16-811 Assignment 3

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1 Problem 1

1.1 Part (a)

The Taylor series expansion in general (centered about 0) is:

$$p(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... \tag{1}$$

I can calculate the first, second and third derivatives of the function $f(x) = \frac{1}{3} + 2sinh(x)$ as follows:

$$f'(x) = 0 + 2cosh(x) \tag{2}$$

$$f''(x) = 2sinh(x) \tag{3}$$

$$f'''(x) = 2cosh(x) \tag{4}$$

And so on - the derivatives continue to alternate between $2sinh(x)$ and $2cosh(x)$. Now this allows me to write a Taylor series centered at 0 as:

$$p(x) = \frac{1}{3} + 2sinh(0) + 2cosh(0)x + \frac{2sinh(0)}{2!}x^2 + \frac{2cosh(x)}{3!}x^3 + ... \tag{5}$$

Solving out, we can see that only the odd terms remain and I can write my final answer in a general form as:

$$p(x) = \frac{1}{3} + 2x + \sum \frac{2}{n!}x^n \tag{6}$$

where n is an odd number from 3 to infinity.

1.2 Part (b)

This plot was produced using code in code/q1.m

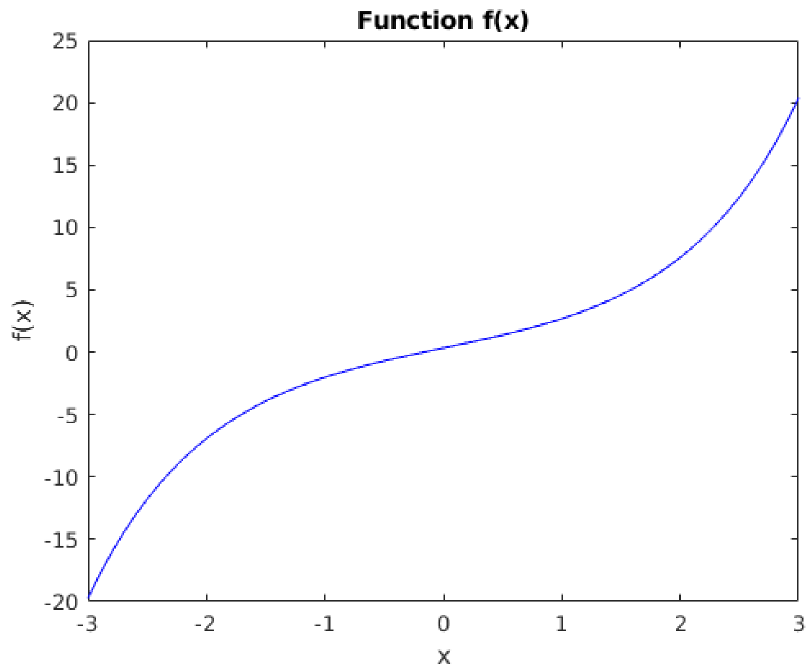

Figure 1: Plot of $f(x)$ over $[-3, 3]$.

1.3 Part (c)

If I want to use a quadratic function to approximate $f(x)$, then I need a polynomial with $n = 2$, and therefore $n + 2 = 4$ points that meet the requirement:

$$(-1)^i[f(x_i) - p(x_i)] = \epsilon||f - p||_\infty \tag{7}$$

I also know (see Part (a)) that the $n + 1 = 3$ derivative of $f(x)$ is $f'''(x) = 2cosh(x)$ which is positive for the entire interval $[-3, 3]$. Therefore, $x_0 = -3$ and $x_3 = 3$. Let me represent my quadratic function as the polynomial $p(x) = a + bx + cx^2$.

As I was working on solving this problem initially, I quickly realized that it is difficult to find 3 equations to find the 3 unknown coefficients in the polynomial expression above. Thanks to a suggestion from the TAs, I propose approximating the function $f(x)$ using a linear polynomial where the coefficient $c = 0$. I will still look for 4 uniformly spaced points and show that they satisfy the error requirement and that therefore a linear polynomial is in fact the best quadratic approximation to the function $f(x)$.

The first step to finding coefficients a and b is to realize that the error at the endpoints of the interval is the same but with opposing signs, thus:

$$e(x_0) = -e(x_3) \tag{8}$$

$$e(x_0) = f(x_0) - p(x_0) = 2sinh(-3) + \frac{1}{3} - (a + b(-3)) = -19.7024 - a + 3b \tag{9}$$

$$e(x_3) = 2sinh(3) + \frac{1}{3} - (a + b(3)) = 20.3691 - a - 3b \tag{10}$$

$$-19.7024 - a + 3b = -(20.3691 - a - 3b) \tag{11}$$

$$a = \frac{1}{3} \tag{12}$$

Now that we have the y-intercept of our linear approximation, I need a second equation to find the slope, b. Notice that I am still assuming that I have uniformly spaced points, and since I know that 2 of them are -3 and 3, I know that the other 2 points must be equally spaced between them. Therefore, $x_1 = -1$ and $x_2 = 1$. Using this information, I can find the slope based on the assumption that the error at x_2 is a local maximum, therefore:

$$e'(x_2) = 2cosh(1) - b(1) = 0 \tag{13}$$

$$b = 3.0862 \tag{14}$$

Now I have my full polynomial, $p(x) = 3.0862x + \frac{1}{3}$.

Please note that I realize this is not entirely correct, because when I calculate the error at all four points, I get the following:

$$e(x_0) = e(x_3) = 8.8742 \tag{15}$$

$$e(x_1) = e(x_2) = 1.4033 \tag{16}$$

The plot of my approximation $p(x)$ against $f(x)$ is shown below, and while it shows that I am close to the correct answer, I clearly am not getting the same error at every point along my interval.

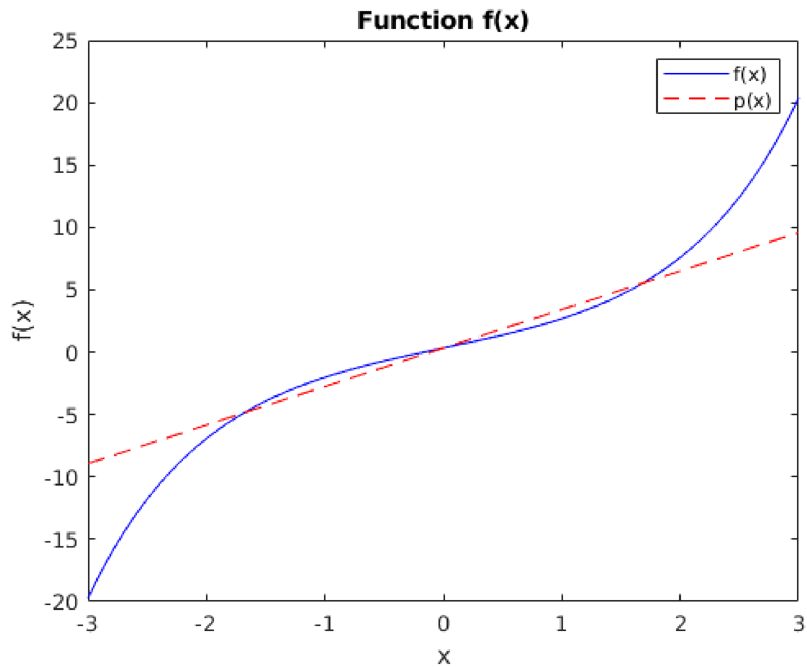

Figure 2: Plot of $f(x)$ and $p(x)$ over $[-3, 3]$.

The L_∞ error is calculated as:

$$L_\infty = \max_{a \leq x \leq b} |f(x) - p(x)| = \max_{-3 \leq x \leq 3} |(\frac{1}{3} + 2sinh(x)) - (\frac{1}{3} + 3.0862x)| = 10.7771 \tag{17}$$

The L_2 error is calculated as:

$$\sqrt{\int_a^b |e(x)|^2 dx} = \sqrt{\int_{-3}^{3} |2cosh(x) - 3.0862|^2 dx} = 15.0079 \tag{18}$$

1.4 Part (d)

To obtain the least squares approximation, we want to minimize the following expression:

$$\int_{-3}^{3} (\frac{1}{3} + 2sinh(x) - p(x))^2 dx \tag{19}$$

where $p(x)$ can be written as:

$$p(x) = \sum_{i=0}^{2} \frac{< f(x), p_i >}{< p_i, p_i >} p_i(x) \tag{20}$$

and

$$p_{i+1}(x) = \left[x - \frac{< xp_i, p_i >}{< p_i, p_i >} \right] p_i(x) - \frac{< p_i, p_i >}{< p_{i-1}, p_{i-1} >} p_{i-1}(x) \tag{21}$$

By definition, I also know that $p_0(x) \equiv 1$ and $p_{-1}(x) \equiv 0$.
I now compute all of the inner products and obtain the following:

$$< p_0, p_0 > = \int_{-3}^{3} 1 \cdot 1 dx = 6 \tag{22}$$

$$< xp_0, p_0 > = \int_{-3}^{3} x dx = 0 \tag{23}$$

$$p_1(x) = [x - 0]p_0(x) - 0 = x \tag{24}$$

$$< p_1, p_1 > = \int_{-3}^{3} x^2 dx = 18 \tag{25}$$

$$< xp_1, p_1 > = \int_{-3}^{3} x^3 dx = 0 \tag{26}$$

$$p_2(x) = [x - 0]p_1(x) - \frac{18}{6} p_0(x) = x^2 - 3 \tag{27}$$

$$< p_2, p_2 > = \int_{-3}^{3} (x^2 - 3)(x^2 - 3) dx = 97.2 \tag{28}$$

Now I can calculate the elements of the polynomial $p(x)$ as follows:

$$< \frac{1}{3} + 2sinh(x), p_0 > = \int_{-3}^{3} \frac{1}{3} + 2sinh(x) dx = 2 \tag{29}$$

$$< \frac{1}{3} + 2sinh(x), p_1 > = \int_{-3}^{3} \left(\frac{1}{3} + 2sinh(x) \right) x dx = 80.7408 \tag{30}$$

$$< \frac{1}{3} + 2sinh(x), p_2 > = \int_{-3}^{3} \left(\frac{1}{3} + 2sinh(x) \right) (x^2 - 3) dx = 0 \tag{31}$$

If we define $d_i \equiv \frac{< f(x), p_i >}{< p_i, p_i >}$ then we can calculate the d_i coefficients as:

$$d_0 = \frac{2}{6} = \frac{1}{3} \tag{32}$$

$$d_1 = \frac{80.7408}{18} = 4.4856 \tag{33}$$

$$d_2 = 0 \tag{34}$$

And so we can finally write the entire polynomial as:

$$p(x) = \frac{1}{3} + 4.4856x \tag{35}$$

It is interesting to note that the least squares approximation also concludes with a linear approximation even when one is working towards obtaining a quadratic polynomial. This result agrees with the argument we made in 1.c), although the coefficients are different.

I used the same expressions for the L_∞ and L_2 errors as shown in 1.c), so I will just provide the numbers here. $L_\infty = 6.5789$ and $L_2 = 5.4091$.

2 Problem 2

I chose to build a 3rd order polynomial to approximate this function. I used a least squares approach with SVD to find the coefficients of the polynomial. The complete code for this problem is given in code/q2.m and the plot of the true function and my approximation is shown below.

I built the matrix A as follows:

$$A = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \tag{36}$$

Where:

$$\phi_1 = \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix} \tag{37}$$

$$\phi_2 = \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix} \tag{38}$$

$$\phi_3 = (\phi_2)^2, \phi_4 = (\phi_2)^3 \tag{39}$$

Then I used SVD to obtain the S, V, U matrices and calculated the coefficients as:

$$\bar{x} = V \cdot S^{-1} \cdot U' \cdot f_i \tag{40}$$

And I obtained the final expression:

$$p(x) = -3.2033 + 20.9635x - 5.669x^2 + 0.4086x^3 \tag{41}$$

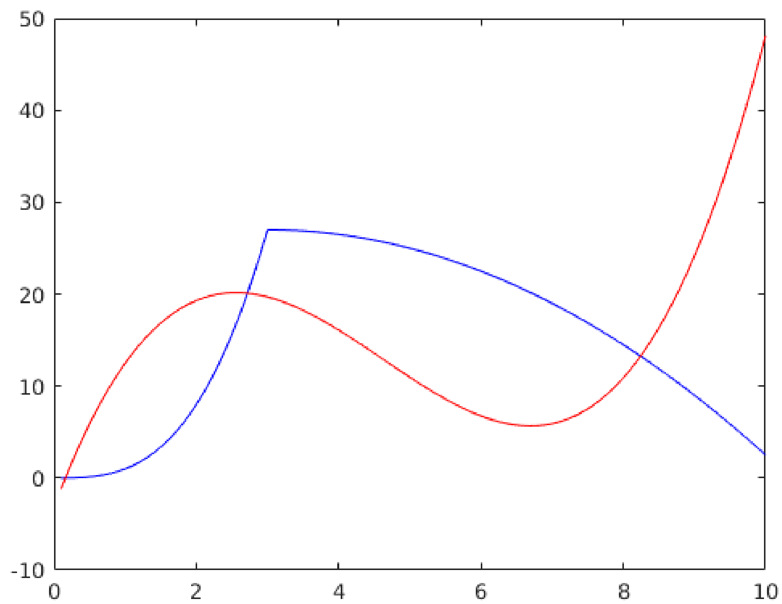

Figure 3: Plot of $f(x)$ and $p(x)$.

3 Problem 3

3.1 Part a)

I need to calculate T_3 and T_4. I know that $T_0(x) = 1$ and $T_1(x) = x$. I will use the recurrence relation to find the higher order terms:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), n > 0 \tag{42}$$

I can calculate $T_2(x)$ as:

$$T_2(x) = 2xT_1(x) - T_0(x) = 2x(x) - 1 = 2x^2 - 1 \tag{43}$$

And then $T_3(x)$ and $T_4(x)$ are:

$$T_3(x) = 2x(2x^2 - 1) - x = 4x^3 - 3x \tag{44}$$

$$T_4(x) = 2x(4x^3 - 3x) - (2x^2 - 1) = 8x^4 - 8x^2 + 1 \tag{45}$$

3.2 Part b)

Now I need to show that T_3 and T_4 are orthogonal to one another, relative to the inner product:

$$< g, h >= \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} g(x)h(x)dx \tag{46}$$

Note that by definition, the inner product of two orthogonal vectors is 0. Let's consider what T_3 and T_4 look like. I have shown a plot below (please note that the figure was generated using code contained in code/q3.m).

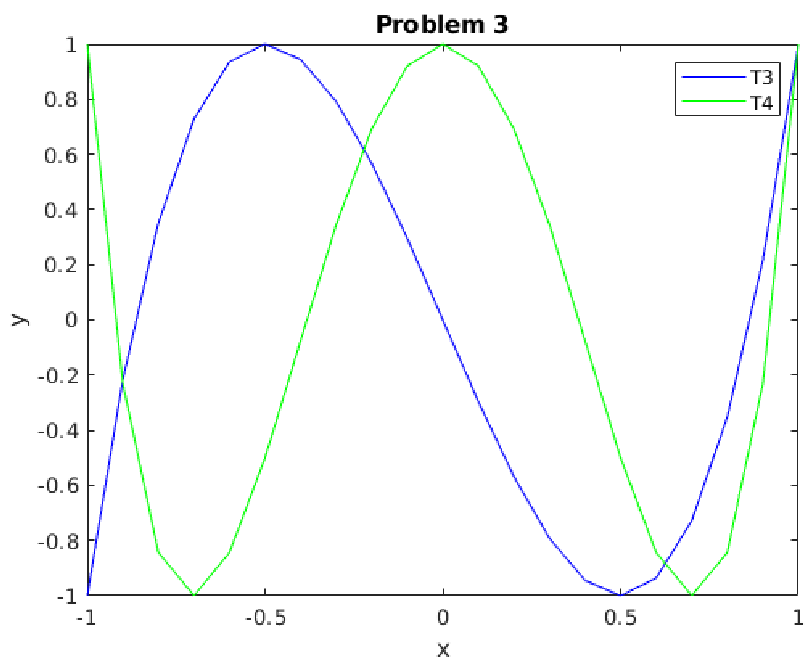

Figure 4: Plot of T_3 and T_4.

We can see that T_3 is an odd function while T_4 is an even function. If I were to integrate an odd function over the interval $[-1, 1]$, the result would be 0. Therefore, the entire expression for the inner product will be 0 and therefore T_3 and T_4 are orthogonal.

3.3 Part c)

The definition of the length of T_n is:

$$||T_n|| = \sqrt{< T_n, T_n >} \tag{47}$$

I can compute the inner product of T_n using the expression shown in part b), but first I will perform a substitution of variables: $x = cos(n\theta)$. This substitution of variables means that my integral now runs from 0 to π and $dx = sin(n\theta)d(n\theta)$.

$$< T_n, T_n >= \int_0^\pi (1 - cos(n\theta)^2)^{-\frac{1}{2}} cos(n\theta)cos(n\theta)sin(n\theta)d(n\theta) \tag{48}$$

$$< T_n, T_n >= \int_0^\pi \frac{cos^2(n\theta)}{sin(n\theta)} sin(n\theta)d(n\theta) = \frac{1}{2}(n\theta + sin(n\theta)cos(n\theta))|_0^\pi = \frac{n\pi}{2} \tag{49}$$

Therefore the length is:

$$||T_n|| = \sqrt{\frac{n\pi}{2}} \tag{50}$$

In this final expression, the value of n does not matter, because for any integer value of n, $cos(\frac{n\pi}{2} = 0$, so the length of T_n is the same for all values of n.

3.4 Part d)

Finally, I want to show that any 2 terms, T_i and T_j, where $i \neq j$, are orthogonal. I can do this by substitute terms into the expression for the inner product as I did in part c):

$$< T_i, T_j >= \int_{-1}^1 T_i \cdot T_j dx = \int_0^\pi cos(i\theta)cos(j\theta)sin\theta d\theta \tag{51}$$

$$< T_i, T_j >= \frac{sin(i\theta)}{i} \cdot \frac{sin(j\theta)}{j} \cdot (-cos\theta)|_0^\pi = 0 \tag{52}$$

Therefore, T_i and T_j are always orthogonal.

4 Problem 4

Please note that the code for all parts of this problem is contained in code/q4.m.

4.1 Part a)

My fitted plane and the data is shown in the figure below. I calculated the average distance of a point in the data set to the fitted plane as the least squares error:

$$E(c) = \sum_{i=1}^n \left(f_i - F(x, y, z)\right)^2 \tag{53}$$

For this problem I found $E(c) = 24.1876$.

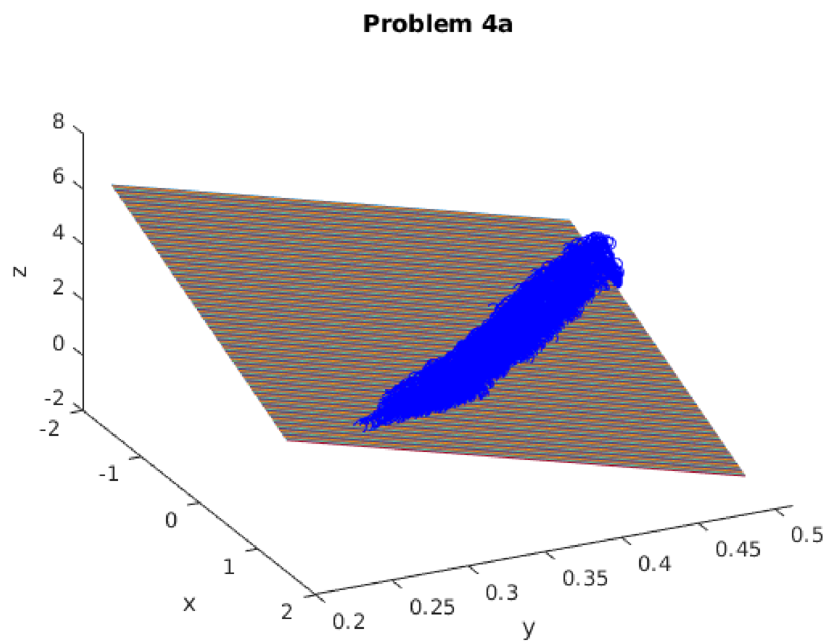

Figure 5: Plot for 4a).

4.2 Part b)

The same fitted plane function that I used in part a) fails to appropriately approximate the dataset in part b) because there is a large cluster of data that lies outside of the true plane of the table. This cluster of points weights the plane-fitting function in the wrong direction, resulting in an erroneous approximation, as shown below.

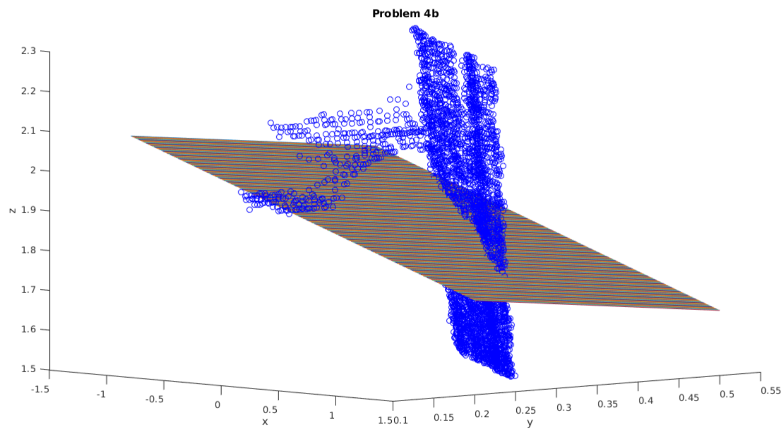

Figure 6: Plot for 4b).

4.3 Part c)

I added a step to my code which fit a line to the x, y data provided for the table plane using least squares. I then filtered out points that were more than a certain distance away from this line. I calculated the distance of a point, (m, n), from the fitted line, $(Ax + By + C = 0)$, as:

$$d = \frac{|Am + Bn + C|}{\sqrt{A^2 + B^2}} \tag{54}$$

The resultant fitted plane for the filtered dataset is a much better fit as shown in the figure below.

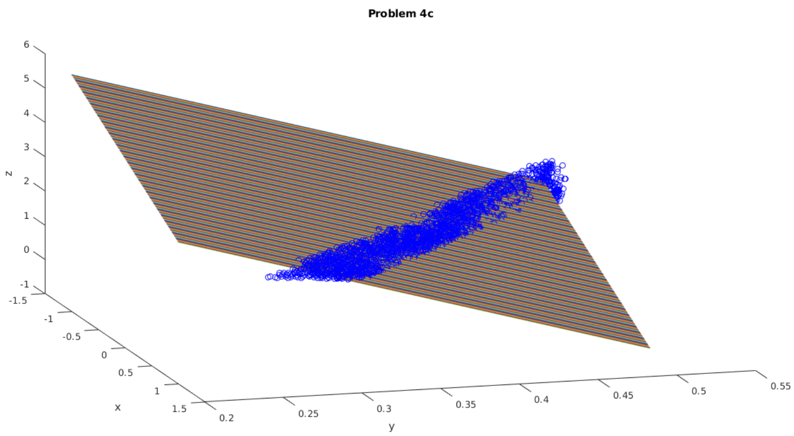

Figure 7: Plot for 4c).

4.4 Part d)

I was not able to find a way to sort the points into 4 distinct groups based on which wall they corresponded to; instead I just noted in the data file that there were an approximately equal number of points for each wall. My result is shown below.

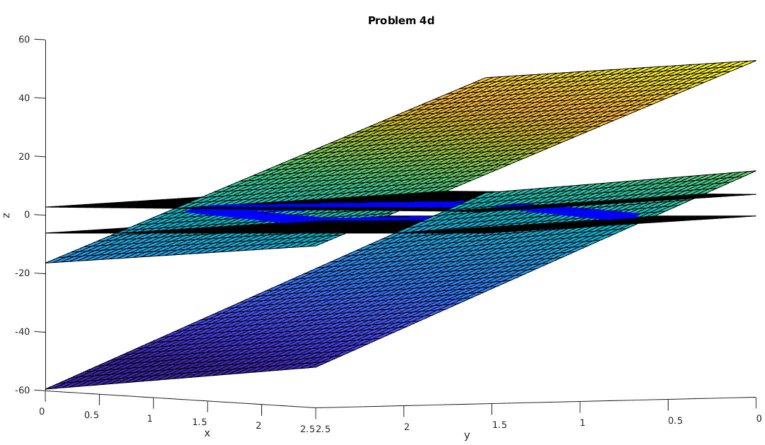

Figure 8: Plot for 4d).

4.5 Part e)

I tried to fit a line to the first n points in the data file, and then reject points that were too far from that fitted line, in order to build 4 clusters of points for the 4 walls. This did not work very well, as can be seen in my result shown below.

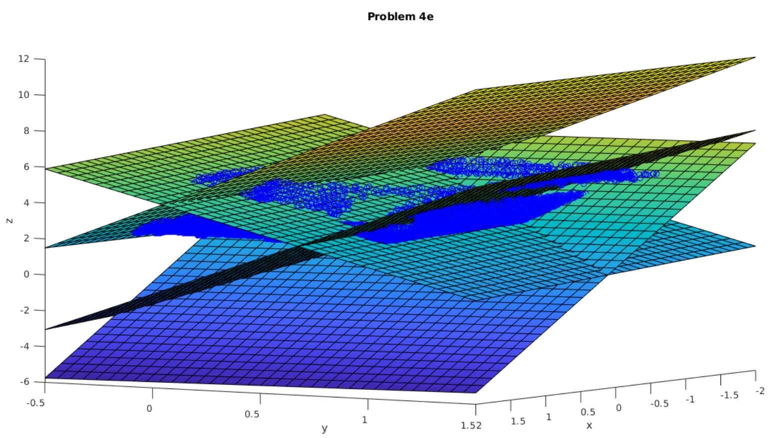

Figure 9: Plot for 4e).