

EE604A - DIGITAL IMAGE PROCESSING ASSIGNMENT

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Image Sources

Sunset (high.jpg) : <https://www.flickr.com/photos/mediaflex/4190084346>

Carving (low.jpg) : <https://www.flickr.com/photos/30440933@N06/2847993403>

Dog (small.jpg) : https://res.cloudinary.com/rover-com/image/upload/a_exif,c_fill,f_jpg,fl_progressive,g_face:center,h_100,q_80,w_100/remote/images/pets/4NpPz08N/50e4a023d9/original.jpg

Face (divyat.jpg) : Own work. Taken with permission.

Solution 1

(a)

Code has been written in two MATLAB®/GNU Octave files :

- Q1/lloyd_max_quantizer.m : The file containing the function.
- Q1/Q1.m : The script to perform the required tasks.

(b)

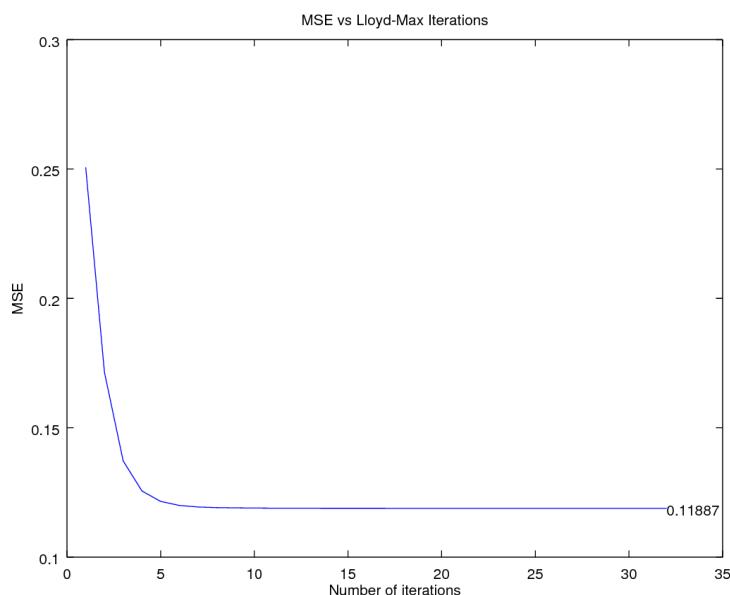
The Lloyd-Max Quantizer was run over a signal sampled from a gaussian distribution of zero mean and unit variance.

The length of the signal was taken to be 10000.

The 4 representation levels are : -1.51440 -0.43642 0.50108 1.57091

The corresponding transition levels are : $-\infty$ -0.975410 0.032332 1.035995 ∞

(c)



(d)

Experiments

The length of the signal was taken to be 10000 in all the experiments.

μ	σ^2	l_1	l_2	l_3	l_4	m_1	m_2	m_3	m_4	m_5	MSE
0	1	-1.52388	-0.45637	0.45432	1.51567	$-\infty$	-0.9901230	-0.0010213	0.9849981	∞	0.11692
0	0.5	-1.05210	-0.31372	0.31439	1.06498	$-\infty$	-0.068291	-0.00033543	0.68968	∞	0.0572
0	2	-2.11742	-0.59929	0.66589	2.14136	$-\infty$	-1.358353	-0.033301	1.403622	∞	0.23691
2.3	1	0.76315	1.82872	2.73656	3.79202	$-\infty$	1.2959	2.2826	3.2643	∞	0.11589
2.3	0.5	1.2408	1.9876	2.6390	3.4106	$-\infty$	1.64142	2.3133	3.0248	∞	0.059457
2.3	2	0.15636	1.66794	2.98038	4.48356	$-\infty$	0.91215	2.32416	3.73197	∞	0.23956

Observations

1. On adding a $\Delta\mu$ to the mean, while keeping σ^2 constant, the representation and transition levels obtained appear to be shifted by $\Delta\mu$ as well.
2. The MSE value appears to be independent of the mean.
3. As $\sigma \uparrow$, MSE \uparrow and $|l_{i+1} - l_i| \uparrow$

(e)

Let the signal be given by S and the pdf of S be $p_S(s)$.

$$\text{Now, } p_S(s) = \begin{cases} \frac{1}{b-a} & a \leq s < b \\ 0 & \text{otherwise} \end{cases}$$

Let us assume that there 'L' representation levels, a_i be the representation levels and m_i be the transition levels.

We begin by initializing m_i uniformly :

$$m_i = a + \frac{b-a}{L} \times (i-1) \quad \forall i \in [1, L+1]$$

We next set a_i :

$$\begin{aligned} a_i &= \frac{\int_{m_i}^{m_{i+1}} s \times p_S(s) ds}{\int_{m_i}^{m_{i+1}} p_S(s) ds} \quad \forall i \in [1, L] \\ &= \frac{\frac{m_{i+1}^2 - m_i^2}{2(b-a)}}{\frac{m_{i+1} - m_i}{b-a}} \quad \forall i \in [1, L] \\ &= \frac{m_i + m_{i+1}}{2} \quad \forall i \in [1, L] \\ &= a + \frac{b-a}{L} \times \frac{i+i-1}{2} \quad \forall i \in [1, L] \\ &= a + \frac{(b-a)(2i-1)}{2L} \quad \forall i \in [1, L] \end{aligned}$$

We now set m_i :

$$\begin{aligned} m_i &= \frac{a_{i-1} + a_i}{2} \quad \forall i \in [2, L] \\ &= a + \frac{b-a}{4L} \times (2(i+i-1) - 2) \quad \forall i \in [2, L] \\ &= a + \frac{b-a}{L} \times (i-1) \quad \forall i \in [2, L] \end{aligned}$$

m_i do not change in the above step. Hence, the Lloyd-Max Quantizer has converged to a final solution.

Solution 2

A C++ function, EE604A::histogram_matching(), for histogram matching of cv::Mat images has been developed. The code required for the matching function is present in :

- Q2/src/histogram_matching.cxx
- Q2/src/histogram_matching.h

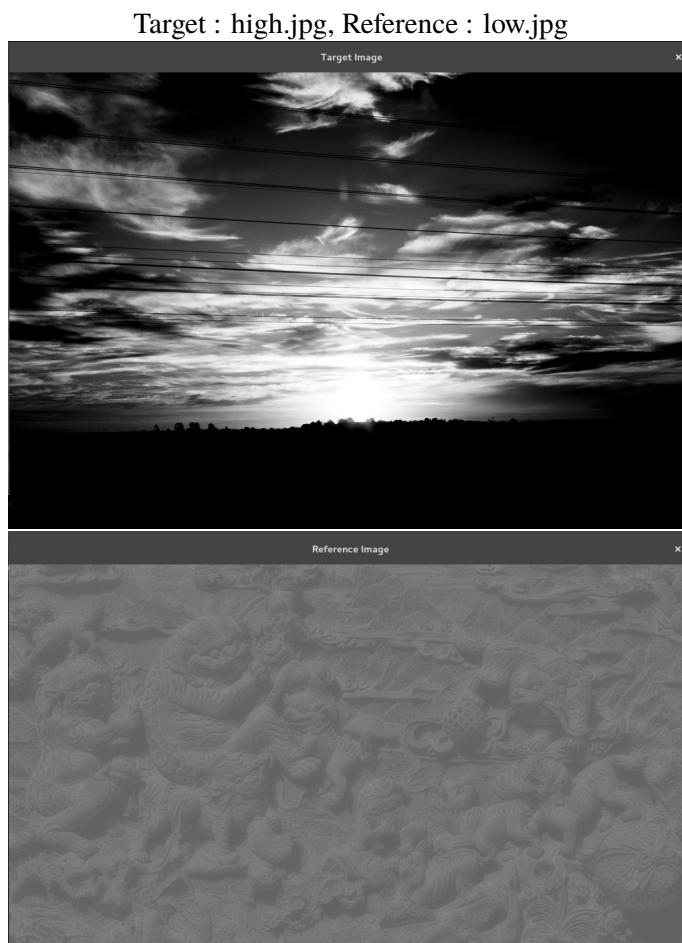
A small program which uses the histogram matching function is present in Q2/src/Q2.cxx

In order to build and run the code, the following steps need to be followed :

1. Enter Q2/src through terminal.
2. run build.sh (OpenCV needs to be installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q2 (the binary file) with the reference and target relative/absolute image paths. Eg. [./Q2 ../../images/high.jpg ../../images/low.jpg] or [./Q2 ../../images/low.jpg ../../images/high.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing 'q' when one of the image windows is open.

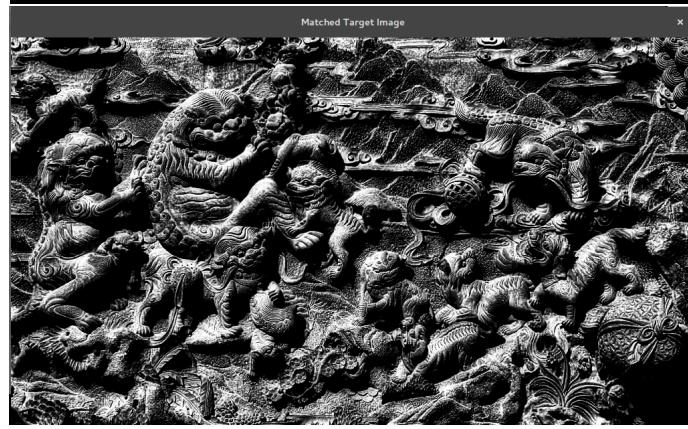
REMARK : A simple plot of the histograms can be obtained on the terminal by toggling the third argument of the function EE604A::histogram_matching() to true.

Results





Target : low.jpg, Reference : high.jpg



Solution 3

Let the 8-connect neighborhood of a point be given by

v_{11}	v_{12}	v_{13}
v_{21}	v_{22}	v_{23}
v_{31}	v_{32}	v_{33}

In bilinear interpolation, we fit a function $f(x, y)$ using weights a_{ij} in the following manner :

$$f(x, y) = \sum_{i=0}^1 \sum_{j=0}^1 a_{ij} x^i y^j \quad (1)$$

Let x_1, x_2, x_3 and y_1, y_2, y_3 be the x and y coordinates of the points in the neighborhood. It is clear that $x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_1$, $y_1 \neq y_2$, $y_2 \neq y_3$ and $y_3 \neq y_1$. Also, let e be the error vector.

$$\text{We have, } \begin{bmatrix} f(x_1, y_1) \\ f(x_1, y_2) \\ f(x_1, y_3) \\ f(x_2, y_1) \\ f(x_2, y_2) \\ f(x_2, y_3) \\ f(x_3, y_1) \\ f(x_3, y_2) \\ f(x_3, y_3) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_1 & y_3 & x_1 y_3 \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_2 & y_3 & x_2 y_3 \\ 1 & x_3 & y_1 & x_3 y_1 \\ 1 & x_3 & y_2 & x_3 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} + e \quad (2)$$

$$\Rightarrow \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_1 & y_3 & x_1 y_3 \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_2 & y_3 & x_2 y_3 \\ 1 & x_3 & y_1 & x_3 y_1 \\ 1 & x_3 & y_2 & x_3 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} + e \quad (3)$$

Our aim is to determine $\begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$ while minimizing the the MSE, $e^T e$.

$$\text{Let } f = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}, X = \begin{bmatrix} 1 & x_1 & y_1 & x_1 y_1 \\ 1 & x_1 & y_2 & x_1 y_2 \\ 1 & x_1 & y_3 & x_1 y_3 \\ 1 & x_2 & y_1 & x_2 y_1 \\ 1 & x_2 & y_2 & x_2 y_2 \\ 1 & x_2 & y_3 & x_2 y_3 \\ 1 & x_3 & y_1 & x_3 y_1 \\ 1 & x_3 & y_2 & x_3 y_2 \\ 1 & x_3 & y_3 & x_3 y_3 \end{bmatrix} \text{ and } W = \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$$

$$\therefore e = f - XW \quad (4)$$

The best estimate for \mathbf{W} is given by :

$$\hat{\mathbf{W}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{f} \quad \text{on minimizing the MSE, } \mathbf{e}^T \mathbf{e}. \quad (5)$$

We need $\mathbf{X}^T \mathbf{X}$ to be invertable for a solution to exist for the given interpolation problem. Let us first show that $\dim(\text{nullspace of } \mathbf{X}) = 0$. We're going to prove that the column vectors of \mathbf{X} are linearly independent, i.e. $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3 + c_4 \mathbf{x}_4 = 0 \implies c_1 = c_2 = c_3 = c_4 = 0$ where $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3 \ \mathbf{x}_4]$

On taking $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3 + c_4 \mathbf{x}_4 = 0$,

$$\text{We have, } c_1 + c_2 x_1 + c_3 y_1 + c_4 x_1 y_1 = 0 \quad (6)$$

$$c_1 + c_2 x_1 + c_3 y_2 + c_4 x_1 y_2 = 0 \quad (7)$$

$$\implies c_3(y_1 - y_2) + c_4 x_1(y_1 - y_2) = 0 \quad \text{from (6) - (7)} \quad (8)$$

$$\implies (y_1 - y_2)(c_3 + c_4 x_1) = 0 \quad (9)$$

$$\implies c_3 + c_4 x_1 = 0 \quad \because y_1 \neq y_2 \quad (10)$$

$$\text{Similarly, using } c_1 + c_2 x_2 + c_3 y_1 + c_4 x_2 y_1 = 0 \quad (11)$$

$$c_1 + c_2 x_2 + c_3 y_2 + c_4 x_2 y_2 = 0 \quad (12)$$

$$\text{we get, } c_3 + c_4 x_2 = 0 \quad (13)$$

$$\implies c_3 = c_4 = 0 \quad \text{from (10), (13) and } \because x_1 \neq x_2 \quad (14)$$

$$\text{Also, } c_1 = c_2 = 0 \text{ from (6), (11), (14) and } \because x_1 \neq x_2 \quad (15)$$

$$\therefore c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3 + c_4 \mathbf{x}_4 = 0 \implies c_1 = c_2 = c_3 = c_4 = 0$$

$\therefore \dim(\text{columnspace of } \mathbf{X}) = 4$ as the column vectors are linearly independent.

$$\text{Hence, } \dim(\text{nullspace of } \mathbf{X}) = 4 - 4 = 0 \quad (16)$$

We're now going to prove that $\text{nullspace of } \mathbf{X} = \text{nullspace of } \mathbf{X}^T \mathbf{X}$.

Let $y \in \text{nullspace of } \mathbf{X}$

$$\begin{aligned} & \therefore \mathbf{X}y = 0 \\ \implies & \mathbf{X}^T \mathbf{X}y = 0 \\ \implies & y \in \text{nullspace of } \mathbf{X}^T \mathbf{X} \\ \therefore & \text{nullspace of } \mathbf{X} \subseteq \text{nullspace of } \mathbf{X}^T \mathbf{X} \end{aligned}$$

Let $y \in \text{nullspace of } \mathbf{X}^T \mathbf{X}$

$$\begin{aligned} & \therefore \mathbf{X}^T \mathbf{X}y = 0 \\ \implies & y^T \mathbf{X}^T \mathbf{X}y = 0 \\ \implies & (Xy)^T Xy = 0 \\ \implies & Xy = 0 \\ \implies & y \in \text{nullspace of } \mathbf{X} \\ \therefore & \text{nullspace of } \mathbf{X}^T \mathbf{X} \subseteq \text{nullspace of } \mathbf{X} \end{aligned}$$

$$\therefore \text{nullspace of } \mathbf{X}^T \mathbf{X} = \text{nullspace of } \mathbf{X} \quad (17)$$

Hence, $\dim(\text{nullspace of } X^T X) = \dim(\text{nullspace of } X) = 0$ using (17) and (18).

$X^T X$ is a 4×4 square matrix. It has full rank since its nullspace is 0. $\therefore X^T X$ is invertible and hence bilinear interpolation in an 8-neighborhood can always be used to get a solution for W and (1) can be constructed.

Solution 4

We are assuming that $\eta_i(x_1, y_1)$ and $\eta_j(x_2, y_2)$ are independent if $i \neq j$ or $x_1 \neq x_2$ or $y_1 \neq y_2$.

$$\therefore \text{if } i \neq j, E[\eta_i(x, y)\eta_j(x, y)] = 0 \quad (18)$$

$$\text{We have, } g_i(x, y) = f_i(x, y) + \eta_i(x, y)$$

$$\text{Also, } f_i(x, y) = f(x, y)$$

$$\begin{aligned} \text{Now, } \hat{g}(x, y) &= \frac{1}{K} \sum_{i=1}^K g_i(x, y) \\ &= \frac{1}{K} K f(x, y) + \frac{1}{K} \sum_{i=1}^K \eta_i(x, y) \\ &= f(x, y) + \frac{1}{K} \sum_{i=1}^K \eta_i(x, y) \end{aligned}$$

$$\begin{aligned} \text{Now, noise variance, } E[(\hat{g}(x, y) - f(x, y))^2] &= E[(\frac{1}{K} \sum_{i=1}^K \eta_i(x, y))^2] \\ &= \frac{1}{K^2} \sum_{i=1}^K \sum_{j=1}^K E[\eta_i(x, y)\eta_j(x, y)] \\ &= \frac{1}{K^2} \sum_{i=1}^K E[\eta_i(x, y)^2] \quad \text{from (18)} \\ &= \frac{1}{K^2} \sum_{i=1}^K \sigma^2 \\ &= \frac{\sigma^2}{K} \quad \because E[\eta_i(x, y)^2] = \sigma^2 \end{aligned}$$

q.e.d

Solution 5

Let $f(x, y, z) : \mathbb{R}^3 \rightarrow \mathbb{R}$. Let $[u \ v \ w]^T$ be the position of $[x \ y \ z]^T$ in a rotated coordinate frame. The relationship between $[u \ v \ w]^T$ and $[x \ y \ z]^T$ is given by :

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (19)$$

$$\text{where, } R = [R_1 \ R_2 \ R_3] \quad (20)$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \quad (21)$$

$$\text{Known properties of } R, |R_1| = 1 \quad (22)$$

$$|R_2| = 1 \quad (23)$$

$$R_1^T R_2 = 0 \quad (24)$$

$$R_3 = R_1 \times R_2 \quad (25)$$

$$\implies |R_3| = 1 \quad (26)$$

$$R_1^T R_3 = R_2^T R_3 = 0 \quad (27)$$

$$R^{-1} = R^T \quad (28)$$

$$\text{We can also write, } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R^T \begin{bmatrix} u \\ v \\ w \end{bmatrix} \quad \text{from (19), (28)} \quad (29)$$

$$\text{Now, } \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} \quad (30)$$

$$= R_{11} \frac{\partial f}{\partial x} + R_{12} \frac{\partial f}{\partial y} + R_{13} \frac{\partial f}{\partial z} \quad \text{from (21), (29)} \quad (31)$$

$$\text{Similarly, } \frac{\partial f}{\partial v} = R_{21} \frac{\partial f}{\partial x} + R_{22} \frac{\partial f}{\partial y} + R_{23} \frac{\partial f}{\partial z} \quad (32)$$

$$\frac{\partial f}{\partial w} = R_{31} \frac{\partial f}{\partial x} + R_{32} \frac{\partial f}{\partial y} + R_{33} \frac{\partial f}{\partial z} \quad (33)$$

$$\begin{aligned} \therefore \frac{\partial^2 f}{\partial^2 u} &= R_{11}\left(\frac{\partial^2 f}{\partial^2 x} \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x} \frac{\partial z}{\partial u}\right) \\ &\quad + R_{12}\left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial^2 y} \frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x} \frac{\partial z}{\partial u}\right) \\ &\quad + R_{13}\left(\frac{\partial^2 f}{\partial x \partial z} \frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial z} \frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial^2 z} \frac{\partial z}{\partial u}\right) \end{aligned} \quad \text{from (31) and chain rule} \quad (34)$$

$$\begin{aligned} &= R_{11}^2 \frac{\partial^2 f}{\partial^2 x} + R_{11}R_{12} \frac{\partial^2 f}{\partial y \partial x} + R_{11}R_{13} \frac{\partial^2 f}{\partial z \partial x} \\ &\quad + R_{12}R_{11} \frac{\partial^2 f}{\partial x \partial y} + R_{12}^2 \frac{\partial^2 f}{\partial^2 y} + R_{12}R_{13} \frac{\partial^2 f}{\partial z \partial y} \\ &\quad + R_{13}R_{11} \frac{\partial^2 f}{\partial x \partial z} + R_{13}R_{12} \frac{\partial^2 f}{\partial y \partial z} + R_{13}^2 \frac{\partial^2 f}{\partial^2 z} \end{aligned} \quad \text{from (21), (29)} \quad (35)$$

$$\begin{aligned} \text{Similarly, } \frac{\partial^2 f}{\partial^2 v} &= R_{21}^2 \frac{\partial^2 f}{\partial^2 x} + R_{21}R_{22} \frac{\partial^2 f}{\partial y \partial x} + R_{21}R_{23} \frac{\partial^2 f}{\partial z \partial x} \\ &\quad + R_{22}R_{21} \frac{\partial^2 f}{\partial x \partial y} + R_{22}^2 \frac{\partial^2 f}{\partial^2 y} + R_{22}R_{23} \frac{\partial^2 f}{\partial z \partial y} \\ &\quad + R_{23}R_{21} \frac{\partial^2 f}{\partial x \partial z} + R_{23}R_{22} \frac{\partial^2 f}{\partial y \partial z} + R_{23}^2 \frac{\partial^2 f}{\partial^2 z} \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{\partial^2 f}{\partial^2 w} &= R_{31}^2 \frac{\partial^2 f}{\partial^2 x} + R_{31}R_{32} \frac{\partial^2 f}{\partial y \partial x} + R_{31}R_{33} \frac{\partial^2 f}{\partial z \partial x} \\ &\quad + R_{32}R_{31} \frac{\partial^2 f}{\partial x \partial y} + R_{32}^2 \frac{\partial^2 f}{\partial^2 y} + R_{32}R_{33} \frac{\partial^2 f}{\partial z \partial y} \\ &\quad + R_{33}R_{31} \frac{\partial^2 f}{\partial x \partial z} + R_{33}R_{32} \frac{\partial^2 f}{\partial y \partial z} + R_{33}^2 \frac{\partial^2 f}{\partial^2 z} \end{aligned} \quad (37)$$

On performing (35) + (36) + (37), we get using (22), (23), (23), (26) and (27)

$$\frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} \quad (38)$$

$$\implies \nabla^2 f(u, v, w) = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} \quad (39)$$

$$= \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} \quad \text{from (38)} \quad (40)$$

$$= \nabla^2 f(x, y, z) \quad (41)$$

\implies Laplacian (∇^2) is isotropic (rotation invariant).

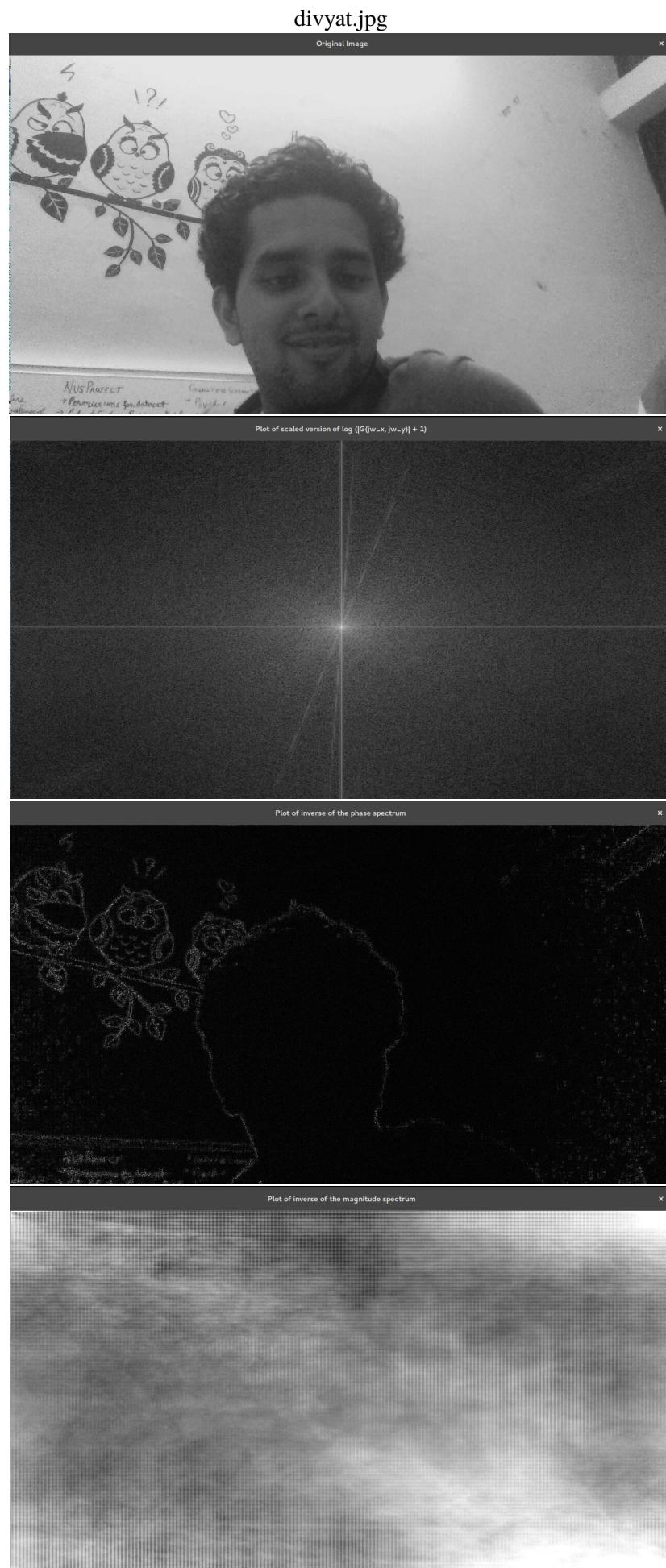
Solution 6

A C++ file, Q6/src/Q6.cxx, contains the code required to perform the required task.

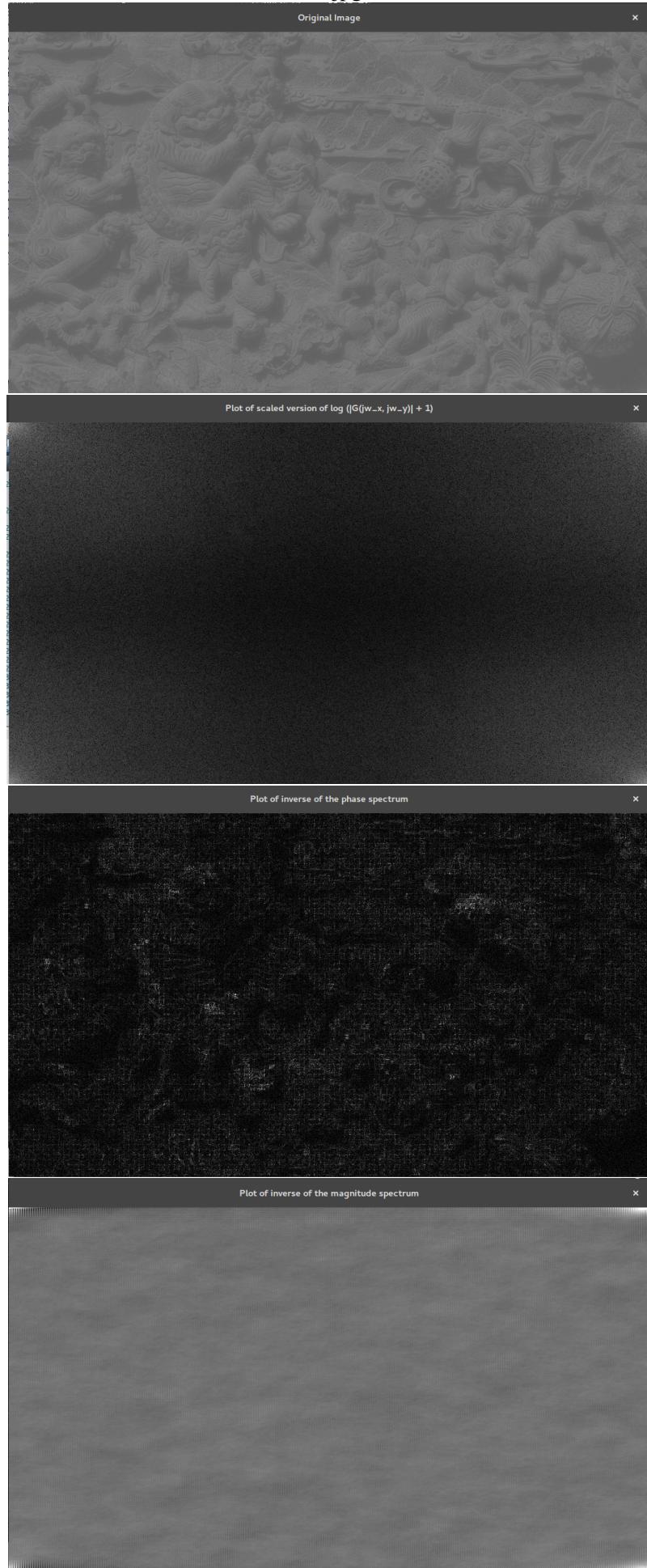
In order to build and run the code, the following steps need to be followed :

1. Enter Q6/src through terminal.
2. run build.sh (OpenCV needs to be installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q6 (the binary file) with the relative/absolute image path. Eg. [./Q6 ../../images/divyat.jpg] or [./Q6 ../../images/low.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing ‘q’ when one of the image windows is open.

Results



low.jpg



Observations

Reconstruction using only phase spectrum

From the above the tests, it can be inferred that on performing reconstruction using only phase information, edges are being preserved and we are losing all the grey-levels of homogenous regions. From the spectrum plots, we can infer that the lower frequency components tend to have greater values than the higher frequency ones in general images. Thus, when we remove the magnitude information, the lower frequency components are getting more suppressed than the higher frequency components, i.e. the operation is similar to a high-pass filter. This premise explains the above images obtained using only the phase spectrum.

Reconstruction using only magnitude spectrum

The reconstruction using only magnitude spectrum appears to only contain information regarding the prominent greylevels present in the image.

Solution 7

Code has been written in two .cxx files :

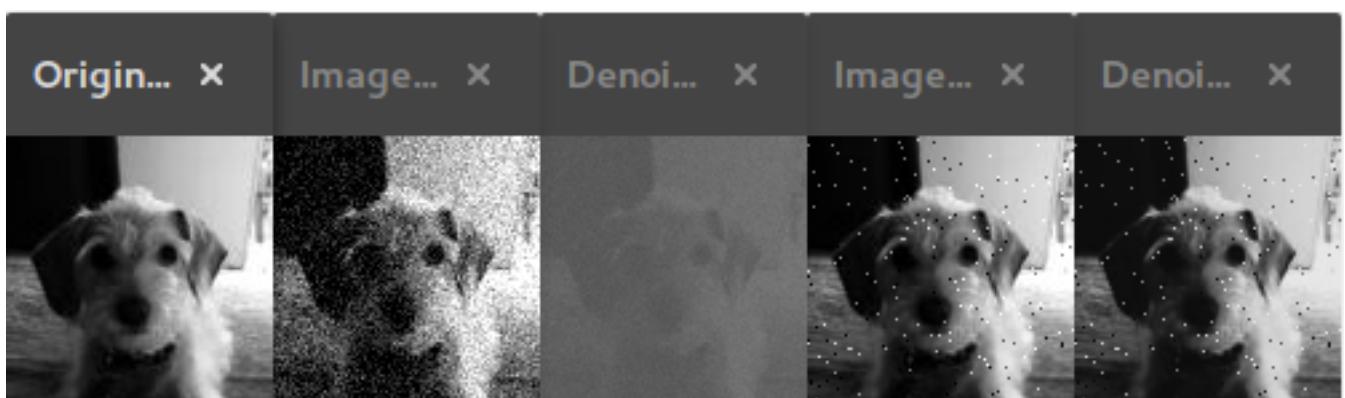
1. Q7/src/pixel_nlm.cxx : Contains the function, EE604A::pixel_nlm(), used to perform neighborhood filter using non-local pixel values.
2. Q7/src/Q7.cxx : Calls the contained function, task(), which adds noise as needed and evaluated the neighborhood filtering method.

There is also a header file present for the EE604A::pixel_nlm() function :

1. Q7/include/pixel_nlm.cxx

In order to build and run the code, the following steps need to be followed :

1. Enter Q7/src through terminal.
2. run build.sh (OpenCV needs to be installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q7 (the binary file) with the relative/absolute image path. Eg. [./Q7 ../../images/small.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing 'q' when one of the image windows is open.



The images from left to right are as follows : (a) The original image (small.jpg), (b) The images with added zero-mean gaussian noise with $\sigma^2 = 0.01$, (c) The reconstructed image [b], (d) The image with impulse noise, (e) The reconstructed image [d]