MA-102: Elementary Real Analysis

Assignment - 1

1. Let $a,b\in\mathbb{R}$. Show that if $a\le b+1/n\ \forall n\in\mathbb{N}$, then $a\le b$.
2. Let A and B be two non-empty subsets of $\mathbb{R}$ such that $\sup A, \inf A$, $\sup B$ and $\inf B$ exist. Define $C=\{x+y: x\in A,\ y\in B\}$ and $D=\{x-y: x\in A,\ y\in B\}$, and prove the following: $\sup C=\sup A+\sup B$, $\inf C=\inf A+\inf B$, $\sup D=\sup A-\inf B$ and $\inf D=\inf A-\sup B$
3. Find the *sup* and *inf* of the following sets of real numbers:

(i) $\{x:(x-a)(x-b)(x-c)(x-d)<0\}$, where $a<b<c<d$

(ii) $\{x: x=2^{-p}+3^{-q}+5^{-r}\}$, where p, q and r take on all positive integer values.

4. Find the largest $\varepsilon>0$ such that S contains the $\varepsilon-$neighborhood of x_0:
   $(i)\ x_0=3/4,\ S=[1/2,1)\quad (ii)\ x_0=5,\ S=(-1,\infty)$
5. Find the limit points of the following sets and classify them as open or closed sets:

$(i)\ \left\{\frac{1}{m}+\frac{1}{n}: m,n\in\mathbb{N}\right\}\ (ii)\ \left]m-\frac{1}{2},\ m+\frac{1}{2}\right[\ (iii)\ [0,1]\cup[2,3]\ (iv)\ \left\{1,-1,\frac{1}{2},-\frac{1}{2},\frac{1}{3},-\frac{1}{3},...\right\}$

6. For $n\ge 1$, let $I_n=\left[\frac{1}{2n+1},\frac{1}{2n}\right]$ and $S=\bigcup_{n=1}^{\infty} I_n$. Then find the set of limit points, boundary points and exterior of S.
7. Prove that a set is closed if and only if it contains all its limit points.
8. Prove that: if S is closed and bounded, then *sup*S and *inf* S are both in S.
9. Prove the following by using mathematical induction:
   (i) If $s_0>0$ and $s_n=1-e^{-s_{n-1}},\ n\ge 1$, then $0<s_n<1,\ n\ge 1$.
   (ii) If $R, x_0>0$ and $x_{n+1}=\frac{1}{2}\left(\frac{R}{x_n}+x_n\right), n\ge 0$, then
   $$x_n>x_{n+1}>\sqrt{R} \text{ and } x_n-\sqrt{R}\le \frac{1}{2^n}\frac{(x_0-\sqrt{R})^2}{x_0}, n\ge 1.$$
   (iii) For what integer n, $\frac{1}{n!}>\frac{8^n}{(2n)!}$. Prove it by induction.
10. Prove that if $x_n\to a$ and $|y_n-x_n|\le\frac{1}{n}\ \forall n\in\mathbb{N}$, then $y_n\to a$.
11. Find limit of the following sequences
    $(i)\ s_n=\sqrt{n}(\sqrt{4n+3}-2\sqrt{n})\ (ii)\ s_n=\frac{2.4.6...2n}{2.5.8...(3n-1)}\ (iii)\ s_n=\frac{1}{n^5}\sum_{k=0}^{5}\binom{n}{k}2^k\ (iv)\ s_n=\sum_{k=0}^{5}\frac{k}{n^2}$
12. Find the limit inferior and limit superior of the sequences $\{a_n\}$, where
    $(i)\ a_n=\sin\frac{n\pi}{2}+\frac{(-1)^n}{n}$ (ii) $a_n=(-1)^n\left(1+\frac{1}{n}\right)$ (iii) $a_n=\frac{(-1)^n}{n^2}$
13. Check convergence of the following sequences using the Cauchy general principle.

(i) $\left\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{(-1)^{n-1}}{n}\right\}$, (ii) $\left\{1+\frac{1}{4}+\frac{1}{7}+...+\frac{1}{3n-2}\right\}$.

14. Test the following sequence for convergence and find its limit, if exists

$$(i)\, s_1 = 5, \text{ and } s_{n+1} = \sqrt{4+s_n} \quad \forall n \geq 1 \quad (ii)\ s_1 = 1 \text{ and } s_{n+1} = \frac{s_n^{\ 2}+3}{2s_n} \quad \forall n \geq 1$$

15. Show that

$$\text{(i) } \lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\dots+\frac{1}{\sqrt{n^2+n}}\right]=1 \quad \text{(ii) } \lim_{n\to\infty}\frac{(n!)^{1/n}}{n+1}\to\frac{1}{e}$$

$$\text{(iii) } a_n = \frac{3n!}{(n!)^3}, \text{ then } \left\{a_n^{1/n}\right\} \text{ converges, find its limit.}$$

$$\text{(iv) } s_n = \frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+n} \quad \forall n \in \mathbb{N}, \text{ is convergent.}$$

16. Show that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ converges to e (Euler's number).

17. Show that the following sequences are divergent:

$$(i)\left\{\sqrt{\frac{n^2+1}{5}}\right\} \qquad (ii)\quad \{n^2-n^3\} \qquad (iii)\quad \left\{\frac{c^n}{n}\right\},\ c>1.$$

18. Prove that every increasing unbounded sequence diverges to ∞.

Answers

3. $(i)\sup = d$, $\inf = a$ (ii) $\sup = 31/30$, $\inf = 0$ 4. (i) $1/4$ (ii) 6

5. $\text{(i)}\{0\}\cup\{n^{-1}: n\in\mathbb{N}\}$, neither open nor closed $\text{(ii)}\left[m-\frac{1}{2}, m+\frac{1}{2}\right]$, open $\text{(iii)}[0,1]\cup[2,3]$, closed

$\text{(iv)}\{0\}$, neither open nor closed 6. S, $\{0, 1/n: n\geq 2\}$, $(-\infty,0)\cup\left[\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+2},\frac{1}{2n+1}\right)\right]\cup(1/2,\infty)$

11. (i) $3/4$ (ii) 0 (iii) $4/15$ (iv) 0 12. (i) $-1,1$ (ii) $-1,1$ (iii) 0 14. (i) $\frac{1+\sqrt{17}}{2}$ (ii) $\sqrt{3}$