Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in Data Science and Engg.

II Semester 2020-21

Course Number	DSECL ZC416
Course Name	Mathematical Foundation for Data Science
Nature of Exam	Open Book
Weightage for grading	30%
Duration	120 minutes
Date of Exam	04/07/2021 (10:00 - 12:00)

# Pages	3
# Questions	4

Instructions

1. All questions are compulsory
2. Questions are to be answered in the order in which they appear in this paper and in the page numbers mentioned before each of them.

Q1 Answer the following questions with justifications.

a) If $\mathbf{A}$ is a 823×823 matrix and one wishes to do a decomposition of the form $\mathbf{A} = LU$ with Doolittle's method, compute separately, the number of divisions, multiplications and additions required in the computation of L and computation of U. (3.5)

b) A student X is interested in solving the system $\mathbf{A}_{n\times n}\mathbf{x} = \mathbf{b}$ using Crout's method. However, a friend of her, Y has already worked out the LU decomposition for the matrix $P\mathbf{A}$ where P is a preconditioner matrix (typically a non-singular matrix) used in biological applications. Can X still use this information to solve the original system? If so, how many divisions, multiplications and additions would be required? If not, justify. (2)

c) Let $\mathbf{A}$ be a $m \times m$ matrix with elements $a_{jk} = j + k - \theta$ for a fixed θ such that $\pi \leq \theta \leq 2\pi$. Compute the rank of $\mathbf{A}$. (1)

d) If the systems $\mathbf{A}_{n\times n}\mathbf{x} = \mathbf{b}_1$ and $\mathbf{A}_{n\times n}\mathbf{y} = \mathbf{b}_2$ are consistent, does that mean that $\mathbf{Az} = \mathbf{b}_1 + \mathbf{b}_2$ is consistent? Justify. (1)

Q2 Answer the following questions with justifications.

a) A student enrolled in machine learning course came across a $m \times m$ matrix H which could be written in the form

$$H = \sum_{k=1}^{n} \alpha^k p_k p_k^T,$$

where $p_k \in \mathbb{R}^m$ is a column vector and $\alpha \in \mathbb{R}$ is a constant such that $2 \leq \alpha \leq 3$. Upon further investigation, it becomes evident that the set $\{p_1, p_2, \ldots, p_n\}$ is orthonormal. With this information, would the student be able to find the n eigenvalues and eigenvectors of H? Can the orthonormality be replaced by a slightly weaker condition, which would enable him to find the eigenvalues and eigenvectors? If so, what is it? If not, why? (2)

b) A professor teaching linear algebra starts with a square matrix G whose QR decomposition is given by $G = Q_0R_0$. He then defines matrices $H = R_0Q_0$ which has a QR decomposition of the form $H = Q_1R_1$ and a matrix $M = R_1Q_1$. Given this, he asks the students to answer the following (2 x 1 = 2)
   i) Do the matrices G and H have the same eigenvalues? Justify.
   ii) Prove or disprove that G and M have the same eigenvalues.

c) Given $B_{n\times n} = LU$, we find that there are n^2 degrees of freedom in B and in the right hand side we have $(n^2-n)/2$ for L and $(n^2+n)/2$ for U (using Doolittle's method) and hence totaling to n^2 degrees of freedom in the right hand side. Does the same hold for the matrix $A_{m\times n}$ with $m < n$ and having linearly independent columns with a QR decomposition of the form $A = Q_1R_1$? (1.5)

d) Let γ be the real root of a polynomial equation of degree 9 with integer coefficients. Construct the matrix (1)

$$A = \begin{pmatrix} 2 & \gamma & 0 \\ \gamma & 2 & \gamma \\ 0 & \gamma & 2 \end{pmatrix}$$

With this information, is it possible to
   i) derive all the possible values of γ so that A has all non-zero eigenvalues?
   ii) calculate the necessary condition on γ so that all the eigenvalues of A are positive?

e) Consider an $n > 122$ and a non-zero column vector $p \in \mathbb{R}^n$. If $B = pp^T$, comment on the singular values of B. In case the given data is not sufficient, you may state your assumptions in getting the singular values. (1)

Q3 Answer the following questions with justifications.

a) Given an LPP

$$\text{Min } Z = \mathbf{cx}$$

subject to

$$\mathbf{A}_{m\times n}\mathbf{x} \leq \mathbf{b} \text{ with } 2m < n$$

,

$$\mathbf{b} \geq 0, \mathbf{x} \geq 0,$$

compute the least upper bound on the number of divisions, multiplications and additions required to solve by the simplex method, assuming that a unique optimal solution is obtained after k steps excluding the initial table. (3.5)

b) Convert the following LPP to standard problem (1.5)

$$\text{Min } Z = 3x_1 + 2x_2 - 4x_3$$

subject to

$$\begin{aligned} 2x_1 - x_2 + x_3 \quad &\geq 2 \\ -4x_1 - 2x_2 - 5x_3 \quad &\leq -7 \\ x_1 \geq 0, x_2 \geq 0, x_3 &\text{ is unrestricted in sign} \end{aligned}$$

c) Without using the simplex method, solve the LPP (1.5)

$$\text{Max } Z = \sum_{j=1}^{n} j x_j$$

subject to the n conditions

$$\sum_{k=1}^{i} x_k \leq i \text{ for } 1 \leq i \leq n$$

d) How many constraints (including the non-negativity constrains) would be there in the currency arbitrage problem in case there are Q currencies in total and US \$ is one of them. Assume that there is a restriction on the maximum daily transaction limit on currencies. (1)

Q4 Answer with justifications.

a) Solve the following system of equations using the Gauss elimination method with and without partial pivoting and three digit rounding arithmetic and compare the results. (2+2)

$$\begin{aligned} 4.03x_1 + 2.19x_2 + 1.23x_3 &= -4.35 \\ 6.21x_1 + 3.61x_2 - 2.46x_3 &= -7.16 \\ 7.92x_1 + 5.11x_2 + 3.29x_3 &= 12.83 \end{aligned}$$

b) Construct, if possible, a linear transformation $T : V \to W$, where $\dim(V)$ is of the form $n^2 + n$ (where $n > 11$), $\dim(\text{Range}(T))$ is of the form $m(m + 3)$ and $\text{Nullity}(T)$ is of the form $2k^2 + 1$ for a suitable choice of n, m and k. Justify. (1.5)

c) If V is a finite dimensional vector space over the field F and $S = v_1, v_2, \ldots, v_m$ is a subset of elements of V such that $\text{Span(S)} = V$, what can be said about the linear independence of the elements in S? Provide proper justifications. (1)

d) Let P be the vector space of polynomials of degree less than or equal to 4 over $\mathbb{R}$. Find if $W = (1 - x, x - x^2, x^2 - x^3, x^4)$ can span P? (1)