a) $m\ddot{x} = \nabla B\, x - c_h \dot{x}$ $\qquad x(0) = 1$

$\qquad\qquad\qquad\qquad \dot{x}(0) = 0$

$$m\ddot{x} + c_h\dot{x} - \nabla B x = 0$$

$$m\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + c_h\left(sX(s) - x(0)\right) - \nabla B\, X(s) = 0$$

$$ms^2X(s) - msx(0) - \cancel{m\dot{x}(0)}^{\,0} + c_h s X(s) - c_h x(0) - \nabla B\, X(s) = 0$$

$$X(s)\left(ms^2 + c_h s - \nabla B\right) = ms + c_h$$

$$\boxed{X(s) = \frac{ms + c_h}{ms^2 + c_h s - \nabla B}}$$

b) $m = 1;\ c_h = 1;\ \nabla B = 20:$

$$X(s) = \frac{s+1}{s^2+s-20} = \frac{s+1}{(s+5)(s-4)} = \frac{a}{s+5} + \frac{b}{s-4}$$

$$a(s-4) + b(s+5) = s+1$$

$$s = +4:\quad 0 + 9b = \cancel{3}\ 5 \Rightarrow b = \cancel{1/3}\ 5/9$$

$$s = -5:\quad -9a = \cancel{-6}\ -4 \Rightarrow a = \cancel{2/3}\ 4/9$$

$$X(s) = \frac{\cancel{2/3}\ 4/9}{s+5} + \frac{\cancel{1/3}\ 5/9}{s-4} \Rightarrow \mathcal{L}^{-1}\{X(s)\} = \cancel{\tfrac{2}{3}}\,\tfrac{4}{9}\, e^{-5t} + \cancel{\tfrac{1}{3}}\,\tfrac{5}{9}\, e^{4t}$$

$$\boxed{x(t) = \cancel{\tfrac{2}{3}}\,\tfrac{4}{9}\, e^{-5t} + \cancel{\tfrac{1}{3}}\,\tfrac{5}{9}\, e^{4t}}$$

c) $X(s) = \frac{s+1}{s^2+s-20}$; $x(t) = \frac{2}{3}e^{-5t} + \frac{1}{3}e^{4t}$

Initial Value Theorem: $\lim_{s\to\infty} s \cdot X(s) = \lim_{t\to 0} x(t)$

$$\lim_{s\to\infty} s \cdot \frac{s+1}{s^2+s-20} = \lim_{s\to\infty} \frac{s^2+s}{s^2+s-20} = \boxed{1}$$

✓agree

$$\lim_{t\to 0}\left(\overset{4/9}{\cancel{\tfrac{2}{3}}} e^{-5t} + \overset{5/9}{\cancel{\tfrac{1}{3}}} e^{4t}\right) = \overset{4/9}{\cancel{\tfrac{2}{3}}} e^{(0)} + \overset{5/9}{\cancel{\tfrac{1}{3}}} e^{(0)} = \overset{4/9}{\cancel{\tfrac{2}{3}}} + \overset{5/9}{\cancel{\tfrac{1}{3}}} = \boxed{1}$$

IVT shows that we did our math right in both time & frequency domains.

d) Final Value Theorem only applies if the system is <u>STABLE</u>. To check this, we look at the poles of the transfer function $X(s)$:

$$s^2+s-20 = (s+5)(s-4) \Rightarrow \text{roots: } -5, +4$$

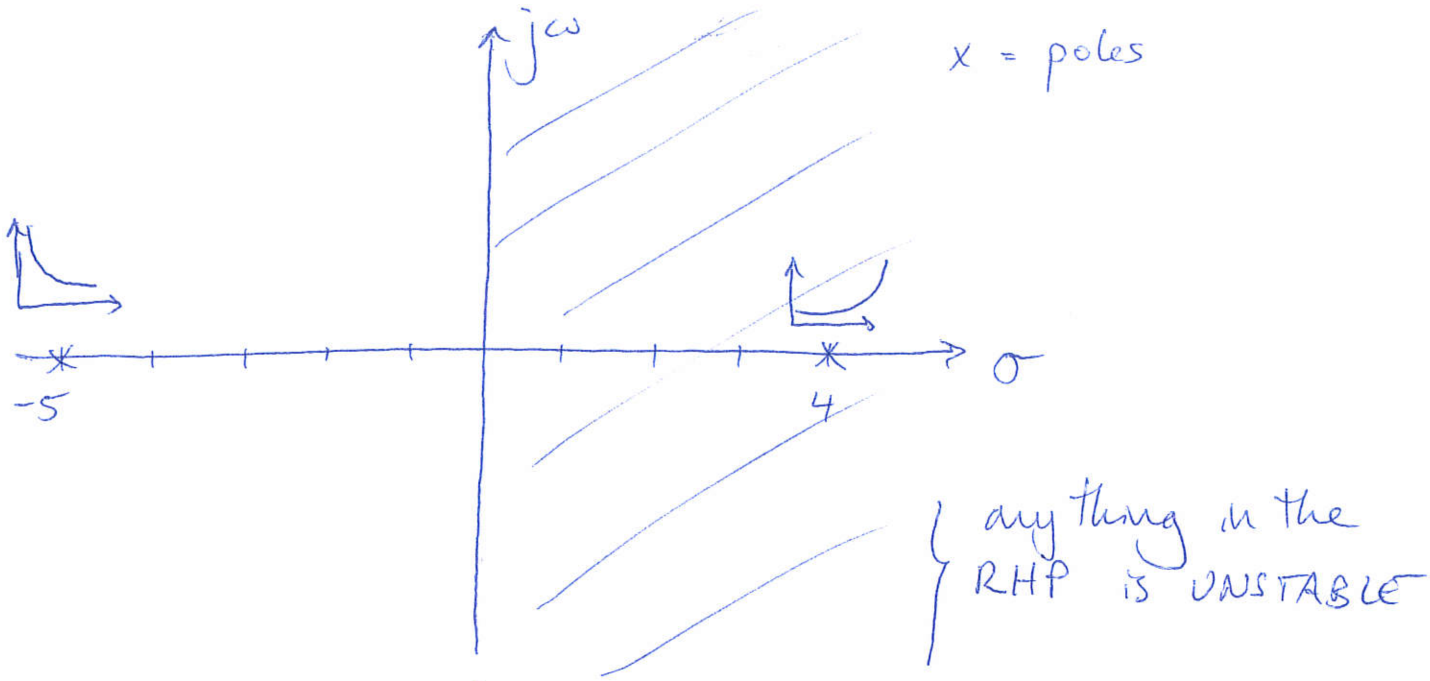
A positive real root means that our system is unstable. Why? Let's look in the time domain!

As $t \to \infty$, $\underbrace{\overset{4/9}{\cancel{\tfrac{2}{3}}} e^{-5t}}_{\text{this term goes to } 0} + \underbrace{\overset{5/9}{\cancel{\tfrac{1}{3}}} e^{4t}}_{\text{this term goes to } \infty}$

this term goes to 0
↳ STABLE

this term goes to ∞
↳ UNSTABLE.

We can also look @ frequency domain:



anything on the Imaginary axis is "Marginally stable" and according to Bounded Input Bounded Output (BIBO) definition, that means poles on the Im axis are <u>also</u> UNSTABLE

e) Let $m=1$, $c_h = 5$, $\nabla B = -6$:

$$X(s) = \frac{s+5}{s^2+5s+6} = \frac{s+5}{(s+2)(s+3)}$$

⇒ now both roots are negative, $s = -2, -3$, so our system is <u>stable</u> and we can apply FVT:

$$\lim_{s \to 0} s \cdot X(s) = \lim_{t \to \infty} x(t)$$

$$\boxed{\lim_{s \to 0} s \cdot \frac{s+5}{s^2+5s+6} = 0}$$

time domain

$$X(s) = \frac{s+5}{(s+2)(s+3)} = \frac{a}{s+2} + \frac{b}{s+3}$$

$$a(s+3) + b(s+2) = s+5$$

$$s=-3: \quad 0 + b(-1) = \overset{2}{\cancel{-8}} \Rightarrow b = \cancel{8}\,-2$$

$$s=-2: \quad a + 0 = \overset{3}{\cancel{-7}} \Rightarrow a = \cancel{-7}\ 3$$

$$X(s) = \frac{\overset{3}{\cancel{-7}}}{s+2} + \frac{\overset{-2}{\cancel{8}}}{s+3} \Rightarrow \mathcal{L}^{-1}\{X(s)\} = \overset{3}{\cancel{-7}}e^{-2t} + \overset{-2}{\cancel{8}}e^{-3t}$$

$$\boxed{\lim_{t\to\infty} \left(\overset{3}{\cancel{-7}}e^{-2t} + \overset{-2}{\cancel{8}}e^{-3t}\right) = 0}$$

⇒ time & frequency domains agree steady state value = 0.

Discussion to give intuition:

→ In Case 1, the magnetic force is pulling the swimmer while the drag resists its motion. The magnetic force is increasing at a faster rate so the swimmer continues to move in the +x-direction @ an ~~exponentially~~ increasing rate.

→ in case 2, the magnetic force & the drag force are acting in the same direction. ~~Actually, with the given IC's, the swimmer never moves~~ When the swimmer reaches $x=0$, there is no F_B, and no $\dot{x}$, and it stops moving.