16-811 Assignment 5

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1 Problem 1

References:
Dawkins, P. Paul's Online Notes: Section 2-2: Surface Area. 05/30/2018. `http://tutorial.math.lamar.edu/Classes/CalcII/SurfaceArea.aspx` Visited 11/02/2019.

Nonlinear Science Group at the George Mason University. Phys 705: Classical Mechanics. Calculus of Variations I. `http://complex.gmu.edu/www-phys/phys705/notes/005%20Calculus%20of%20Variations%20I.pdf` Visited 11/02/2019.

The objective for this problem is to minimize the surface area of a revolved curve. We can write this minimization as:

$$L = \int_{x_0}^{x_1} da \tag{1}$$

We can write $da = 2\pi y ds$ where ds is the hypotenuse of the right angled triangle formed by dx and dy. Therefore we can write:

$$da = 2\pi y\sqrt{(dx)^2 + (dy)^2} = 2\pi y\sqrt{1 + (y')^2}dx \tag{2}$$

Plugging this back into the equation for L, we can see that F is equal to the integrand:

$$L = 2\pi \int_{x_0}^{x_1} y\sqrt{1 + (y')^2}dx \tag{3}$$

$$F = y\sqrt{1 + (y')^2} \tag{4}$$

Since F does not explicitly depend on x, we can use an alternative form of the Euler-Lagrange equation, where:

$$y'\frac{\partial F}{\partial y'} - F = c \tag{5}$$

We can calculate $\frac{\partial F}{\partial y'}$ as follows:

$$\frac{\partial F}{\partial y'} = y\frac{1}{2}(1+(y')^2)^{-\frac{1}{2}}2y' = \frac{yy'}{\sqrt{1+(y')^2}} \tag{6}$$

Plugging the partial back into the Euler-Lagrange equation, we get:

$$\frac{(y')^2 y}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = c \tag{7}$$

$$\frac{y(y')^2 - y(1+(y')^2)}{\sqrt{1+(y')^2}} = \frac{-y}{\sqrt{1+(y')^2}} = c \tag{8}$$

We can rearrange this expression to be in terms of y':

$$y' = \sqrt{\frac{y^2}{c^2} - 1} \tag{9}$$

Integrating this expression with respect to dx, we can find an expression for y:

$$x = \int dx = \int \frac{dy}{\frac{1}{c}\sqrt{y^2 - c^2}} = ccosh^{-1}(\frac{y}{c}) + b \tag{10}$$

$$y = ccosh(\frac{x-b}{c}) \tag{11}$$

This expression for y depends on the fixed points, x_0 and x_1.

If the function y(x) is not twice differentiable, and therefore not the C^2 solution, then we can find the Gold Schmidt solution which is just a line between the two endpoints. Alternatively, if we were to fix only one end point, then all the solution curves would be tangent to some envelope function (which would have a parabolic shape).

2 Problem 2

The solution to this problem uses the fundamental principle of the Calculus of Variations that:

$$\frac{f(x+h) - f(x) - \nabla f(x)^T h}{h} = 0 \tag{12}$$

where the third term is df and h is a small perturbation. In this problem, we can characterize the rectangular coordinates (x,y,z) as functions of small perturbations of the circular coordinates (u,v). In this case, we would say that:

$$h = \begin{bmatrix} du \\ dv \end{bmatrix} \tag{13}$$

This allows us to write the terms dx, dy, dz in the expression for arclength as functions of perturbations in (u,v). For example, we can say $dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv$. We can write the expressions for dx, dy, dz as follows:

$$dx = -Rsin(v)sin(u)du + Rcos(v)cos(u)dv \tag{14}$$

$$dy = Rsin(v)cos(u)du + Rcos(v)sin(u)dv \tag{15}$$

$$dz = -Rsin(v)dv \tag{16}$$

Therefore, we can cast the expression for the arclength in terms of u and v as follows:

$$\begin{aligned}\sqrt{dx^2 + dy^2 + dz^2} = \Big[&\big(- Rsin(v)sin(u)du + Rcos(v)cos(u)dv\big)^2 + \\ &\big(Rsin(v)cos(u)du + Rcos(v)sin(u)dv\big) + \big(- Rsin(v)dv\big)^2 \Big]^{\frac{1}{2}}\end{aligned} \tag{17}$$

After some algebraic manipulations we get to this expression:

$$\begin{aligned}= \big[&R^2du^2(sin^2(v)sin^2(u) + sin^2(v)cos^2(u)) \\ &+ R^2dv^2(cos^2(v)cos^2(u) + cos^2(v)sin^2(u) + sin^2(v))\big]^{\frac{1}{2}}\end{aligned} \tag{18}$$

We can greatly simplify this expression by using the identity $cos^2\theta + sin^2\theta = 1$:

$$= \big[R^2du^2sin^2(v) + R^2dv^2\big]^{\frac{1}{2}} = R\big[sin^2(v) + (v')^2\big]^{\frac{1}{2}}du \tag{19}$$

If we were to minimize this expression, then the integrand is:

$$F = \big[sin^2(v) + (v')^2\big]^{\frac{1}{2}} \tag{20}$$

Since F does not explicitly depend on u, we can use the alternate form of the Euler-Lagrange equation:

$$v'\frac{\partial F}{\partial v'} - F = c \tag{21}$$

Where we can write $\frac{\partial F}{\partial v'}$ as:

$$\frac{\partial F}{\partial v'} = \frac{v'}{\sqrt{sin^2(v) + (v')^2}} \tag{22}$$

Plugging this expression back into the Euler-Lagrange equation and solving for v' we get:

$$v' = \frac{1}{c}\sqrt{sin^4(v) - c^2sin^2(v)} \tag{23}$$

Now I can solve for u by integrating du:

$$\int du = \int \frac{c}{\sqrt{sin^4(v) - c^2 sin^2(v)}} dv \tag{24}$$

Therefore we get the final expression:

$$u(v) = -sin^{-1}\left(\frac{cot(v)}{\sqrt{\frac{1}{c^2} - 1}}\right) + k \tag{25}$$

3 Problem 3

References:
`http://mathworld.wolfram.com/BrachistochroneProblem.html`

To solve this problem I need the integrand from the Brachistochrone problem, listed below:

$$F = \sqrt{\frac{1 + (y')^2}{2gy}} \tag{26}$$

I need to plug this expression into this modified Euler-Lagrange equation:

$$F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0 \tag{27}$$

I can find $F_{y'}$:

$$F_{y'} = \frac{y'}{\sqrt{\frac{1+(y')^2}{2gy}}} \tag{28}$$

I want to solve the Euler-Lagrange equation for y', and after some algebraic manipulation and application of the quadratic formula, I find:

$$y' = \frac{-gyg_x \pm \sqrt{g^2y^2g_x^2 + 2gy(g_y)^2 - (g_y)^2}}{g_y(2gy - 1)} \tag{29}$$

To prove orthogonality I need to show:

$$< y', \nabla g >= y'^T \nabla g = 0 \tag{30}$$

Where:

$$\nabla g = \begin{bmatrix} g_x \\ g_y \end{bmatrix} \tag{31}$$

I was not able to proceed from here, however, because I could not find a way to prove that the two expressions are orthogonal to one another - the algebra did not work out.

4 Problem 4

4.1 Part (a)

In this section we are going to derive the relationships between the joint torques and the angular states for the balanced manipulator using Lagrangian Dynamics. Please note that I forgot there was no gravity in this example. This derivation assumes that there is gravity in the y-direction and I correct the assumption at the end of the solution by writing out the solution without the gravity terms.

We need to write the Lagrangian in terms of the kinetic and potential energies: $L = T - V$. We can write T and V as follows:

$$V = m_1 g h_1 + m_2 g h_2 + m_2 g h_3 \tag{32}$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} m_2 v_2^2 \tag{33}$$

I can write the heights h_i as follows:

$$h_1 = l_1 sin(\theta_1) \tag{34}$$
$$h_2 = l_1 sin(\theta_1) - l_2 cos(\theta_1 + \theta_2) \tag{35}$$
$$h_3 = l_1 sin(\theta_1) + l_2 sin(\theta_1 + \theta_2) \tag{36}$$

I can write the velocities as follows:

$$v_1 = l_1 \dot{\theta}_1 \tag{37}$$
$$v_2^2 = \dot{x}^2 + \dot{y}^2 \tag{38}$$

We can write $\dot{x}$ and $\dot{y}$ by first writing x and y and then taking the derivatives:

$$x = l_1 cos(\theta_1) + l_2 cos(\theta_1 + \theta_2) \tag{39}$$
$$y = l_1 sin(\theta_1) + l_2 sin(\theta_1 + \theta_2) \tag{40}$$

The derivatives are:

$$\dot{x} = -l_1 sin(\theta_1)(\dot{\theta}_1) - l_2 sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \tag{41}$$
$$\dot{y} = l_1 cos(\theta_1)(\dot{\theta}_1) + l_2 cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \tag{42}$$

Now I can write an expression for v_2^2 as follows:

$$\begin{aligned} v_2^2 = l_1^2 sin^2(\theta_1)(\dot{\theta}_1^2) &+ 2 l_1 l_2 sin(\theta_1) sin(\theta_1 + \theta_2)(\dot{\theta}_1)(\dot{\theta}_1 + \dot{\theta}_2) \\ &+ l_2^2 sin^2(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 + l_1^2 cos^2(\theta_1)(\dot{\theta}_1^2) \\ &+ 2 l_1 l_2 cos(\theta_1) cos(\theta_1 + \theta_2)(\dot{\theta}_1)(\dot{\theta}_1 + \dot{\theta}_2) \\ &+ l_2^2 cos^2(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2)^2 \end{aligned} \tag{43}$$

After some algebraic manipulations we can get v_2^2 in a slightly shorter format:

$$v_2^2 = l_1^2(\dot{\theta}_1^2) + 2l_1l_2(\dot{\theta}_1)(\dot{\theta}_1 + \dot{\theta}_2)cos(\theta_2) + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \tag{44}$$

Now I can write out the entire Lagrangian equation as:

$$\begin{aligned} L = \frac{1}{2}m_1l_1\dot{\theta}_1 + m_2l_1^2\dot{\theta}_1^2 + 2m_2l_1l_2(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)cos(\theta_2) + m_2l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ -(m_1gl_1sin(\theta_1)+m_2gl_1sin(\theta_1)-m_2gl_2cos(\theta_1+\theta_2)+m_2gl_1sin(\theta_1)+m_2gl_2sin(\theta_1+\theta_2)) \end{aligned} \tag{45}$$

In general, I want to write out the torques on each joint as:

$$\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} \tag{46}$$

So I need to write out the terms in this expression for torque. Let's start with $\frac{\partial L}{\partial \theta_i}$:

$$\frac{\partial L}{\partial \theta_1} = -\big((m_1gl_1 + 2m_2gl_1)cos(\theta_1) + m_2gl_2sin(\theta_1 + \theta_2) - m_2gl_2cos(\theta_1 + \theta_2)\big) \tag{47}$$

$$\frac{\partial L}{\partial \theta_2} = -2m_2l_1l_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)sin(\theta_2) - (m_2gl_2sin(\theta_1 + \theta_2) + m_2gl_2cos(\theta_1 + \theta_2)) \tag{48}$$

Now let's write the expressions for $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i}$:

$$\frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{2}m_1l_1 + m_2l_1^2(2\dot{\theta}_1) + 2m_2l_1l_2(2\dot{\theta}_1 + \dot{\theta}_2)cos(\theta_2) + m_2l_22(\dot{\theta}_1 + \dot{\theta}_2) \tag{49}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = 2m_2l_1l_2(\dot{\theta}_1)cos(\theta_2) + m_2l_22(\dot{\theta}_1 + \dot{\theta}_2) \tag{50}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_1}) = 2m_2l_1^2\ddot{\theta}_1 + 2m_2l_1l_2cos(\theta_2)(2\ddot{\theta}_1 + \ddot{\theta}_2) + 2m_2l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \tag{51}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_2}) = 2m_2l_1l_2cos(\theta_2)(\ddot{\theta}_1) + 2m_2l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \tag{52}$$

Now I can write the expressions for τ_1 and τ_2:

$$\begin{aligned} \tau_1 = \Big[(2m_2l_1^2 + 4m_2l_1l_2cos(\theta_2) + 2m_2l_2)\ddot{\theta}_1 + (2m_2l_1l_2cos(\theta_2) + 2m_2l_2)\ddot{\theta}_2\Big] \\ + \Big[(m_1gl_1 + 2m_2gl_1)cos(\theta_1) + m_2gl_2sin(\theta_1 + \theta_2) - m_2gl_2cos(\theta_1 + \theta_2)\Big] \end{aligned} \tag{53}$$

$$\tau_2 = \Bigg[(2m_2l_1l_2cos(\theta_2) + 2m_2l_2)\ddot{\theta}_1 + 2m_2l_2\ddot{\theta}_2\Bigg]$$
$$-\Bigg[(-2m_2l_1l_2(\dot{\theta}_1)^2 - 2m_2l_1l_2\dot{\theta}_1\dot{\theta}_2)sin(\theta_2) - m_2gl_2sin(\theta_1+\theta_2) + m_2gl_2cos(\theta_1+\theta_2)\Bigg] \tag{54}$$

Now I can write out one equation in vector form to show the relationships between the torques and the state variables:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}2m_2l_1^2 + 4m_2l_1l_2cos(\theta_2) + 2m_2l_2 & 2m_2l_1l_2cos(\theta_2) + 2m_2l_2\\ 2m_2l_1l_2cos(\theta_2) + 2m_2l_2 & 2m_2l_2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}$$
$$+\begin{bmatrix}0 & 0 & 0\\ 2m_2l_1l_2sin(\theta_2) & 2m_2l_1l_2sin(\theta_2) & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_2^2\end{bmatrix}$$
$$+\begin{bmatrix}m_1gl_1cos(\theta_1) + 2m_2gl_1cos(\theta_1) + m_2gl_2sin(\theta_1+\theta_2) - m_2gl_2cos(\theta_1+\theta_2)\\ m_2gl_2sin(\theta_1+\theta_2) - m_2gl_2cos(\theta_1+\theta_2)\end{bmatrix} \tag{55}$$

Please note that I forgot there was no gravity in this example. This means that the correct answer in the absence of gravity is the same as above without the gravity terms, i.e. the correct answer is:

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}2m_2l_1^2 + 4m_2l_1l_2cos(\theta_2) + 2m_2l_2 & 2m_2l_1l_2cos(\theta_2) + 2m_2l_2\\ 2m_2l_1l_2cos(\theta_2) + 2m_2l_2 & 2m_2l_2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}$$
$$+\begin{bmatrix}0 & 0 & 0\\ 2m_2l_1l_2sin(\theta_2) & 2m_2l_1l_2sin(\theta_2) & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_2^2\end{bmatrix} \tag{56}$$

4.2 Part (b)

When $\ddot{\theta}_2 = 0$, that means that the second link is moving at a constant velocity. In that case, τ_1 is related to $\ddot{\theta}_1$ by contributions from the mass of link 2 and the current position of link 2 relative to link 1. τ_1 in this case is no longer receiving a second contribution from $\ddot{\theta}_2$ with respect to the mass of link 2 and its current position.