16-811 Assignment 5: Resubmission 1

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1 Problem 2

The solution to this problem uses the fundamental principle of the Calculus of Variations that:

$$\frac{f(x+h) - f(x) - \nabla f(x)^T h}{h} = 0 \tag{1}$$

where the third term is df and h is a small perturbation. In this problem, we can characterize the rectangular coordinates (x,y,z) as functions of small perturbations of the circular coordinates (u,v). In this case, we would say that:

$$h = \begin{bmatrix} du \\ dv \end{bmatrix} \tag{2}$$

This allows us to write the terms dx, dy, dz in the expression for arclength as functions of perturbations in (u,v). For example, we can say $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$. We can write the expressions for dx, dy, dz as follows:

$$dx = -Rsin(v)sin(u)du + Rcos(v)cos(u)dv \tag{3}$$

$$dy = Rsin(v)cos(u)du + Rcos(v)sin(u)dv \tag{4}$$

$$dz = -Rsin(v)dv \tag{5}$$

Therefore, we can cast the expression for the arclength in terms of u and v as follows:

$$\sqrt{dx^2 + dy^2 + dz^2} = \Big[\big(-Rsin(v)sin(u)du + Rcos(v)cos(u)dv \big)^2 + \\ \big(Rsin(v)cos(u)du + Rcos(v)sin(u)dv \big) + \big(-Rsin(v)dv \big)^2 \Big]^{\frac{1}{2}} \tag{6}$$

After some algebraic manipulations we get to this expression:

$$= \left[R^2du^2(sin^2(v)sin^2(u) + sin^2(v)cos^2(u)) \right.$$
$$\left. + R^2dv^2(cos^2(v)cos^2(u) + cos^2(v)sin^2(u) + sin^2(v))\right]^{\frac{1}{2}} \quad (7)$$

We can greatly simplify this expression by using the identity $cos^2\theta + sin^2\theta = 1$:

$$= \left[R^2du^2sin^2(v) + R^2dv^2\right]^{\frac{1}{2}} = R\left[sin^2(v) + (v')^2\right]^{\frac{1}{2}}du \quad (8)$$

If we were to minimize this expression, then the integrand is:

$$F = \left[sin^2(v) + (v')^2\right]^{\frac{1}{2}} \quad (9)$$

Since F does not explicitly depend on u, we can use the alternate form of the Euler-Lagrange equation:

$$v'\frac{\partial F}{\partial v'} - F = c \quad (10)$$

Where we can write $\frac{\partial F}{\partial v'}$ as:

$$\frac{\partial F}{\partial v'} = \frac{v'}{\sqrt{sin^2(v) + (v')^2}} \quad (11)$$

Plugging this expression back into the Euler-Lagrange equation and solving for v' we get:

$$v' = \frac{1}{c}\sqrt{sin^4(v) - c^2sin^2(v)} \quad (12)$$

Now I can solve for u by integrating du:

$$\int du = \int \frac{c}{\sqrt{sin^4(v) - c^2sin^2(v)}}dv \quad (13)$$

Therefore we get the final expression for u as a function of v as:

$$u(v) = -sin^{-1}\left(\frac{cot(v)}{\sqrt{\frac{1}{c^2} - 1}}\right) + k \quad (14)$$

To show that this path is on the arc of a great circle, we can show that this function actually describes a plane passing through the origin, which would intersect the sphere along a great arc of radius R. Therefore, the path itself is on the arc of the great circle. We can do this by rewriting the expression for $u(v)$ as the expression for a plane. First I will simplify the expression for $u(v)$:

$$u(v) = -sin^{-1}\left(\frac{cot(v)}{\sqrt{\frac{1}{c^2} - 1}}\right) + k \quad (15)$$

$$-(u-k) = sin^{-1}\left(\frac{cot(v)}{\sqrt{\frac{1}{c^2}-1}}\right) \tag{16}$$

$$sin(k-u) = \left(\frac{cot(v)}{\sqrt{\frac{1}{c^2}-1}}\right) \tag{17}$$

Now let's write the expression for a plane in (x, y, z) coordinates and then convert to (u, v, R) coordinates:

$$Ax + By + Cz + D = 0 \tag{18}$$

Since the plane is going to pass through the origin, we can set the intercept, D, to 0, and then plug in our expressions for (x, y, z):

$$A(Rsin(v)cos(u)) + B(Rsin(v)sin(u)) + C(Rcos(v)) = 0 \tag{19}$$

$$sin(v)(ARcos(u) + BRsin(u)) = -CRcos(v) \tag{20}$$

We can cancel out R:

$$Acos(u) + Bsin(u) = -Ccot(v) \tag{21}$$

Using the trigonometric identity: $sin(\theta + \phi) = \frac{A}{C}sin(\theta) + \frac{B}{C}cos(\theta)$, we can rewrite the left hand side as follows (where E is just a constant):

$$Esin(u + k) = -Ccot(v) \tag{22}$$

If we compare this expression with the rearranged expression for $u(v)$ above (Eq 17), we can see that they match. (The constant coefficients in the two expressions can be redefined to be equal without loss of accuracy.) Therefore we have shown that the path u(v) does indeed run along an arc of the great circle of the sphere.

2 Problem 3

References:
(1) Weisstein, Eric W. "Brachistochrone Problem." From MathWorld – A Wolfram Web Resource. `http://mathworld.wolfram.com/BrachistochroneProblem.html`
(2) Department of Mathematics, Oregon State University. "The Gradient and Directional Derivative." 1996. `https://math.oregonstate.edu/home/programs/undergrad/CalculusQuestStudyGuides/vcalc/grad/grad.html` Visit 11/21/2019.

To solve this problem I need the integrand from the Brachistochrone problem, listed below:

$$F = \frac{1}{\sqrt{-2g}} \frac{\sqrt{1+(y')^2}}{\sqrt{y_0 - y}} \tag{23}$$

I can find $F_{y'}$:

$$F_{y'} = \frac{1}{\sqrt{-2g}} \frac{y'}{\sqrt{(y_0 - y)(1+(y')^2)}} \tag{24}$$

I need to plug this expression into this modified Euler-Lagrange equation:

$$F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0 \tag{25}$$

After plugging in we get:

$$\frac{1}{\sqrt{-2g}} \frac{y'}{\sqrt{(y_0 - y)(1+(y')^2)}} - \frac{g_y}{g_x + g_y y'} \frac{1}{\sqrt{-2g}} \frac{\sqrt{1+(y')^2}}{\sqrt{y_0 - y}} = 0 \tag{26}$$

I can move the second term to the other side and square both sides:

$$\frac{(y')^2}{(-2g)(y_0 - y)(1-(y')^2)} = \frac{g_y^2}{(g_x + g_y y')^2} \frac{1}{(-2g)} \frac{1+(y')^2}{y_0 - y} \tag{27}$$

We multiply the denominators against the opposite sides of the equations, cancel some common terms and simplify the expressions to get:

$$(y')^3(2g_x g_y) + (y')^2(g_x^2 - 2g_y^2) - g_y^2 = 0 \tag{28}$$

If we divide through by g_x^2 we can rewrite this expression in terms of the gradient of $g(x, y)$, i.e. in terms of g':

$$(y')^3(2g') + (y')^2(1 - 2(g')^2) - (g')^2 = 0 \tag{29}$$

Now we can solve for y' and find that one of the roots of y' is g', i.e.:

$$y' = g' = \frac{g_y}{g_x} \tag{30}$$

Recall that $g(x, y)$ is a contour and therefore the gradient of g is perpendicular to the contour itself. Since $y' = g'$, the slope of the optimizing curve $y(x)$ is parallel to the slope of the gradient of the contour, which is perpendicular to the contour itself. Therefore, we know that the optimizing curve $y(x)$ is also orthogonal to the contour.

3 Problem 4

References:

(1) Hyperphysics. "Coriolis Force." `http://hyperphysics.phy-astr.gsu.edu/hbase/corf.html` Visited 11/21/2019.

(2) Sutherland, Mark. Brock University. "Coriolis and Centrifugal Forces." `http://www.physics.brocku.ca/fun/NEWT3D/PDF/CORIOLIS.PDF` Visited 11/21/2019.

3.1 Part (a)

In this section we are going to derive the relationships between the joint torques and the angular states for the balanced manipulator using Lagrangian Dynamics.

We need to write the Lagrangian in terms of the kinetic and potential energies: $L = T - V$. We can write T and V as follows:

$$V = m_1gh_1 + m_2gh_2 + m_2gh_3 \tag{31}$$

$$T = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_2v_2^2 \tag{32}$$

Notice that in this problem, there is no gravity acting on the arm so we will neglect the potential energy term, V, from the Lagrangian.

I can write the heights h_i as follows:

$$h_1 = l_1sin(\theta_1) \tag{33}$$
$$h_2 = l_1sin(\theta_1) - l_2cos(\theta_1 + \theta_2) \tag{34}$$
$$h_3 = l_1sin(\theta_1) + l_2sin(\theta_1 + \theta_2) \tag{35}$$

I can write the velocities as follows:

$$v_1 = l_1\dot{\theta}_1 \tag{36}$$
$$v_2^2 = \dot{x_2}^2 + \dot{y_2}^2 \tag{37}$$
$$v_3^2 = \dot{x_3}^2 + \dot{y_3}^2 \tag{38}$$

We can write $\dot{x}_i$ and $\dot{y}_i$ by first writing x_i and y_i and then taking the derivatives:

$$x_2 = l_1cos(\theta_1) + l_2cos(\theta_1 + \theta_2) \tag{39}$$
$$y_2 = l_1sin(\theta_1) + l_2sin(\theta_1 + \theta_2) \tag{40}$$
$$x_3 = l_1cos(\theta_1) - l_2cos(\theta_1 + \theta_2) \tag{41}$$
$$y_3 = l_1sin(\theta_1) - l_2sin(\theta_1 + \theta_2) \tag{42}$$

The derivatives are:

$$\dot{x}_2 = -l_1sin(\theta_1)(\dot{\theta}_1) - l_2sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \tag{43}$$
$$\dot{y}_2 = l_1cos(\theta_1)(\dot{\theta}_1) + l_2cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \tag{44}$$
$$\dot{x}_3 = -l1sin(\theta_1)(\dot{\theta}_1) + l_2sin(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \tag{45}$$
$$\dot{y}_3 = l_1cos(\theta_1)\dot{\theta}_1 - l_2cos(\theta_1 + \theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \tag{46}$$

Now I can write simplified expressions for v_i^2 as follows:

$$v_2^2 = l_1^2(\dot{\theta}_1^2) + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2l_1 l_2 cos(\theta_2)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \tag{47}$$

$$v_3^2 = l_1^2(\dot{\theta}_1^2) + l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 - 2l_1 l_2 cos(\theta_2)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2) \tag{48}$$

Now I can write out the entire Lagrangian equation as:

$$\begin{aligned} L = \frac{1}{2}m_1 l_1 \dot{\theta}_1 + \frac{1}{2}m_2\Bigg(l_1^2\dot{\theta}_1^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2l_1 l_2 cos(\theta_2)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)\Bigg) \\ + \frac{1}{2}m_2\Bigg(l_1^2\dot{\theta}_1^2 + l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 - 2l_1 l_2 cos(\theta_2)(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2)\Bigg) \end{aligned} \tag{49}$$

In general, I want to write out the torques on each joint as:

$$\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} \tag{50}$$

So I need to write out the terms in this expression for torque. Let's start with $\frac{\partial L}{\partial \theta_i}$:

$$\frac{\partial L}{\partial \theta_1} = 0 \tag{51}$$

$$\frac{\partial L}{\partial \theta_2} = 0 \tag{52}$$

Now let's write the expressions for $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i}$:

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2\Bigg(2l_1^2\dot{\theta}_1 + 2l_2^2(\dot{\theta}_1 + \dot{\theta}_2)\Bigg) \tag{53}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = 2m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) \tag{54}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_1}) = m_1 l_1^2 \ddot{\theta}_1 + 2m_2 l_1^2 \ddot{\theta}_1 + 2m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \tag{55}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_2}) = 2m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \tag{56}$$

Now I can write the expressions for τ_1 and τ_2:

$$\tau_1 = \Bigg(m_1 l_1^2 + 2m_2 l_1^2 + m_2 l_2^2\Bigg)\ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_2 \tag{57}$$

$$\tau_2 = 2m_2 l_2^2 \ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_2 \tag{58}$$

Now I can write out one equation in vector form to show the relationships between the torques and the state variables:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 l_1^2 + 2m_2 l_1^2 + 2m_2 l_2^2 & 2m_2 l_2^2 \\ 2m_2 l_2^2 & 2m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \tag{59}$$

3.2 Part (b)

When $\ddot{\theta}_2 = 0$, that means that the second link is moving at a constant velocity. In that case, τ_1 is related to $\ddot{\theta}_1$ by three terms, $m_1 l_1^2$, $2m_2 l_1^2$ and $2m_2 l_2^2$.

The first term, $m_1 l_1^2$, represents the torque equivalent of the centripetal force required to pull one mass of mass m_1 in towards the center of rotation.

The second term, $2m_2 l_1^2$, represents the torque equivalent of the centripetal force required to pull 2 masses of mass m_2 towards the center of rotation.

The third term, $2m_2 l_2^2$, represents the torque equivalent of the Coriolis force applied to the 2 masses of mass m_2 which are moving at a constant velocity. Since these 2 masses are rotating about a point (m_1) which is itself accelerating in a circular direction. The rotation of m_1 relative to the rotating masses m_2 imparts a Coriolis force on them with a lever arm of length l_2.