Assignment-6

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1 Assignments

1. Consider the identity

$$\frac{d}{dx} xJ_1(x) = xJ_0(x)$$

   (a) Write a function to generate

$$f(x) = xJ_1(x)$$

   (b) Generate the Chebyshev fit to this function over the interval (0, 5) by sampling at 50 points. Plot the magnitudes of the coefficients in a semilog plot. Determine a good cutoff for the series.

   (c) Generate the Chebyshev approximation of the function. Obtain the maximum error over the interval.

   (d) Now use `chder` to compute the Chebyshev series for

$$\frac{d}{dx}(xJ_1(x))$$

   Determine the error of the Chebyshev series by comparing it to $xJ_0(x)$.

   (e) Now try to do the same thing using differences of samples

$$\frac{df}{dx} \approx \frac{f(x+\Delta/2) - f(x-\Delta/2)}{\Delta}$$

   Obtain the accuracy of the RS in the above equation as a function of Δ. Repeat for the "non-centered" derivative:

$$\frac{df}{dx} \approx \frac{f(x+\Delta) - f(x)}{\Delta}$$

   Plot both errors vs Δ in a log-log plot, and discuss the accuracy obtained by the two methods. Note that exactly the same result can be got from Fourier, for periodic functions. Also, discuss the computational cost of obtaining the derivative this way. Is it a desirable approach?

2. Consider the function $\sin(\pi x)$ for $-1 < x \leq 1$. Obtain a chebyshev fit for the function and plot the truncated sum upto the 10th term and determine where the errors are to be found. Is the error uniform?

3. The following five functions are given, for $-1 < x \leq 1$

$$f(x) = \exp(x) \tag{1}$$

$$g(x) = \frac{1}{x^2 + \delta^2} \tag{2}$$

$$h(x) = \frac{1}{\sin^2(\pi x/2) + \delta^2} \tag{3}$$

$$u(x) = \exp(-|x|) \tag{4}$$

$$v(x) = \sqrt{x + 1.1} \tag{5}$$

The goal is to fit series approximations to the five functions and to study how well the series converge to the functions. Note that the five functions have the following distinct properties:

- $f(x)$ is a rapidly increasing function that is neither even nor periodic.
- $g(x)$ is a rational, even, aperiodic function that is smooth.
- $h(x)$ is an even, periodic function that has continuous derivatives to all orders.
- $u(x)$ is an even function that has a discontinuous derivative at $x = 0$.
- $v(x)$ has a branch cut at $x = -1.1$. It is however, regular in $-1 < x < 1$.

(a) Fit Eqs. (1), (2), (3), (4) and (5) to Chebyshev series. Determine the error of the fits as a function of the number of terms kept, for different δ. In the case of $u(x)$, try breaking the range into two parts $-1 \leq x \leq 0$ and $0 < x \leq 1$ and fit. How do the fits compare to fitting $u(x)$ over the whole range?

(b) Fit Eq. (1), (2), (3), (4) and (5) to Fourier series, and study the rate at which the coefficients decay in magnitude, for $\delta = 3$. Can you predict this decay rate from properties of the function? For $\delta = 1$ and $\delta = 0.3$ how do the series change? Plot the magnitude of the coefficients vs n for all cases on a log-log plot.

(c) Use Clenshaw to approximate the functions for a truncated Chebyshev series. Similarly do the same for Fourier. Compare the fits and compare the cost to obtain the function values. Is a Fourier approximation worth it?

2 Chebyshev and Fourier fits

Fourier Fit

A Fourier fit is a mathematical technique used to approximate a given function by representing it as a sum of sinusoidal functions. The general form of a Fourier series is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right) \right]$$

Here, a_0, a_n, and b_n are the Fourier coefficients that determine the amplitude of the corresponding cosine and sine terms. T is the period of the function.

The coefficients can be determined by integrating the product of the given function and the basis functions (cosine and sine terms) over one period and solving the resulting integrals.

Chebyshev Fit

A Chebyshev fit involves approximating a function using Chebyshev polynomials, which are defined on a specific interval, typically [-1, 1]. The Chebyshev polynomials of the first kind, $T_n(x)$, are given by the recurrence relation:

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

A Chebyshev series for a function $f(x)$ on the interval [a, b] is given by:

$$f(x) \approx \sum_{n=0}^{N} c_n T_n\left(\frac{2x-(a+b)}{(b-a)}\right)$$

Here, c_n are the Chebyshev coefficients, and N is the degree of the approximation. The choice of N determines the accuracy of the approximation.

The Chebyshev coefficients can be computed using the following integral:

$$c_n = \frac{2}{b-a}\int_a^b f(x) T_n\left(\frac{2x-(a+b)}{(b-a)}\right) dx$$

Both Fourier and Chebyshev fits are widely used in numerical analysis and signal processing to represent functions and data sets efficiently. The choice between the two depends on the characteristics of the data and the specific requirements of the application.

3 Solutions

3.1 Question 1

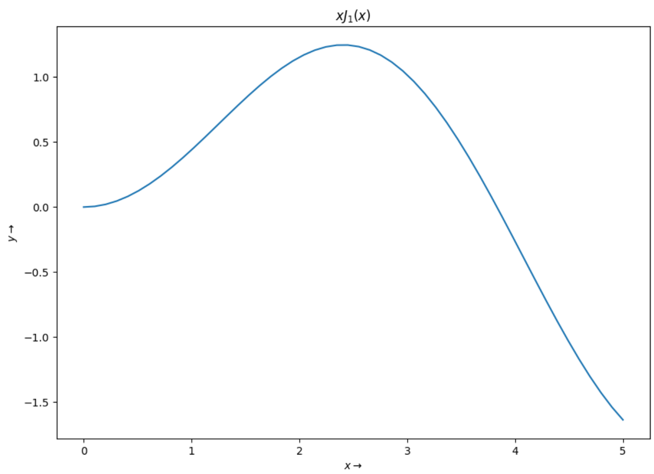

Figure 1: Function Plot

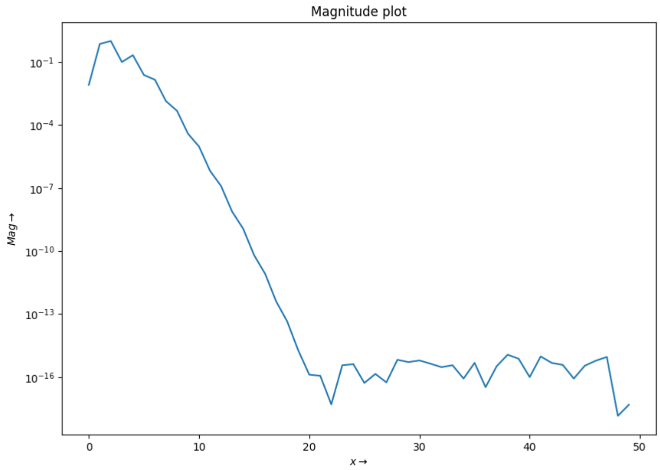

Figure 2: Magnitude of Chebyshev Coeffs.

We plot the function $f(x) = xJ_1(x)$ in Figure: 1. We next calculate 50 coefficients of the Chebyshev summation and plot their magnitude in Figure: 2. We notice that around the 20th coefficient, the magnitude of the further coefficients is insignificant.

We use the Clenshaw recursion for evaluation the Chebyshev approximation of the function. We get a maximum absolute error of 3.025e-15.

We repeat the same things for $\frac{d}{dx}(x(J_1(x))$ and we get a maximum mean absolute error of 4.443e-13.

Finally, we compare the error in the derivative calculated using the centered difference and un-centered difference formula in Figure: 3.

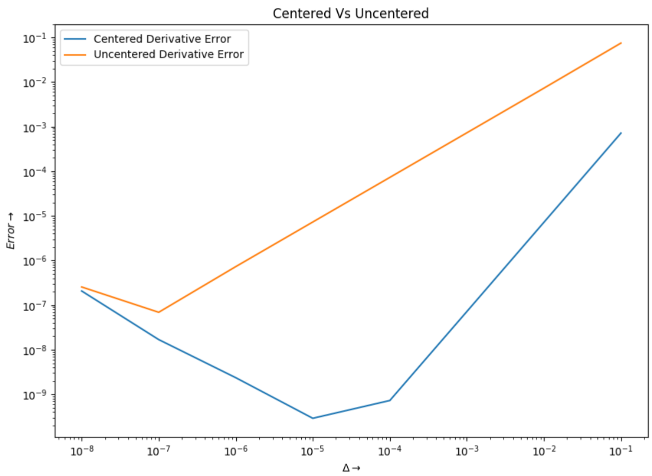

Figure 3: Centered vs Uncentered Estimate

3.2 Question 2

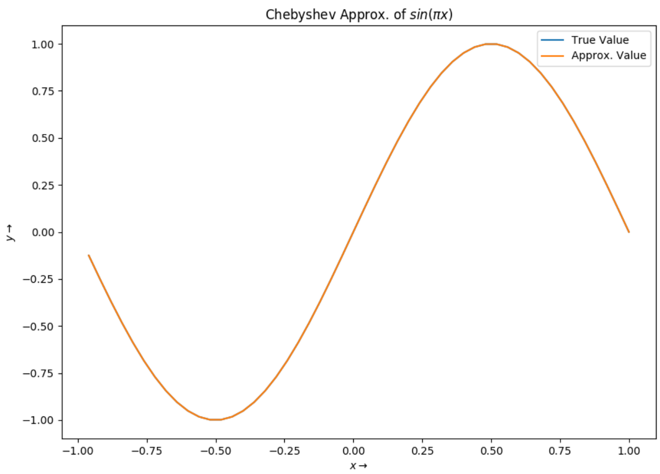

Figure 4: Fitted vs Actual Function

By first estimating 50 coefficients for $\sin(\pi x)$ we see that the alternate coefficients oscillate and around 20, the magnitude of coefficients becomes insignificant. Next, we use these 20 coefficients to obtain the Chebshev approximation and plot that along with the actual values in Figure: 4. We also plot the error in linear scale in Figure: 5. We see that the error is uniform.

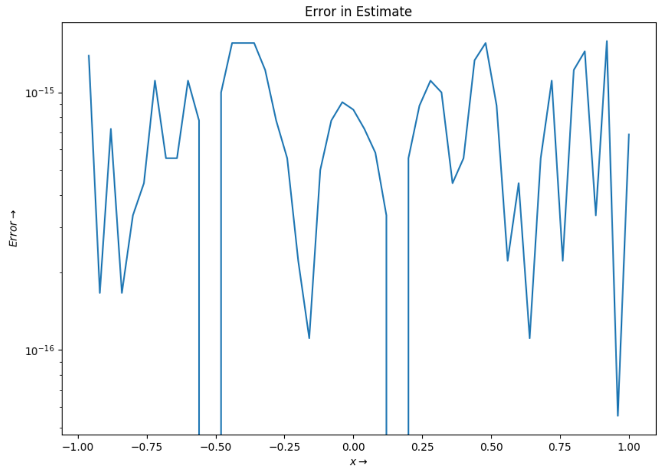

Figure 5: Error in Chebyshev fit

3.3 Question 3

We fit Eqs. (1), (2), (3), (4) and (5) to Chebyshev series. After obtaining the coefficients we plot the error of the fits as a function of the number of terms kept. Additionally, we also break $u(x)$ at 0(Kink Point) and then fit it separately. These are plotted in Figure: 6, 7, 8, 9, 10, 11

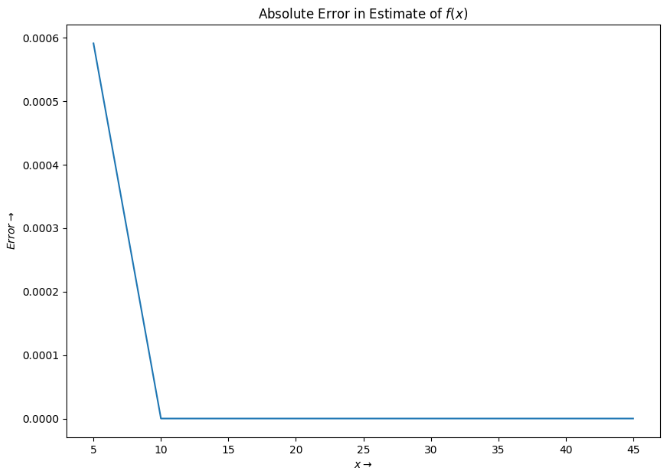

Figure 6: Chebyshev coeff. of $f(x)$

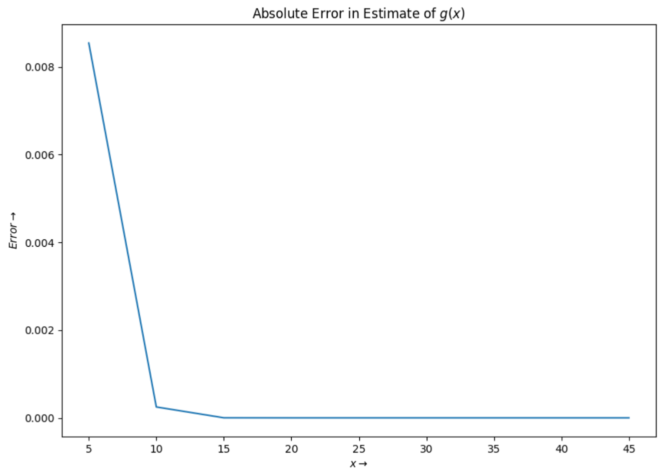

Figure 7: Chebyshev coeff. of $g(x)$

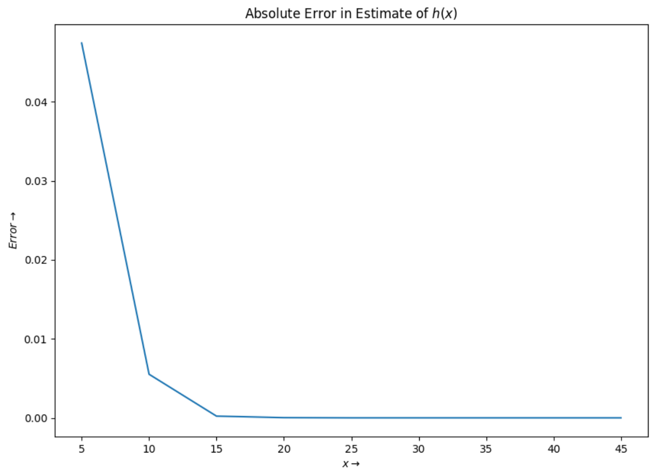

Figure 8: Chebyshev coeff. of $h(x)$

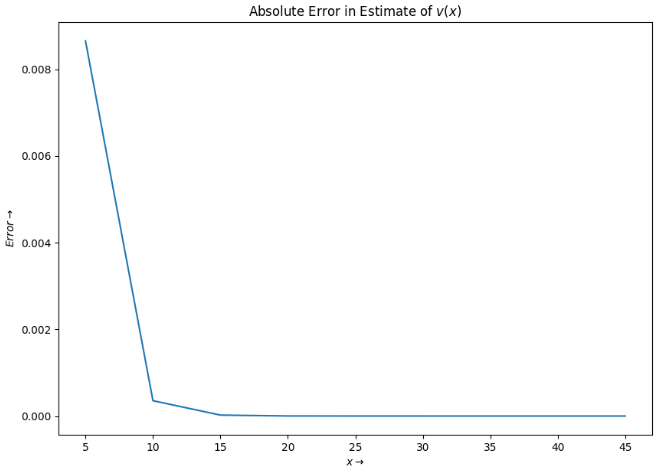

Figure 9: Chebyshev coeff. of $v(x)$

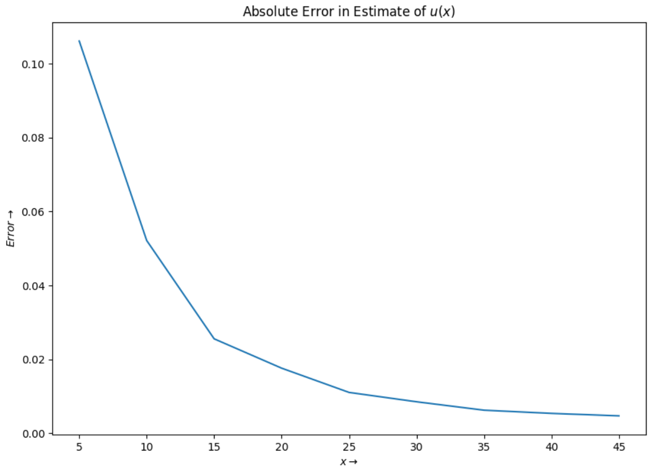

Figure 10: Chebyshev coeff. of $u(x)$

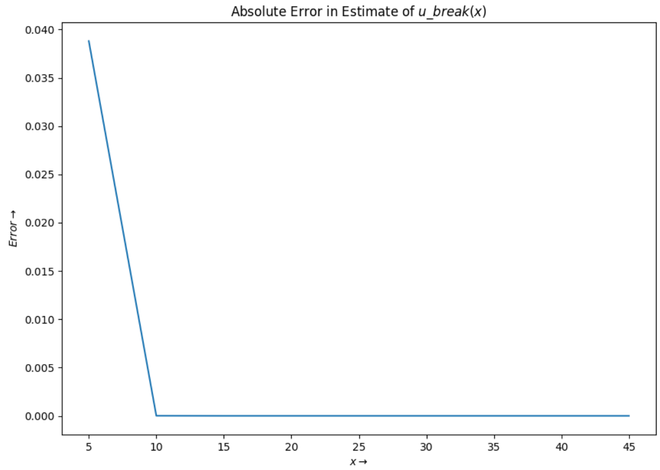

Figure 11: Chebyshev coeff. of $u\_break(x)$

We see in Figure: 10, 11 that the error of the fit of $u(x)$ dropped significantly we get almost 0 error at $n = 10$ which was not possible if we fitted $u(x)$ as it is.

Next we fit Eqs. (1), (2), (3), (4) and (5) to Fourier series. After obtaining the coefficients we plot their magnitude in a log log plot. These are plotted in Figure: 12, 13, 14, 15, 16, 17

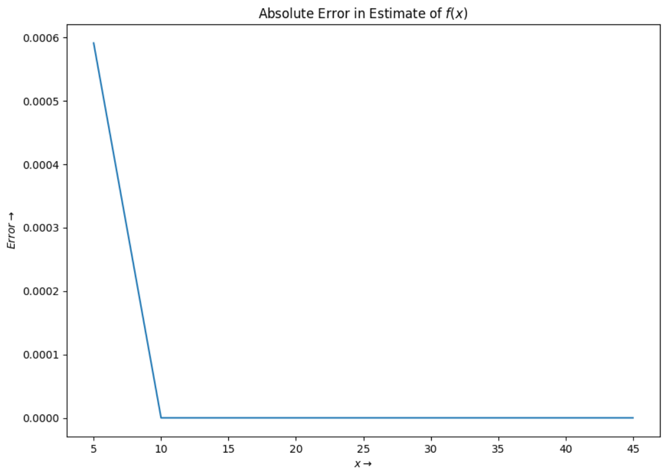

Figure 12: Chebyshev fit of $f(x)$

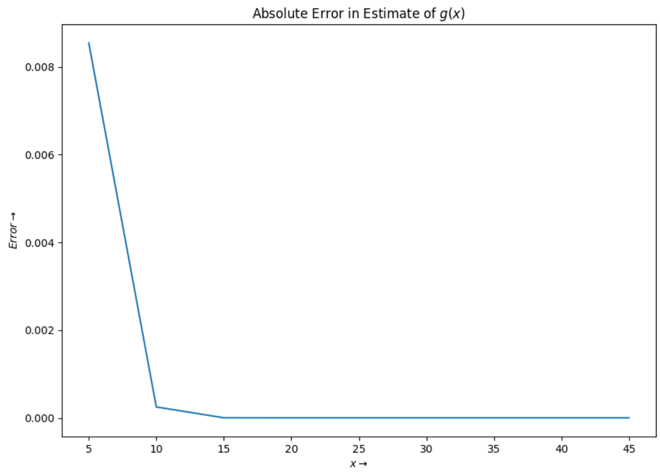

Figure 13: Chebyshev fit of $g(x)$

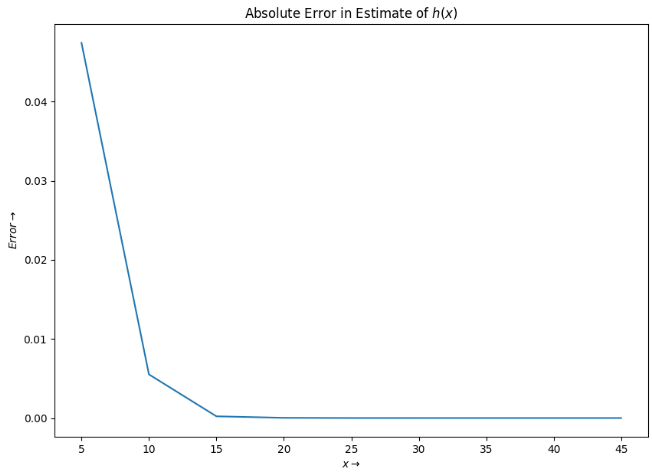

Figure 14: Chebyshev fit of $h(x)$

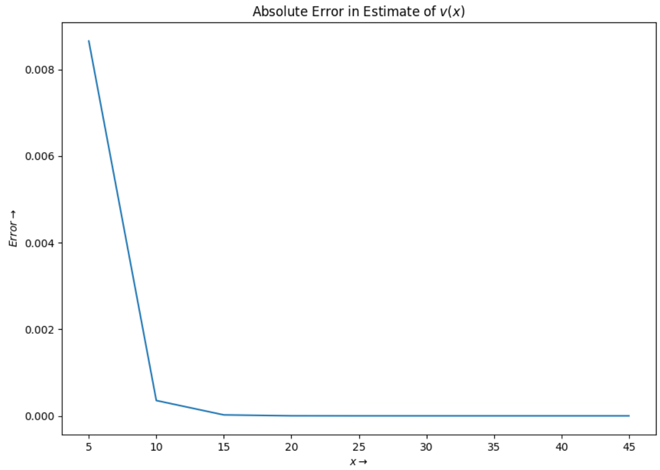

Figure 15: Chebyshev fit of $v(x)$

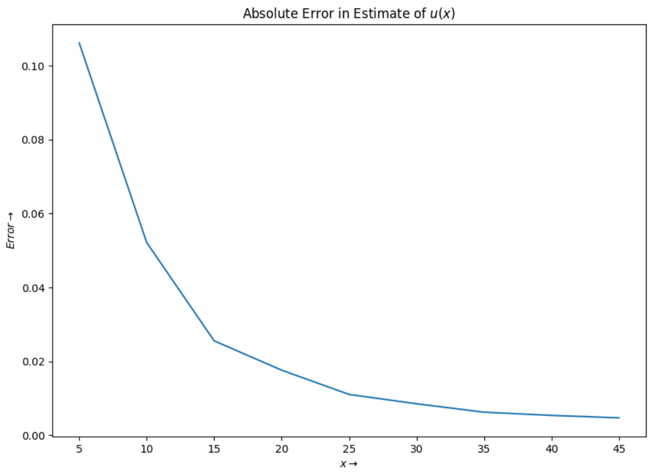

Figure 16: Chebyshev fit of $u(x)$

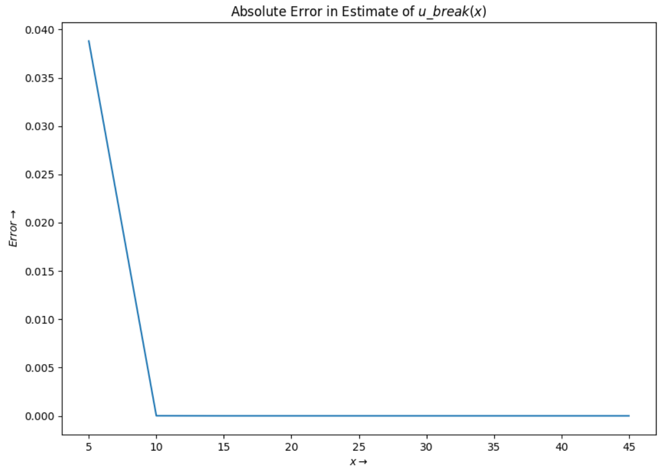

Figure 17: Chebyshev fit of $u\_break(x)$

Finally, we use Clenshaw Summation to evaluate the Fourier Approximations of the function and we plot the error vs x graph in Figure: 18, 19, 20, 21, 22.

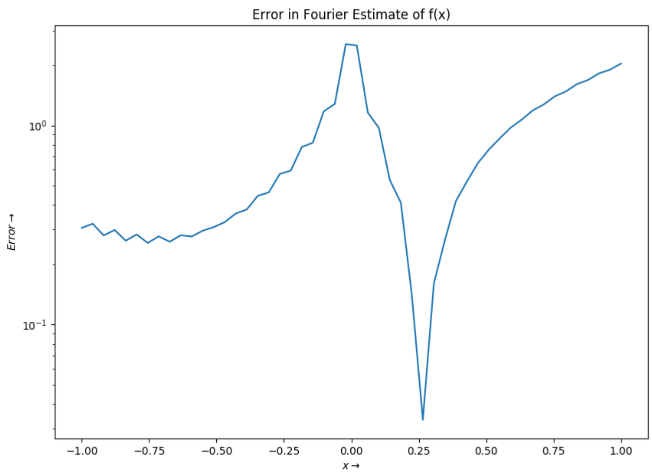

Figure 18: Fourier fit of $f(x)$

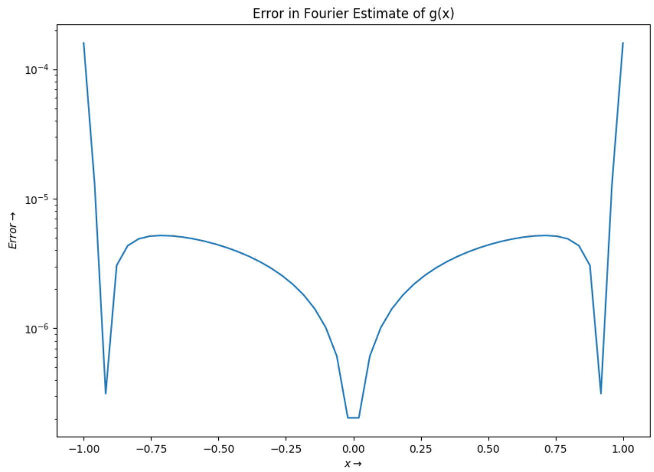

Figure 19: Fourier fit of $g(x)$

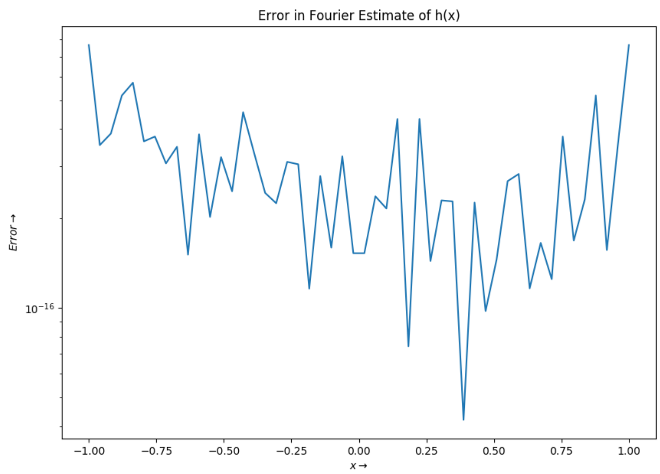

Figure 20: Fourier fit of $h(x)$

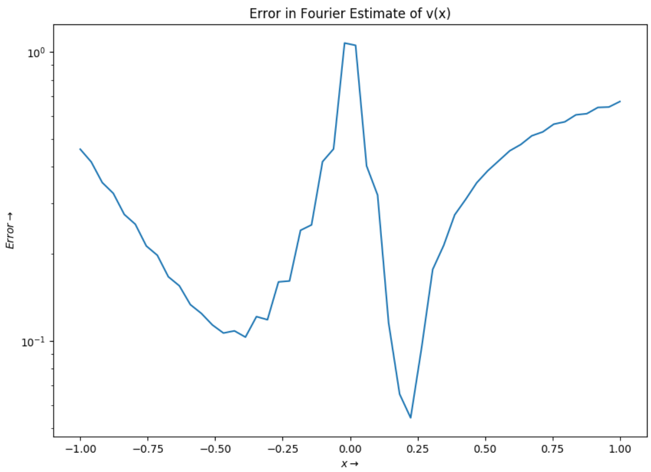

Figure 21: Fourier fit of $v(x)$

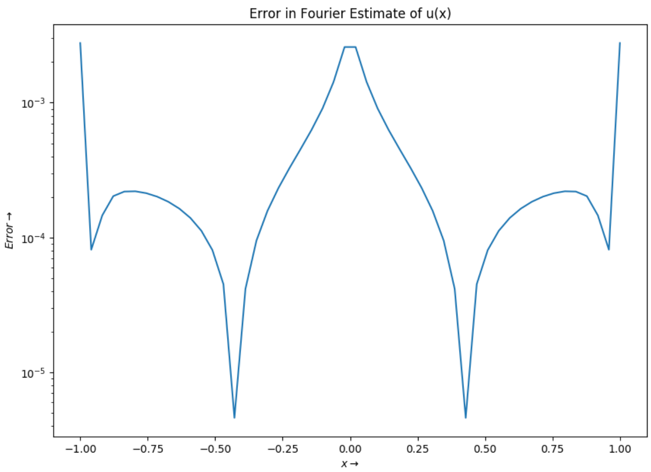

Figure 22: Fourier fit of $u(x)$

We see that Fourier fits only the periodic graph perfectly and for the rest Chebyshev performs the best if the function is analytic in the region.