EE604A - Digital Image Processing Assignment

Satya Prakash Panuganti, 14610 31 August, 2017

Image Sources

Sunset (high.jpg) : https://www.flickr.com/photos/mediaflex/4190084346
Carving (low.jpg) : https://www.flickr.com/photos/30440933@N06/2847993403
Dog (small.jpg) : https://res.cloudinary.com/rover-com/image/upload/a_exif,c_fill,f_jpg,fl_progressive g_face:center,h_100,q_80,w_100/remote/images/pets/4NpPzO8N/50e4a023d9/original.jpg
divyat.jpg : Own work. Taken with permission.

Solution 1

(a)

Code has been written in two MATLAB® files :

- Q1\lloyd_max_quantizer.m : The file contating the function.
- Q1\Q1.m : The script to perform the required tasks.

(b)

The 4 representation levels are :

The corresponding transition levels are :

(c)

(d)

(e)

Solution 2

A C++ function, EE604A::histogram_matching(), for histogram matching of cv::Mat images has been developed. The code required for the matching function is present in :

- Q2\src\histogram_matching.cxx
- Q2\src\histogram_matching.h

A small program which uses the histogram matching function is present in Q2\src\Q2.cxx

In order to build and run the code, the following steps need to be followed :

1. Enter Q2\src.
2. run build.sh (OpenCV needs to installed and a version of g++ supporting c++14 needs to be present)

3. Execute Q2 (the binary file) with the reference and target relative/absolute image paths. Eg. [./Q2 ../../images/high.jpg ../../images/low.jpg] or [./Q2 ../../images/low.jpg ../../images/high.jpg] (On Ubuntu)

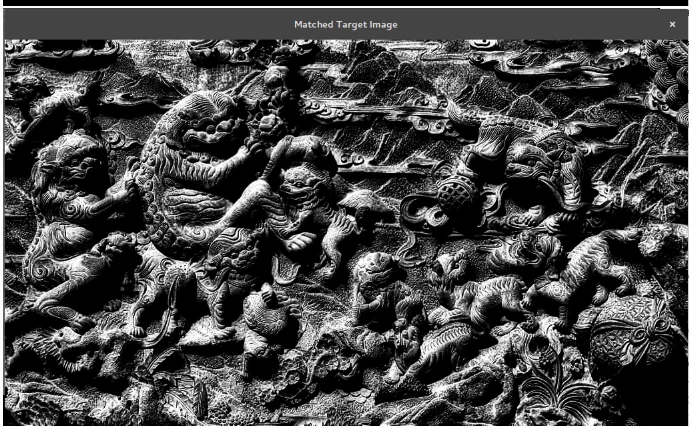
4. The program can be closed by Ctrl+C on the terminal or by pressing 'q' when one of the image windows is open.

REMARK : A simple plot of the the histograms can be obtained by toggling the third argument of the function EE604A::histogram_matching() to true.

Results

Target : high.jpg, Reference : low.jpg

Target : low.jpg, Reference : high.jpg



Solution 3

Let the 8-connect neighborhood of a point be given by

v11	v12	v13
v21	v22	v23
v31	v32	v33

.

In blinear interpolation, we fit a function $f(x, y)$ using weights a_{ij} in the following manner :

$$f(x, y) = \Sigma_{i=0}^{1} \Sigma_{j=0}^{1} a_{ij} x^i y^j \tag{1}$$

Let x_1, x_2, x_3 and y_1, y_2, y_3 be the x and y coordinates of the points in the neighborhood. It is clear that $x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_1$, $y_1 \neq y_2$, $y_2 \neq y_3$ and $y_3 \neq y_1$. Also, let e be the error vector.

$$We\ have, \begin{bmatrix} f(x_1,y_1) \\ f(x_1,y_2) \\ f(x_1,y_3) \\ f(x_2,y_1) \\ f(x_2,y_2) \\ f(x_2,y_3) \\ f(x_3,y_1) \\ f(x_3,y_2) \\ f(x_3,y_3) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_2 & y_3 & x_2y_3 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_3 & y_2 & x_3y_2 \\ 1 & x_3 & y_3 & x_3y_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} + e \tag{2}$$

$$\implies \begin{bmatrix} v11 \\ v12 \\ v13 \\ v21 \\ v22 \\ v23 \\ v31 \\ v32 \\ v33 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_2 & y_3 & x_2y_3 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_3 & y_2 & x_3y_2 \\ 1 & x_3 & y_3 & x_3y_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} + e \tag{3}$$

Our aim is to determine $\begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$ while minimizing the the MSE, e^Te.

Let $f = \begin{bmatrix} v11 \\ v12 \\ v13 \\ v21 \\ v22 \\ v23 \\ v31 \\ v32 \\ v33 \end{bmatrix}$, $X = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_2 & y_3 & x_2y_3 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_3 & y_2 & x_3y_2 \\ 1 & x_3 & y_3 & x_3y_3 \end{bmatrix}$ and $W = \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$

$$\therefore\ e = f - XW \tag{4}$$

The best estimate for W is given by :

$$\hat{W} = (X^TX)^{-1}X^Tf \qquad on\ minimizing\ the\ MSE,\ e^Te. \tag{5}$$

We need X^TX to be invertable for a solution to exist for the given interpolation problem. Let us first show that $dim(nullspace\ of\ X) = 0$. We're going to prove that the column vectors of X are linearly independent, i.e. $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0 \implies c_1 = c_2 = c_3 = c_4 = 0$ where $X = [X_1\ X_2\ X_3\ X_4]$

On taking $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$,

$$
\begin{aligned}
We\ have,\ c_1 + c_2x_1 + c_3y_1 + c_4x_1y_1 &= 0 && (6)\\
c_1 + c_2x_1 + c_3y_2 + c_4x_1y_2 &= 0 && (7)\\
\implies c_3(y_1 - y_2) + c_4x_1(y_1 - y_2) &= 0 \qquad from\ (6) - (7) && (8)\\
\implies (y_1 - y_2)(c_3 + c_4x_1) &= 0 && (9)\\
\implies c_3 + c_4x_1 &= 0 \qquad \because\ y_1 \neq y_2 && (10)\\
Similiarly,\ using\ c_1 + c_2x_2 + c_3y_1 + c_4x_2y_1 &= 0 && (11)\\
c_1 + c_2x_2 + c_3y_2 + c_4x_2y_2 &= 0 && (12)\\
we\ get,\ c_3 + c_4x_2 &= 0 && (13)\\
\implies c_3 = c_4 &= 0 \qquad from\ (10),\ (13)\ and\ \because\ x_1 \neq x_2 && (14)\\
Also,\ c_1 = c_2 = 0\ from\ (6),\ (11),\ (14)\ and\ \because\ x_1 \neq x_2 & && (15)
\end{aligned}
$$

$\therefore\ c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0 \implies c_1 = c_2 = c_3 = c_4 = 0$
$\therefore\ dim(columnspace\ of\ X) = 4$ as the column vectors are linearly independent.

$$Hence,\ dim(nullspace\ of\ X) = 4 - 4 = 0 \tag{16}$$

We're now going to prove that $nullspace\ of\ X = nullspace\ of\ X^TX$.

$$
\begin{aligned}
&Let\ y \in nullspace\ of\ X\\
&\qquad \therefore Xy = 0\\
&\qquad \implies X^TXy = 0\\
&\qquad \implies y \in nullspace\ of\ X^TX\\
&\qquad \therefore nullspace\ of\ X \subseteq\ nullspace\ of\ X^TX
\end{aligned}
$$

$$
\begin{aligned}
&Let\ y \in nullspace\ of\ X^TX\\
&\qquad \therefore X^TXy = 0\\
&\qquad \implies y^TX^TXy = 0\\
&\qquad \implies (Xy)^TXy = 0\\
&\qquad \implies Xy = 0\\
&\qquad \implies y \in nullspace\ of\ X\\
&\qquad \therefore nullspace\ of\ X^TX \subseteq\ nullspace\ of\ X
\end{aligned}
$$

$$\therefore\ nullspace\ of\ X^TX = nullspace\ of\ X \tag{17}$$

Hence, $dim(nullspace\ of\ X^TX) = dim(nullspace\ of\ X) = 0$ using (17) and (18).
X^TX is a 4×4 square matrix. It has full rank since its nullspace is 0. $\therefore$ X^TX is invertible and hence bilinear interpolation in an 8-neighborhood can always be used to get a solution for W and (1) can be constructed.

Solution 4

We are assuming that $\eta_i(x_1, y_1)\ and\ \eta_j(x_2, y_2)$ are independent if i $\neq j\ or\ x_1 \neq x_2\ or\ y_1 \neq y_w$.

$$\therefore\ if\ i \neq j,\ E[\eta_i(x,y)\eta_j(x,y)] = 0 \tag{18}$$

$$\begin{aligned}
We\,have,\ g_i(x,y) &= f_i(x,y) + \eta_i(x,y) \\
Also,\ f_i(x,y) &= f(x,y) \\
Now,\ \hat{g}(x,y) &= \frac{1}{K}\Sigma_{i=1}^{K} g_i(x,y) \\
&= \frac{1}{K}Kf(x,y) + \frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x,y) \\
&= f(x,y) + \frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x,y) \\
Now,\ noise\ variance,\ E[(g(\hat{x,y}) - f(x,y))^2] &= E[(\frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x,y))^2] \\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}\Sigma_{j=1}^{K}E[\eta_i(x,y)\eta_j(x,y)] \\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}E[\eta_i(x,y)^2] && from\ (1) \\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}\sigma^2 && \because\ E[\eta_i(x,y)^2] = \sigma^2 \\
&= \frac{\sigma^2}{K} \\
q.e.d
\end{aligned}$$

Solution 5

Let $f(x, y, z) : \Re^3 \rightarrow \Re$. Let $[u\ v\ w]^T$ be the position of $[x\ y\ z]^T$ in a rotated cooridnate frame. The relationship between $[u\ v\ w]^T$ and $[x\ y\ z]^T$ is given by :

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{19}$$

$$where,\ R = [R_1\ R_2\ R_3] \tag{20}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{21}$$

$$Known\ properties\ of\ R,\ |R_1| = 1 \tag{22}$$

$$|R_2| = 1 \tag{23}$$

$$R_1^T R_2 = 0 \tag{24}$$

$$R_3 = R_1 \times R_2 \tag{25}$$

$$\implies |R_3| = 1 \tag{26}$$

$$R_1^T R_3 = R_2^T R_3 = 0 \tag{27}$$

$$R^{-1} = R^T \tag{28}$$

$$We\ can\ also\ write,\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R^T \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad from\ (2),\ (9) \tag{29}$$

$$Now,\ \nabla^2 f(u, v, w) = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} \tag{30}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \tag{31}$$

$$= R_{11}\frac{\partial f}{\partial x} + R_{12}\frac{\partial f}{\partial y} + R_{13}\frac{\partial f}{\partial z} \qquad from\ (4), (10) \tag{32}$$

$$Similarily,\ \frac{\partial f}{\partial v} = R_{21}\frac{\partial f}{\partial x} + R_{22}\frac{\partial f}{\partial y} + R_{23}\frac{\partial f}{\partial z} \tag{33}$$

$$\frac{\partial f}{\partial w} = R_{31}\frac{\partial f}{\partial x} + R_{32}\frac{\partial f}{\partial y} + R_{33}\frac{\partial f}{\partial z} \tag{34}$$

$$
\begin{aligned}
\therefore\ \frac{\partial^2 f}{\partial^2 u} &= R_{11}(\frac{\partial^2 f}{\partial^2 x}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial u}) \\
&+ R_{12}(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial^2 y}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial u}) \\
&+ R_{13}(\frac{\partial^2 f}{\partial x \partial z}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial z}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial^2 z}\frac{\partial z}{\partial u}) && from\ (13)\ and\ chain\ rule \quad (35) \\
&= R_{11}^2\frac{\partial^2 f}{\partial^2 x} + R_{11}R_{12}\frac{\partial^2 f}{\partial y \partial x} + R_{11}R_{13}\frac{\partial^2 f}{\partial z \partial x} \\
&+ R_{12}R_{11}\frac{\partial^2 f}{\partial x \partial y} + R_{12}^2\frac{\partial^2 f}{\partial^2 y} + R_{12}R_{13}\frac{\partial^2 f}{\partial z \partial y} \\
&+ R_{13}R_{11}\frac{\partial^2 f}{\partial x \partial z} + R_{13}R_{12}\frac{\partial^2 f}{\partial y \partial z} + R_{13}^2\frac{\partial^2 f}{\partial^2 z} && from\ (4),(10) \quad (36) \\
Similarily,\ \frac{\partial^2 f}{\partial^2 v} &= R_{21}^2\frac{\partial^2 f}{\partial^2 x} + R_{21}R_{22}\frac{\partial^2 f}{\partial y \partial x} + R_{21}R_{23}\frac{\partial^2 f}{\partial z \partial x} \\
&+ R_{22}R_{21}\frac{\partial^2 f}{\partial x \partial y} + R_{22}^2\frac{\partial^2 f}{\partial^2 y} + R_{22}R_{23}\frac{\partial^2 f}{\partial z \partial y} \\
&+ R_{23}R_{21}\frac{\partial^2 f}{\partial x \partial z} + R_{23}R_{22}\frac{\partial^2 f}{\partial y \partial z} + R_{23}^2\frac{\partial^2 f}{\partial^2 z} && (37) \\
\frac{\partial^2 f}{\partial^2 w} &= R_{31}^2\frac{\partial^2 f}{\partial^2 x} + R_{31}R_{32}\frac{\partial^2 f}{\partial y \partial x} + R_{31}R_{33}\frac{\partial^2 f}{\partial z \partial x} \\
&+ R_{32}R_{31}\frac{\partial^2 f}{\partial x \partial y} + R_{32}^2\frac{\partial^2 f}{\partial^2 y} + R_{32}R_{33}\frac{\partial^2 f}{\partial z \partial y} \\
&+ R_{33}R_{31}\frac{\partial^2 f}{\partial x \partial z} + R_{33}R_{32}\frac{\partial^2 f}{\partial y \partial z} + R_{33}^2\frac{\partial^2 f}{\partial^2 z} && (38)
\end{aligned}
$$

On performing (17) + (18) + (19), we get using (5), (6), (7), (9) and (10)

$$
\begin{aligned}
\frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} &= \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} && (39) \\
\implies \nabla^2 f(u,v,w) &= \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} && (40) \\
&= \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} && from\ (22) \quad (41) \\
&= \nabla^2 f(x,y,z) && (42)
\end{aligned}
$$

$q.e.d$

Solution 6

A C++ file containds the code required to perform the required task.

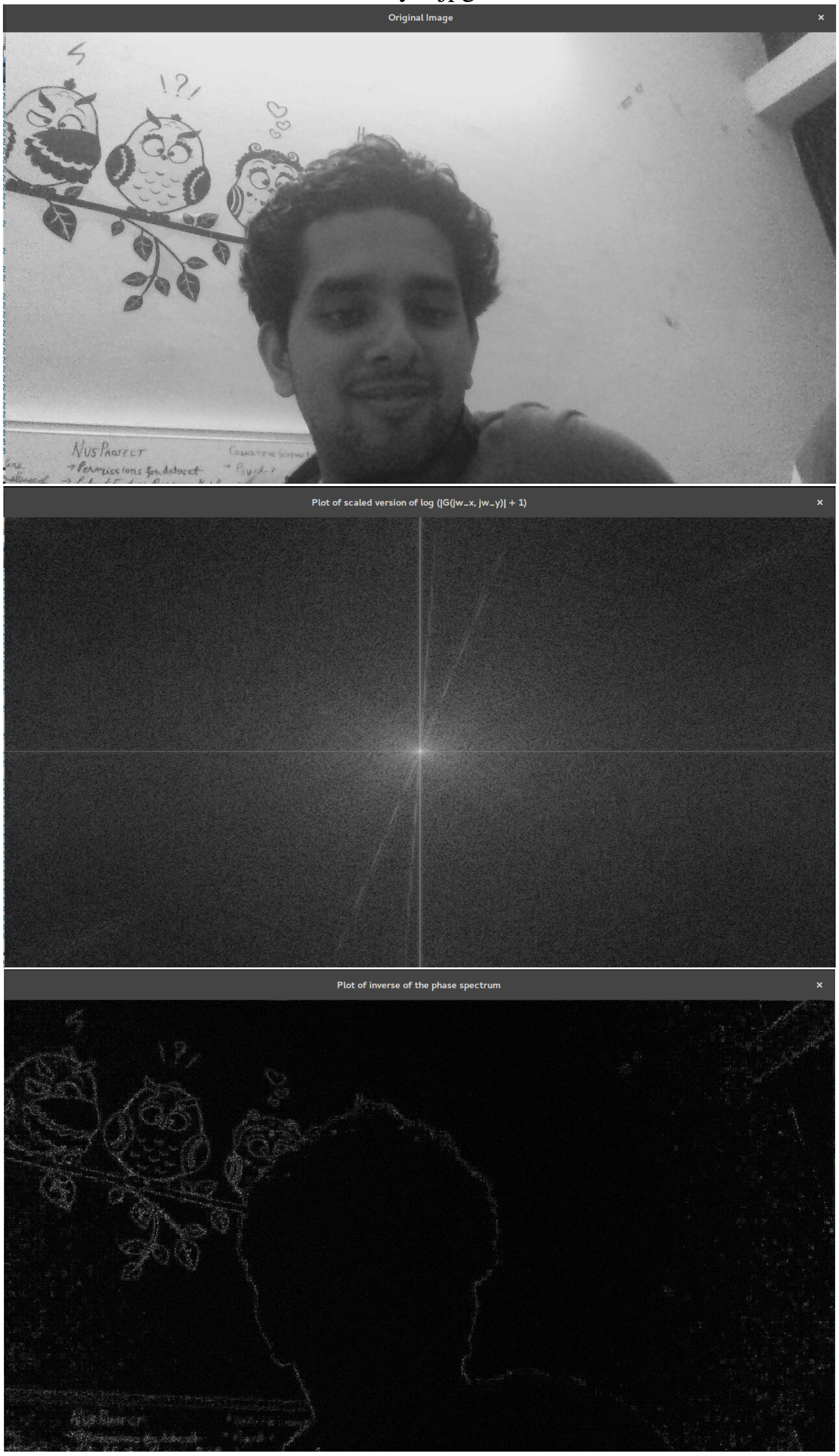
In order to build and run the code, the following steps need to be followed :

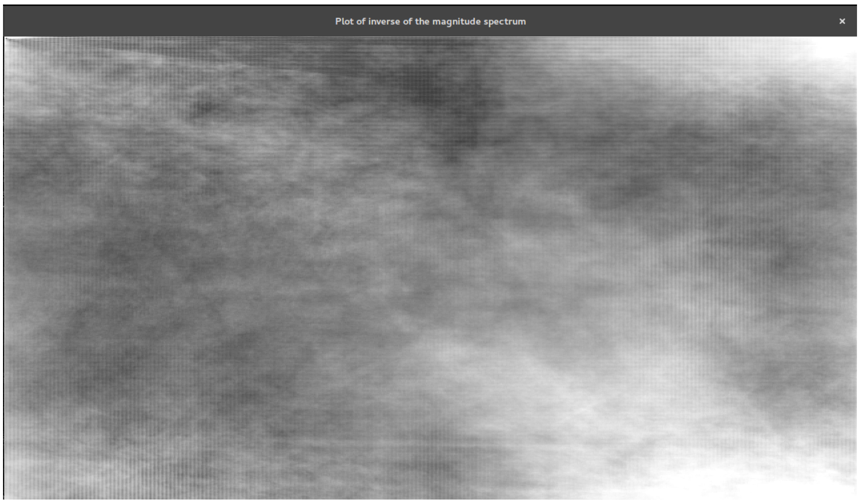
1. Enter Q6\src.
2. run build.sh (OpenCV needs to installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q6 (the binary file) with the relative/absolute image path. Eg. [./Q6 ../../images/high.jpg] or [./Q2 ../../images/low.jpg ../../images/high.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing ‘q’ when one of the image windows is open.

REMARK : A simple plot of the the histograms can be obtained by toggling the third argument of the function EE604A::histogram_matching() to true.

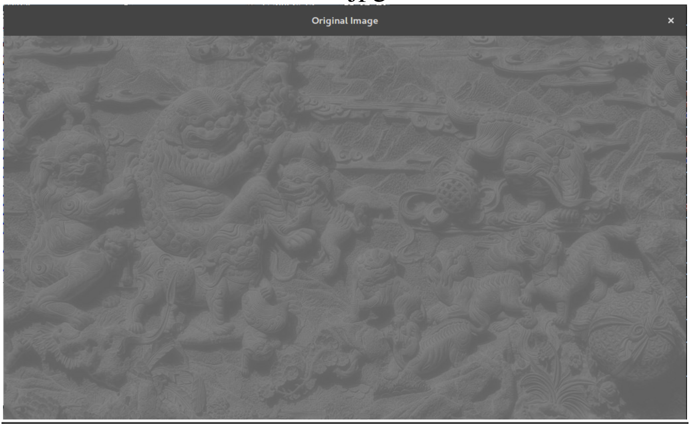
Results

divyat.jpg




Plot of inverse of the magnitude spectrum

low.jpg


Original Image


Plot of scaled version of log (|G(jw_x, jw_y)| + 1)


Plot of inverse of the phase spectrum

Observations

Reconstruction using only phase spectrum

From the above the tests, it can be inferred that performing reconstruction using only phase information, edges are being preserved and we are losing all grey-levels of homogenous regions. From, the spectrum plots, we can infer that the lower frequency components tend to have greater values than the higher frequency ones in general images. Thus, when we remove the magnitude information, the lower frequency components are getting more suppressed than the higer frequency components, i.e. the operation is similar to a high-pass filter. This premise explains the above images obatined using only the phase spectrum.

Reconstruction using only magnitude spectrum

The reconstruction using only magnitude spectrum appears to only contain information regarding the prominent greylevels present in the image.

Solution 7

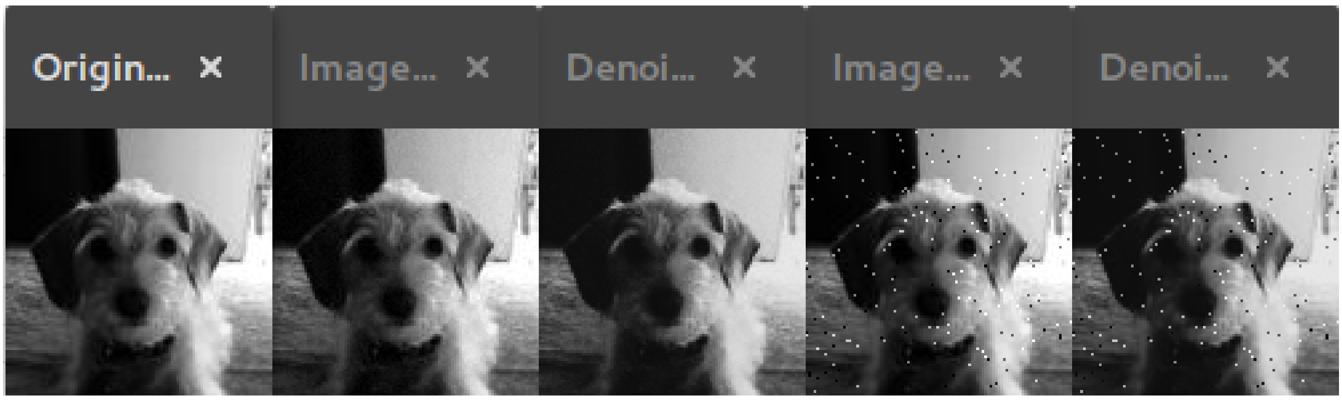


Figure 1: The images from left to right are as follows : (a) The original image (small.jpg), (b) The images with added zero-mean gaussian noise with $\sigma^2 = 0.1$, (c) The reconstucted image [b], (d) The image with impulse noise, (e) The reconstructed image [d]