$$\begin{bmatrix} a_{11} & \boxed{a_{12}} & \cdots & a_{1n} \\ \vdots & & & \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \boxed{} & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc} & e & & \\ \end{array}$$

$$= \begin{bmatrix} 1 & \boxed{3} & -2 & 6 \\ 2 & 0 & -1 & -2 \\ 3 & 1 & \boxed{2} & 4 \\ -5 & 7 & -4 & 8 \end{bmatrix} \quad d,\ (3,3),\ (4,3)$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 4 & 5 \end{bmatrix}; \quad M_1 : \begin{bmatrix} -21 & +2 & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

Minor
↓
Cofactor

d

$$C_1 : \begin{bmatrix} - & - & e \\ \alpha & \beta & f \\ - & - & g \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$D = 3e + 4f + 5g$$

$$D = 2\cdot\alpha + (-1)\beta + 4\cdot f$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 0 & 6 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & -1 & 4 & 1 \\ 3 & 0 & 6 & 7 \end{array}\right]$$

$$R_2 \to R_2 - 2R_1 ; \quad R_3 \to R_3 - 3R_1$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -5 & -2 & -7 \\ 0 & -6 & -3 & -5 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ & \cdot & -1 & 1\cdot 6 \end{array}\right] \quad R_2 \to R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0.4 & 1.4 \\ 0 & -6 & -3 & -5 \end{array}\right] \quad R_2 \to \frac{R_2}{-5}$$

$$R_3 \to R_3 + 6R_2$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0.4 & 1.4 \\ 0 & 0 & \boxed{-0.6} & 3.4 \end{array}\right]$$

$$r(A) = 3, \quad r(\tilde{A}) = 3$$

$$-0.6\, x_3 = 3.4 \quad \text{or} \quad x_3 = \frac{-3.4}{.6}$$

x_2 also

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & -1 & 4 & 3 \\ \boxed{4 \quad 1 \quad 7} & & & 7 \end{array}\right]$$

$$R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 - 4R_1$$

$$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -7 & -5 & -9 \\ 0 & -7 & -5 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -7 & -5 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad r(A) = r(\tilde{A}) = 2$$

$\to$ 1

$$\begin{array}{c} \quad \alpha \quad \beta \\ \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \end{array}$$

Let $x_3 = \alpha = 1$

$\alpha = 0.2$

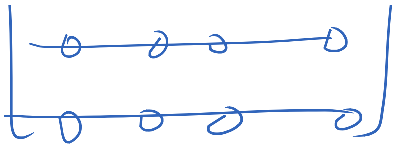
$\alpha = \pi$

$x_1 \sim \alpha$

$$-7x_2 - 5x_3 = -9$$

$$-7x_2 = -9 + 5\alpha$$

$$x_2 = \frac{-9 + 5\alpha}{-7}$$



$x_1 \sim \alpha$ — -7

$x_1 = 4 - 2\underline{\underline{\alpha}} - 3\underline{\underline{\beta}}$

$x_2 = \underline{\underline{\alpha}}$

$x_3 = \beta$

$A_{n \times n}\, x = b$

$$\boxed{\text{rank}(A) = r < n}$$

$(n-r)$ variables arbitrary values & find the other r in terms of $(n-r)$ variables.

$$A : \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cancel{a_{21}}\ 0 & \cdots & a_{2n} \\ \vdots & & \\ \cancel{a_{n1}}\ 0 & \cdots & a_{nn} \end{bmatrix} \qquad R_j \to R_j - \frac{a_{j1}}{a_{11}} R_1 \quad \text{for } j = 2 \ldots n$$

$$\left(\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0.4003 & \boxed{-1.502} & 2.501 \end{array}\right)$$

$$R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1 \qquad \frac{.4003}{.0004} \overset{45}{=} 1001$$

$$\begin{array}{cc|c} 0.0004 & 1.402 & 1.406 \\ 0 & & \end{array}$$

$-1.502 - 1001\,(1.402)$

$-1.502 - 1403$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & a_{1\,n+1} \\ 0 & \boxed{a_{22}} & \cdots & a_{2n} & a_{2\,n+1} \\ 0 & a_{32} & \cdots & a_{3n} & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} & a_{n\,n+1} \end{array}\right]$$

$a_{i,n+1} = b_i$;

$- \quad a_{22} - \frac{a_{21}}{a_{11}} a_{12}$

$- \quad a_{23} - \frac{a_{21}}{a_{11}} a_{13}$

Division, $\boxed{\text{Addition, Subtr}}$ & Multiplication

$(n-1)n$. $1(n)$.

$R_j \to R_j - \frac{a_{j1}}{a_{11}} R_1$

for $j = 2$ to n

2^{nd} row — 1 division $\quad (n-1)\,n \; 1\,A$

3^{rd} row — 1 div $\quad (n-2)(n-1)$

$\vdots \quad (n-k)\,(n-k+1)$

n^{th} row — 1 div

$\boxed{(n-1) \text{ division}}$

$(n-2)$

$\cdots \quad (n \cdot k)$

$\therefore$ Total no. of divisions $= (n-1) + (n-2) + \cdots + 1$

$$D = \frac{(n-1) \cdot n}{2}$$

$$M + A = 2 \sum_{k=1}^{n-1} (n-k)\,(n-k+1)$$

Let $n - k = s$, when $k = 1$, $s = n-1$

when $k = n-1$, $s = 1$

$$M + A = 2 \sum_{s=1}^{n-1} (s)\,(s+1)$$

$$= 2\left(\sum_{s=1}^{n-1} s^2 + s\right)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$M+A = 2\left[\frac{(n-1)(n)(2n-1)}{6} + \frac{(n-1)n}{2}\right]$$

$$\text{Total} = D+M+A = \frac{(n-1).n}{2} + 2\left[\frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2}\right]$$

$$= \frac{(n-1).n}{2}\left[1+2+\frac{2}{3}(2n-1)\right]$$

$$= \frac{(n-1).n}{2}\left[\frac{4n+7}{3}\right]$$

$$= \frac{1}{6}(n^2-n)(4n+7)$$

$$= \frac{1}{6}\left[4n^3+3n^2-7n\right] = \frac{4}{6}n^3+\frac{n^2}{2}-\frac{7n}{6}$$

$$= \frac{2}{3}n^3 + \boxed{\frac{n^2}{2} - \frac{7}{6}n}$$

$$\sim O\left(\underline{2/3\, n^3}\right)$$

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & \cdots & c_{1n} & \alpha_1 \\ 0 & c_{22} & \cdots & c_{2n} & \alpha_2 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \boxed{c_{nn}} & \alpha_n \end{array}\right]$$

	D	M	A
x_n	1	0	0
x_{n-1}	1	1	1
x_{n-2}	1	2	2

$L \quad 0 \quad \cdots \quad \boxed{c_{nn} \mid d_n}$

x_{n-1}	1	1	1
x_{n-2}	1	2	2
$\vdots$			
x_1	1	$n-1$	$n-1$
	n	$\frac{(n-1)n}{2}$	$\frac{(n-1)n}{2}$

$\therefore$ Total for BS $= n + \frac{(n-1)n}{2} \cdot 2$

$= n + n^2 - n = n^2$

$O(n^2)$

Problem: A fixed matrix $A_{n \times n}$ $n \sim 10000$, $k = 25$

✓ $A_{n\times n} x_1 = b_1$. $A_{n\times n} x_2 = b_2$ $\cdots$ $A_{n \times n} x_k = b_k$

11 min   11 m

FE $\sim 25 \times 11$ min   11 min

B.S 0.1×25

$\boxed{275 \text{ minutes}}$

276 min

LU decomposition

$A = LU$ (L is lower triangular, U is upper triangular)

$$L = \begin{array}{c} \\ 1 \\ 2 \\ \\ n-1 \end{array} \begin{pmatrix} 1 & & & 0 \\ l_{21} & 1 & & \\ l_{31} & l_{32} & \ddots & \\ l_{n1} & l_{n2} \cdots & l_{nn-1} & 1 \end{pmatrix}$$

Lower triangular

$\frac{(n-1) \cdot n}{2}$

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & & \ddots & \\ 0 & \cdots & & u_{nn} \end{pmatrix} \begin{array}{c} n \\ \vdots \\ 2 \\ 1 \end{array}$$

upper triangular

$\frac{n(n+1)}{2}$

u_{nn}), $\frac{n(n+1)}{2}$

$A_{n\times n} = n^2$ $\qquad \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$

$$\begin{bmatrix} 2 & 3 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l_{21} \; 5.5 & 1 \end{bmatrix} \begin{bmatrix} \cancel{u_{11}}\; 2 & \cancel{u_{12}}\; 3 \\ 0 & \cancel{u_{22}}\; -12.5 \end{bmatrix}$$

$A = LU$ & store L & U values

$Ax = b \Rightarrow LUx = b$

Set $\underline{Ux = y}$, then $\underset{O(n^3/3)}{L}\,\underset{O(n^2)}{y} = b$ $\sim y$

$Ux = y \rightarrow O(n^2)$

$$\underset{O(n^3/3)}{\boxed{L}}, \underset{O(n^3/3)}{\boxed{U}}, \quad \underset{O(n^2)}{y}, \quad \underset{O(n^2)}{x}$$