EE604A - Digital Image Processing Assignment

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Solution 1

Solution 2

Solution 3

Solution 4

We are assuming that $\eta_i(x_1, y_1)\ and\ \eta_j(x_2, y_2)$ are independent if i $\neq j\ or\ x_1 \neq x_2\ or\ y_1 \neq y_w$.

$$\therefore\ if\ i \neq j,\ E[\eta_i(x, y)\eta_j(x, y)] = 0 \tag{1}$$

$$\begin{aligned}
We\ have,\ g_i(x, y) &= f_i(x, y) + \eta_i(x, y)\\
Also,\ f_i(x, y) &= f(x, y)\\
Now,\ \hat{g}(x, y) &= \frac{1}{K}\Sigma_{i=1}^{K} g_i(x, y)\\
&= \frac{1}{K} K f(x, y) + \frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x, y)\\
&= f(x, y) + \frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x, y)\\
Now,\ noise\ variance,\ E[(g(\hat{x}, y) - f(x, y))^2] &= E[(\frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x, y))^2]\\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}\Sigma_{j=1}^{K} E[\eta_i(x, y)\eta_j(x, y)]\\
&= \frac{1}{K^2}\Sigma_{i=1}^{K} E[\eta_i(x, y)^2] && from\ (1)\\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}\sigma^2 && \because\ E[\eta_i(x, y)^2] = \sigma^2\\
&= \frac{\sigma^2}{K}\\
q.e.d
\end{aligned}$$

Solution 5

Let $f(x, y, z) : \Re^3 \rightarrow \Re$. Let $[u\ v\ w]^T$ be the position of $[x\ y\ z]^T$ in a rotated cooridnate frame. The relationship between $[u\ v\ w]^T$ and $[x\ y\ z]^T$ is given by :

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2}$$

$$where,\ R = [R_1\ R_2\ R_3] \tag{3}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{4}$$

$$Known\ properties\ of\ R,\ |R_1| = 1 \tag{5}$$

$$|R_2| = 1 \tag{6}$$

$$R_1^T R_2 = 0 \tag{7}$$

$$R_3 = R_1 \times R_2 \tag{8}$$

$$\implies |R_3| = 1 \tag{9}$$

$$R_1^T R_3 = R_2^T R_3 = 0 \tag{10}$$

$$R^{-1} = R^T \tag{11}$$

$$We\ can\ also\ write,\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R^T \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad from\ (2),\ (9) \tag{12}$$

$$Now,\ \nabla^2 f(u, v, w) = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} \tag{13}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \tag{14}$$

$$= R_{11}\frac{\partial f}{\partial x} + R_{12}\frac{\partial f}{\partial y} + R_{13}\frac{\partial f}{\partial z} \qquad from\ (4), (10) \tag{15}$$

$$Similarily,\ \frac{\partial f}{\partial v} = R_{21}\frac{\partial f}{\partial x} + R_{22}\frac{\partial f}{\partial y} + R_{23}\frac{\partial f}{\partial z} \tag{16}$$

$$\frac{\partial f}{\partial w} = R_{31}\frac{\partial f}{\partial x} + R_{32}\frac{\partial f}{\partial y} + R_{33}\frac{\partial f}{\partial z} \tag{17}$$

$$\therefore \ \frac{\partial^2 f}{\partial^2 u} = R_{11}(\frac{\partial^2 f}{\partial^2 x}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial u})$$
$$+ R_{12}(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial^2 y}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial u})$$
$$+ R_{13}(\frac{\partial^2 f}{\partial x \partial z}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial z}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial^2 z}\frac{\partial z}{\partial u}) \qquad from\ (13)\ and\ chain\ rule \tag{18}$$

$$= R_{11}^2\frac{\partial^2 f}{\partial^2 x} + R_{11}R_{12}\frac{\partial^2 f}{\partial y \partial x} + R_{11}R_{13}\frac{\partial^2 f}{\partial z \partial x}$$
$$+ R_{12}R_{11}\frac{\partial^2 f}{\partial x \partial y} + R_{12}^2\frac{\partial^2 f}{\partial^2 y} + R_{12}R_{13}\frac{\partial^2 f}{\partial z \partial y}$$
$$+ R_{13}R_{11}\frac{\partial^2 f}{\partial x \partial z} + R_{13}R_{12}\frac{\partial^2 f}{\partial y \partial z} + R_{13}^2\frac{\partial^2 f}{\partial^2 z} \qquad from\ (4), (10) \tag{19}$$

$$Similarily,\ \frac{\partial^2 f}{\partial^2 v} = R_{21}^2\frac{\partial^2 f}{\partial^2 x} + R_{21}R_{22}\frac{\partial^2 f}{\partial y \partial x} + R_{21}R_{23}\frac{\partial^2 f}{\partial z \partial x}$$
$$+ R_{22}R_{21}\frac{\partial^2 f}{\partial x \partial y} + R_{22}^2\frac{\partial^2 f}{\partial^2 y} + R_{22}R_{23}\frac{\partial^2 f}{\partial z \partial y}$$
$$+ R_{23}R_{21}\frac{\partial^2 f}{\partial x \partial z} + R_{23}R_{22}\frac{\partial^2 f}{\partial y \partial z} + R_{23}^2\frac{\partial^2 f}{\partial^2 z} \tag{20}$$

$$\frac{\partial^2 f}{\partial^2 w} = R_{31}^2\frac{\partial^2 f}{\partial^2 x} + R_{31}R_{32}\frac{\partial^2 f}{\partial y \partial x} + R_{31}R_{33}\frac{\partial^2 f}{\partial z \partial x}$$
$$+ R_{32}R_{31}\frac{\partial^2 f}{\partial x \partial y} + R_{32}^2\frac{\partial^2 f}{\partial^2 y} + R_{32}R_{33}\frac{\partial^2 f}{\partial z \partial y}$$
$$+ R_{33}R_{31}\frac{\partial^2 f}{\partial x \partial z} + R_{33}R_{32}\frac{\partial^2 f}{\partial y \partial z} + R_{33}^2\frac{\partial^2 f}{\partial^2 z} \tag{21}$$

On performing $(17) + (18) + (19)$, we get using (5), (6), (7), (9) and (10)

$$\frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} \tag{22}$$
$$\implies \nabla^2 f(u,v,w) = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} \tag{23}$$
$$= \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} \qquad from\ (22) \tag{24}$$
$$= \nabla^2 f(x,y,z) \tag{25}$$

$q.e.d$

Solution 6