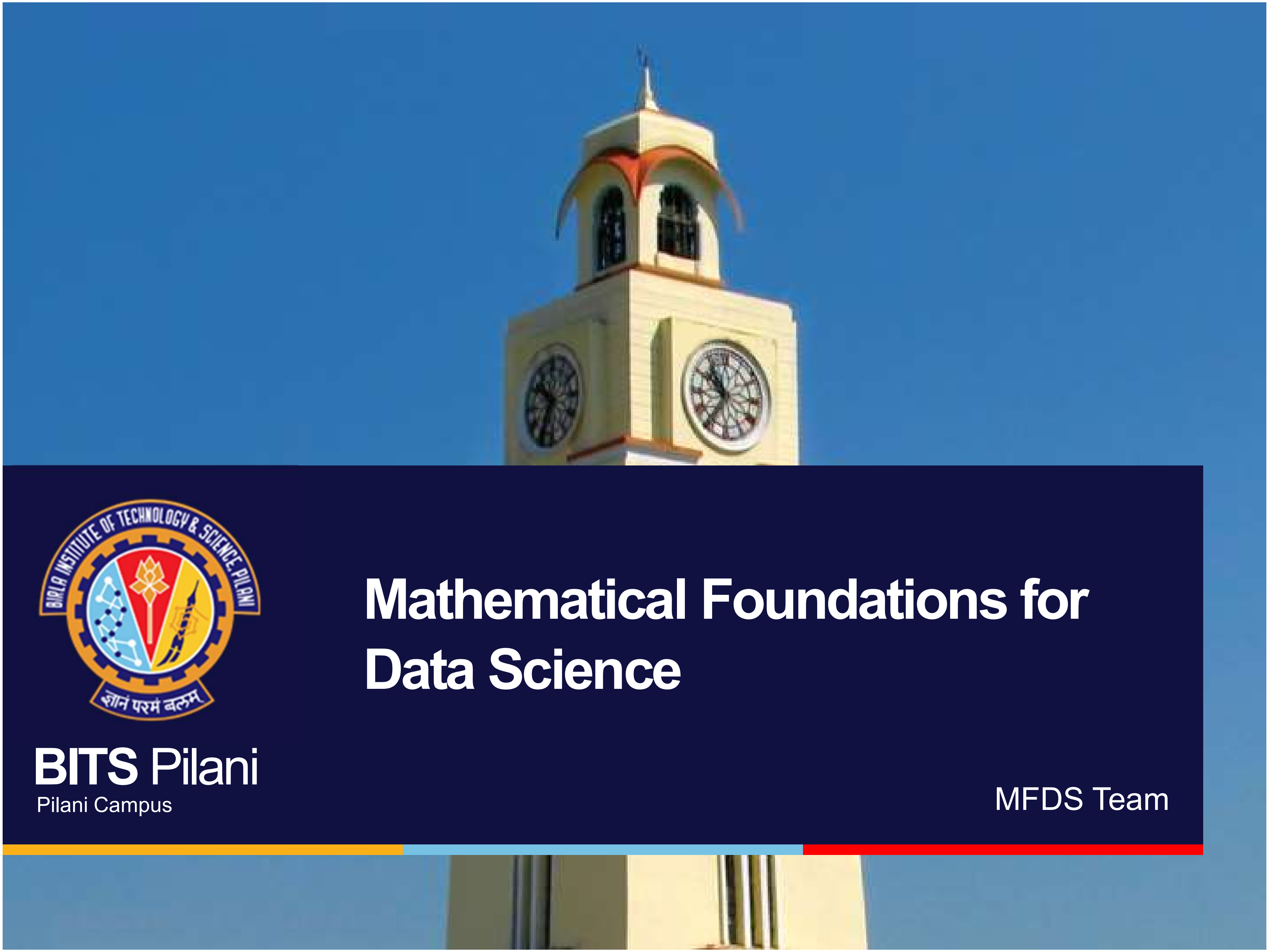

Mathematical Foundations for Data Science

BITS Pilani
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MFDS Team

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Webinar#1

Agenda

Discussion on

- Gauss-Elimination Method (MIDSEM 2021)
- Traffic Flow Problem
- Convergence of Gauss-Seidel and Gauss Jacobi Method
- Operations Count on LU- Decomposition Method
- Cholesky's Method
- Example for Vector Space (MIDSEM 2021)
- Problems on Rank Nullity Theorem(MIDSEM 2021)

Gauss-Elimination Method (MIDSEM 2021)

Solve the following system of equations using the Gauss elimination method with and without partial pivoting and three digit rounding arithmetic and compare the results

$$4.03x_1 + 2.19x_2 + 1.23x_3 = -4.35$$
$$6.21x_1 + 3.61x_2 - 2.46x_3 = -7.16$$
$$7.92x_1 + 5.11x_2 + 3.29x_3 = 12.83$$

Solution:

Gauss Elimination without Pivoting:

Step1 :R2-(a21/a11)R1, R3-(a31/a11)R1, $\begin{bmatrix} 4.03 & 2.19 & 1.23 & -4.35 \\ 0 & 0.24 & -4.35 & -0.46 \\ 0 & 0.800 & 0.87 & 21.4 \end{bmatrix}$

Step 2: R3-(a32/a22)R1 $\begin{bmatrix} 4.03 & 2.19 & 1.23 & -4.35 \\ 0 & 0.24 & -4.35 & -0.46 \\ 0 & 0 & 15.4 & 22.9 \end{bmatrix}$

Then we after back substitution we get, x1=-15.2, x2= 25.1, x3=1.49

Gauss Elimination with Pivoting:

Step 1: R13

$$\begin{bmatrix} 7.92 & 5.11 & 3.29 & 12.8 \\ 6.21 & 3.61 & -2.46 & -7.16 \\ 4.03 & 2.19 & 1.23 & -4.35 \end{bmatrix}$$

Step2 : R2-(a21/a11)R1, R3-(a31/a11)R1,

$$\begin{bmatrix} 7.92 & 5.11 & 3.29 & 12.8 \\ 0 & -0.400 & -5.04 & -17.2 \\ 0 & -0.41 & -0.440 & -10.9 \end{bmatrix}$$

Step 3: R23

$$\begin{bmatrix} 7.92 & 5.11 & 3.29 & 12.8 \\ 0 & -0.41 & -0.440 & -10.9 \\ 0 & -0.40 & -5.04 & -17.2 \end{bmatrix}$$

Step 4: R3-(a32/a22)R1

$$\begin{bmatrix} 7.92 & 5.11 & 3.29 & 12.8 \\ 0 & -0.41 & -0.44 & -10.9 \\ 0 & 0 & -4.61 & -6.60 \end{bmatrix}$$

Then we after back substitution we get, x1=-15.2, x2= 25.1, x3=1.43

Traffic Flow Problem

Modelling of electrical / traffic networks would lead to a linear system Ax=b.

Refer to the text book / other resources and construct a network which

has the following properties

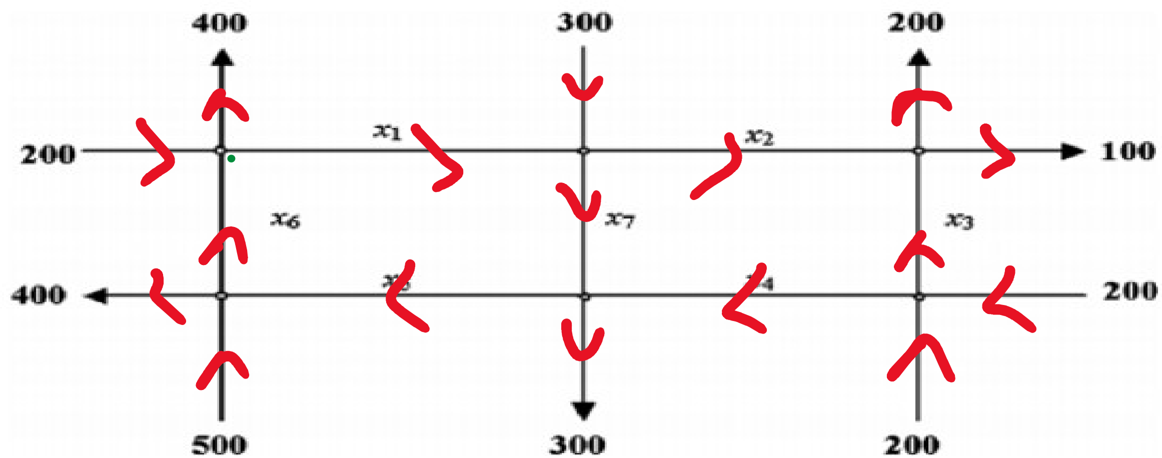
a) the number of equations is 6

b) A has rank 5

c) the system is consistent.

Solution:

Consider the following traffic flow diagram and construct the linear system AX=b. Hence solve.



On modelling, using the fact that, the out-flow traffic= inflow traffic we get,

Inflow =Outflow

$$
\begin{aligned}
200 + x_6 &= 400 + x_1 \\
x_1 + 300 &= x_2 + x_7 \\
x_2 + x_3 &= 200 + 100 \\
200 + 200 &= x_3 + x_4 \\
x_7 + x_4 &= x_5 + 300 \\
500 + x_5 &= 400 + x_6
\end{aligned}
$$

on rewriting the equations

$$
\begin{aligned}
x_1 - x_6 &= -200 \\
x_1 - x_2 - x_7 &= -300 \\
x_2 + x_3 &= 300 \\
x_3 + x_4 &= 400 \\
x_4 - x_5 + x_7 &= 300 \\
x_5 - x_6 &= -100
\end{aligned}
$$

The Augmented matrix for the above system is

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix}1&0&0&0&0&-1&0&:&-200\\1&-1&0&0&0&0&-1&:&-300\\0&1&1&0&0&0&0&:&300\\0&0&1&1&0&0&0&:&400\\0&0&0&1&-1&0&1&:&300\\0&0&0&0&1&-1&0&:&-100\end{bmatrix}$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix}1&0&0&0&0&-1&0&:&-200\\0&1&0&0&0&-1&1&:&100\\0&1&1&0&0&0&0&:&300\\0&0&1&1&0&0&0&:&400\\0&0&0&1&-1&0&1&:&300\\0&0&0&0&1&-1&0&:&-100\end{bmatrix}R'_2=-R_2+R_1$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix}1&0&0&0&0&-1&0&:&-200\\0&1&0&0&0&-1&1&:&100\\0&0&1&0&0&1&-1&:&200\\0&0&1&1&0&0&0&:&400\\0&0&0&1&-1&0&1&:&300\\0&0&0&0&1&-1&0&:&-100\end{bmatrix}R'_3=R_3-R_2$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix}1&0&0&0&0&-1&0&:&-200\\0&1&0&0&0&-1&1&:&100\\0&0&1&0&0&1&-1&:&200\\0&0&0&1&0&-1&1&:&200\\0&0&0&1&-1&0&1&:&300\\0&0&0&0&1&-1&0&:&-100\end{bmatrix}R'_4=R_4-R_3$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix}1&0&0&0&0&-1&0&:&-200\\0&1&0&0&0&-1&1&:&100\\0&0&1&0&0&1&-1&:&200\\0&0&0&1&0&-1&1&:&200\\0&0&0&0&1&-1&0&:&-100\\0&0&0&0&1&-1&0&:&-100\end{bmatrix} R'_5=-R_5+R_4 \quad [\mathbf{A}:\mathbf{B}]=\begin{bmatrix}1&0&0&0&0&-1&0&:&-200\\0&1&0&0&0&-1&1&:&100\\0&0&1&0&0&1&-1&:&200\\0&0&0&1&0&-1&1&:&200\\0&0&0&0&1&-1&0&:&-100\\0&0&0&0&0&0&0&:&0\end{bmatrix} R'_6=R_6-R_7$$

$$Rank\ of\ [A:B]=Rank\ of\ [A]=5\neq 7\ (Number\ of\ unknowns)$$

So, 7-5 =2 arbitrary value can be assumed to solve the above system

If $x_6=k,$

$x_5=k-100$

$x_1=k-200$

$x_2+x_7=k+100$

$x_3-x_7=-k+200$

$x_4+x_7=k+200$

Again if $x_7=l$

$x_2=k-l+100$

$x_3=-k+l+200$

$x_4=k-l+200$

is the required solution.

The above system will have infinitely many solution , Since rank of [A:B] and A =5 and which is less than no of Unknowns i.e equal to 7. You can check the answers by changing values of k and l.

Convergence of Gauss-Seidel and Gauss Jacobi Method

Consider the following matrix of order 4 and do the following.

$$A = \begin{bmatrix} 7 & 1 & -2 & 1 \\ 1 & 8 & 1 & 0 \\ -2 & 1 & 5 & -1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \quad \textbf{and} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Check whether the system is diagonally dominant or not ?

a) If no, give reason

b) If yes, check the convergence criteria for both Gauss-Seidel and Gauss Jacobi Method using suitable norm.

Solution:

The system is diagonally dominant.

Since in each row every a_{ij} > sum of all the other

Elements of the same row for i=j.

$$\mathit{Row}\ 1:\ 1+|-2|+1<7$$

$$\mathit{Row}\ 2:\ 1+0+1<8$$

$$\mathit{Row}\ 3:\ |-2|+1+|-1|<5$$

$$\mathit{Row}\ 4:\ 1+0+|-1|<3$$

Convergence of Gauss Seidel Method

The condition for Convergence of Gauss Seidel method is $P = \left\|(I+L)^{-1}U\right\|$
Where P , is any one norm out of L-1 norm , L-2 norm (Frobenius norm) and L-inf norm should be less than 1.

$$A = \begin{bmatrix} 7 & 1 & -2 & 1 \\ 1 & 8 & 1 & 0 \\ -2 & 1 & 5 & -1 \\ 1 & 0 & -1 & 3 \end{bmatrix} \qquad \tilde{A} = \begin{bmatrix} 1 & 1/7 & -2/7 & 1/7 \\ 1/8 & 1 & 1/8 & 0 \\ -2/5 & 1/5 & 1 & -1/5 \\ 1/3 & 0 & -1/3 & 1 \end{bmatrix}$$

$$A = I + L + U$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/8 & 0 & 0 & 0 \\ -2/5 & 1/5 & 0 & 0 \\ 1/3 & 0 & -1/3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1/7 & -2/7 & 1/7 \\ 0 & 0 & 1/8 & 0 \\ 0 & 0 & 0 & -1/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(I+L)^{-1}U=\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}0&0&0&0\\1/8&0&0&0\\-2/5&1/5&0&0\\1/3&0&-1/3&0\end{bmatrix}\right)^{-1}\begin{bmatrix}0&1/7&-2/7&1/7\\0&0&1/8&0\\0&0&0&-1/5\\0&0&0&0\end{bmatrix}$$

$$(I+L)^{-1}U=\begin{bmatrix}1&0&0&0\\1/8&1&0&0\\-2/5&1/5&1&0\\1/3&0&-1/3&1\end{bmatrix}^{-1}\begin{bmatrix}0&1/7&-2/7&1/7\\0&0&1/8&0\\0&0&0&-1/5\\0&0&0&0\end{bmatrix}$$

$$(I+L)^{-1}U=\begin{bmatrix}0&0.1429&-0.2857&0.1429\\0&-0.0179&0.1607&-0.0179\\0&0.0607&-0.1464&-0.1393\\0&-0.0274&0.0464&-0.0940\end{bmatrix}$$

$$L-1\ norm=P_1=\left\|(I+L)^{-1}U\right\|=0.6393$$

$$L-2\ norm=Frobenius\ norm=P_2=\left\|(I+L)^{-1}U\right\|=0.4530$$

$$L-\infty\ norm=P_\infty=\left\|(I+L)^{-1}U\right\|=0.5714$$

Convergence of Gauss Jacobi Method

The condition for Convergence of Gauss Jacobi method is

$$P = \left\| -D^{-1}(L+U) \right\|$$

Where P , is any one norm out of L-1 norm , L-2 norm (Frobenius norm) and L-inf norm should be less than 1.

$$A = L + D + U$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-D^{-1}(L+U)=-\begin{pmatrix}7&0&0&0\\0&8&0&0\\0&0&5&0\\0&0&0&3\end{pmatrix}^{-1}\left(\begin{bmatrix}0&0&0&0\\1&0&0&0\\-2&1&0&0\\1&0&-1&0\end{bmatrix}\begin{bmatrix}0&1&-2&1\\0&0&1&0\\0&0&0&-1\\0&0&0&0\end{bmatrix}\right)$$

$$-D^{-1}(L+U)=-\begin{bmatrix}1/7&0&0&0\\0&1/8&0&0\\0&0&1/5&0\\0&0&0&1/3\end{bmatrix}\begin{bmatrix}0&1&-2&1\\1&0&1&0\\-2&1&0&-1\\1&0&-1&0\end{bmatrix}$$

$$-D^{-1}(L+U)=-\begin{bmatrix}0&-0.1429&0.2857&-0.1429\\-0.1250&0&-0.1250&0\\0.4000&-0.2000&0&0.2000\\-0.3333&0&0.3333&0\end{bmatrix}$$

$$L-1\ norm=P_1=\left\|(I+L)^{-1}U\right\|=0.8583$$

$$L-2\ norm=Frobenius\ norm=P_2=\left\|(I+L)^{-1}U\right\|=0.7848$$

$$L-\infty\ norm=P_\infty=\left\|(I+L)^{-1}U\right\|=0.8000$$

Operations Count on LU- Decomposition Method

- <u>Doolittle LU Decomposition</u>
- [A] = [L][U]
- If we factorize in such a way that

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

i.e. diagonal elements of [L] are 1, then the approach is Doolittle's method.

- If

$$[A] = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ 0 & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

the approach is called Crout's method.

Algorithm for Doolittle's LU Decomposition

- Doolittle algorithm is developed using knowledge of Gauss elimination.
- Recall Gauss elimination at any step *k*

$$\left.\begin{aligned} a_{ij}^{(k)} &= a_{ij}^{(k-1)} - l_{ik}a_{kj}^{(k-1)} \\ l_{ik} &= a_{ik}^{(k-1)} / a_{kk}^{(k-1)} \end{aligned}\right\} k = 1,2,3,\ldots,n-1;\ j = k,k+1,\ldots,n;\ i = k+1,k+2,\ldots,n$$

- You can now write

$$a_{ij}^{(k)} - a_{ij}^{(k-1)} = -l_{ik}a_{kj}^{(k-1)}$$

$$\text{i.e., } a_{ij}^{(k)} - a_{ij}^{(k-2)} = -l_{ik}a_{kj}^{(k-1)} - l_{i(k-1)}a_{(k-1)j}^{(k-2)}$$

$$\text{Extending, } a_{ij}^{(k)} - a_{ij} = -\sum_{m=1}^{k} l_{im}a_{mj}^{(m-1)}$$

$$\text{or, } a_{ij} = a_{ij}^{(k)} + \sum_{m=1}^{k} l_{im}a_{mj}^{(m-1)};\ i = k+1,k+2,\ldots,n;\ j = k,k+1,\ldots,n$$

- This is nothing but

$$[A]=[L][U]$$

$$\Rightarrow a_{ij}^{(k)} = a_{ij} - \sum_{m=1}^{k} l_{im} a_{mj}^{(m-1)}$$

where $l_{ij} \rightarrow$ elements of lower triangular matrix

$$\text{such that } l_{ij} = \begin{cases} l_{ik} & ;\text{ for } i > k; k = 1,2,3,\dots,n-1 \\ 1 & ;\text{ for } i = j \\ 0 & ;\text{ for } i < j \end{cases}$$

- So you get

$$[L]=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{pmatrix}, [U]=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{pmatrix}$$

where $l_{21} = a_{21}/a_{11}, l_{32} = a_{32}^{(1)}/a_{22}^{(1)}$, etc.

- So the steps involved are:
  - No. of steps as in Gauss elimination

    k = 1,2,3,...,n-1
  - At any *k* , *i= k+1,k+2,...,n* and *j=k,k+1,...,n*

$$l_{kk} = 1$$
$$l_{ik} = 0; i < k$$
$$u_{ij} = a_{ij}^{(k)} = a_{ij}^{(k-1)} - l_{ik} a_{kj}^{(k-1)}$$
$$u_{ij} = 0$$
$$l_{ik} = a_{ik}^{(k-1)} / a_{kk}^{(k-1)} ; i > k$$

- Forward substitution for *c*

$$c_1 = b_1$$
$$c_i = b_i - \sum_{m=1}^{k-1} l_{im} c_m ; i = 2, 3, \ldots, n$$

- Back substitution for *x*

$$x_n = c_n / u_{nn}$$
$$x_{n-1} = \frac{c_{n-1} - u_{(n-1)n} x_n}{u_{(n-1)(n-1)}}$$
$$\text{In genral, } x_i = \frac{c_i - \sum_{j=i+1}^{n} u_{ij} x_j}{u_{ii}} ; i = (n-1), (n-2), \ldots, 2, 1$$

Operation Count:

Let A be square matrix of order n. To find LU

- Reduce A in to REF, we get U

This requires

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots\cdots & a_{2n} \\ . & & & & \\ . & & & & \\ a_{n1} & a_{n1} & \cdots & \cdots\cdots & a_{nn} \end{bmatrix}$$

Fix first row

Note: To covert first element of each row below a_{11} to zero no operation required. Similarly after first set of row transformations to convert elements below a_{22} to zero no operation required and so on.

Division Count

- At the first set of operations for each row from second row onwards required 1 division. There are n-1 rows hence n-1 divisions are required .
- At the second set of operations for each row from Third row onwards required 1 division. There are n-2 rows hence n-2 divisions are required .
- At the Third set of operations for each row from fourth row onwards required 1 division. There are n-3 rows hence n-3 divisions are required .
- And so on
- At the (n-1)th set of operations 1 division. There will be 1 only one row left hence 1 division is required .

Hence total number of divisions are :

$$(n-1)+(n-2)+(n-2)+\text{.........}+2+1=\sum_{k=1}^{n-1} k=\frac{n(n-1)}{2}$$

Note: This is sum of n-1 natural numbers

Multiplication Count

- At the first set of operations to compute each element in second row(from second column onwards) requires 1 multiplication. Since there are n-1 elements in second row it requires n-1 multiplications. Similarly for each row requires n-1 multiplication

  Total multiplications at this stage $=(n-1)+(n-1)+(n-1)+\ldots\ldots\ldots+(n-1)=(n-1)(n-1)$

- At the second set of operations to compute each element in third row (from third column onwards) requires 1 multiplication. Since there are n-2 elements in third row it requires n-1 multiplications. Similarly for each row requires n-2 multiplication

  Total multiplications at this stage $=(n-2)+(n-2)+(n-2)+\ldots\ldots\ldots+(n-2)=(n-2)(n-2)$

- At the Third set of operations= $(n-3)+(n-3)+(n-3)+\ldots\ldots\ldots+(n-3)=(n-3)(n-3)$

- And so on

- At the (n-1)th set of operations. There will be 1 only one row left hence 1 multiplication is required .

Hence total number of multiplications are :

$$(n-1)^2+(n-2)^2+(n-3)^2+\ldots\ldots\ldots+1.1=\sum_{k=1}^{n-1}(n-k)^2$$

$$=\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$$

Addition Count

- At the first set of operations to compute each element in second row(from second column onwards) requires 1 addition. Since there are n-1 elements in second row it requires n-1 addition s. Similarly for each row requires n-1 multiplication

  Total addition s at this stage = $(n-1)+(n-1)+(n-1)+\dots\dots\dots+(n-1)=(n-1)(n-1)$

- At the second set of operations to compute each element in third row (from third column onwards) requires 1 multiplication. Since there are n-2 elements in third row it requires n-1 addition s. Similarly for each row requires n-2 addition

  Total addition s at this stage = $(n-2)+(n-2)+(n-2)+\dots\dots\dots+(n-2)=(n-2)(n-2)$

- At the Third set of operations= $(n-3)+(n-3)+(n-3)+\dots\dots\dots+(n-3)=(n-3)(n-3)$

- And so on

- At the (n-1)th set of operations. There will be 1 only one row left hence 1 addition is required .

Hence total number of addition s are :

$$(n-1)^2+(n-2)^2+(n-3)^2+\dots\dots\dots+1.1=\sum_{k=1}^{n-1}(n-k)^2$$

$$=\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$$

Total number of operations:

=Additions +Multiplications +Divisions

$$= 2\left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}\right) + \frac{n(n-1)}{2}$$

$$= \frac{2n^3}{3} - \frac{2n^2}{2} + \frac{2n}{6} + \frac{n^2}{2} - \frac{n}{2}$$

$$= \frac{2n^3}{3} - \frac{n^2}{2} - \frac{n}{6}$$

- <u>Number of operations</u>
- In Doolittle's algorithm,
  - to evaluate l_{ik} for any k^{th} step for $i > k$, it takes $(n-k)$ operations.
  - To evaluate $u_{ij} = a_{ij}^{(k)}$ it takes $2(n-k)^2$ operations.
  - In $(n-k)$ elimination steps to form [L] and [u], it takes, $\sum_{p=1}^{n-1}(n-p)(2n-2p+1)$

$$= \frac{n}{6}(n-1)(4n+1) = \frac{2}{3}n^3 - \frac{n^2}{2} - \frac{n}{6}$$

- To perform forward substitution,

  as $c_1 = b_1$ (No operation required)

$$c_i = b_i - \sum_{m=1}^{i-1} l_{im} c_m ; i = 2,3,4,\ldots,n$$

There are *2i* operations for each i.

So, no. of operations = $\sum_{i=2}^{n} 2i = n^2 - n$

- To perform backward substitution, it requires n^2 operations (as in Gauss)
- Total no. of operations = $\frac{2}{3}n^3 - \frac{n^2}{2} - \frac{n}{6} + 2n^2 - n$

$$= \frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$$

- No. of operations is same as Gauss elimination method.
- However if there are many systems involving [A], then you may need to just add the no. of operations for forward & backward substitution for each system.

Cholesky's Method

Cholesky's method

Cholesky's factorization is a decomposition of a Hermitian, Positive Definite matrix into the product of Lower Triangular matrix and its conjugate transpose which is useful for efficient numerical solutions.

Let the system be $Ax = b$ ....$\langle 1 \rangle$

We decompose $A = LL^T$ ....$\langle 2 \rangle$

Using $\langle 2 \rangle$ in $\langle 1 \rangle$, we get

$$LL^T x = b \text{........................} \langle 3 \rangle$$

We take

$$L^T x = c \text{........................} \langle 4 \rangle$$

Using $\langle 4 \rangle$ in $\langle 3 \rangle$, we get

$$Lc = b \text{........................} \langle 5 \rangle$$

Using $\langle 5 \rangle$, we get c and with $\langle 4 \rangle$ we get x.

ALGORITHM:

We can use the following algorithm to factorize a matrix by Cholesky's method.

Notations:

L_{k-1}: The $(k-1)\times(k-1)$ upper left corner of L.

a_k: The first $(k-1)$ entries in column k of A.

l_k: The first $(k-1)$ entries in column k of L^T

a_{kk}& l_{kk}: The kk entries of A and L respectively.

1. Initialize $L_1 = \sqrt{a_{11}}$

2. For $k = 2 \ldots \ldots \ldots . n$

   2.1. Solve $L_{k-1} l_k = a_k$ for l_k

   2.2. $l_{kk} = \sqrt{a_{kk} - l_k^T l_k}$

   2.3. $L_k = \begin{bmatrix} L_{k-1} & 0 \\ l_k^T & l_{kk} \end{bmatrix}$

Example: Solve the following system of equations by Cholesky's method:

$$16x_1 + 4x_2 + 4x_3 - 4x_4 = 32$$

$$4x_1 + 10x_2 + 4x_3 + 2x_4 = 26$$

$$4x_1 + 4x_2 + 4x_3 - 2x_4 = 20$$

$$-4x_1 + 2x_2 - 2x_3 + 4x_4 = -6$$

Sol:- Let $A = LL^T$

Where $L = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}$, $L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix}$

$$\begin{bmatrix} 16 & 4 & 4 & -4 \\ 4 & 10 & 4 & 2 \\ 4 & 4 & 6 & -2 \\ -4 & 2 & -2 & 4 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & l_{41} \\ 0 & l_{22} & l_{32} & l_{42} \\ 0 & 0 & l_{33} & l_{43} \\ 0 & 0 & 0 & l_{44} \end{bmatrix}$$

$l_{11}^2 = 16$, $l_{11} l_{21} = 4$, $l_{11} l_{31} = 4$, $l_{11} l_{41} = -4$

$\boxed{l_{11} = 4}$ $\boxed{l_{21} = 1}$ $\boxed{l_{31} = 1}$ $\boxed{l_{41} = -1}$

$(l_{21})^2 + (l_{22})^2 = 10 \Rightarrow \boxed{l_{22} = 3.}$

$l_{21} l_{31} + l_{22} l_{32} = 4 \Rightarrow \boxed{l_{32} = 1}$

$l_{21} l_{41} + l_{22} l_{42} = 2 \Rightarrow \boxed{l_{42} = 1}$

$(l_{31})^2 + (l_{32})^2 + (l_{33})^2 = 6 \Rightarrow \boxed{l_{33} = 2}$

$l_{31} l_{41} + l_{32} l_{42} + l_{33} l_{43} = -2$

$$\boxed{l_{43} = -1}$$

$$(l_{41})^2 + (l_{42})^2 + (l_{43})^2 + (l_{44})^2 = 4$$

$$\boxed{l_{44} = 1}$$

$$AX = B \Rightarrow L\underbrace{L^T X}_{Y} = B \quad \therefore LY = B \text{ where } Y = L^T X$$

Consider $LY = B$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ 20 \\ -6 \end{bmatrix}$$

$$4y_1 = 32 \Rightarrow y_1 = 8$$

$$y_1 + 3y_2 = 26 \Rightarrow y_2 = 6$$

$$y_1 + y_2 + 2y_3 = 20 \Rightarrow y_3 = 3.$$

$$-y_1 + y_2 - y_3 + y_4 = -6 \Rightarrow y_4 = -1$$

Consider $L^T x = C$

$$\begin{bmatrix} 4 & 1 & 1 & -1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ 3 \\ -1 \end{bmatrix}$$

$x_4 = -1$

$2x_3 - x_4 = 3 \Rightarrow x_3 = 1$

$3x_2 + x_3 + x_4 = 6 \Rightarrow x_2 = 2$

$4x_1 + x_2 + x_3 - x_4 = 8 \Rightarrow x_1 = 1$

$$\Rightarrow X = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

Example for Vector Space

Show that the set $p_n(t)$ of all polynomials of degree $\leq$ n of the form $p(t) = a_0 x^n + a_1 x^{n-1} + \ldots\ldots + a_n$ where $n \in Z^+$ along with the zero polynomial is a vector space over the field of real numbers.

Solution : For all p, q, r polynomials of degree less than or equal to n

Let $p(x) = \sum_{k=0}^{\infty} p_k x^k$, $q(x) = \sum_{k=0}^{\infty} q_k x^k$, $r(x) = \sum_{k=0}^{\infty} q_k x^k$

$p+q$ is a polynomial of degree less than or equal to n i.e $\sum_{k=0}^{\infty} p_k x^k + \sum_{k=0}^{\infty} q_k x^k = \sum_{k=0}^{\infty} (p_k + q_k) x^k$

$p+q = q+p$, $(p+q)+r = p+(q+r)$

$p+0 = 0+p = p$

For real numbers a, b ap is a polynomila of degree less than or equal to n

$1\ p = p$, $a(p+q) = ap + aq$, $(a+b)p = ap + bp$

Hence given polynomial is a vectorspace

If V is a finite dimensional vector space over the field F and S = $v_1, v_2, . . . , v_m$ is a subset of elements of V such that Span(S) = V , what can be said about the linear independence of the elements in S? Provide proper justifications.

Solution:

Let the dimension of V=n.

Case 1:

If m=n, The set S is a basis and hence the vectors are linearly independent

Case 2:

If m>n, The vectors are linearly dependent

Problems on Rank Nullity Theorem

Let V and W be vector spaces of dimensions n and m respectively over $\mathbb{R}$ and $T : V \to W$ be a linear transformation. Aravind starts with the basis $\{u_1, u_2, \ldots, u_r\}$, $(r < n)$ of $\mathrm{Ker}(T)$ and adds a few more elements of V namely, $\{u_{r+1}, u_{r+2}, \ldots, u_n\}$ such that the combined set $\{u_1, u_2, \ldots, u_n\}$ becomes a basis for V. How should Aravind proceed to prove that $\mathrm{Range}(T)$ has dimension $n - r$, in order that the rank-nullity theorem is satisfied? Assumptions, if any, are to be stated clearly.

Solution:

Let $b \in Range(T)$and $v \in V$ and let $T(v) = b$. Since $\{u_1, u_2, \ldots, u_r, u_{r+1} \ldots \ldots, u_n\}$ is a basis for V, we can write

$$v = \sum_{i=1}^{n} \alpha_i u_i = \sum_{i=1}^{\gamma} \alpha_i u_i + \sum_{i=r+1}^{n} \alpha_i u_i .$$

Hence,

$$b = T(v) = T(\textstyle\sum_{i=r+1}^{n} \alpha_i u_i)$$

since $T(u_i)_{i=1...r} = 0$.

Also, since the set $\{u_i\}_{r+1}^{n}$ is linearly independent,

$\{T\{u_i\}_{i=r+1}^{n}\}$ *are also linearly independent*.

Therefore, the dimension of $Range(T) = n - r$.

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(a, b, c, d) = (2a + 4b + c - d, 3a + b - 2c, a + 5c + 4d)$$

Verify the rank-nullity theorem and find out if there exists integers a, b, d such that $(a, b, 7, d) \in \text{Ker}(T)$

Solution:

The given transformation can be represented by the matrix

$$A = \begin{bmatrix} 2 & 4 & 1 & -1 \\ 3 & 1 & -2 & 0 \\ 1 & 0 & 5 & 4 \end{bmatrix}$$

Reducing it in to REF $A = \begin{bmatrix} 1 & 0 & 0 & \frac{41}{59} \\ 0 & 1 & 0 & -\frac{45}{59} \\ 0 & 0 & 1 & \frac{39}{59} \end{bmatrix} \Rightarrow Rank(A) = 3$

The solution for Rx=0 gives x = c$\begin{bmatrix} -41/59 \\ 45/59 \\ -39/59 \\ 1 \end{bmatrix}$ where c is a constant.

Therefore Nullity =1.

So rank + Nullity = 4 = No. of columns.

Since the nullspace contains elements of the form c$\begin{bmatrix} -41/59 \\ 45/59 \\ -39/59 \\ 1 \end{bmatrix}$, it will not contain any elements of the form $\begin{bmatrix} a \\ b \\ 7 \\ d \end{bmatrix}$.

Let P be the vector space of polynomials of degree less than or equal to 4 over R. Find whether

$W = (1 - x, x - x^2, x^2 - x^3, x^4)$ can span P?

We find

$$Span(W) = c_1(1-x) + c_2(x-x^2) + c_3(x^2-x^3) + c_4(x^4)$$

$$= c_4x^4 - c_3x^3 + (c_3-c_2)x^2 + (c_2-c_1)x + c_1$$

Equating it with a general degree 4 polynomial,

$$A x^4 + B x^3 + C x^2 + D x + E = c_4 x^4 - c_3 x^3 + (c_3 - c_2) x^2 + (c_2 - c_1) x + c_1$$

We get the following matrix equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \\ E \end{bmatrix}$$

The rank of the matrix =4. Hence the W will not span P.

Let $T : R3 \rightarrow R3$ be defined by
$T(x1,x2,x3)=(x1-x2+2x3,2x1+x2,-x1-2x2+2x3)$ then,

i. Show that T is a linear transformation.

ii. What are the conditions on a, b, c such that (a, b, c) is in the null space of T. Also, find the nullity of T.

Solution :

i. To Prove T is a Linear Transformation

We need to Check the following conditions

$T(x+y)=T(x)+T(y) \quad \forall x,y \in R3$

$T(cx)=c\,T(x) \qquad \forall x \in R3$ and for any scalar 'c' $\in R$

Let x=(x1,x2,x3) and y=(y1,y2,y3)

Consider

T(x+y)=T[(x1,x2,x3)+(y1,y2,y3)]

=T[(x1+y1,x2+y2,x3+y3)]

=[{(x1+y1)-(x2+y2)+2(x3+y3)},{2(x1+y1)+(x2+y2)},

{-(x1+y1)−2(x2+y2)+2(x3+y3)}]

=[(x1−x2+2x3)+(y1−y2+2y3),(2x1+x2)+(2y1+y2),

(−x1−2x2+2x3)+(−y1−2y2+2y3)]

=[(x1−x2+2x3),(2x1+x2),(−x1−2x2+2x3)]+

[(y1−y2+2y3),(2y1+y2),(−y1−2y2+2y3)]

=T[(x1,x2,x3)]+T[(y1,y2,y3)]

T(x+y)=T(x)+T(y)

Now $T[cx]=T[c(x1,x2,x3)]$

$=T[(cx1,cx2,cx3)]$

$=[(cx1-cx2+2cx3),(2cx1+cx2),(-cx1-2cx2+2cx3)]$

$=[c(x1-x2+2x3),c(2x1+x2),c(-x1-2x2+2x3)]$

$=c[(x1-x2+2x3),(2x1+x2),(-x1-2x2+2x3)]$

$=cT[(x1,x2,x3)]$

$T[cx]=cT[x]$

ii.

The conditions on a, b, c such that (a, b, c) is in the null space of T is given by $T(a,b,c)=(0,0,0)$

$$\left[(a-b+2c),(2a+b),(-a-2b+c)\right]=(0,0,0)$$

$$\begin{aligned} a-b+2c&=0\\ 2a+b+0c\;\;&=0\\ -a-2b+2c&=0 \end{aligned}$$

To solve the homogeneous equation, We write

$$[A:B]=\begin{bmatrix} 1 & -1 & 2 & : & 0\\ 2 & 1 & 0 & : & 0\\ -1 & -2 & 2 & : & 0 \end{bmatrix}$$

$$[A:B]=\begin{bmatrix}1 & -1 & 2 & : & 0\\ 0 & 3 & -4 & : & 0\\ 0 & -3 & 4 & : & 0\end{bmatrix} \begin{matrix} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{matrix} = \begin{bmatrix}1 & -1 & 2 & : & 0\\ 0 & 3 & -4 & : & 0\\ 0 & 0 & 0 & : & 0\end{bmatrix} R_3' = R_3 + R_1$$

-

Nullity of T=n=1=Dim(N(T))

Thus we have, a-b+2c=0, 3b-4c=0
Setting , c=k, b=4k/3 and a=-2k/3.
Therefore (a,b,c) ϵNull space if c=k, b=4k/3 and a=-2k/3.

From rank nullity theorem:
Rank(T)+Nullity of (T)=Dim(R3)
2+1=3
Hence, rank nullity theorem verified.

Verify Rank Nullity Theorem (For Traffic Problem)

Let V be a finite dimension and let F: $R^n \to R^m$ be linear.
Then Dim(V)=dim(Im(F))+dim(Ker(F))
=Rank(F)+Nullity(F)

In the above traffic flow problem matrix A: $R^7 \to R^6$

Dimension of (R7)=7
Rank of the matrix (A) =5
Nullity =dimension of the Null space=dimension of Solution basis of the corresponding homogeneous system AX=0

$$[A,B]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

Echelon form of the system

$$[A,B]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since rank = 5, number of free variables = 2, no. of arbitrary values can be assumed to find the solution

$$\text{Let } x_6 = k \text{ and } x_7 = l,$$

$$\Rightarrow x_5 = k, x_4 = k - l, x_3 = -k + l, x_2 = k - l, x_1 = k$$

$$\text{Then the solution } X = \begin{bmatrix} k \\ k-l \\ -k+l \\ k-l \\ k \\ k \\ l \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + l \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \textit{The Solution Basis} = \{(1,1,-1,1,1,1,0),(0,-1,1,-1,0,0,1)\}$$

dimension of the solution of the homogeneous system = 2=Nullity

Since dim(A)=Rank(A)+Nullity $\Rightarrow$ 7=5-2

Hence, Rank-Nullity Theorem is verified.

Construct, if possible, a linear transformation $T : V \rightarrow W$, where $\dim(V)$ is of the form $n^2 + n$ (where $n > 11$), $\dim(\text{Range}(T))$ is of the form $m(m + 3)$ and $\text{Nullity}(T)$ is of the form $2k^2 + 1$ for a suitable choice of n,m and k. Justify.

Solution: Using rank – nullity theorem, $\text{rank} = \dim(V) - \text{nullity}(T) = (n^2 + n) - (2k^2 + 1)$

Also rank=dim(Range(T))

Hence $(n^2 + n) - (2k^2 + 1) = m(m + 3)$ (where $n > 11$)

$m(m + 3) + (2k^2 + 1) = (n^2 + n)$

Even + odd = even.

This is not possible. Hence, we cannot construct such a transformation

THANK YOU