Assignment-1

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1 Assignments

1. In python sample $\sin(x + x^2)$ from 0 to 1 at 5 points. Use these points as your table and do fourth-order interpolation on

   ```
   xx=linspace(-0.5, 1.5, 200)
   ```

   Since all the points in the table are used for 4^{th} order interpolation, this allows you to see what the effect of choosing a window that is not centered about the desired xx value.

2. Sample the same function at 30 points from 0 to 1. Now you have to choose the nearest 5 points and do fourth-order interpolation. How does the accuracy change? What is the change due to?

3. With the same table of values, vary the order of interpolation. How does the error vary? Plot the error vs x for different orders in a semi-log plot. Explain the curves you get.

4. Vary the interpolation order n from 3 to 20 and determine the way the maximum error varies with order.

5. We require a 6 digit accurate method to compute the function

   $$f(x) = \frac{\sin(\pi x)}{\sqrt{1 - x^2}} \tag{1}$$

   between 0.1 and 0.9. The function is known exactly (to 15 digits) at certain locations, $x_k = x_0 + kdx, k = 0, ..., n$ where n is the order of interpolation required.

   (a) Convert the function to a table, spaced 0.05 apart, sampling it from 0.1 to 0.9.

   (b) In Python, plot this function and determine its general behavior. Is it analytic in that region? What is the radius of convergence? What is the nature of the function's behavior near ± 1??

(c) Use the `polintroutine` to interpolate the function at a thousand points between 0.1 and 0.9 for different orders. What order gives 6-digit accuracy? Explain the convergence in terms of the table spacing and the ROC.

2 Polynomial Interpolation

Polynomial interpolation involves finding a polynomial function that passes through a given set of data points (x_i, y_i), where $i = 0, 1, 2, \ldots, n$. The goal is to find a polynomial $P(x)$ of degree m that satisfies:

$$P(x_i) = y_i \text{ for all } i = 0, 1, 2, \ldots, n.$$

The general form of a polynomial of degree m is:

$$P(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m$$

To determine the coefficients $(a_0, a_1, a_2, \ldots, a_m)$, you can set up a system of linear equations based on the given data points.

Using the data points, you have the equations:

$$\begin{aligned}
P(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_m x_0^m = y_0 \\
P(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_m x_1^m = y_1 \\
P(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + \ldots + a_m x_2^m = y_2 \\
&\ldots \\
P(x_n) &= a_0 + a_1 x_n + a_2 x_n^2 + \ldots + a_m x_n^m = y_n
\end{aligned}$$

You have $n+1$ data points, and you want to determine the coefficients $(a_0, a_1, a_2, \ldots, a_m)$ such that the polynomial $P(x)$ fits all these equations simultaneously.

This system of linear equations can be written in matrix form as:

$$X \cdot A = Y$$

Where: - X is a matrix of size $(n+1) \times (m+1)$ with elements $X_{ij} = x_i^j$ (powers of x for each data point). - A is a column vector of size $(m+1)$ with elements $A_i = a_i$ (coefficients of the polynomial). - Y is a column vector of size $(n+1)$ with elements $Y_i = y_i$ (the y-values of the data points).

To find the coefficients A, you can solve this system of linear equations, which will yield the coefficients needed to construct the interpolating polynomial $P(x)$.

Once you've determined the coefficients A, you can construct the interpolating polynomial:

$$P(x) = A_0 + A_1 x + A_2 x^2 + \ldots + A_m x^m$$

This polynomial will approximate the original function or data points within the specified degree of the polynomial (degree m) and can be used to estimate values at any desired point within the range of the given data.

3 Solutions

3.1 Question 1

We sample the function $\sin\left(x + x^2\right)$ at 5 points in between 0 to 1 and perform 4^{th} order polynomial interpolation. We get the following plot:

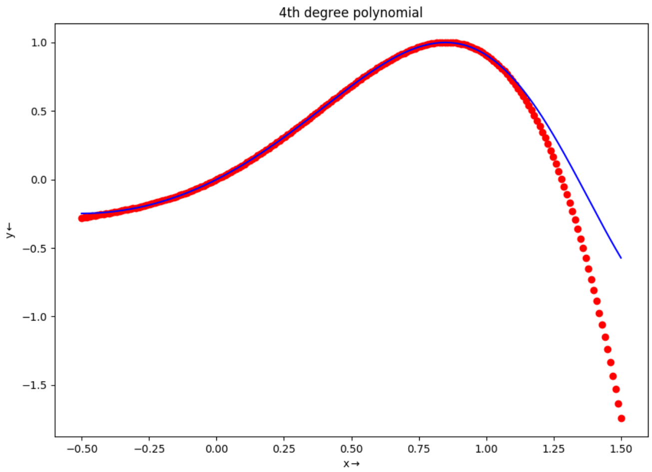


Figure 1: $4^{th} orderinterpolationof \sin\left(x + x^2\right) sampledat5points$

We plot the error in interpolation for each point in the plot below.

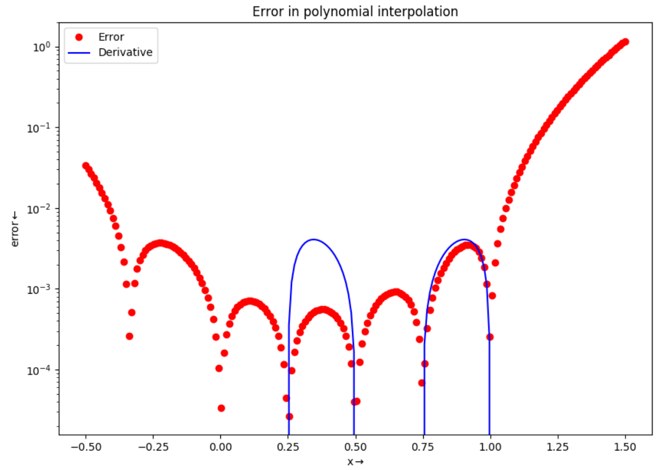


Figure 2: $Errorinestimation$

3.2 Question 2

Now we sample the $\sin(x + x^2)$ at 30 points instead of 5 and we plot the same graphs as question 1 below:

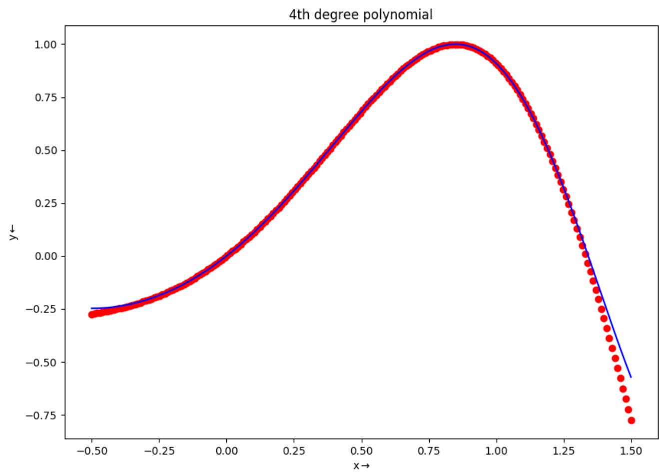


Figure 3: $4^{th} order interpolation of \sin(x + x^2) sampled at 30 points$

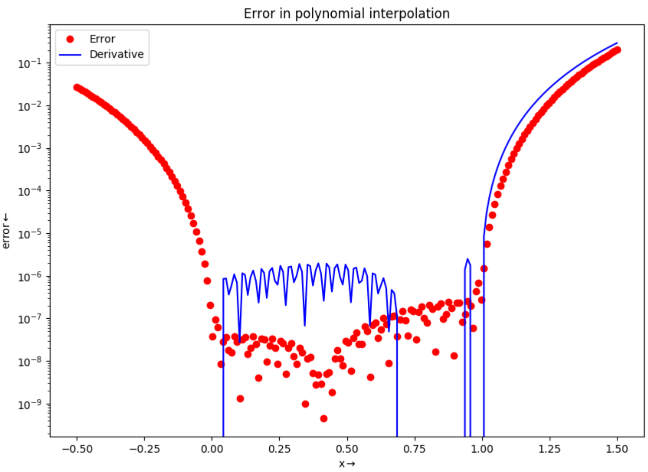


Figure 4: $Error in estimation$

3.3 Question 3&4

We now vary the order of interpolation from 3 to 20 and plot the average and maximum error for all of them below:

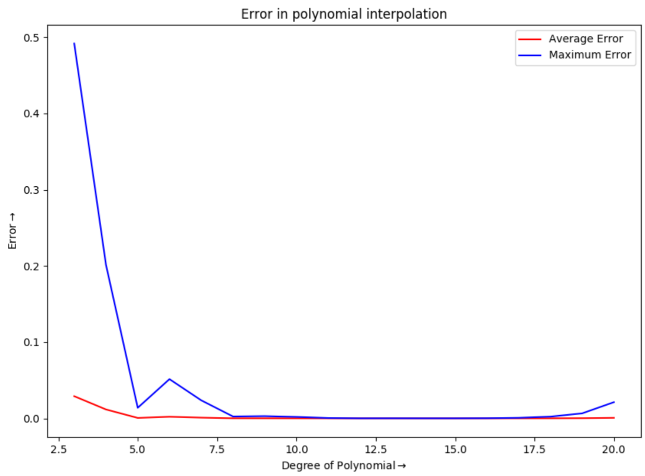
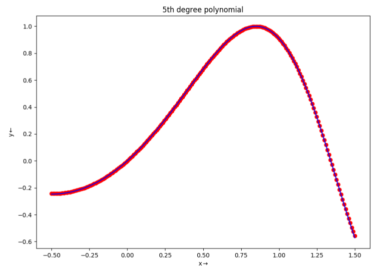
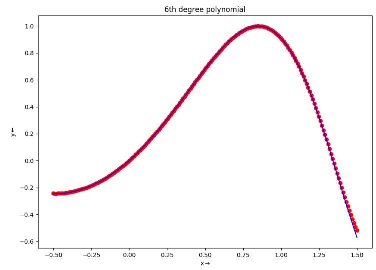


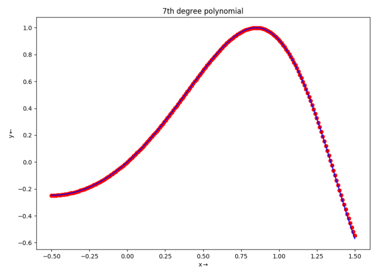
Figure 5: $Error vs Order of estimate$

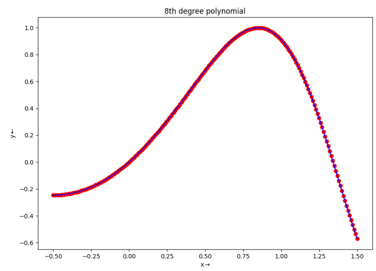
We see that as the order of polynomial increases the error in interpolation decreases. This is expected, as increasing the order of polynomial results in a better fit of the data.

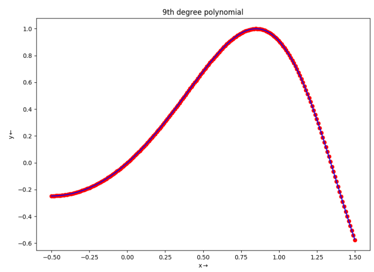
Below we plot the fitted polynomial and error for each point on semilog plot:

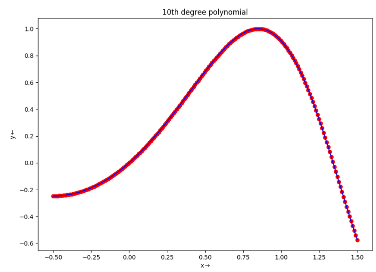


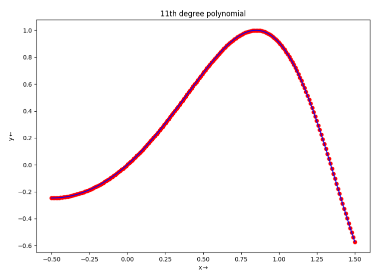


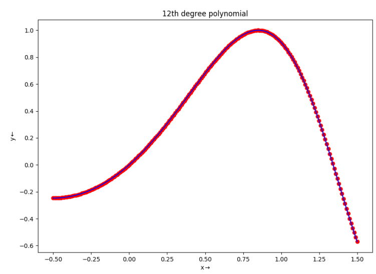












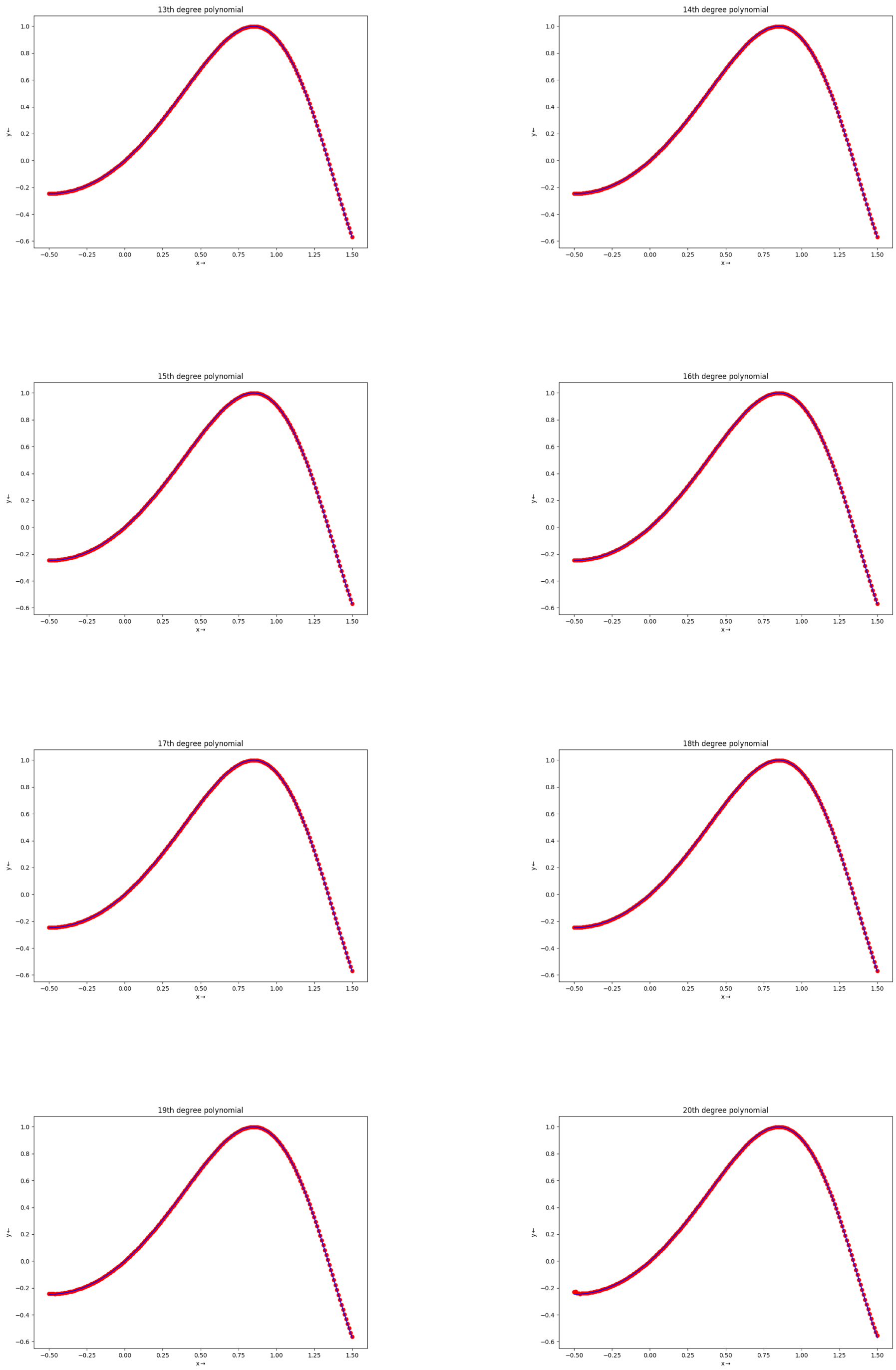
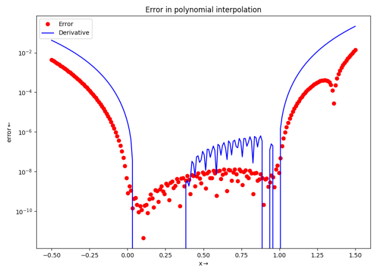
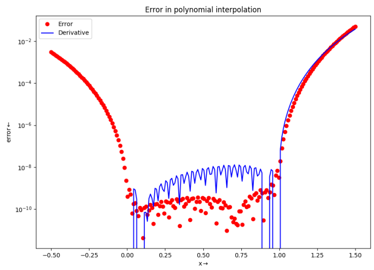


Figure 6: Polynomial Estimates


Error in polynomial interpolation
Error
Derivative
error→
x→


Error in polynomial interpolation
Error
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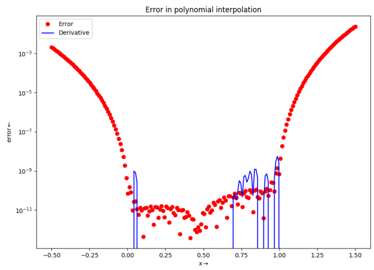
Error in polynomial interpolation
Error
Derivative
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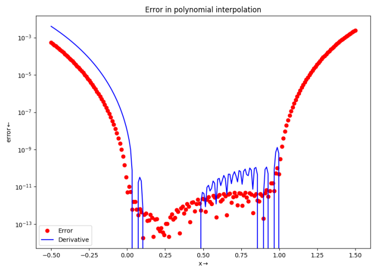
Error in polynomial interpolation
Error
Derivative
error→
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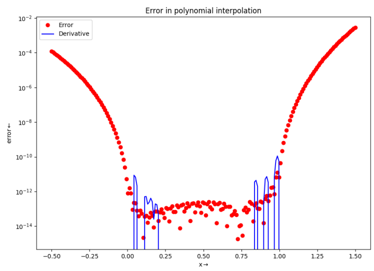
Error in polynomial interpolation
Error
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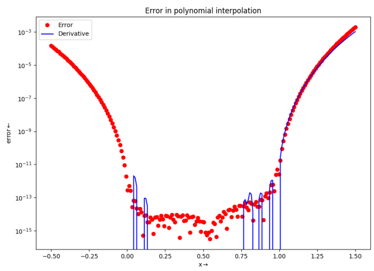
Error in polynomial interpolation
Error
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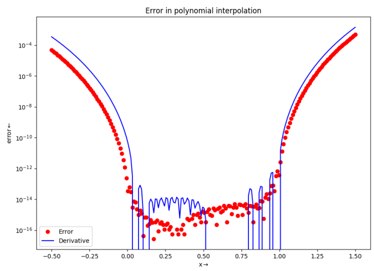
Error in polynomial interpolation
Error
Derivative
error→
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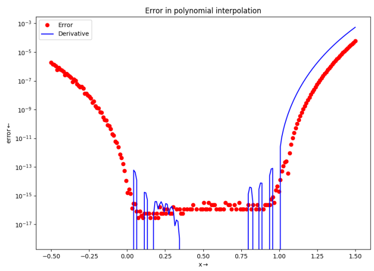

Error in polynomial interpolation
Error
Derivative
error→
x→

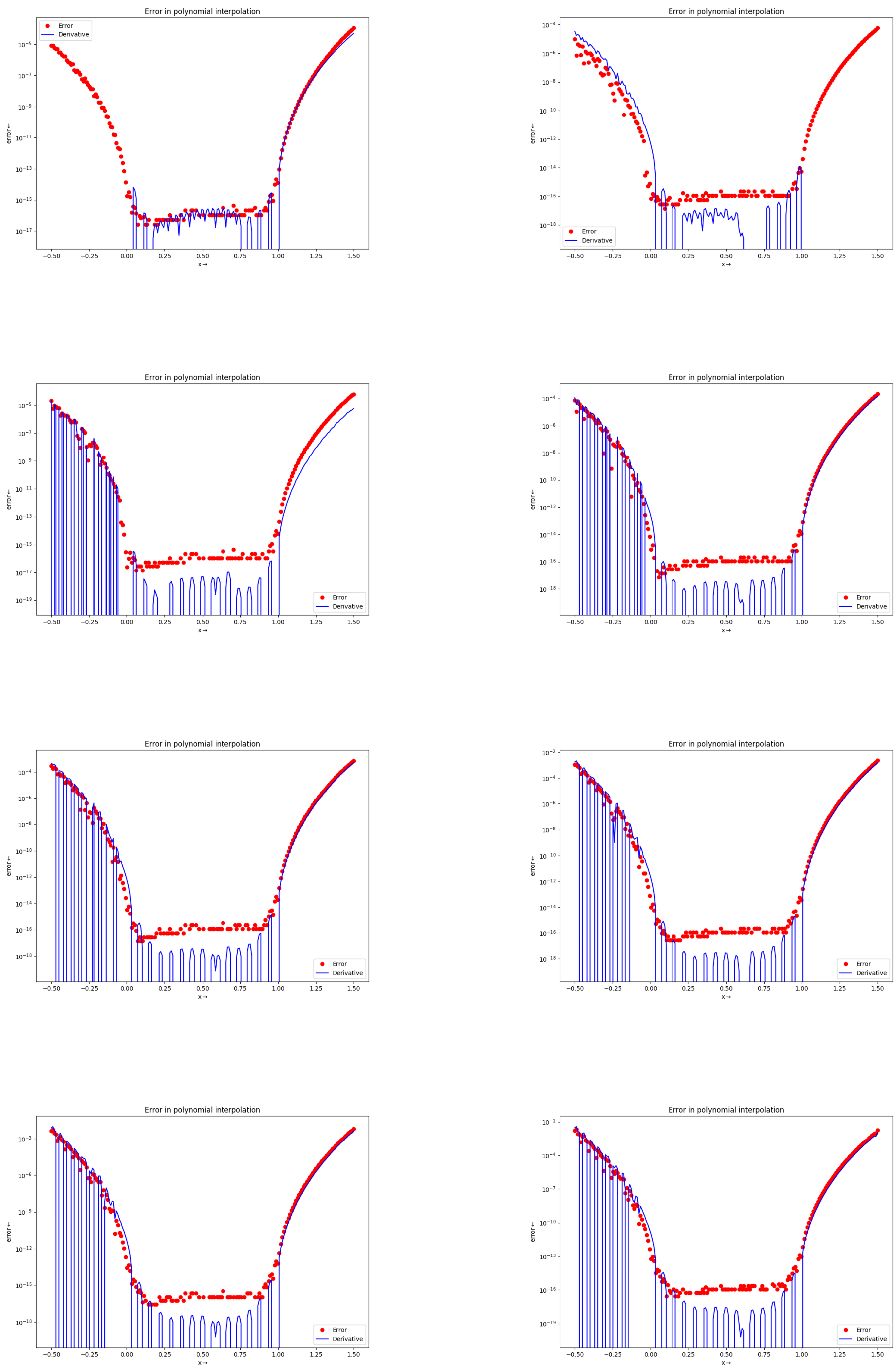


Figure 7: Error in Estimates

We see a similar trend here as well.

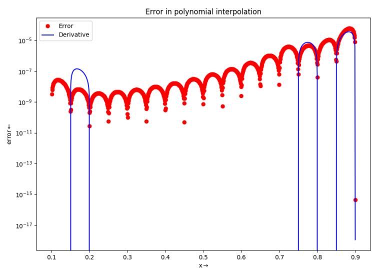
3.4 Question 5

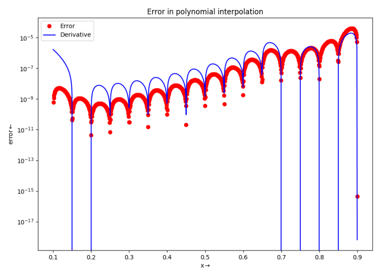
We interpolated the function

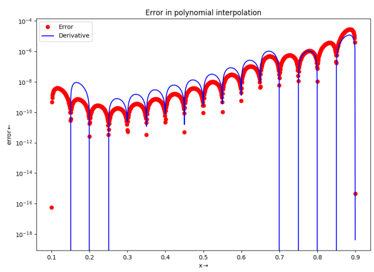
$$f(x) = \frac{\sin(\pi x)}{\sqrt{1 - x^2}}$$

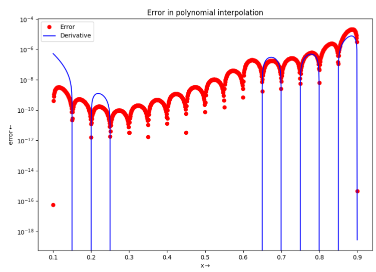
The function is sampled from 0.1 to 0.9 with a step size of 0.05. We perform polynomial interpolation for various degrees to find the degree for which the error is less than 10^{-5} which is an accuracy of 6 digits.

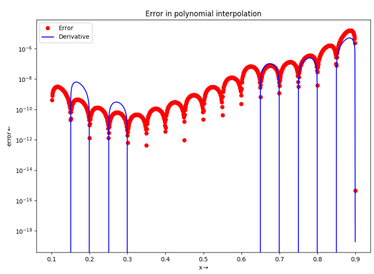
We vary the degree from 6 to 15. Below are the error and interpolation plots:

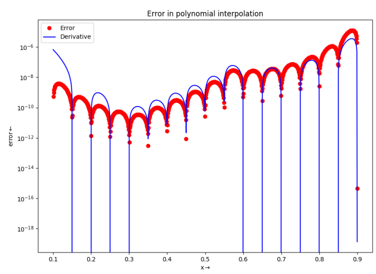












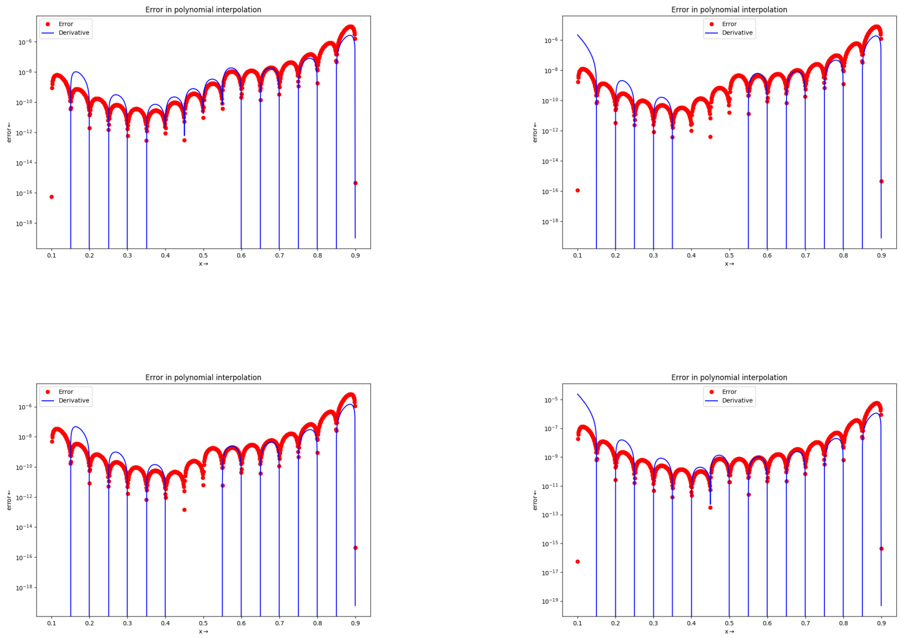
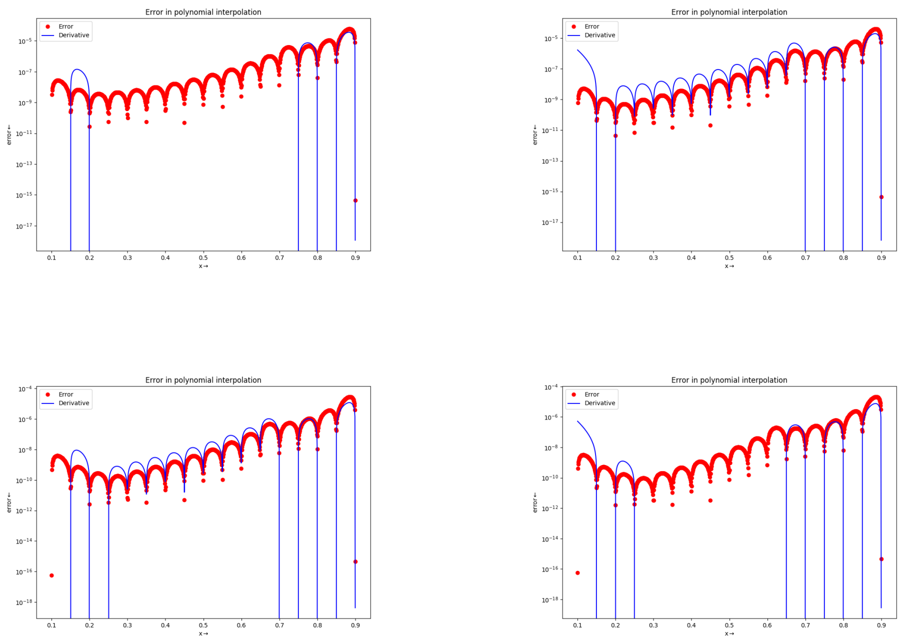


Figure 8: Error in Estimates



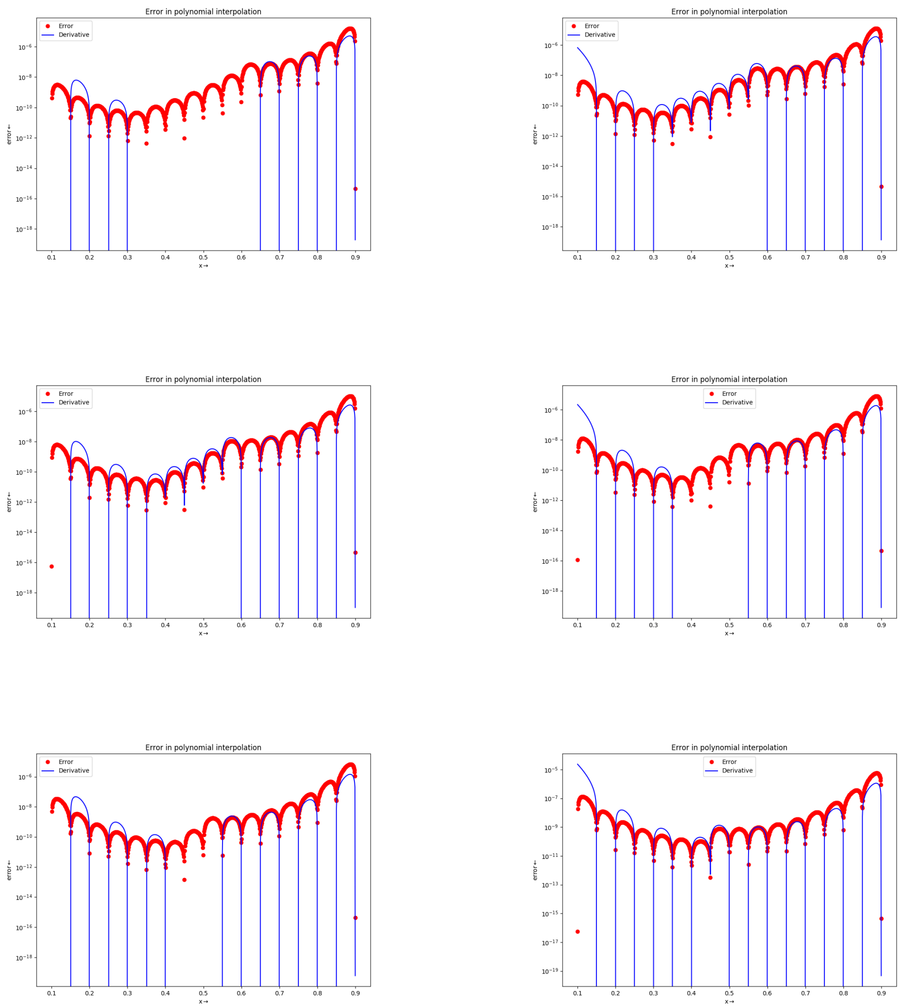


Figure 9: Polynomial Estimates

The function $f(x)$ is undefined at $x = \pm 1$ and is not defined in real plane for $|x| > 1$. $f(x)$ is well defined for (-1, 1) and hence is analytic in that region with a radius of convergence(R) = 1, i.e. the function is differentiable everywhere in the region (-1, 1) and can be represented by power series.

For the behaviour of $f(x)$ around $x = \pm 1$, by using limits and L'Hôpital's Rule we see that as $x \to \pm 1$, $f(x) \to 0$.

As for the order of polynomial that gives the error of less than 10^{-5} we use the maximum error in each order and then report the order that gives an error less than 10^{-5}.

4 Conclusion

- We studied the effects of increasing the degree of polynomial when interpolating.
- We also how the sampling rate affects the accuracy of interpolation.