EE604A - Digital Image Processing Assignment

Satya Prakash Panuganti, 14610 6 September, 2017

Image Sources

Sunset (high.jpg) : https://www.flickr.com/photos/mediaflex/4190084346
Carving (low.jpg) : https://www.flickr.com/photos/30440933@N06/2847993403
Dog (small.jpg) : https://res.cloudinary.com/rover-com/image/upload/a_exif,c_fill,f_jpg,fl_progressive,g_face:center,h_100,q_80,w_100/remote/images/pets/4NpPzO8N/50e4a023d9/original.jpg
Face (divyat.jpg) : Own work. Taken with permission.

Solution 1

(a)

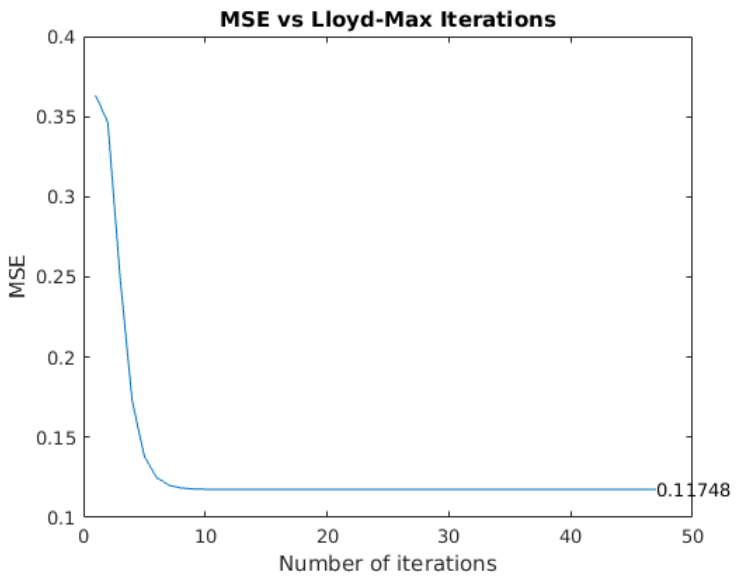
Code has been written in three MATLAB® files :

- Q1/lloyd_max_quantizer_function.m : The file containing the function which returns the mean square errors, representation levels and transition levels given the pdf of a signal.
- normal.m : File containing function which returns the pdf value at a point.
- Q1/Q1.m : The script to perform the required tasks.

(b)

The Lloyd-Max Quantizer was run over a signal with zero mean and unit variance normal distribution. The 4 representation levels are : -1.5104 -0.4528 0.4528 1.5104
The corresponding transition levels are : $-\infty$ -0.9816 0.0000 0.9816 ∞

(c)



(d)

Experiments

In all of the experiments, the transition levels were intialy choosen uniformaly between -10 and 10. The smallest and largest transition levels were than forced to $-\infty$ and ∞ respectively before the LLoyd-Max Quantizer iterations were carried out.

μ	σ^2	l_1	l_2	l_3	l_4	m_1	m_2	m_3	m_4	m_5	MSE
0	1	-1.5104	-0.4528	0.4528	1.5104	$-\infty$	-0.9816	0.0000	0.9816	∞	0.11748
0	0.5	-1.0680	-0.3202	0.3202	1.0680	$-\infty$	-0.6941	0.0000	0.6941	∞	0.041536
0	2	-2.1361	-0.6403	0.6403	2.1361	$-\infty$	-1.3882	0.0000	1.3882	∞	0.33229
2.3	1	0.7896	1.8472	2.7528	3.8104	$-\infty$	1.3184	2.3000	3.2816	∞	0.11748
2.3	0.5	1.2320	1.9798	2.6202	3.3680	$-\infty$	1.6059	2.3000	2.9941	∞	0.041536
2.3	2	0.1639	1.6597	2.9403	4.4361	$-\infty$	0.9118	2.3000	3.6882	∞	0.33229

Observations

1. On adding a $\Delta\mu$ to the mean, while keeping σ^2 constant, the representation and transition levels obatined get shifted by $\Delta\mu$ as well.
2. The MSE value are independent of the mean.
3. As $\sigma \uparrow$, MSE $\uparrow$ and $|l_{i+1} - l_i| \uparrow$

(e)

Let the signal be denoted by S and the pdf of S be $p_S(s)$.

$$Now,\ p_S(s) = \begin{cases} \dfrac{1}{b-a} & a \leq s < b \\ 0 & \text{otherwise} \end{cases}$$

Let us assume that there 'L' representation levels, a_i are the representation levels and m_i are the transition levels.
We begin by initializing m_i uniformly :

$$m_i = a + \frac{b-a}{L} \times (i-1)\ \forall\ i \in [1, L+1]$$

We next set a_i :

$$\begin{aligned} a_i &= \frac{\int_{m_i}^{m_{i+1}} s \times p_S(s) ds}{\int_{m_i}^{m_{i+1}} p_S(s) ds}\ \forall\ i \in [1, L] \\ &= \frac{\frac{m_{i+1}^2 - {m_i}^2}{2(b-a)}}{\frac{m_{i+1}-m_i}{b-a}}\ \forall\ i \in [1, L] \\ &= \frac{m_i + m_{i+1}}{2}\ \forall\ i \in [1, L] \\ &= a + \frac{b-a}{L} \times \frac{i+i-1}{2}\ \forall\ i \in [1, L] \\ &= a + \frac{(b-a)(2i-1)}{2L}\ \forall\ i \in [1, L] \end{aligned}$$

We now set m_i :

$$\begin{aligned} m_i &= \frac{a_{i-1} + a_i}{2} \; \forall \; i \in [2, L] \\ &= a + \frac{b-a}{4L} \times (2(i+i-1) - 2) \; \forall \; i \in [2, L] \\ &= a + \frac{b-a}{L} \times (i-1) \; \forall \; i \in [2, L] \end{aligned}$$

m_i do not change in the above step. Hence, the Lloyd-Max Quantizer has converged to a final solution.

Solution 2

A C++ function, EE604A::histogram_matching(), for histogram matching of cv::Mat images has been developed. The code required for the matching function is present in :

- Q2/src/histogram_matching.cxx
- Q2/src/histogram_matching.h

A small program which uses the histogram matching function is present in Q2/src/Q2.cxx

In order to build and run the code, the following steps need to be followed :

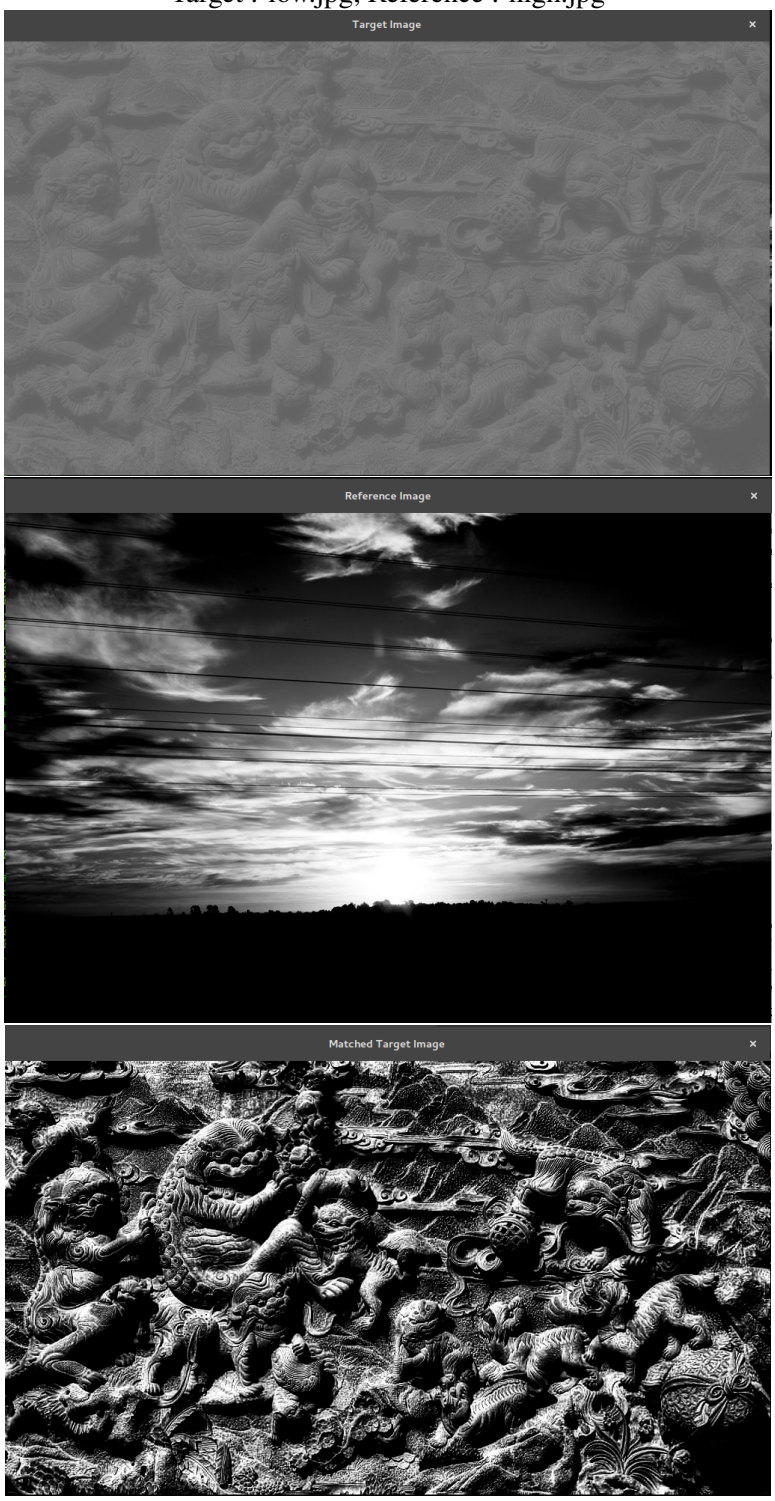
1. Enter Q2/src through terminal.
2. run build.sh (OpenCV needs to installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q2 (the binary file) with the reference and target relative/absolute image paths. Eg. [./Q2 ../../images/high.jpg ../../images/low.jpg] or [./Q2 ../../images/low.jpg ../../images/high.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing 'q' when one of the image windows is open.

REMARK : A simple plot of the the histograms can be obtained on the terminal by toggling the third argument of the function EE604A::histogram_matching() to true.

Results

Target : high.jpg, Reference : low.jpg

Target : low.jpg, Reference : high.jpg



Solution 3

Let the 8-connect neighborhood of a point be given by

v_{11}	v_{12}	v_{13}
v_{21}	v_{22}	v_{23}
v_{31}	v_{32}	v_{33}

.

In blinear interpolation, we fit a function $f(x, y)$ using weights a_{ij} in the following manner :

$$f(x, y) = \Sigma_{i=0}^{1}\Sigma_{j=0}^{1}a_{ij}x^i y^j \tag{1}$$

Let x_1, x_2, x_3 and y_1, y_2, y_3 be the x and y coordinates of the points in the neighborhood. $x_1 \neq x_2$, $x_2 \neq x_3$, $x_3 \neq x_1$, $y_1 \neq y_2$, $y_2 \neq y_3$ and $y_3 \neq y_1$ since the points belong to different rows and colums. Also, let e be the error vector.

$$We\ have, \quad \begin{bmatrix} f(x_1,y_1) \\ f(x_1,y_2) \\ f(x_1,y_3) \\ f(x_2,y_1) \\ f(x_2,y_2) \\ f(x_2,y_3) \\ f(x_3,y_1) \\ f(x_3,y_2) \\ f(x_3,y_3) \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_2 & y_3 & x_2y_3 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_3 & y_2 & x_3y_2 \\ 1 & x_3 & y_3 & x_3y_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} + e \tag{2}$$

$$\implies \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_2 & y_3 & x_2y_3 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_3 & y_2 & x_3y_2 \\ 1 & x_3 & y_3 & x_3y_3 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix} + e \tag{3}$$

Our aim is to determine $\begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$ while minimizing the the MSE, $e^T e$.

Let $f = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \\ v_{21} \\ v_{22} \\ v_{23} \\ v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}$, $X = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_1 & y_2 & x_1y_2 \\ 1 & x_1 & y_3 & x_1y_3 \\ 1 & x_2 & y_1 & x_2y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ 1 & x_2 & y_3 & x_2y_3 \\ 1 & x_3 & y_1 & x_3y_1 \\ 1 & x_3 & y_2 & x_3y_2 \\ 1 & x_3 & y_3 & x_3y_3 \end{bmatrix}$ and $W = \begin{bmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{bmatrix}$

$$\therefore\ e = f - XW \tag{4}$$

The best estimate for W is given by :

$$\hat{W} = (X^T X)^{-1} X^T f \qquad on\ minimizing\ the\ MSE,\ e^T e. \tag{5}$$

We need $X^T X$ to be invertable for a solution to exist for the given interpolation problem. Let us first show that $dim(nullspace\ of\ X) = 0$. We're going to prove that the column vectors of X are linearly independent, i.e. $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0 \implies c_1 = c_2 = c_3 = c_4 = 0$ where $X = [X_1\ X_2\ X_3\ X_4]$

On taking $c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0$,

$$We\ have, c_1 + c_2x_1 + c_3y_1 + c_4x_1y_1 = 0 \tag{6}$$

$$c_1 + c_2x_1 + c_3y_2 + c_4x_1y_2 = 0 \tag{7}$$

$$\implies c_3(y_1 - y_2) + c_4x_1(y_1 - y_2) = 0 \qquad from\ (6) - (7) \tag{8}$$

$$\implies (y_1 - y_2)(c_3 + c_4x_1) = 0 \tag{9}$$

$$\implies c_3 + c_4x_1 = 0 \qquad \because\ y_1 \neq y_2 \tag{10}$$

$$Similiarly,\ using\ c_1 + c_2x_2 + c_3y_1 + c_4x_2y_1 = 0 \tag{11}$$

$$c_1 + c_2x_2 + c_3y_2 + c_4x_2y_2 = 0 \tag{12}$$

$$we\ get, c_3 + c_4x_2 = 0 \tag{13}$$

$$\implies c_3 = c_4 = 0 \qquad from\ (10),\ (13)\ and\ \because\ x_1 \neq x_2 \tag{14}$$

$$Also,\ c_1 = c_2 = 0\ from\ (6),\ (11),\ (14)\ and\ \because\ x_1 \neq x_2 \tag{15}$$

$\therefore\ c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 = 0 \implies c_1 = c_2 = c_3 = c_4 = 0$

$\therefore\ dim(columnspace\ of\ X) = 4$ as the column vectors are linearly independent.

$$Hence,\ dim(nullspace\ of\ X) = 4 - 4 = 0 \tag{16}$$

We're now going to prove that $nullspace\ of\ X = nullspace\ of\ X^TX$.

$Let\ y \in nullspace\ of\ X$

$$\therefore Xy = 0$$
$$\implies X^TXy = 0$$
$$\implies y \in nullspace\ of\ X^TX$$
$$\therefore nullspace\ of\ X \subseteq\ nullspace\ of\ X^TX$$

$Let\ y \in nullspace\ of\ X^TX$

$$\therefore X^TXy = 0$$
$$\implies y^TX^TXy = 0$$
$$\implies (Xy)^TXy = 0$$
$$\implies Xy = 0$$
$$\implies y \in nullspace\ of\ X$$
$$\therefore nullspace\ of\ X^TX \subseteq\ nullspace\ of\ X$$

$$\therefore\ nullspace\ of\ X^TX = nullspace\ of\ X \tag{17}$$

Hence, $dim(nullspace\ of\ X^TX) = dim(nullspace\ of\ X) = 0$ using (17) and (18).
X^TX is a 4×4 square matrix. It has full rank since its nullspace is 0. $\therefore$ X^TX is invertible and hence bilinear interpolation in an 8-neighborhood can always be used to get a solution for W and (1) can be constructed.
The linear system is overdetermined, but it always has a rank of 4. Hence, an approximate solution can always be obtained through the pseudoinverse. The approximate solution is given by (5).

Solution 4

We are assuming that $\eta_i(x_1, y_1)\ and\ \eta_j(x_2, y_2)$ are independent if i $\neq j$ *or* $x_1 \neq x_2$ *or* $y_1 \neq y_2$.

$$\therefore\ if\ i \neq j,\ E[\eta_i(x,y)\eta_j(x,y)] = 0 \tag{18}$$

$$\begin{aligned}
We\ have,\ g_i(x,y) &= f_i(x,y) + \eta_i(x,y) \\
Also,\ f_i(x,y) &= f(x,y) \\
Now,\ \hat{g}(x,y) &= \frac{1}{K}\Sigma_{i=1}^{K} g_i(x,y) \\
&= \frac{1}{K}Kf(x,y) + \frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x,y) \\
&= f(x,y) + \frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x,y) \\
Now,\ noise\ variance,\ E[(\hat{g}(x,y) - f(x,y))^2] &= E[(\frac{1}{K}\Sigma_{i=1}^{K}\eta_i(x,y))^2] \\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}\Sigma_{j=1}^{K}E[\eta_i(x,y)\eta_j(x,y)] \\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}E[\eta_i(x,y)^2] && from\ (18) \\
&= \frac{1}{K^2}\Sigma_{i=1}^{K}\sigma^2 && \because\ E[\eta_i(x,y)^2] = \sigma^2 \\
&= \frac{\sigma^2}{K} \\
q.e.d
\end{aligned}$$

Hence, the noise variance gets reduced K times by averaging.

Solution 5

Let $f(x, y, z) : \Re^3 \rightarrow \Re$. Let $[u\ v\ w]^T$ be the position of $[x\ y\ z]^T$ in a rotated cooridnate frame. The relationship between $[u\ v\ w]^T$ and $[x\ y\ z]^T$ is given by :

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{19}$$

$$where,\ R = [R_1\ R_2\ R_3] \tag{20}$$

$$= \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \tag{21}$$

$$Known\ properties\ of\ R,\ |R_1| = 1 \tag{22}$$

$$|R_2| = 1 \tag{23}$$

$$R_1^T R_2 = 0 \tag{24}$$

$$R_3 = R_1 \times R_2 \tag{25}$$

$$\implies |R_3| = 1 \tag{26}$$

$$R_1^T R_3 = R_2^T R_3 = 0 \tag{27}$$

$$R^{-1} = R^T \tag{28}$$

$$We\ can\ also\ write,\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R^T \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad from\ (19),\ (28) \tag{29}$$

$$Now,\ \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \qquad using\ chain\ rule \tag{30}$$

$$= R_{11}\frac{\partial f}{\partial x} + R_{12}\frac{\partial f}{\partial y} + R_{13}\frac{\partial f}{\partial z} \qquad from\ (21), (29) \tag{31}$$

$$Similarily,\ \frac{\partial f}{\partial v} = R_{21}\frac{\partial f}{\partial x} + R_{22}\frac{\partial f}{\partial y} + R_{23}\frac{\partial f}{\partial z} \tag{32}$$

$$\frac{\partial f}{\partial w} = R_{31}\frac{\partial f}{\partial x} + R_{32}\frac{\partial f}{\partial y} + R_{33}\frac{\partial f}{\partial z} \tag{33}$$

$$\therefore \frac{\partial^2 f}{\partial^2 u} = R_{11}(\frac{\partial^2 f}{\partial^2 x}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial u})$$
$$+ R_{12}(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial^2 y}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial z \partial x}\frac{\partial z}{\partial u})$$
$$+ R_{13}(\frac{\partial^2 f}{\partial x \partial z}\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y \partial z}\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial^2 z}\frac{\partial z}{\partial u}) \qquad from\ (31)\ and\ chain\ rule \qquad (34)$$
$$= R_{11}^2\frac{\partial^2 f}{\partial^2 x} + R_{11}R_{12}\frac{\partial^2 f}{\partial y \partial x} + R_{11}R_{13}\frac{\partial^2 f}{\partial z \partial x}$$
$$+ R_{12}R_{11}\frac{\partial^2 f}{\partial x \partial y} + R_{12}^2\frac{\partial^2 f}{\partial^2 y} + R_{12}R_{13}\frac{\partial^2 f}{\partial z \partial y}$$
$$+ R_{13}R_{11}\frac{\partial^2 f}{\partial x \partial z} + R_{13}R_{12}\frac{\partial^2 f}{\partial y \partial z} + R_{13}^2\frac{\partial^2 f}{\partial^2 z} \qquad from\ (21), (29) \qquad (35)$$

$$Similarily,\ \frac{\partial^2 f}{\partial^2 v} = R_{21}^2\frac{\partial^2 f}{\partial^2 x} + R_{21}R_{22}\frac{\partial^2 f}{\partial y \partial x} + R_{21}R_{23}\frac{\partial^2 f}{\partial z \partial x}$$
$$+ R_{22}R_{21}\frac{\partial^2 f}{\partial x \partial y} + R_{22}^2\frac{\partial^2 f}{\partial^2 y} + R_{22}R_{23}\frac{\partial^2 f}{\partial z \partial y}$$
$$+ R_{23}R_{21}\frac{\partial^2 f}{\partial x \partial z} + R_{23}R_{22}\frac{\partial^2 f}{\partial y \partial z} + R_{23}^2\frac{\partial^2 f}{\partial^2 z} \qquad (36)$$

$$\frac{\partial^2 f}{\partial^2 w} = R_{31}^2\frac{\partial^2 f}{\partial^2 x} + R_{31}R_{32}\frac{\partial^2 f}{\partial y \partial x} + R_{31}R_{33}\frac{\partial^2 f}{\partial z \partial x}$$
$$+ R_{32}R_{31}\frac{\partial^2 f}{\partial x \partial y} + R_{32}^2\frac{\partial^2 f}{\partial^2 y} + R_{32}R_{33}\frac{\partial^2 f}{\partial z \partial y}$$
$$+ R_{33}R_{31}\frac{\partial^2 f}{\partial x \partial z} + R_{33}R_{32}\frac{\partial^2 f}{\partial y \partial z} + R_{33}^2\frac{\partial^2 f}{\partial^2 z} \qquad (37)$$

On performing $(35) + (36) + (37)$, we get using (22), (23), (24), (26) and (27)

$$\frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} \qquad (38)$$
$$\implies \nabla^2 f(u,v,w) = \frac{\partial^2 f}{\partial^2 u} + \frac{\partial^2 f}{\partial^2 v} + \frac{\partial^2 f}{\partial^2 w} \qquad (39)$$
$$= \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z} \qquad from\ (38) \qquad (40)$$
$$= \nabla^2 f(x,y,z) \qquad (41)$$
$$\implies Laplacian\ (\nabla^2)\ is\ isotropic\ (rotation\ invariant)\ since$$
$$it\ doesn't\ change\ on\ rotating\ the\ coordinate\ frame.$$

Solution 6

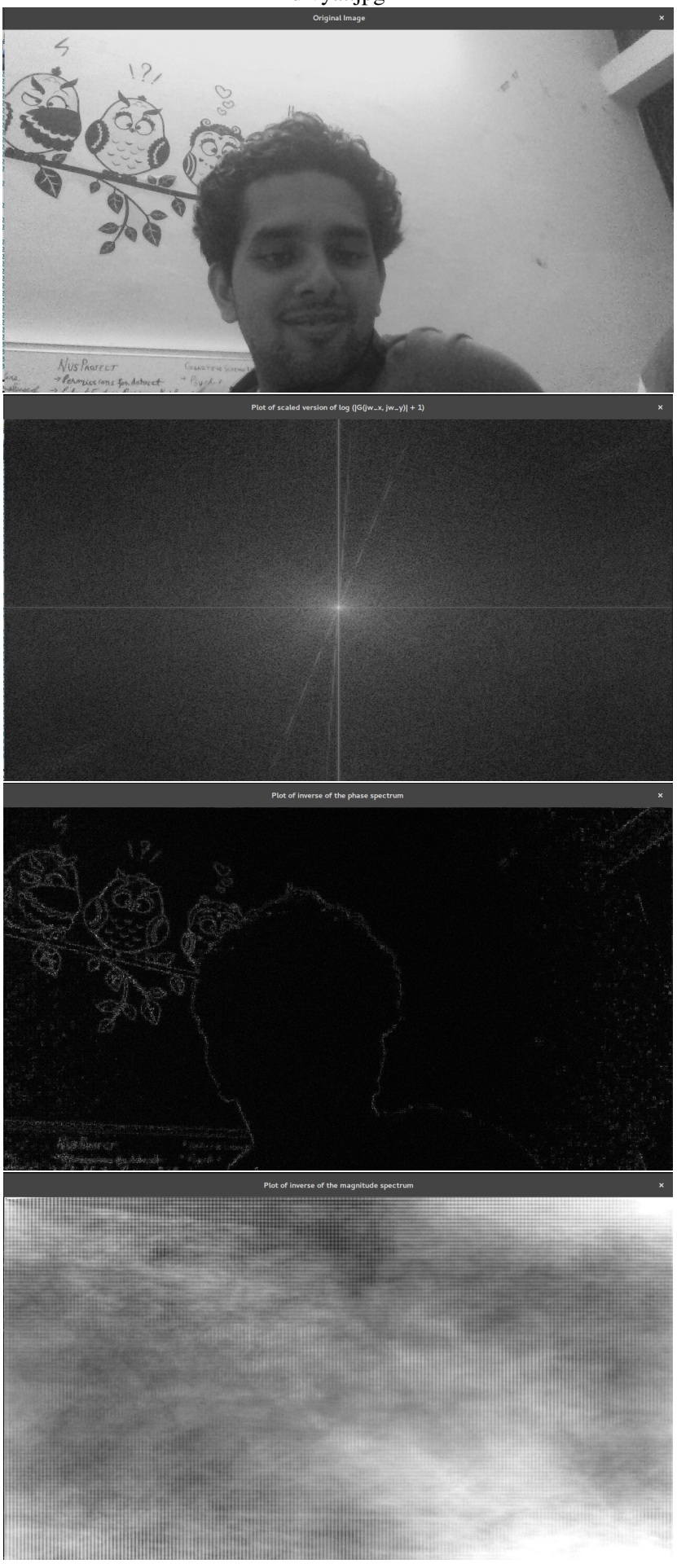
A C++ file, Q6/src/Q6.cxx, contains the code required to perform the required task.

In order to build and run the code, the following steps need to be followed :

1. Enter Q6/src through terminal.
2. run build.sh (OpenCV needs to installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q6 (the binary file) with the relative/absolute image path. Eg. [./Q6 ../../images/divyat.jpg] or [./Q6 ../../images/low.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing 'q' when one of the image windows is open.

Results

divyat.jpg



low.jpg


Original Image
Plot of scaled version of log (|G(jw_x, jw_y)| + 1)
Plot of inverse of the phase spectrum
Plot of inverse of the magnitude spectrum

Observations

Structure of the image seems to be stored in the phase spectrum and it appears as if the magnitude spectrum on its own does not contain useful information.

Reconstruction using only phase spectrum

From the above the tests, it can be inferred that on performing reconstruction using only phase information, edges are being preserved and we are losing all the grey-levels of homogenous regions. From the spectrum plots, we can infer that the lower frequency components tend to have greater values than the higher frequency ones in general images. Thus, when we remove the magnitude information, the lower frequency components are getting more suppressed than the higer frequency components, i.e. the operation is similar to a high-pass filter. This premise explains the above images obatined using only the phase spectrum.

Reconstruction using only magnitude spectrum

The reconstruction using only magnitude spectrum appears to only contain information regarding the prominent greylevels present in the image. Removing phase information disturbs the structure of the image.

Solution 7

Code has been written in two .cxx files :

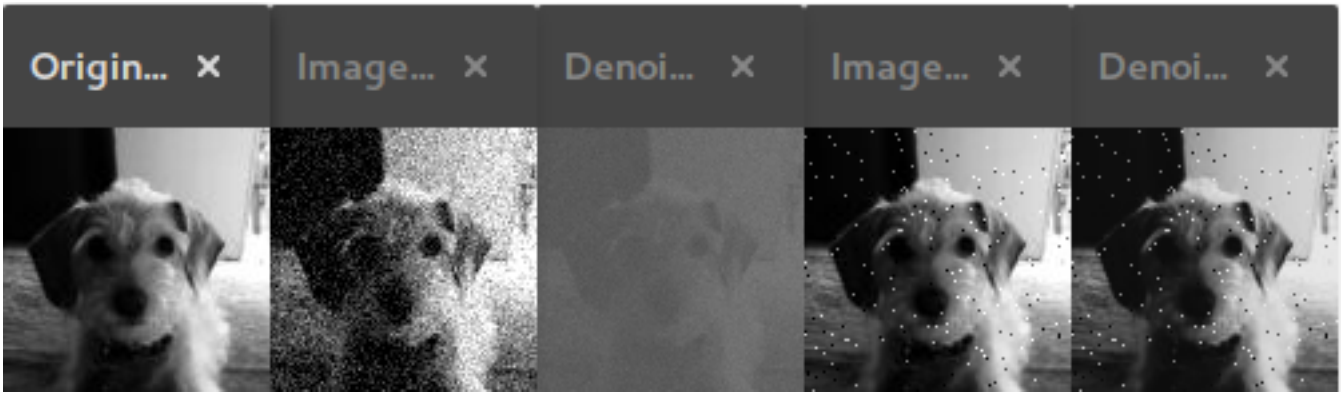
1. Q7/src/pixel_nlm.cxx : Contains the function, EE604A::pixel_nlm(), used to perform neighborhood filter using non-local pixel values.
2. Q7/src/Q7.cxx : Calls the contained function, task(), which adds noise as needed and evaluated the neighborhood filtering method.

There is also a header file present for the EE604A::pixel_nlm() function :

1. Q7/include/pixel_nlm.cxx

In order to build and run the code, the following steps need to be followed :

1. Enter Q7/src through terminal.
2. run build.sh (OpenCV needs to installed and a version of g++ supporting c++14 needs to be present)
3. Execute Q7 (the binary file) with the relative/absolute image path. Eg. [./Q7 ../../images/small.jpg] (On Ubuntu)
4. The program can be closed by Ctrl+C on the terminal or by pressing ‘q’ when one of the image windows is open.



The images from left to right are as follows : (a) The original image (small.jpg), (b) The images with added zero-mean gaussian noise with $\sigma^2 = 0.01$, (c) The reconstucted image [b], (d) The image with impulse noise, (e) The reconstructed image [d]