow 6$

$\therefore Q \,\&\, R \rightarrow 12$

$A \rightarrow 12$

$Av \rightarrow$ rotates v

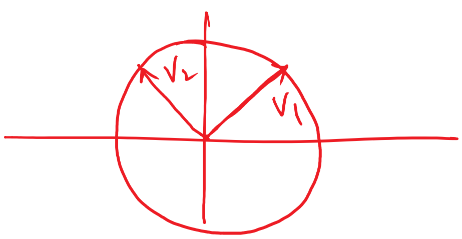
stretches / compresses v.



$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

$\alpha > 1$

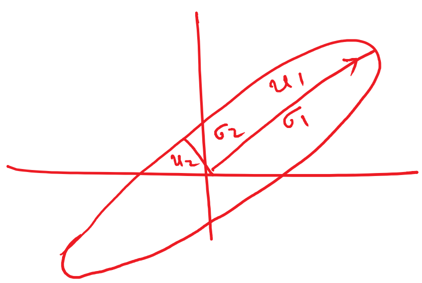
$A \rightarrow$ orthogonal $\rightarrow$ only angle is changed



v_1 is orthogonal to v_2

$Av_1 = \sigma_1 u_1$

$Av_2 = \sigma_2 u_2$



ellipse

$\underbrace{u_1, u_2, \ldots, u_n}$

$$\begin{aligned} A\boxed{v_1} &= \sigma_1 \boxed{u_1} \\ A v_2 &= \sigma_2 u_2 \\ A v_n &= \sigma_n u_n \end{aligned}$$

$$A_{m\times n} = U\Sigma V^T \;;\; A^T_{n\times m} \qquad (xy)^T = y^T x^T$$

$$A^T = (U\Sigma V^T)^T = V\,\Sigma^T U^T$$

$$(A^T A)_{n\times n} = V\Sigma^T \underbrace{U^T . U} \Sigma V^T \qquad U^T = U^{-1}$$

$$V^T = V^{-1}$$

$$= V\,\Sigma^T I . \Sigma . V^T$$

$$(A^T A) = V . \Sigma^T\Sigma . V^T$$

$$\therefore (A^T A)\, V = V\,\Sigma^T\Sigma . \underbrace{V^T . V} = V\,\Sigma^T\Sigma .$$

$$(A^T A)\, V = (\Sigma^T\Sigma)\, V \quad (\text{special value of } \Sigma).$$

Eigenvectors of $A^T A \to V$.

eigenvalues are $\Sigma^T\Sigma$.