EE5011: Fitting of Functions

September 24, 2018

1 Reading Portion

Chapter 5 on function evaluation of Numerical Recipes.

2 Aliasing

A Fourier or Chebyshev fit is exact for a function that is expressible as a finite sum of the basis functions. But generally, we have extra terms that are discarded. What this means is that the function is not band-limited.

As we know from DSP, if a function that is not band-limited is sampled, the digital representation suffers from aliasing. Why is this the case? After all, suppose a function is given by

$$f_i = \sum_{k=0}^{\infty} c_k p_k(x_i)$$

where $\{p_k\}$ are an orthonormal set. Then, as we proved in class,

$$\langle f|p_j\rangle = \left\langle \sum_{k=0}^{\infty} c_k p_k|p_j \right\rangle = c_j$$

which does not care if modes other than p_j are present in f or not. So why is c_k corrupted in the discrete case?

The answer lies in the evaluation of $\langle f|p_k\rangle$. In the analog domain, this inner product has an exact definition as

$$\int_a^b w(x)f(x)p_k(x)dx$$

So the analog representation does not suffer from aliasing. But the *digital inner product depends on N*. i.e.,

$$\langle f|p_k\rangle_N = \sum_{m=1}^{N} w_m f(x_m)p_k(x_m)$$

If I have more terms in my representation (i.e., higher sampling rate), *I am sampling at a different set of points, and indeed I am sampling at more points. So our estimate of c_k depends on N.*

Since Gaussian quadratures tells us that the digital representation is exact for polynomials $f(x)$ of order $2N-1$ or less, $\langle f|p_k\rangle_N$ becomes more an more accurate as N is made larger.

In the Fourier context this use of a larger N is nothing more than over sampling. In order to avoid aliasing we use a large enough number of samples so that $f(x)$, being band limited, has no components beyond a certain $p_N(x)$. The Gaussian Quadrature equivalent is that f is of finite polynomial order in x. And when it is, the correct approach is to sample at a sufficiently order so that all the details of $f(x)$ are captured.

What does this imply for the problems in this assignment? If you want to get an N term representation (either Fourier or Chebyshev), you need to start with an M term series, $M \gg N$ and then discard the higher order terms. This is because the functions in the problems *are not band limited* and have components to all order. In DSP language that means:

- Oversample the analog signal.
- Digitally low pass filter the signal
- Subsample the filtered digital signal

We do all this when signal processing in the Fourier domain. But here, the last step is not needed since we are generating an *analog* approximation to the signal, i.e., the output is not f_i but $f(x)$ for any x.

3 Generating a Fourier Fit

For Chebyshev, the routines to compute the coefficients and to evaluate the function are given.

For the fourier fit, you have that

$$\begin{aligned}\int_{-1}^{1} f(x)\cos\left(m(x+1)\pi/2\right)dx &= c_m\int_{-1}^{1}\cos^2\left(m(x+1)\pi/2\right)dx \\ &= c_m\end{aligned}$$

This involves M integrations, which is slow compared to the FFT which costs $\mathcal{O}(N\log_2 N)$ multiplications. But it is arbitrarily accurate and suffers no aliasing error.

For extra credit, try using the FFT to oversample the function and obtain the coefficients and compare the results.

Once we have the coefficients we can compute the Fourier series at a given x using direct evaluation. Next week we will study the Clenshaw algorithm for fast evaluation, that will use the identity

$$\cos(nx) = 2\cos x\cos((n-1)x) - \cos((n-2)x)$$

4 Programming Portion

1. Consider the identity
$$\frac{d}{dx}(xJ_1(x)) = xJ_0(x)$$
   (a) Write a function to generate
$$f(x) = xJ_1(x)$$

(b) Generate the Chebyshev fit to this function over the interval $(0,5)$ by sampling at 50 points. Plot the magnitudes of the coefficients in a semilog plot. Determine a good cutoff for the series.

(c) Generate the Chebyshev approximation of the function. Obtain the maximum error over the interval.

(d) Now use `chder` to compute the Chebyshev series for

$$\frac{d}{dx}\left(xJ_1(x)\right)$$

Determine the error of the Chebyshev series by comparing it to $xJ_0(x)$.

(e) Now try to do the same thing using differences of samples

$$\frac{df}{dx} \approx \frac{f(x+\Delta/2)-f(x-\Delta/2)}{\Delta}$$

Obtain the accuracy of the RS in the above equation as a function of Δ. Repeat for the "non-centered" derivitive:

$$\frac{df}{dx} \approx \frac{f(x+\Delta)-f(x)}{\Delta}$$

Plot both errors vs Δ in a log-log plot, and discuss the accuracy obtained by the two methods. Note that exactly the same result can be got from Fourier, for periodic functions. Also discuss the computational cost of obtaining the derivitive this way. Is it a desirable approach?

Note that if there is measurement noise, things can get quite complicated and the advantage of chebyshev fitting is no longer clear.

2. Consider the function $\sin(\pi x)$ for $-1 < x \leq 1$. Obtain a chebyshev fit for the function and plot the truncated sum upto the 10^{th} term and determine where the errors are to be found. Is the error uniform?

3. The following five functions are given, for $-1 < x < 1$.

$$\begin{aligned}
f(x) &= \exp(x) && (1)\\
g(x) &= \frac{1}{x^2+\delta^2} && (2)\\
h(x) &= \frac{1}{\sin^2(\pi x/2)+\delta^2} && (3)\\
u(x) &= \exp(-|x|) && (4)\\
v(x) &= \sqrt{x+1.1} && (5)
\end{aligned}$$

The goal is to fit series approximations to the five functions and to study how well the series converge to the functions. Note that the five functions have the following distinct properties:

- $f(x)$ is a rapidly increasing function that is neither even nor periodic.
- $g(x)$ is a rational, even, aperiodic function that is smooth.

- $h(x)$ is an even, periodic function that has continuous derivatives to all orders.
- $u(x)$ is an even function that has a discontinuous derivative at $x = 0$.
- $v(x)$ has a branch cut at $x = -1.1$. It is, however, regular in $-1 < x < 1$.

(a) Fit Eqs. (1), (2), (3), (4) and (5) to Chebyshev series. Determine the error of the fits as a function of the number of terms kept, for different δ. In the case of $u(x)$, try breaking the range into two parts $-1 \leq x \leq 0$ and $0 < x \leq 1$ and fit. How do the fits compare to fitting $u(x)$ over the whole range?

(b) Fit Eq. (1), (2), (3), (4) and (5) to Fourier series, and study the rate at which the coefficients decay in magnitude, for $\delta = 3$. Can you predict this decay rate from properties of the function? For $\delta = 1$ and $\delta = 0.3$ how do the series change? Plot the magnitude of the coefficients vs n for all cases on a log-log plot (you may have to exclude zero coefficients to keep the log-log plot routine from complaining).

Note: We expect Fourier to do well with a truly periodic function, so $h(x)$ should be handled well. The others will give trouble.

Note: Remember aliasing! If you wish to use `FFT`, sample the function at far more points than required, take the transform and then truncate. Else, even $h(x)$ will not show quick convergence of coefficients. (see the section above. You are probably better off using the exact integral to compute coefficients.)

(c) Use Clenshaw to approximate the functions for a truncated Chebyshev series. Similarly do the same for Fourier. Compare the fits and *compare the cost to obtain the function values.* Is a Fourier approximation worth it?

I have dropped Rational Chebyshev from this assignment. However, rational chebyshev and continued fractions are, in fact, the method of choice for function fits. Hence, you have to study these topics even if there is no assignment on those methods.

5 General Rules of Thumb

- If the function is analytic and periodic, use Fourier. Nothing works as well.
- If the function is analytic and aperiodic, use Chebyshev.
- If the function is piecewise smooth, use Chebyshev on each piece separately. Trying to fit the $e^{|x|}$ does not work well, while fitting e^x did work well. So just fit e^x in $0 < x < 1$ and e^{-x} in $-1 < x < 0$.
- Even if the function has singularities near the range of fitting, the Chebyshev series coefficients do fall off exponentially. However, there should not be a singularity within the range. Then, no matter how large an M we choose, we always have slowly decreasing coefficients, and are unable to truncate. This is true even when the function is

  $$f(x) = \sqrt{x+1}$$

  which is well defined $(-1, 1)$. However, its derivitive is not nice at -1, despite the function being continuous. So the Chebyshev series does not do a good job.

- The rate at which the coefficients decay is clearly linked to the distance of the poles to the region, i.e., to the radius of convergence.

- Sometimes we have a tricky function to fit. Suppose we have a function like

  $$f(x) = e^{-x} + e^{-0.01x}, \quad 0 < x < 100$$

  Then, over most of the range, it is $e^{-0.01x}$ that we see, yet the series has to have enough terms to capture the e^{-x} for small x. This can lead to a large number of terms in the Chebyshev series. Such a problem can be solved by breaking the domain into $0 < x < 5$ and $5 \leq x < 100$. Now series in both regions will converge quickly.

- If $f(x)$ has a chebyshev series,

  $$f(x) = \sum_{k=0}^{N} c_k T_k(x)$$

  we can deduce the chebyshev series for the indefinite integral of $f(x)$. This is given by

  $$\int^x f(x)dx = C + \sum_{k=1}^{N-1} \frac{c_{k-1} - c_{k+1}}{2k} T_k(x) + \frac{c_{N-1}}{2N} T_N(x)$$

  where C is the constant of integration.

- There is a similar formula for the derivative. Perhaps the most important thing about the dervitive is that unlike even the fourier series where the higher order coefficients get amplified and make the signal very choppy, the derivitive is one order lower than the fit. So it is actually smoother in some sense than the original function. The derivitive recursion formula is (for coefficients starting with index 1)

  $$d_{i+1} - d_{i-1} = -2\,(i-1)\,c_i, \quad i = 2, 3, \ldots n-1$$

  As usual, it is best to implement this using Clenshaw algorithm and get the new series.