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Dynamical Properties of the Hénon Mapping

Wadia Faid Hassan Al-Shameri

Department of Mathematics, Faculty of Applied Science
Thamar University, Yemen
wfha61@yahoo.com

Abstract

The Hénon map is an iterated discrete-time dynamical system that exhibits chaotic behavior in two-dimension. In this paper, we investigate the dynamical properties of the Hénon map which exhibit transitions to chaos through period doubling route. However, we focus on the mathematics behind the map. Then, we analyze the fixed points of the Hénon map and present algorithm to obtain Hénon attractor. Implementation of that dynamical system will be done using MATLAB programs are used to plot the Hénon attractor and bifurcation diagram in the phase space.

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Keywords: Strange attractor, fixed points, bifurcation

1. INTRODUCTION

Various definitions of chaos were proposed. A system is chaotic in the sense of Devaney [1] if it is sensitive to initial conditions and has a dense set of periodic points.

The chaotic behaviour of low-dimensional maps and flows has been extensively studied and characterized [10]. Another feature of chaos theory is the strange attractor which first appears in the study of two- dimensional discrete dynamical systems.

A strange attractor is a concept in chaos theory that is used to describe the behaviour of chaotic systems. Roughly speaking, an attracting set for a dynamical system is a closed subset A_τ of its phase space such that for "many" choices of initial point the system will evolve towards A_τ . The word attractor will be reserved for an attracting set which satisfies some supplementary conditions, so that it cannot be split into smaller pieces [12]. In the case of an iterated map, with discrete time steps, the simplest attractors are attracting fixed points.

The Hénon map presents a simple two-dimensional invertible iterated map with quadratic nonlinearity and chaotic solutions called strange attractor. Strange attractors are a link between the chaos and the fractals. Strange attractors generally have noninteger dimensions. Hénon's attractor is an attractor with a non-integer dimension (so-called fractal dimension [6]). The fractal dimension is a useful quantity for characterizing strange attractors. The Hénon map gives the strange attractor with a fractal structure [11]. The Hénon map is proposed by the French astronomer and mathematician Michel Hénon [5] as a simplified model of the Poincare map that arises from a solution of the Lorenz equations. The Hénon map is given by the following pair of first-order difference equations:

$$x_{n+1} = 1 - ax_n^2 + y_n$$
$$y_{n+1} = bx_n$$

where a and b are (positive) bifurcation parameters.

Since the second equation above can be written as $y_n = x_{n-1}$, the Hénon map can be written in terms of a single variable with two time delays [9]:

$$x_{n+1} = 1 - ax_n^2 + bx_{n-1}$$

The parameter b is a measure of the rate of area contraction, and the Hénon map is the most general two-dimensional quadratic map with the property that the contraction is independent of x and y. For $b = 0$, the Hénon map reduces to the quadratic map which follows period doubling route to chaos [2, 7]. Bounded solutions exist for the Hénon map over a range of a and b values.

The paper is organized as follows. In Section 2 we formulate the main mathematical properties of the Hénon map. Section 3 illustrates the fixed points and derives some results related to the existence of fixed points in the Hénon map. In Section 4 we present algorithm for computer investigations to obtain Hénon attractor, create a bifurcation diagram in the phase space that shows the solutions of the Hénon map and discuss the results of investigations. Section 5 presents the main conclusions of the work.

2. MATHEMATICAL PROPERTIES OF HENON MAP

The Hénon map has yielded a great deal of interesting characteristics as it was studied. At their core, the Hénon map is basically a family of functions defined from $\mathbb{R}^2 \to \mathbb{R}^2$ and denoted by:

$$H_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix},$$

where $a,\ b \in \mathbb{R}$ (the set of real numbers). As a whole, this family of maps is sometimes represented by the letter H , and are referred to collectively as just the Hénon map. Usually, a and b are taken to be not equal to 0, so that the map is always two-dimensional. If a is equal to 0, then it reduces to a one-dimensional logistic equation. By plotting points or through close inspection, it can be seen that H is just a more generalized form of another family of functions of the form $F_c(x) = 1 - cx^2$, where c is a constant. Therefore, one can visualize the graph of the Hénon map as being similar to a sideways parabola opening to the left, with its vertex somewhere on the x-axis, in general close to $(0,1)$.

Although it appears to be just a single map, the Hénon map is actually composed of three different transformations [8], usually denoted H_1, H_2 *and* H_3. These transformations are defined below:

$$H_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 1 - ax^2 + y \end{pmatrix}, H_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} bx \\ y \end{pmatrix} \text{ and } H_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

From the above definitions, we have that $H_{ab} = H_3 \circ H_2 \circ H_1$. To imagine visually how a parabola of the form $F_c(x) = 1 - cx^2$ could be formed from applying the three transformations above, first assume that $a > 1$, and we begin with an ellipse centred at $(0,0)$ on the real plane. The transformation defined by H_1 is a nonlinear bending in the y-axis and then H_2 contract the ellipse along the x-axis (the contraction factor is given by the parameter b) and stretch it along the y-axis, and elongate the edges of the half below the x-axis so it looks like an upright arch. Lastly, H_3 then takes the ensuing figure and reflects it along the line $y = x$. This resultant shape looks like a parabola opening to the left with an enlarged section near the vertex, which is very similar to the family of curves we defined earlier.

Next we will find the Jacobian of H_{ab} .

Theorem 1. The Hénon map has the following Jacobian: $DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$ with $\det DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = -b$ *for fixed real numbers a and b and for all* $x, y \in \mathbb{R}^2$. If $a^2x^2 + b \geq 0$, then the eigenvalues of $DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix}$ are the real numbers $\lambda = -ax \pm \sqrt{a^2x^2 + b}$.

Proof [3]. Since the coordinates function of H_{ab} are given by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = 1 - ax^2 + y \text{ and } g\begin{pmatrix} x \\ y \end{pmatrix} = bx.$$

We find that

$$DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix},$$

so that

$$\det DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = \det\begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} = -b.$$

To determine the eigenvalues of $DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix}$ we observe that

$$\det\left(DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix} - \lambda I \right) = \det\begin{pmatrix} -2ax-\lambda & 1 \\ b & -\lambda \end{pmatrix} = \lambda^2 + 2ax\lambda - b.$$

Therefore λ is an eigenvalues of $DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix}$ if $\lambda^2 + 2ax\lambda - b = 0.$ This means that

$$\lambda = \frac{-2ax \pm \sqrt{4a^2x^2 + 4b}}{2} = -ax \pm \sqrt{a^2x^2 + b}.$$

Thus, the eigenvalues are real if $a^2x^2 + b \geq 0.$ ■

Next we will show that H_{ab} is one-to-one.

Theorem 2. H_{ab} is one-to-one.

Proof [3]. Let x, y, z, and w be real numbers. Now, in order for H_{ab} to be one-to-one, we must have $H_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = H_{ab}\begin{pmatrix} z \\ w \end{pmatrix}$ if and only if $\begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} = \begin{pmatrix} 1 - az^2 + w \\ bz \end{pmatrix}$. In other words, H_{ab} must map each ordered pair of x and y must map to a unique pair of x and y. This means that we want $1 - ax^2 + y = 1 - az^2 + w$ and $bx = bz$. Now, since b is not allowed to be 0, it follows that $x = z$. Then, y must equal w as well, and so we have $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix}$. As a result, H_{ab} is one-to-one. ■

Another interesting property of the Hénon map is that it is invertible. It is not obvious just from inspection, but it is possible to derive an exact expression for H_{ab}^{-1}.

Theorem 3. For $b \neq 0$, the inverse of H_{ab} is $H_{ab}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{b}y \\ -1 + \frac{a}{b^2}y^2 + x \end{pmatrix}$ and it is one-to-one.

Proof [3]. We could show that H_1, H_2 *and* H_3 are invertible, and then that

$$H_{ab}^{-1} = (H_3 \circ H_2 \circ H_1)^{-1} = H_1^{-1} \circ H_2^{-1} \circ H_3^{-1}:$$

Simply computing $\left(H_{ab} \circ H_{ab}^{-1}\right)\begin{pmatrix} x \\ y \end{pmatrix}$ and verifying that it is equal to $\begin{pmatrix} x \\ y \end{pmatrix}$ would show that this is the inverse of the Hénon map H_{ab}.■

3. FIXED POINTS OF H_{ab}

This system's fixed points depend on the values of a and b. In general, the process to find fixed points of a function f involves solving the equation $f(p) = p$. For the Hénon map, that means we must solve: $H_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \to y = bx$ and $x = 1 - ax^2 + y$. After doing some basic substitutions, we find an expression for x, which is $x = \frac{1}{2a}\left(b - 1 \pm \sqrt{(1-b)^2 + 4a}\right)$. From that, we can deduce that unless $a = 0$, any fixed points would be real if $a \geq -\frac{1}{4}(1-b)^2$.

In the event that H_{ab} has two fixed points p and q. They are given

$$p = \begin{Bmatrix} \frac{1}{2a}\left(b - 1 + \sqrt{(1-b)^2 + 4a}\right) \\ \frac{b}{2a}\left(b - 1 + \sqrt{(1-b)^2 + 4a}\right) \end{Bmatrix},\ q = \begin{Bmatrix} \frac{1}{2a}\left(b - 1 - \sqrt{(1-b)^2 + 4a}\right) \\ \frac{b}{2a}\left(b - 1 - \sqrt{(1-b)^2 + 4a}\right) \end{Bmatrix} \tag{1}$$

Since we know the fixed point of H_{ab} and the eigenvalues of $DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix}$ for all x and y, we can determine conditions under which the fixed point p is attracting.

Theorem 4. The fixed point p of H_{ab} is attracting provided that $0 \neq a \in (-\frac{1}{4}(1-b)^2, \frac{3}{4}(1-b)^2)$.

Proof [3]. Using the fact that " if p is a fixed point of F and if the derivative matrix $DF(p)$ exists, with eigenvalues λ_1 *and* λ_2 such that $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ then p is attracting"; this fact tells us that p is attracting if the eigenvalues of $|DH_{ab}\begin{pmatrix} x \\ y \end{pmatrix}| < 1$. Letting $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$, we know from equation (1) that

$$p_1 = \frac{1}{2a}\left(b - 1 + \sqrt{(1-b)^2 + 4a}\right) \tag{2}$$

so that $2ap_1 = b - 1 + \sqrt{(1-b)^2 + 4a}$. Therefore

$$2ap_1 > b - 1, \text{ or equivalently, } 2ap_1 + 1 > b \tag{3}$$

By theorem1 the eigenvalues of $| DH_{ab}(p) | < 1$ if $| -ap_1 \pm \sqrt{a^2 p_1^{\ 2} + b} | < 1$. We will show that if $a \in (-\frac{1}{4}(1-b)^2, \frac{3}{4}(1-b)^2)$, then

$$0 \le -ap_1 + \sqrt{a^2 p_1^{\ 2} + b} < 1 \tag{4}$$

On the one hand, because $b > 0$ we have

On the other hand, $a > \dfrac{-(1-b)^2}{4}$ by hypothesis, so that $(1-b)^2 + 4a > 0$.

Consequently $p_1 \in \mathbb{R}$ by equation (2), and by equation (3),

$$(ap_1 + 1)^2 = a^2 p_1^{\ 2} + 2ap_1 + 1 > a^2 p_1^{\ 2} + b > 0.$$

It follows that $ap_1 + 1 > \sqrt{a^2 p_1^{\ 2} + b}$, so that $-ap_1 + \sqrt{a^2 p_1^{\ 2} + b} < 1$. Therefore, inequality (4) is proved. An analogous argument proves that $-1 < -ap_1 - \sqrt{a^2 p_1^{\ 2} + b} < 0$.

Consequently the eigenvalues of $| DH_{ab}(p) | < 1$, so that p is an attracting fixed point. ■

Using Theorem 4, we can derive the following results related to the existence of fixed points in the Hénon map:

Value of the parameter a in terms of the parameter b	Fixed/Periodic points of H_{ab}
$a < -\frac{1}{4}(1-b)^2$	None
$-\frac{1}{4}(1-b)^2 < a < \frac{3}{4}(1-b)^2$	Two fixed points: one attracting, one saddle
$a > \frac{3}{4}(1-b)^2$	Two attracting period-2 points

Table 1.

Since the derivative matrix for this map is $\begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$ and both a and b are real numbers, so we have $\det\begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} = -b$, and also it can be seen that the map has a constant Jacobian. Solving for the eigenvalues of this matrix gives us $\lambda^2 + 2a\lambda x - b = 0$, and so we solve for λ and get $\lambda = -ax \pm \sqrt{a^2x^2 + b}$. So, the eigenvalues are real if and only if $a^2x^2 + b \geq 0$, and any values of x and y can give us at most two eigenvalues. Now, we know that if λ_1 and λ_2 are our two eigenvalues, then their values will determine whether a fixed point is attractive, repelling, or a saddle point. For example, if we let $a = b = 1$, then, as expected, we get two fixed points for H_{ab}, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$. By using the formula $\lambda = -ax \pm \sqrt{a^2x^2 + b}$ to compute the eigenvalues of both of these fixed points, we discover that since we have $|\lambda_1| < 1$ and $|\lambda_2| > 1$ for both points, (without loss of generality), then they both are saddle points of that particular mapping.

4. HENON ATTRACTOR AND BIFURCATION DIAGRAM

We consider the Hénon attractor which arises from the two parameter mapping defined by

$$H_{ab}(x, y) = (1 - ax^2 + y,\ bx).$$

The Hénon attractor is denoted by A_H , and is defined as the set of all points for which the iterates of every point in a certain quadrilateral Q surrounding A_H approach a point in the set. It is an example of a quadratic strange attractor, since the highest power in its formulas is 2. The Hénon map does not have a strange attractor for all values of the parameters a and b , where the parameters a and b controls the nonlinearity and the dissipation. For $a = 1.4$ and $b = 0.3$ this map shows chaotic behaviour by iterating the equations

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + y_n \\ y_{n+1} &= bx_n \end{aligned}$$

This chaotic behaviour is known as Hénon attractor is the orbit of the iteration.

The following pseudo-code algorithm can be used to explore the Hénon attractor on the computer.

- Define the parameters a and b for the Hénon map;
- Specify the initial conditions

  $x = x_0 ; y = y_0$

  $x_1 = x_0 ; y_1 = y_0 ;$
- Iterates the Hénon map

  $for\ \ i = 1\ \ to\ \ \max iter$

  $x_{i+1} = 1 - a * x_i^2 + y_i ;$

  $y_{i+1} = b * x_i ;$

  $x = x_{i+1} ; y = y_{i+1}$

  end
- Plot Hénon attractor

  $plot\ \ (x, y)$

The standard (typical) parameter [9] values of the Hénon map H_{ab} has $a = 1.4$ and $b = 0.3$. This Hénon map has a chaotic strange attractor. The result of computation is shown in Figure 1 created by Matlab program1 and listed in the Appendix. If we zoom in on portions of this attractor, we can see a fractal structure.

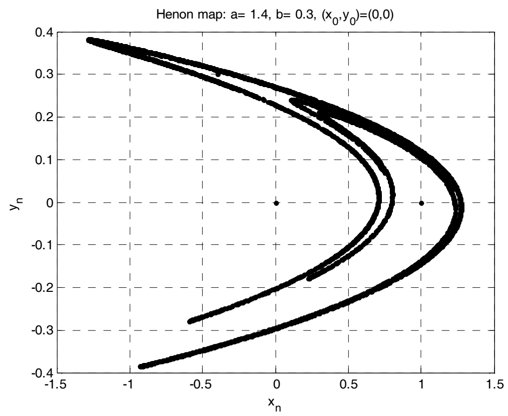

Figure 1. The Hénon attractor.

In his analysis of this map Michel Hénon [5] defined a trapping quadrilateral and showed that all points on and inside this quadrilateral did not escape to infinity as they were iterated. Instead they remained inside the quadrilateral forever. At each iteration the quadrilateral is stretched and folded by the Hénon map until the geometrical attractor is obtained.

In Figure 1, it can be observed the existence of a strange attractor, very popular, known under the name of Hénon attractor. Thus, except for the first few points, we plot the points in the orbit. The picture that "develops" is called the Hénon attractor. The orbit points wander around the attractor in a random fashion. The orbits are very sensitive to the initial conditions, a sign of chaos, but the attractor appears to be a stable geometrical object that is not sensitive to initial conditions.

To study the evolution of that dynamic system, we plot the bifurcation diagram in the phase space, using Matlab program 2 listed in the Appendix. That diagram allows to visualise the bifurcation phenomena which is the transition of the orbit structure. Clearly, H_{ab} has a period-doubling bifurcation when $a = \frac{3}{4}(1-b)^2$. For different value of the parameter a, we plot a set of converged values of x, that means, we plot the Hénon map bifurcation diagram when $b = 0.3$ and the initial conditions $x_0 = y_0 = 0$ are within the basin of attraction for this map.

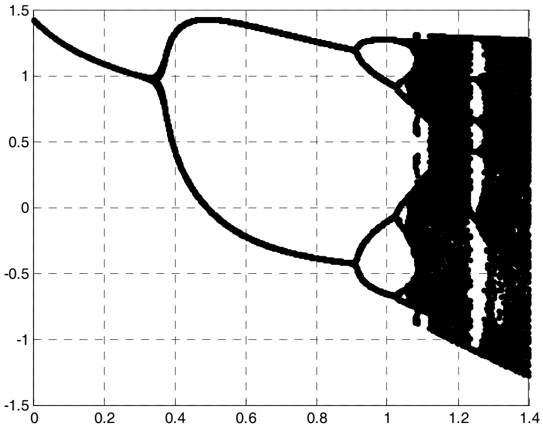

Figure 2. The bifurcation diagram.

This Hénon map receives a real number between 0 and 1.4, then returns a real number in $[-1.5,\ 1.5]$ again. The various sequences are yielded depending on the parameter a and the initial values x_0, y_0. We can see that if the parameter a is taken between 0 and about 0.32, the sequence $\{x_n\}$ converges to a fixed point x_f independent on the initial value x_0 and y_0. But what happens to the sequence $\{x_n\}$ when the parameter exceeds 0.32? As you see with the help of the previous graph, the sequence converges to a periodic orbit of period-2. Such situation happens when the parameter a is taken between about 0.32 and 0.9. If the parameter $b = 0.4$, then there are points of periods one (when $a = 0.2$), two (when $a = 0.5$), and four (when $a = 0.9$), If you make the parameter a larger, the period of the periodic orbit will be doubled, i.e. $8, 16, 32, \ldots$

This is called period doubling cascade, and beyond this cascade, the attracting periodic orbit disappears and we will see chaos if the parameter $a > 1.42720$. As you see above, the transition of the orbit structure is in accordance with the change of parameter is called bifurcation phenomena. At least, the following graph (Figure 3) shows a zoom on the first lower branch of the bifurcation diagram (Figure 2):

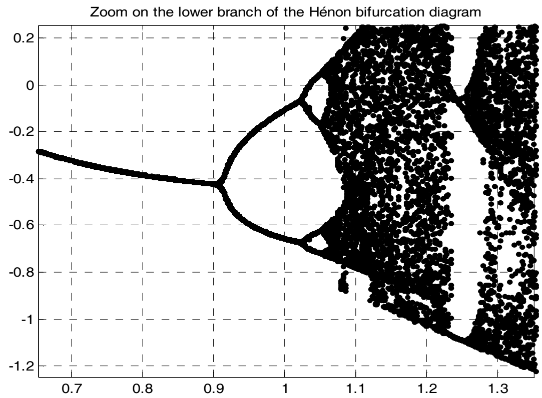

Figure 3. Zoom on the bifurcation diagram.

We can reasonably think that the Hénon attractor is an iterated fractal structure and there are usual phenomena associated with bifurcation diagrams. However, for the Hénon map, different chaotic attractors can exist simultaneously for a range of parameter values of a. This system also displays hysteresis for certain parameter values.

5. CONCLUSIONS

In this research paper we have presented a discrete two-dimensional dynamical system. The Hénon map is considered a representative for this class of dynamical system. We formulate the main mathematical properties of the Hénon map, obtain fixed points and derive some results related to the existence of the fixed points, create Hénon attractor, build a bifurcation diagram that shows the solutions of the Hénon map and give detailed characterization of the bifurcation diagram structure of the Hénon map as well as related analytical computations.

From the analysis of the results, we can conclude that all the dynamical properties we have studied are present in Hénon attractor. So, a subset of the phase space is a strange attractor if only it is an attractor which has a great sensibility to the initial conditions possessing fractal structure and which is indivisible in another attractor.

APPENDIX

Program 1

```
% Matlab code (see [4]) to demonstrate Hénon map strange attractor
clc; clear all;
% define the parameters
a=input('a = ');
```

```
b=input('b = ');
% specify the initial conditions.
x0=input('x0 = ');
y0=input('y0 = ');
n=input('Maximum number of iterations = ');
x=zeros(1,n+1);
y=zeros(1,n+1);
x(1)=x0;
y(1)=y0;
% main routine
for i=1:n         % iterates the Hénon map
 x(i+1)=1-a*(x(i)^2)+y(i);
 y(i+1)=b*x(i);
end
plot(x,y,'.k','LineWidth',.5,'MarkerSize',5);
xlabel('x_n'); ylabel('y_n');
title(['Henon map: a= ',num2str(a),', b= ',num2str(b),',
(x_0,y_0)=(',num2str(x0),',',num2str(y0),')']);
grid
zoom
```

Program 2

```
%  Matlab code (see [4]) to demonstrate bifurcation diagram for Hénon map
%  x=f(a), b=0.3  and  xo=yo=0
clc; clear all;
n = input('number of iterations = ');
% fix the parameter b and vary the parameter a
b=0.3;
a=0:0.001:1.4;
% initialization a zero for x and y
% x(0)=y(0)=0
x(:,1)=zeros(size(a,2),1);
y(:,1)=zeros(size(a,2),1);
% iterate the Hénon map
for k=1 : size(a,2)
   for i=1:130
      y(k,i+1)=b*x(k,i);
      x(k,i+1)=1+y(k,i)-a(k)*x(k,i)^2;
   end
end
% display module of the last 50 values of x:
r=a(1,1)*ones(1,51);
m=x(1,80:130);
for k=2 : size(a,2)
   r=[r,a(1,k)*ones(1,51)];
   m=[m,x(k,80:130)];
end
plot(r,m,'.k');
grid;
zoom;
```

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