Question 1

1. Starting with the gravitational force and centripetal force balance for the two masses:
$$\frac{Gm_1m_2}{r^2} = m_1\omega^2 r_1 = m_2\omega^2 r_2$$
Simplifying, we get:
$$\frac{Gm_2}{r^2} = \omega^2 r_1, \quad \frac{Gm_1}{r^2} = \omega^2 r_2$$
$$m_1 r_1 = m_2 r_2$$
$$r_1 + r_2 = r$$
$$\frac{G(m_1+m_2)}{r^2} = \omega^2 r$$
Which gives us:
$$\omega = \sqrt{\frac{G(m_1+m_2)}{r^3}} \tag{1}$$

2. (a) The kinetic energy T of the system is:
$$T = \frac{1}{2}m_1\omega^2 r_1^2 + \frac{1}{2}m_2\omega^2 r_2^2$$
Substituting r_1 and r_2 as:
$$r_1 = \frac{m_2 r}{m_1+m_2}, \quad r_2 = \frac{m_1 r}{m_1+m_2}$$
The kinetic energy becomes:
$$T = \frac{1}{2}\omega^2 m_1 \left(\frac{m_2 r}{m_1+m_2}\right)^2 + \frac{1}{2}\omega^2 m_2 \left(\frac{m_1 r}{m_1+m_2}\right)^2$$
Simplifying further:
$$T = \frac{1}{2}\omega^2 m_1 m_2 r^2 \left(\frac{1}{m_1+m_2}\right)^2 (m_1+m_2)$$
Finally:
$$T = \frac{1}{2}\omega^2 \frac{m_1 m_2}{m_1+m_2} r^2$$
(b) The potential energy U is given by:
$$U = -\frac{Gm_1m_2}{r}$$

The total energy E of the system is given by the sum of kinetic and potential energy:

$$\begin{aligned} E &= T + U \\ &= \frac{1}{2}\omega^2 \frac{m_1 m_2 r^2}{m_1 + m_2} - \frac{G m_1 m_2}{r} \end{aligned}$$

Replacing r by $\left(\frac{G(m_1+m_2)}{\omega^2}\right)^{1/3}$, we get:

$$\begin{aligned} E &= \frac{1}{2}\omega^2 \frac{m_1 m_2}{m_1 + m_2} \left(\frac{G(m_1+m_2)}{\omega^2}\right)^{2/3} - \frac{G m_1 m_2}{\left(\frac{G(m_1+m_2)}{\omega^2}\right)^{1/3}} \\ &= \frac{1}{2} G^{2/3} \omega^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} - G^{2/3}\omega^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \\ &= -\frac{1}{2} G^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \omega^{2/3} \end{aligned}$$

$$E = -\frac{1}{2} G^{2/3} \frac{m_1 m_2}{(m_1+m_2)^{1/3}} \omega^{2/3} \tag{2}$$

The negative sign in the expression for E indicates that the binary system's total energy is negative, signifying that the bodies are bound by gravity.

Question 2

1. Larmor's formula for electromagnetic power:

$$P = \frac{1}{4\pi\epsilon_0} \frac{2}{3c^3} \dot{p}^2$$

where $\dot{p}$ represents the time derivative of the electric dipole moment.

For gravitational radiation, the lowest order radiation term is the quadrupole term. The extension of Larmor's formula to the quadrupole term gives:

$$P \propto \left(\frac{\partial^3 Q}{\partial t^3}\right)^2$$

where Q is the quadrupole moment of the mass distribution and G is analogous to $\frac{1}{4\pi\epsilon_0}$ in the equation for electromagnetic power
we have:

$$\begin{aligned} [P] &= M L^2 T^{-3} \\ [G] &= M^{-1} L^3 T^{-2} \\ [c] &= L T^{-1} \\ [\dddot{Q}] &= M L^2 T^{-3} \end{aligned}$$

Thus, the dimensions of power are related to the constants and the second time derivative of the quadrupole moment by:

$$[P] = [G]^a [c]^b \left[\frac{\partial^3 Q}{\partial t^3}\right]^2$$

$$\begin{aligned} ML^2T^{-3} &= \left(M^1 L^3 T^{-2}\right)^a \left(LT^{-1}\right)^b \left(ML^2T^{-3}\right)^2 \\ &= M^{-a+2} L^{3a+b+4} T^{-2a-b-6} \end{aligned}$$

Equating the exponents of like terms, we get the system:

$$\begin{aligned} -a + 2 = 1 &\implies a = 1 \\ 3a + b + 4 = 2 &\implies b = -5 \\ -2a - b - 6 = -3 &\implies \text{Satisfied with } a = 1 \text{ and } b = -5 \end{aligned}$$

Therefore, the power radiated is given by:

$$P = N \frac{G}{c^5} \left(\frac{\partial^3 Q}{\partial t^3}\right)^2$$

where N is the linear GR value and is equal to $\frac{32}{5}$

$$P = \frac{32}{5} \frac{G}{c^5} \left(\frac{\partial^3 Q}{\partial t^3}\right)^2 \tag{3}$$

Relation between $\omega_{\mathbf{orb}}$ and $\omega_{\mathbf{GW}}$

Due to the nature of the binary orbit, the quadrupole moment undergoes a full oscillation, returning to its initial state twice for each complete orbit. This is because the quadrupole moment depends not just on the separation of the masses but also on their positions relative to the observer, which change twice per orbit due to the binary motion.

As a direct consequence, the frequency of the gravitational waves generated by the system is twice the orbital frequency. The gravitational waves are fundamentally connected to the rate of change of the quadrupole moment, and since this rate of change completes two full cycles per orbital period, the gravitational wave frequency is doubled.

$$\omega_{\text{GW}} = 2\omega_{\text{orb}} \tag{4}$$

Question 3

1. The quadrupole moment Q and its third time derivative are given by:

$$Q = Mr_1^2 + M_2r_2^2 = \mu r^2$$
$$\frac{\partial^3 Q}{\partial t^3} = \mu r^2 \omega^3$$

where μ is the reduced mass and r is the separation between the two masses.

The power P radiated as gravitational waves is:

$$P = \frac{32}{5}\frac{G}{c^5}\mu^2 r^4 \omega^6$$

which is derived from the third time derivative of the quadrupole moment.

$$P = \frac{32}{5}\frac{G}{c^5}\mu^2 r^4 \omega^6$$

Substituting r with the expression in terms of the angular frequency ω:

$$r = \left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{1/3}$$

The power P then becomes:

$$\begin{aligned} P &= \frac{32}{5}\frac{G}{c^5}\mu^2 \left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{4/3} \omega^6 \\ &= \frac{32}{5}\frac{G}{c^5}\left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 \left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{4/3} \omega^6 \\ &= \frac{32}{5}\frac{G^{7/3}}{c^5}\frac{(m_1 m_2)^2}{(m_1 + m_2)^{2/3}}\omega^{10/3} \end{aligned}$$

Simplifying the expression, we obtain:

$$P = \frac{32}{5}\frac{G^{7/3}}{c^5}\frac{\mu^2 (m_1 + m_2)^{4/3}}{\omega^{8/3}}\omega^{10/3}$$

Finally, the simplified form of the power radiated is:

$$P = \frac{32}{5}\frac{G^{7/3}}{c^5}\frac{(m_1 m_2)^2}{(m_1 + m_2)^{2/3}}\omega^{10/3} \tag{5}$$

Question 4

1. From equation (2), the rate of change of total energy is given by:

$$\frac{dE}{dt} = \frac{1}{3}G^{2/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\omega^{-1/3}\dot{\omega} \tag{6}$$

Equating the rate of change of total energy to the power expression obtained, we have:

$$\frac{1}{3}G^{2/3}\frac{m_1 m_2}{(m_1+m_2)^{1/3}}\omega^{-1/3}\dot{\omega} = \frac{32}{5}\frac{G^{7/3}}{c^5}\frac{(m_1 m_2)^2}{(m_1+m_2)^{2/3}}\omega^{10/3}$$

This leads to an expression for $\dot{\omega}$:

$$\dot{\omega} = \frac{96}{5}\frac{(m_1 m_2)}{(m_1+m_2)^{1/3}}\frac{G^{5/3}}{c^5}\omega^{11/3}$$

$$\dot{\omega} = \frac{96}{5}\left(\frac{G\,(m_1 m_2)^{3/5}}{c^3\,(m_1+m_2)^{1/5}}\right)^{5/3}\omega^{11/3}$$

The term $\frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}$ is known as the "chirp mass" $\mathcal{M}$:

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}} \tag{7}$$

$$\dot{\omega} = \frac{96}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}\omega^{11/3} \tag{8}$$

Question 5

1. Starting with the expression for the time derivative of the orbital angular frequency:

$$\dot{\omega} = \frac{96}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}\omega^{11/3}$$

$$2\pi\dot{f} = \frac{96}{5}\left(\frac{G\mathcal{M}}{c^3}\right)^{5/3}(2\pi f)^{11/3}$$

Let f_{GW} be the frequency of the wave:

$$f_{\mathrm{GW}} = 2f_{\mathrm{orb}}$$

Then, the rate of change of the gravitational wave frequency is given by:

$$\frac{\dot{f}_{\mathrm{GW}}}{2} = \frac{96}{5} 2^{8/3} \pi^{8/3} \left(\frac{G\mathcal{M}}{c^3} \right)^{5/3} \left(\frac{f_{\mathrm{GW}}}{2} \right)^{11/3}$$

Solving for $\dot{f}_{\mathrm{GW}}$:

$$\dot{f}_{\mathrm{GW}} = \frac{96}{5} \pi^{8/3} \left(\frac{G\mathcal{M}}{c^3} \right)^{5/3} f_{\mathrm{GW}}^{11/3}$$

we get the relation between chirp mass ($\mathcal{M}$) ,frequency of the gravitational wave (f_{GW}) and its time derivative ($\dot{f}_{\mathrm{GW}}$) as :

$$\mathcal{M} = \frac{c^3}{G} \left[\frac{5}{96} \pi^{-8/3} f_{\mathrm{GW}}^{-11/3} \dot{f}_{\mathrm{GW}} \right]^{3/5} \tag{9}$$