16-811 Assignment 2

Emma Benjaminson

15 October 2019

1 Problem 1

NOTE: I used the following references to help me solve this problem:

(1) Kaw, A. and Keteltas, M. "Newton's Divided Difference Interpolation." 23 Dec 2009. `http://mathforcollege.com/nm/mws/gen/05inp/mws_gen_inp_txt_ndd.pdf`

1.1 Part (a)

Please see code located in code/q1.m for the implementation of my divided differences function.

1.2 Part (b)

Please see code located in code/q1.m for the solution to this problem. I used my divided differences function to calculate an interpolated value of 0.716531556049305 for $x = \frac{1}{3}$, which matches the correct answer, 0.716531310573789, to six decimal places.

1.3 Part (c)

Please see code located in code/q1.m for the full solution to this problem. I have plotted the interpolated and true values against the number of points used to interpolate in Figure 1. The actual value of $f(0.05) = 0.9615$. I found that increasing the number of sample points in this instance worked to improve the accuracy of my interpolated values exponentially. The true values were:

1. n = 2: px = 0.997647058823529
2. n = 4: px = 0.990136470588236
3. n = 40: px = 0.961538461538461

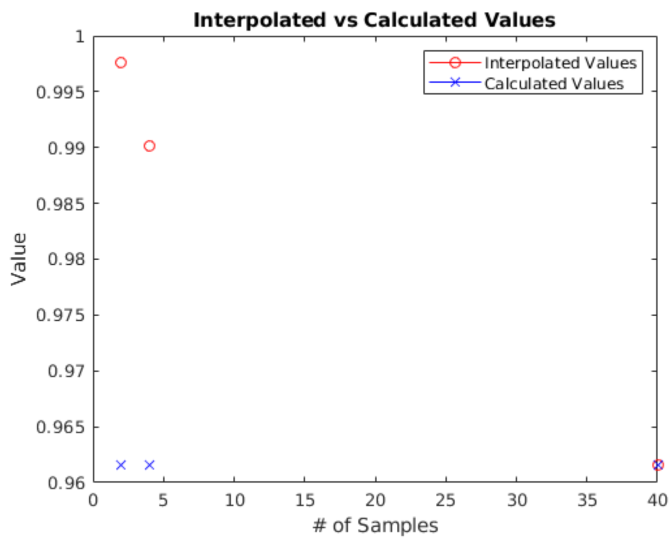

Figure 1: Plot of interpreted and actual values for sample sizes of 2, 4 and 40.

1.4 Part (d)

Please see the code located in code/q1.m for the full solution to this problem. I have plotted the error estimates for the given function against the number of sample points in Figure 2 below. The plot shows that the error is very low (10^{-4}) for sample sizes up to 20, and then at 40, the error estimate increases. Looking at a plot of the actual function, we are trying to find a point on a steeply sloping part of the curve. It makes sense that the smaller the interval between points, the more likely it is that we can see significant (but still on the order of 10^{-4}) changes in the error because we have an increased chance of being slightly further from the true value.
The estimated errors for each sample size are listed below:

1. n = 2: E = $0.000000000000139 \times 10^{-4}$
2. n = 4: E = $0.000000000006800 \times 10^{-4}$
3. n = 6: E = $0.000000000033445 \times 10^{-4}$
4. n = 8: E = $0.000000000108663 \times 10^{-4}$
5. n = 10: E = $0.000000001032369 \times 10^{-4}$
6. n = 12: E = $0.000000004296702 \times 10^{-4}$
7. n = 14: E = $0.000000003728268 \times 10^{-4}$
8. n = 16: E = $0.000000069883960 \times 10^{-4}$

9. n = 18: E = $0.000000077842038 \times 10^{-4}$
10. n = 20: E = $0.000000092688218 \times 10^{-4}$
11. n = 40: E = $0.231813310700296 \times 10^{-4}$

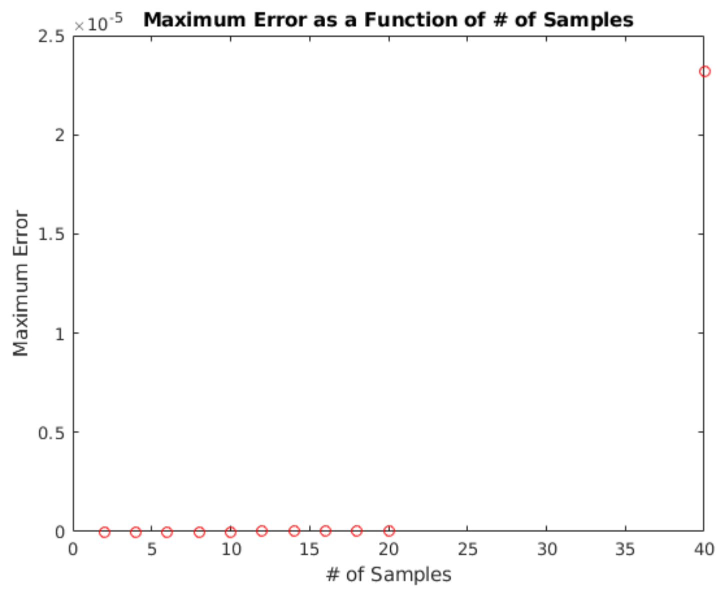

Figure 2: Plot of estimated error for divided differences interpolation function against number of sample points used.

2 Problem 2

NOTE: I used the following references to help solve this problem:

(1) Qiu, Jingmei. "Polynomial Interpolation - Error Analysis." University of Houston course notes. `https://www.math.uh.edu/~jingqiu/math4364/interp_error.pdf`

In this problem we have:

$$f(x) = cos(x), x \in \left[-\frac{\pi}{2}, \frac{3\pi}{2} \right] \tag{1}$$

The general form expression for the error in our interpolation is:

$$e_n(\bar{x}) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (\bar{x} - x_i) \tag{2}$$

In this problem, x is a collection of uniformly spaced points over a range of 2π. If I have n points, then the spacing between adjacent points is $h = \frac{2\pi}{n}$. This allows me to say that every point can be written as:

$$x_i = -\frac{\pi}{2} + ih = -\frac{\pi}{2} + i\left(\frac{2\pi}{n}\right) \tag{3}$$

My objective in this problem is to find the value of h (the spacing between points) required to achieve 6 decimal digit accuracy. First let's solve this problem for the linear interpolation case (with a second order polynomial) and then expand to the quadratic interpolation case (with a third order polynomial).

2.1 Linear Interpolation Solution

Let's rewrite Eq 2 in terms of a linear interpolation. First we can say that:

$$\frac{f''(\xi)}{2!} = -\frac{cos(\xi)}{2} \tag{4}$$

I also know that my product contains 2 terms as follows:

$$(\bar{x} - x_0)(\bar{x} - x_1) \tag{5}$$

And given my expression for x_i in Eq 3, I can rewrite the entire equation as:

$$e_1(\bar{x}) = -\frac{cos(\xi)}{2}(\bar{x} - (-\frac{\pi}{2}))(\bar{x} - (-\frac{\pi}{2} + h)) \tag{6}$$

Now since I cannot find ξ exactly I need to bound the error by finding what it could be in the worst case. The worst case assumes that each component of the equation is maximized. I am going to find the maximum of the cosine component first as:

$$\max_{-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}} \left| -\frac{cos(x)}{2} \right| = \frac{1}{2} \tag{7}$$

Now I need to maximize the product:

$$\max_{\bar{x} \in [x_0, x_1]} |(\bar{x} - x_0)(\bar{x} - x_1)| \leq \max_{y \in [-h,h]} |y(y-h)| \tag{8}$$

$$= \max_{y \in [-h,h]} |y^2 - yh| \tag{9}$$

Looking at the derivative to find a maximum, we can see that:

$$y = \frac{h}{2} \tag{10}$$

Now we can say that:

$$\max_{\bar{x} \in [x_0, x_1]} |(\bar{x} - x_0)(\bar{x} - x_1)| \leq \max_{y \in [-h,h]} |\frac{h^2}{4}| \tag{11}$$

Therefore the worst case error is:

$$e_1(\bar{x}) = \frac{h^2}{8} \tag{12}$$

If our initial requirement is that the error must be smaller than 6 decimal places, I can write that as:

$$\frac{h^2}{8} \leq 5 \times 10^{-7} \tag{13}$$

This seems to indicate that $h = 0.002$ and so the number of sample points is calculated as:

$$n = \frac{2\pi}{0.002} = 3141.6 \tag{14}$$

We round up to 3142 and add 1 because n is the number of intervals, not the number of points. So the total number of points required is 3143.

2.2 Quadratic Interpolation Solution

I can follow the same logic as presented above:

$$\frac{f'''(\xi)}{3!} = \frac{sin(\xi)}{6} \tag{15}$$

And the product contains 3 terms:

$$(\bar{x} - x_0)(\bar{x} - x_1)(\bar{x} - x_2) \tag{16}$$

Putting it all together I get:

$$e_2(\bar{x}) = \frac{sin(\xi)}{6}(\bar{x} - x_0)(\bar{x} - x_1)(\bar{x} - x_2) \tag{17}$$

Let's maximize each element:

$$\max_{-\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}} \left| \frac{sin(\xi)}{6} \right| = \frac{1}{6} \tag{18}$$

$$\max_{\bar{x} \in [x_0, x_2]} |(\bar{x} - x_0)(\bar{x} - x_1)(\bar{x} - x_2)| \leq \max_{y \in [-h,h]} |(y+h)y(y-h)| \tag{19}$$

$$= \max_{y \in [-h,h]} |y^3 - yh^2| \tag{20}$$

We can maximize this expression (as shown in the lecture notes) by taking the derivative and finding that for the derivative to be 0 (a condition for a maximum):

$$3y^2 - h^2 = 0 \tag{21}$$

$$y = \pm \frac{h}{\sqrt{3}} \tag{22}$$

We can plug this value for y back into our expression above and find that the product is maximized at:

$$\frac{2}{3}\frac{1}{\sqrt{3}}h^3 \tag{23}$$

And now I can write the error and compare it to my desired accuracy:

$$e_2(\bar{x}) \leq \frac{1}{9\sqrt{3}}h^3 \leq 5 \times 10^{-7} \tag{24}$$

Solving for h we find that $h = 0.0198$ and so the number of points we need is:

$$n = \frac{2\pi}{0.0198} = 317.3 \tag{25}$$

We round up to 318 and add 1 because n is the number of intervals, not the number of points. So the total number of points required is 319.

3 Problem 5

NOTE: I used the following references to solve this problem and I had help from the TA's, thank you!

(1) Mathews, J. and Fink, K. "Numerical Methods Using Matlab, 4th Edition." Prentice-Hall Inc., 2004. `http://mathfaculty.fullerton.edu/mathews/n2003/mullersmethod/MullersMethodProof.pdf` Visited 09/21/2019.

(2) Velix, Oscar. "Muller's Method." `https://www.youtube.com/watch?v=XIIEjwtkONc` Visited 10/14/2019.

3.1 Part (a)

Please see my code in code/q5.m for my implementation of Muller's method.

3.2 Part (b)

The real root is 3. The two complex conjugate roots are $1 \pm 2i$