16-811 Assignment 4: Resubmission 1

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1 Problem 1

References for this problem:
(1) Drakos, Nikos, Moore, Ross, Robert. "Fourth Order Runge Kutta Method" 2002-01-28.
(2) `http://www.math.ubc.ca/~israel/m215/runge/runge.html` Visited 10/20/2019.
(3) `https://lpsa.swarthmore.edu/NumInt/NumIntFourth.html` Visited 11/10/2019.

1.1 Part (a)

We can rearrange this function as follows:

$$y'(x)y(x) = 1 \tag{1}$$

A possible solution for this type of equation is:

$$y(x) = \sqrt{2}\sqrt{x+c} \tag{2}$$

Where c is some constant. Given the initial condition, we can solve for c:

$$y(2) = \sqrt{2}\sqrt{2+c} = \sqrt{2} \tag{3}$$

$$c = -1 \tag{4}$$

Therefore the exact analytical solution is:

$$y(x) = \sqrt{2}\sqrt{x-1} \tag{5}$$

1.2 Part (b)

Please see my code contained in code/q1.m for the solution to this problem. The plots of both the Euler's method solution and the error over the interval [1,2] are shown in Figure 1 below. The table of solutions is also given below.

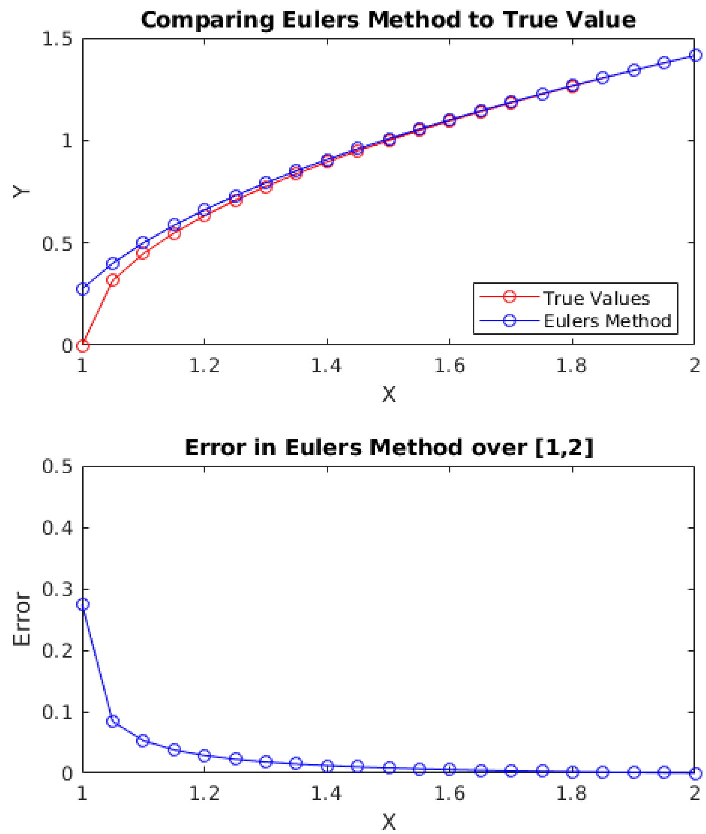
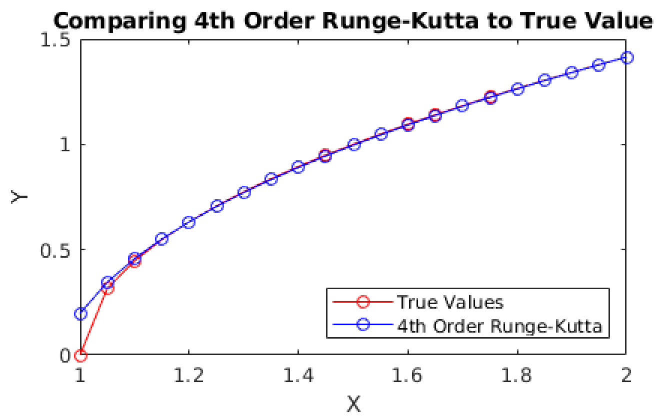


Figure 1: Plot of Euler's method over interval [1,2].

x_i	$y(x_i)$	y_i	Error
2.0000	1.4142	1.4142	0
1.9500	1.3784	1.3789	0.0005
1.9000	1.3416	1.3426	0.0010
1.8500	1.3038	1.3054	0.0015
1.8000	1.2649	1.2671	0.0021
1.7500	1.2247	1.2276	0.0028
1.7000	1.1832	1.1869	0.0036
1.6500	1.1402	1.1447	0.0046
1.6000	1.0954	1.1011	0.0056
1.5500	1.0488	1.0556	0.0068
1.5000	1.0000	1.0083	0.0083
1.4500	0.9487	0.9587	0.0100
1.4000	0.8944	0.9065	0.0121
1.3500	0.8367	0.8514	0.0147
1.3000	0.7746	0.7926	0.0181
1.2500	0.7071	0.7296	0.0225
1.2000	0.6325	0.6610	0.0286
1.1500	0.5477	0.5854	0.0377
1.1000	0.4472	0.5000	0.0528
1.0500	0.3162	0.4000	0.0838
1.0000	0	0.2750	0.2750

1.3 Part (c)

Please see my code contained in code/q1.m for the solution to this problem. The plots of both the 4th order Runge-Kutta method solution and the error over the interval [1,2] are shown in Figure 2 below. The table of solutions is also given below.



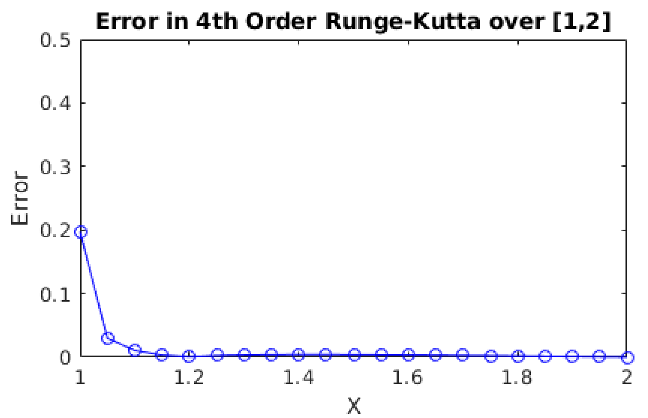


Figure 2: Plot of 4th order Runge-Kutta method over interval [1,2].

x_i	$y(x_i)$	y_i	Error
2.0000	1.4142	1.4142	0
1.9500	1.3784	1.3780	0.0004
1.9000	1.3416	1.3408	0.0009
1.8500	1.3038	1.3025	0.0013
1.8000	1.2649	1.2631	0.0018
1.7500	1.2247	1.2226	0.0022
1.7000	1.1832	1.1806	0.0026
1.6500	1.1402	1.1372	0.0030
1.6000	1.0954	1.0921	0.0033
1.5500	1.0488	1.0452	0.0036
1.5000	1.0000	0.9961	0.0039
1.4500	0.9487	0.9446	0.0041
1.4000	0.8944	0.8904	0.0041
1.3500	0.8367	0.8328	0.0039
1.3000	0.7746	0.7712	0.0034
1.2500	0.7071	0.7047	0.0024
1.2000	0.6325	0.6320	0.0005
1.1500	0.5477	0.5508	0.0031
1.1000	0.4472	0.4578	0.0105
1.0500	0.3162	0.3458	0.0295
1.0000	0	0.1975	0.1975

1.4 Part (d)

Please see my code contained in code/q1.m for the solution to this problem. The plots of both the 4th order Adams Bashforth method solution and the error over the interval [1,2] are shown in Figure 3 below. The table of solutions is also given below.

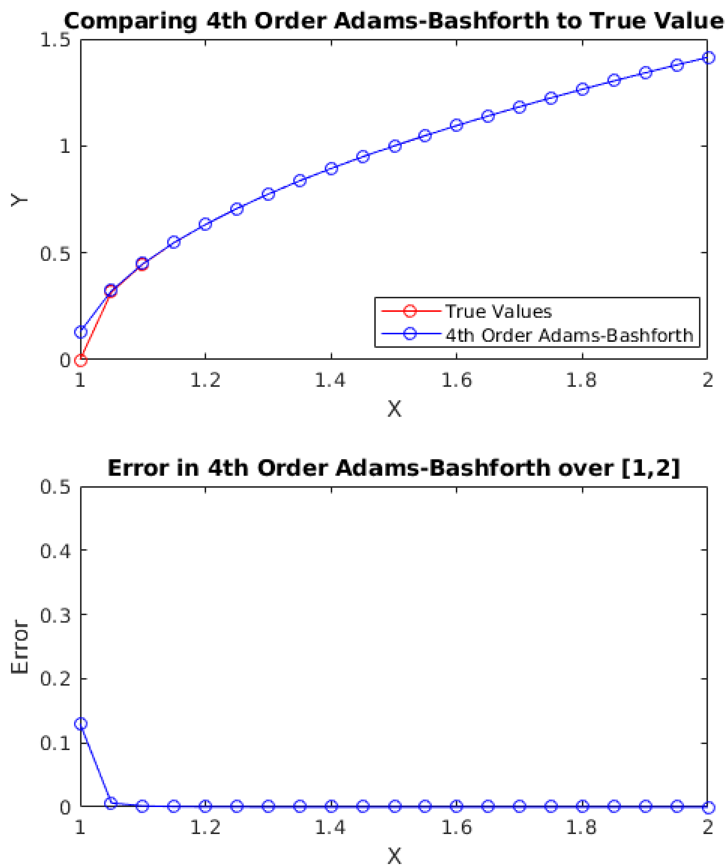


Figure 3: Plot of 4th order Adams Bashforth method over interval [1,2].

x_i	$y(x_i)$	y_i	Error
2.0000	1.4142	1.4142	0.0000
1.9500	1.3784	1.3784	0.0000
1.9000	1.3416	1.3416	0.0000
1.8500	1.3038	1.3038	0.0000
1.8000	1.2649	1.2649	0.0000
1.7500	1.2247	1.2247	0.0000
1.7000	1.1832	1.1832	0.0000
1.6500	1.1402	1.1402	0.0000
1.6000	1.0954	1.0955	0.0000
1.5500	1.0488	1.0488	0.0000
1.5000	1.0000	1.0000	0.0000
1.4500	0.9487	0.9487	0.0000
1.4000	0.8944	0.8945	0.0000
1.3500	0.8367	0.8367	0.0001
1.3000	0.7746	0.7747	0.0001
1.2500	0.7071	0.7073	0.0002
1.2000	0.6325	0.6328	0.0003
1.1500	0.5477	0.5484	0.0007
1.1000	0.4472	0.4489	0.0017
1.0500	0.3162	0.3225	0.0063
1.0000	0	0.1301	0.1301

2 Problem 5

References for this problem:
(1) http://www.ai.mit.edu/courses/6.891-nlp/lagrange.pdf

We can define the function $f(x, y) = xy$ to describe the area of the figure that we are trying to maximize. We can write the constraint as $h(x, y) = 2x + 2y - p$ where p is the fixed perimeter of the figure. Now we can use the method of Lagrangian multipliers to find the length and width of the rectangle with the greatest area for a fixed perimeter:

$$F(x, y, \lambda) = f(x, y) + \lambda h(x, y) = xy + \lambda(2x + 2y - p) \tag{6}$$

We take the gradient of $F(x, y, \lambda)$:

$$\nabla F = \begin{bmatrix} \frac{\partial f}{\partial x} + \lambda \frac{\partial h}{\partial x} \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial h}{\partial y} \\ h \end{bmatrix} = \begin{bmatrix} y + \lambda(2) \\ x + \lambda(2) \\ 2x + 2y - p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{7}$$

Now we have a system of 3 equations that we can solve to find x and y in terms of p, as follows. Given that $y = -2\lambda$ and $x = 2\lambda$, we can solve for $\lambda = \frac{-p}{8}$. We can substitute this value back into the expressions for x and y and find that $y = x = \frac{p}{4}$. In other words, the rectangle with the most area for a given perimeter is a square.

Now let's check the second order sufficiency conditions. The first condition states that: $\exists x^*$ s.t. $h(x^*) = 0$ and $\exists \lambda \in \mathbb{R}$ s.t. $\nabla f(x^*) + \lambda^T \nabla h(x^*) = 0$. If we assume that $x^* = \begin{bmatrix} \frac{p}{4} \\ \frac{p}{4} \end{bmatrix}$ and $\lambda = \frac{-p}{8}$ then:

$$\begin{bmatrix} \frac{p}{4} \\ \frac{p}{4} \end{bmatrix} + (\frac{-p}{8}) \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0 \tag{8}$$

Therefore the first condition has been met. The second condition is $L(x^*) = \nabla^2 f(x^*) + \lambda^T \nabla^2 h(x^*)$ is negative definite over the plane $M = \{y : y^T \nabla h(x^*) = 0\}$ in order to maximize x.

We can calculate the terms in $L(x^*)$ below:

$$\nabla f(x^*) = \begin{bmatrix} y \\ x \end{bmatrix} \tag{9}$$

$$\nabla^2 f(x^*) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{10}$$

$$\nabla h(x^*) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{11}$$

$$\nabla^2 h(x^*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{12}$$

Therefore $L(x^*) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + (\frac{-p}{8}) \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $M = \{y : y^T \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 0\}$. From the definition of M, we can see that one non-trivial solution for y is $y = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. To show that $L(x^*)$ is negative definite over M I can write the following:

$$y^T L(x^*) y < 0 \tag{13}$$

$$\begin{bmatrix} -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \right) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -2 < 0 \tag{14}$$

Therefore we can say that $L(x^*)$ is negative definite over M and so the second condition has been met.