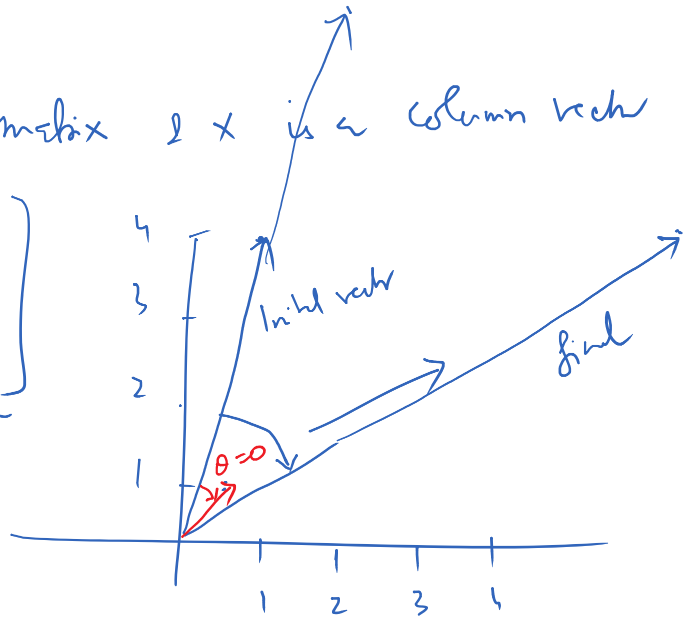
A is a square matrix & x is a column vector

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 11 \end{bmatrix}$$



$$X = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$2x = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

X – vector

λX → Scaling of the vector in the same direction

$$\begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 1 \end{bmatrix}$$

1. Just do the Scaling & no change in angle :

$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} x$ → Scaling no change in angle.

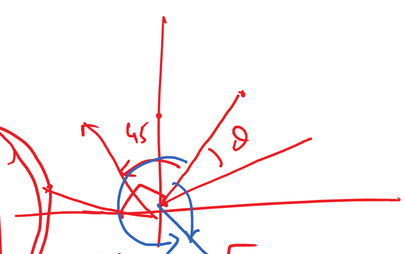
$\alpha > 1$, Stretching

$\alpha < 1$ Contracting

2. Just do the rotation no scaling

Orthogonal matrix

$$\boxed{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$



$\left\| \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\| = \sqrt{17}$

$\theta = 45°$ → $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ \dots & \dots \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} -3/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}$

$\theta = 45^\circ$ $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \rightarrow \begin{bmatrix} -3/\sqrt{2} \\ 5/\sqrt{2} \end{bmatrix}$

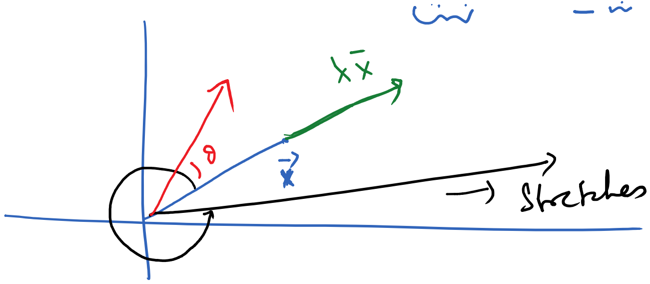
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ $\sqrt{17}$

$\left\| \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\| = \sqrt{17}$

$Ax \rightarrow$ rotates & stretches x

$(\lambda, \vec{x}) \rightarrow$ eigenvalue eigenvector pair

$Ax = \lambda x$ $\quad x \neq 0$



$Ax = \lambda x$

$A_{n \times n}$ → particular

x any vector → $= \lambda x$ nth

scaling

---

nth degree polynomial in λ

$\text{Det}(A - \lambda I) = 0$

all real roots

complex roots appear as conjugate pairs.

$2+3i \quad 2-3i$ is also a root

$z \quad \bar{z}$

$z : a+ib \qquad \bar{z} = a - ib$

$x^3 = 1, \quad (x-1)(1+x+x^2) = 0$

$x = 1, \quad 1+x+x^2 = 0$

$\to \frac{1+\sqrt{3}i}{2}$

$\hookrightarrow = \frac{1-\sqrt{3}i}{2}$

Gerschgorin's theorem

$$Ax = \lambda x$$

$\lambda \to$ eigenvalue , $X \to$ corr. eigenvector

X will have components $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$|z| = \sqrt{a^2+b^2}$

$z = a + ib$

$$\max_{1 \le i \le n} |x_i|$$

Let x_j be that component s.that $|x_j| \geq |x_i|$ $\forall\, i,\ 1 \le i \le n$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}$$

$$a_{j1}x_1 + a_{j2}x_2 \cdots + a_{jn}x_n = \lambda x_j$$

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jj}x_j + \cdots + a_{jn}x_n = \lambda x_j$$

$$\therefore \left|\underbrace{(a_{jj} - \lambda)\, x_j}_{|x_j|}\right| = \left|\underbrace{-a_{j1}x_1 - a_{j2}x_2 - a_{jj-1}x_{j-1} - a_{jj+1}x_{j+1} \cdots - a_{jn}x_n}_{|x_i|}\right|$$

$|a+b| \le |a| + |b|$

$\therefore$ equality

$|a+b| \le |a| + |b|$

triangle inequality

$|x_j|$

$|x_j|$

$$|a_{jj} - \lambda| \left|\frac{x_j}{x_j}\right| \le |a_{j1}| \left|\frac{x_1}{x_j}\right| + |a_{j2}| \left|\frac{x_2}{x_j}\right| + \dots \; |a_{jn}| \left|\frac{x_n}{x_j}\right|$$

$$x \quad \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}$$

$$|a_{jj} - \lambda| \le |a_{j1}| + |a_{j2}| + \dots \; |a_{jn}|$$

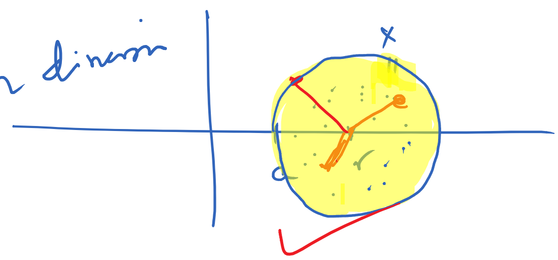
$|a_{jj}|$ will not be present

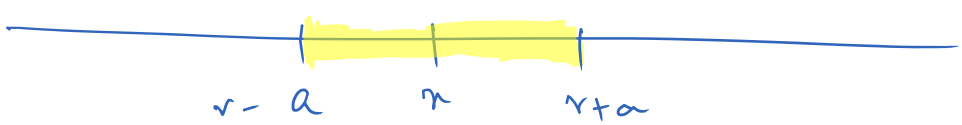
Interpretation of $|\underline{x} - \overset{\text{given}}{\vec{r}}| \le \overset{\text{given}}{\vec{a}}$

Sphere

Hypersphere in n dimension

$n \geq 4$





Complex: $z = a + ib$ a & b are real nos

eg $2.7 + 6.7i$ is a complex no. $i = \sqrt{-1}$

$\bar{z} = a - ib$

$z\bar{z} = (a+ib)(a-ib) = a^2+b^2 = |z|^2$

$z\bar{z} = \bar{z}z$

$\overline{f(z)} = \bar{z}_1$

Defn. (1) A is Hermitian : λ real $\bar{A}^T = A$ (Sym)

(2) Skew Hermitian $\lambda = 0$ or pure img. $\bar{A}^T = -A$ (Sk Sym)

(3) A is Unitary $|\lambda| = 1$ $\bar{A}^T = A^{-1}$ (orthog)

(1) & (2). Start with an eigenvalue & eigenvector pair

$$Ax = \lambda x$$

$$\bar{x}^T A x = \bar{x}^T \lambda x$$

$\bar{x}^T x$: $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}$

$$\bar{x}^T = [\bar{x}_1, \bar{x}_2 \ldots \bar{x}_n]$$

$$\therefore \bar{x}^T x = [\bar{x}_1, \bar{x}_2 \ldots \bar{x}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\therefore \bar{x}^T x = [\ldots] \begin{bmatrix} \vdots \\ x_n \end{bmatrix}$

$= \bar{x}_1 x_1 + \bar{x}_2 x_2 \ldots + \bar{x}_n x_n$

real no. real no ... real no

$\therefore \bar{x}^T x$ is a real no.

$x \neq 0$ as x is an eigen vector.

$\therefore \bar{x}^T \neq 0 \quad \therefore \bar{x}^T x \neq 0$

$$\bar{x}^T A x = \lambda \bar{x}^T x$$

$(AB)^T = B^T A^T$ $\quad \therefore \lambda = \dfrac{\bar{x}^T A x}{\bar{x}^T x}$ $\quad \bar{x}^T A x = \text{scalar}$

$$\lambda = \lambda^T = \frac{(\bar{x}^T A x)^T}{\bar{x}^T x} = \frac{x^T A^T (\bar{x}^T)^T}{\bar{x}^T x}$$

$$= \frac{x^T A^T \bar{x}}{\bar{x}^T x} = \overline{\left(\frac{\bar{x}^T \bar{A}^T x}{\bar{x}^T x}\right)} = \overline{\frac{\bar{x}^T A x}{\bar{x}^T x}}$$

$$= \bar{\lambda}$$

$\cancel{a} + ib = \cancel{a} - ib$

$2ib = 0 \Rightarrow b = 0$

$a + ib$ becomes $a \Rightarrow \lambda$ real.

$\lambda = -\bar{\lambda}, \quad a + ib = -a + ib$

$\Rightarrow 2a = 0 \Rightarrow a = 0$

$\Rightarrow$ If $b \neq 0$ purely imaginary

$b = 0$ $\quad \lambda = 0$

(c) A is Unitary $\quad \bar{A}^T = A^{-1}$

$$\overline{Ax} = \overline{\lambda x} = \bar{A}\bar{x} = \bar{\lambda}\bar{x}$$

$$(\bar{A}\bar{x})^T = (\bar{\lambda}\bar{x})^T$$

$$\bar{x}^T \bar{A}^T = \bar{x}^T \bar{\lambda} = \bar{\lambda}\bar{x}^T$$

$$\bar{x}^T \bar{A}^T \cdot Ax = \bar{\lambda}\bar{x}^T \cdot \lambda x$$

$$\bar{x}^T \cancel{x} = \lambda\bar{\lambda}\bar{x}^T \cancel{x}$$

A

$2 + \frac{1}{2} i$

$$\lambda\bar{\lambda} = 1$$

$$|\lambda|^2 = 1 \Rightarrow |\lambda| = 1$$

$$A = LU$$

$A \to (PA)$ Preconditioner matrix

$$Ax = b \quad \boxed{PA}x = \boxed{Pb}$$

$$\hat{A}x = \hat{b}$$