# Synthesis of Boolean Functions

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# Program Synthesis as an application of QBF

**Program sketching**: [Solar-Lezama'08]

Replace 'holes' (**??**) by an integer such that the assertion is satisfied for all inputs.

```
1 int abs(int x) {
2   int r;
3   if (x > ??) r =  x;
4   else        r = -x;
5   assert(r = |x|); // mathematical notation
6   return r;
7 }
```

An encoding in QBF:

$$\exists y \in \mathbb{B}^{32}\ \forall x \in \mathbb{B}^{32}\ \exists r \in \mathbb{B}^{32}.\ (x > y \rightarrow r = x)\ \wedge\ (x \leq y \rightarrow r = -x)\ \wedge\ r = |x|$$

**Observation:** The Skolem function (mapping $x$ to $r$) represents the program semantics.

# Beyond Certification - Program Synthesis Revisited

**Program sketching with expressions**:

Replace 'holes' (**???**) by an **expression** such that the assertion is satisfied for all inputs.

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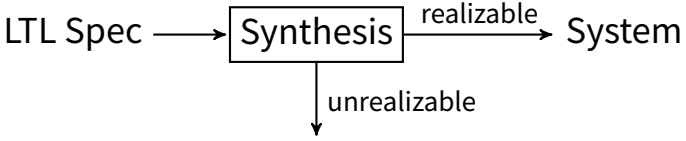
An encoding in QBF:

$$\forall x \in \mathbb{B}^{32}\ \exists c \in \mathbb{B},\ r \in \mathbb{B}^{32}.\ (c \to r = x) \wedge (\neg c \to r = -x) \wedge r = |x|$$

**The certificate for $c$ is the missing part of the implementation!**

# Bounded Synthesis of Reactive Systems

[Schewe, Finkbeiner'07]




Applications: hardware bus systems, motion planning in robotics, …

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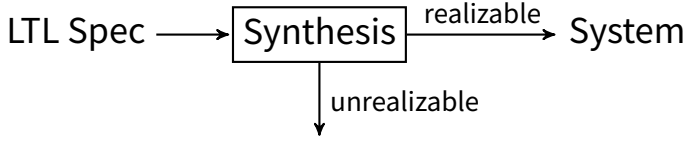


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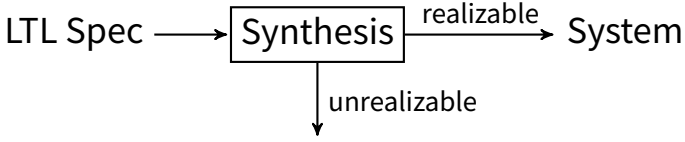


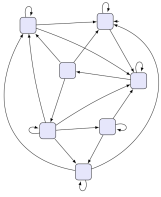

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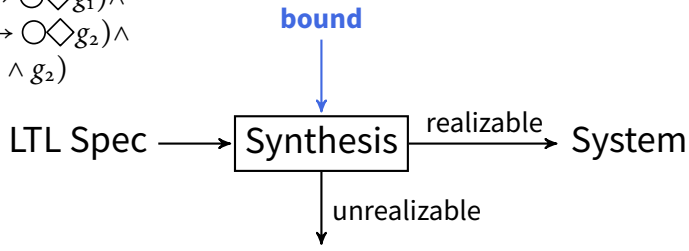


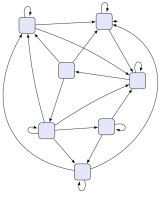

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In general, reactive synthesis needs automata-theoretic algorithms, but if we **bound** the number of states, we can reduce it to solving a constraint system.

# QBF encoding of Bounded Synthesis

[Faymonville et al.'17]

Given an LTL specification $\varphi$, find a system $S$ such that $\varphi \models S$. Let $Q$ be the states of the universal co-Büchi automaton of $\varphi$. Let $S$ have a fixed set of states $T$, a set of input variables $I$, and a set of output variables $O$.

$$\exists \{\lambda^{\mathbb{B}}_{t,q}, \lambda^{\#}_{t,q} \mid t \in T, q \in Q\}$$

$$\forall I$$

$$\exists \{\tau_{t,t'} \mid (t,t') \in T \times T\}$$

$$\exists \{o_t \mid o \in O, t \in T\}$$

$$\lambda^{\mathbb{B}}_{t_0,q_0} \wedge \bigwedge_{t \in T} \bigvee_{t' \in T} \tau_{t,t'}$$

$$\bigwedge_{q \in Q} \bigwedge_{t \in T} \left( \lambda^{\mathbb{B}}_{t,q} \rightarrow \bigwedge_{q' \in Q} \left( \delta_{t,q,q'} \rightarrow \bigwedge_{t' \in T} \left( \tau_{t,t'} \rightarrow \lambda^{\mathbb{B}}_{t',q'} \wedge \lambda^{\#}_{t',q'} \rhd_{q'} \lambda^{\#}_{t,q} \right) \right) \right)$$

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In particular, the encoding has variables $o_t$ for every state $t \in T$. Their Skolem functions are the implementation of the system.

In practice, QBF solvers find small implementations.

# Invariant Synthesis

```
int multiply(int a, int b) { // assume a is positive
  int acc = 0;
  for(int i = 0; i < a; i++) {
    acc += b;
  }
  return acc;
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Find it automatically by solving this formula: $\exists inv : \mathbb{B}^{32\cdot 4} \to \mathbb{B}$.

$$\forall a, b.\ inv(a, b, 0, 0) \ \wedge \qquad \text{loop entry}$$

$$\forall a, b, i, acc.\ inv(a, b, i, acc) \ \wedge\ i = a \ \to\ acc = a \cdot b \ \wedge \qquad \text{loop exit}$$

$$\forall a, b, i, acc, a', b', i', acc'. \qquad \text{inductiveness}$$

$$a = a' \ \wedge\ b = b' \ \wedge\ i' = i + 1 \ \wedge\ acc' = acc + b \ \wedge inv(a, b, i, acc) \to inv(a', b', i', acc')$$

This is not a QBF! But can be expressed in DQBF, an extension of QBF.

# Overview

**QBF with one quantifier alternation**

A Different Take on QBF solving

**2QBF Certification and Functional Synthesis**

From Quantifiers to Functions

**Dependency QBF**

Principled Approaches to Reasoning about Functions

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Negating first leads to a QCNF in 2QBF: $\forall x_1, \ldots, x_{32}\, \exists y_1, \ldots, y_{32}.\ \bigwedge_i x_i = y_i$

$\forall x_1, \ldots, x_{32}\, \exists y_1, \ldots, y_{32}.\ \bigwedge_i (x_i \vee \neg y_i) \wedge (\neg x_i \vee y_i)$

# $\forall x_1, \ldots, x_{32}\ \exists y_1, \ldots, y_{32}\ .\ \bigwedge_i (x_i \vee \neg y_i) \wedge (\neg x_i \vee y_i)$ in QDIMACS

```
p cnf 64 64
a 1 2 [...] 31 32 0
e 33 34 [...] 63 64 0
1 -33 0
-1 33 0
2 -34 0
-2 34 0
[...]
30 -62 0
-30 62 0
31 -63 0
-31 63 0
32 -64 0
```

64 variables and 64 clauses is **tiny** for QBF solvers and the formula is trivial, yet most QBF solvers have problems solving it without preprocessing.

**DEMO**

# A Closer Look at the Problem

What do QCDCL/CEGAR solvers for QBF do for $\forall X \exists Y.\ \varphi(X, Y)$?

1. Start with formula $\psi := \mathit{true}$ over variables $X$.
2. Search a solution $x$ for $\psi$.
3. If there is no such assignment, terminate with *true*.
4. Search an assignment $y$ to $\varphi(x, Y)$.
5. If there is no such assignment, terminate with *false*.
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**Corollary:**
Solving the formula $\forall x_1, \ldots, x_n\ \exists y_1, \ldots, y_n.\ \bigwedge_i f(x_i) = y_i$ takes $|coDom(f)|^n$ iterations.

# A Different Approach - Example

$$\forall x_1, x_2.\ \exists y_1, y_2, y_3, y_4.\ \underbrace{(x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_1 \vee \neg x_2 \vee y_1)}_{f_{y_1}:\ ?} \wedge$$

$$(\neg x_2 \vee y_2) \wedge (\neg y_1 \vee y_2) \wedge (x_2 \vee y_1 \vee \neg y_2)$$

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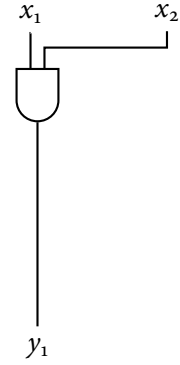

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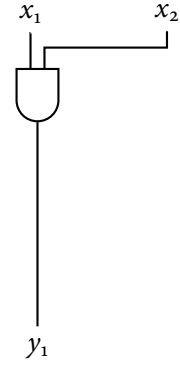

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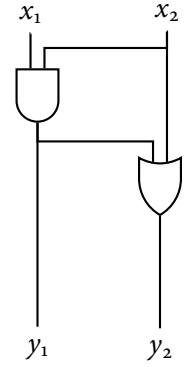

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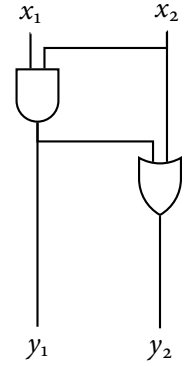

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$$(y_1 \vee \neg y_3) \wedge (y_2 \vee \neg y_3) \wedge \underbrace{(\neg y_1 \vee \neg y_2 \vee y_3)}_{\text{decision clause}} \wedge$$

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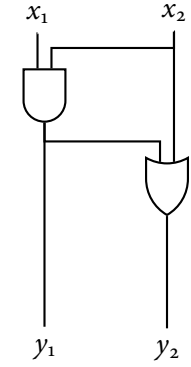

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$$\forall x_1, x_2.\ \exists y_1, y_2, y_3, y_4.\ \underbrace{(x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_1 \vee \neg x_2 \vee y_1)}_{f_{y_1}\colon (x_1, x_2) \mapsto x_1 \wedge x_2} \wedge$$

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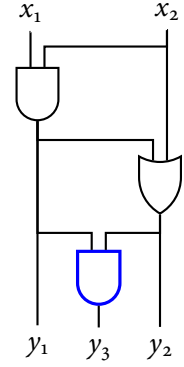
$$(\neg y_1 \vee y_4) \wedge (\neg y_3 \vee \neg y_4)$$



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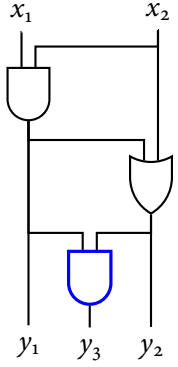

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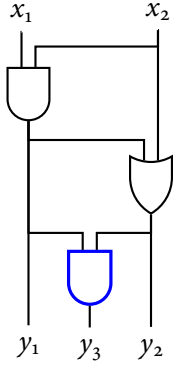

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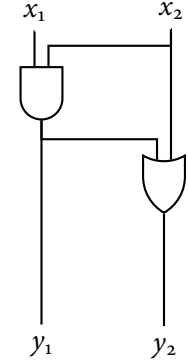

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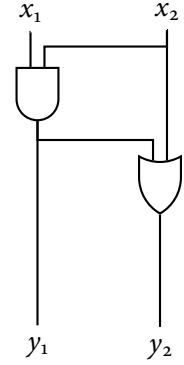

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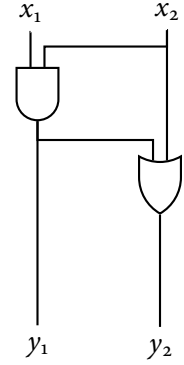

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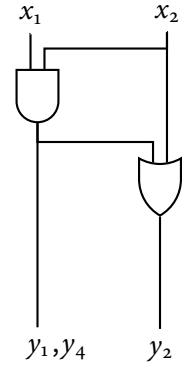

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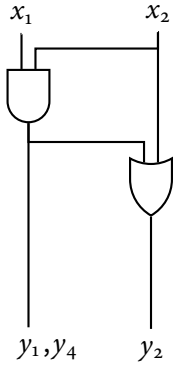

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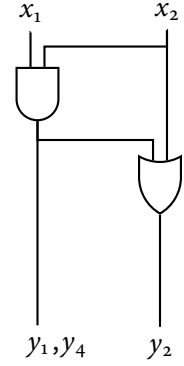


Skolem function has 3 distinct outputs; QCDCL/CEGAR needs 3 iterations.

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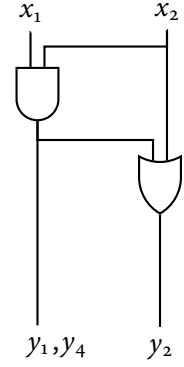


Skolem function has 3 distinct outputs; QCDCL/CEGAR needs 3 iterations.
Exploited **functional dependencies** to derive avoid enumeration of outputs.

# Incremental Determinization [R., Seshia'16]

CDCL in the function space:

| | SAT $\exists Y.\ \varphi$ | QBF $\forall X.\exists Y.\ \varphi$ |
|---|---|---|
| Partial assignment | $Y \to \mathbb{B} \cup \{\top\}$ | $Y \to ((X \to \mathbb{B}) \to \mathbb{B}) \cup \{\top\}$ |
| Decision | literal | constraints/function |
| Conflict | propagation of $y$ and $\neg y$ | $\exists X$. propagation of $y$ and $\neg y$ |
| Learning | clause | clause |
| **Propagation** | unit propagation | **unique Skolem function** |

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Rest of part 1 of this tutorial:

- Propagation, Decisions, Conflict Analysis
- Termination

# Incremental Determinization - An Overview

1: **procedure** IncrementalDeterminization($\forall X.\exists Y.\varphi$)
2: $dlvl \leftarrow 0$; $D \leftarrow \varnothing$
3: **while** true **do**
4: $D, \varphi, conflict, \mathbf{x} \leftarrow$ Propagate($\forall X.\exists Y.\varphi, D, dlvl$)
5: **if** $conflict$ **then**
6: $c \leftarrow$ AnalyzeConflict($\forall X.\exists Y.\ \varphi, \mathbf{x}, D, dlvl$)
7: **if** $c$ only contains variables in $X$ **then**
8: **return** false
9: $dlvl \leftarrow$ (maximal decision level in $c$) $- 1$
10: $\varphi, D \leftarrow$ Backtrack($\varphi, D, dlvl$)
11: $\varphi \leftarrow \varphi \wedge c$
12: **else**
13: **if** $D = Y$ **then**
14: **return** true
15: $v \leftarrow$ PickVar($Y \setminus D$)
16: $dlvl \leftarrow dlvl + 1$
17: $\varphi \leftarrow \varphi \wedge$ Decision($v, \varphi, D$)

# Incremental Determinization - An Overview

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**Main data structure:** a set $D$ of variables.

**Invariant:** For every assignment to $X$ the clauses that contain only variables in $D$ admit exactly one assignment to $D$, which can be computed by Boolean constraint propagation.

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Next up:

1. Propagate
2. Decision
3. AnalyzeConflict

# Unique consequences, unique Skolem functions

Given $\forall X.\exists Y.\varphi$ and a set $D \subseteq Y$ of variables that have a Skolem function already.

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Clause $(l_1 \vee \cdots \vee l_n)$ has **unique consequence** $y = var(l_n)$ if

$$y \in Y \setminus D, \text{ and } var(l_i) \in D \cup X \text{ for } i < n.$$

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Two SAT checks:

- Functional: $\mathcal{D} \wedge \bigwedge (UC_v \setminus \{v, \neg v\})$
- Conflicted: $\mathcal{D} \wedge \bigvee \neg(UC_v[1/v]) \wedge \bigvee \neg(UC_v[0/v])$

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1. Check all clauses for a unique consequence
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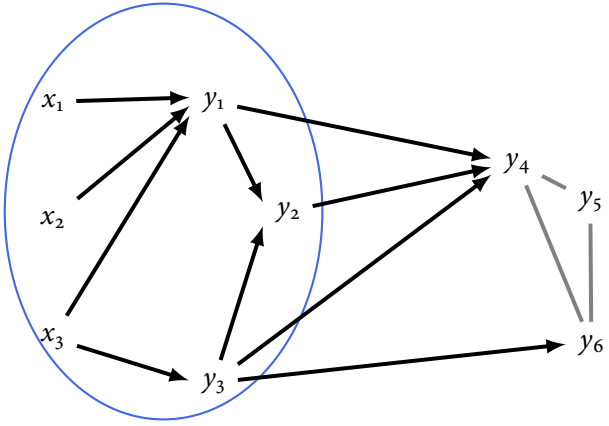
Propagation alone solves $\forall x_1, \ldots, x_{32}\, \exists y_1, \ldots, y_{32}\, .\ \bigwedge_i (x_i \vee \neg y_i) \wedge (\neg x_i \vee y_i)$.

**DEMO**

# Unique Consequences and the Partial Function View

D (variables with
unique Skolem function)



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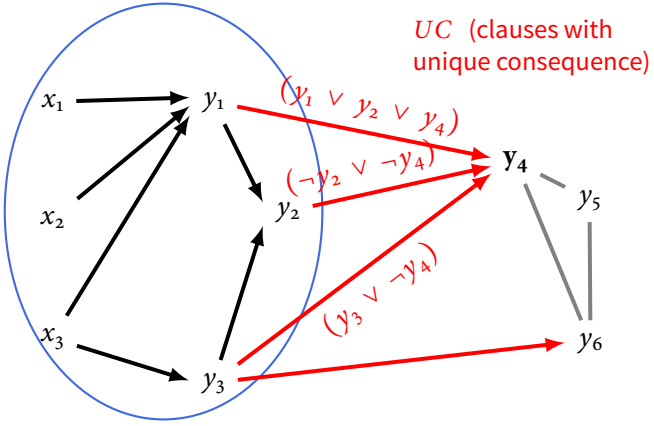
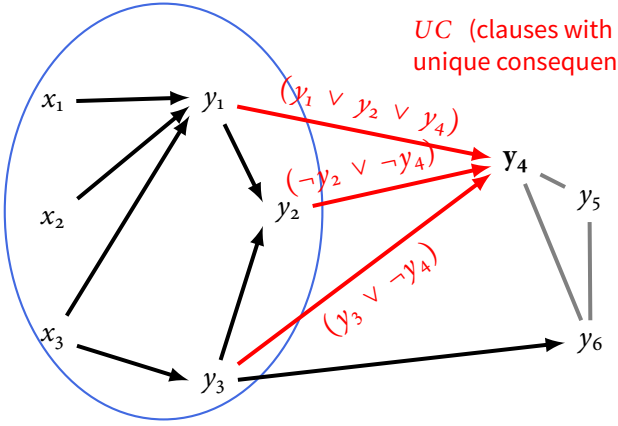
D (variables with unique Skolem function)



# Unique Consequences and the Partial Function View

D (variables with unique Skolem function)

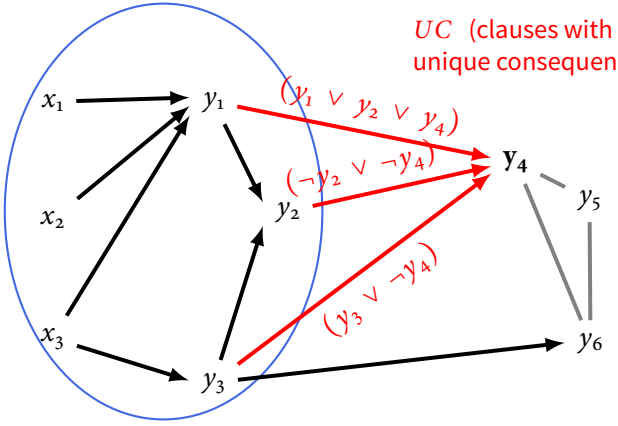
$UC$ (clauses with unique consequence)




| Condition | $y_4$ |
|---|---|
| $\neg y_1 \wedge \neg y_2$ | 1 |
| | ?? |
| $y_2$ | 0 |
| $\neg y_3$ | 0 |

# Unique Consequences and the Partial Function View

D (variables with unique Skolem function)




| Condition | $\mathbf{y_4}$ |
|---|---|
| $\neg y_1 \wedge \neg y_2$ | 1 |
| | ?? |
| $y_2$ | 0 |
| $\neg y_3$ | 0 |

Functional: $?? \stackrel{?}{=} \varnothing$

Conflict: $0 \cap 1 \stackrel{?}{\neq} \varnothing$

Decision:
assign ?? to 0 or 1

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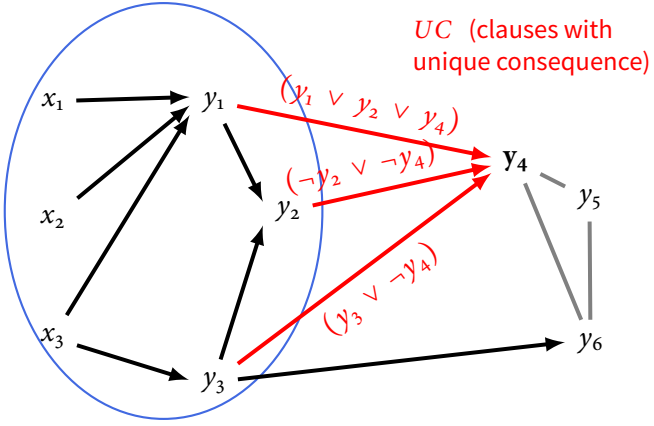


Assume $y_4$ is functional and not conflicted.

$\Rightarrow$ Can propagate $y_4$

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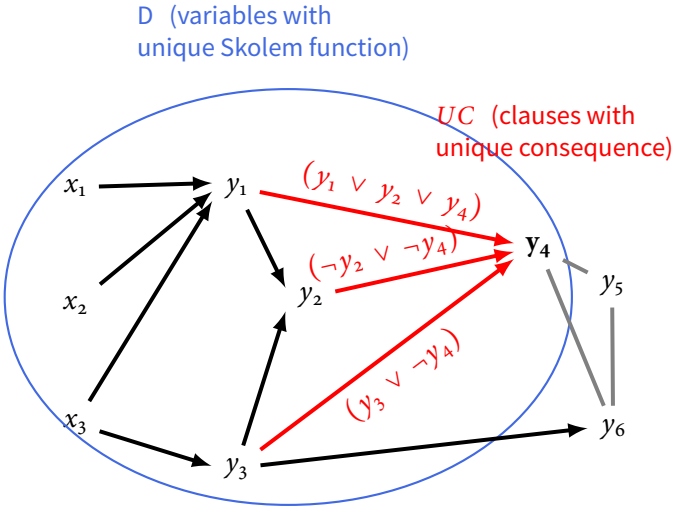


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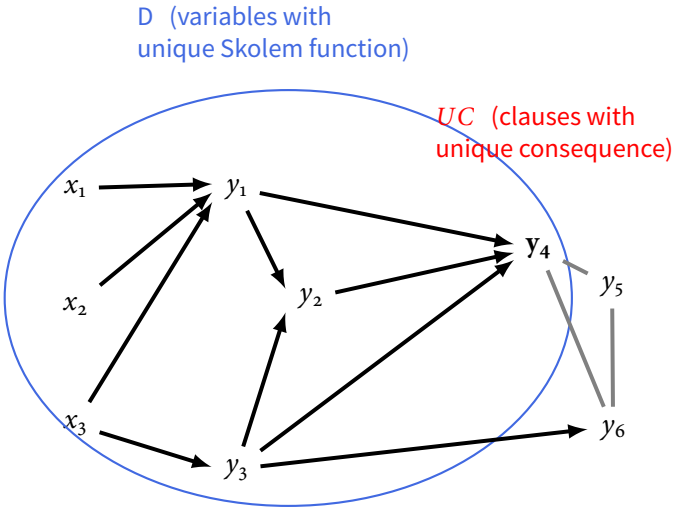


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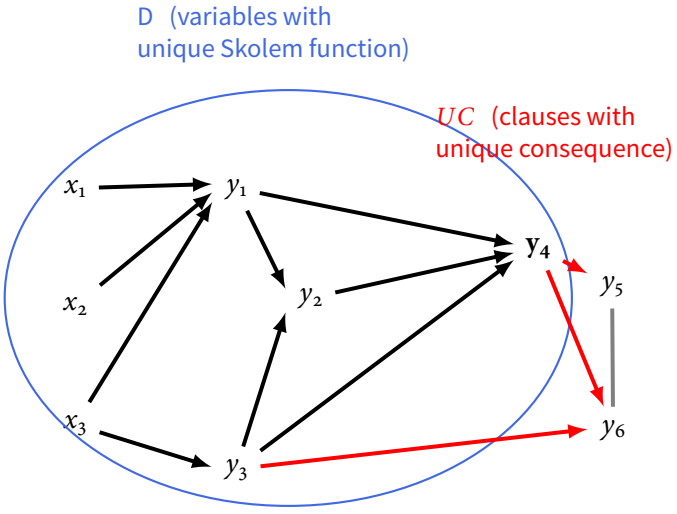



Assume $y_4$ is functional and not conflicted.

$\Rightarrow$ Can propagate $y_4$

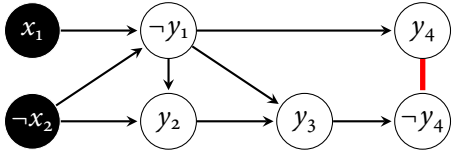
Get new clauses with unique consequence.

# Implication Graphs

**Implication graphs in CDCL for SAT (and in QCDCL):**

1. Add a node for every decision.
2. Add propagation edges: If $l^*$ was propagated (with Boolean constraint propagation) by a clause $(l_1 \vee \cdots \vee l_n \vee l*)$ add an edge $l_i \longrightarrow l'$ for every $i$.
3. A conflict is represented if for some variable both literals occur.

Example: $(x_1 \vee \neg x_2 \vee \neg y_1) \wedge (\neg x_2 \vee y_2) \wedge (\neg y_1 \vee y_2 \vee y_3) \wedge (y_3 \vee \neg y_4) \wedge (\neg y_1 \vee y_4)$



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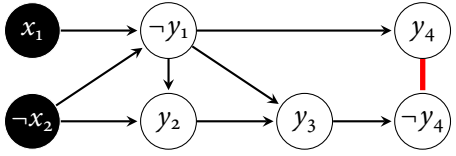


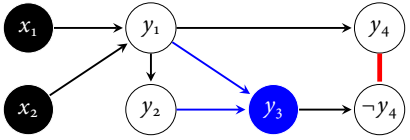

Lifting implication graphs to functions? Not necessary!

- We have the assignment for the universal variables.

# Conflict Analysis in Incremental Determinization

**How do decisions and assignments to the universal variables interact?**

- Propagating the universals assigns all variables in $D$ and the conflicted variable.




$$\forall x_1, x_2.\ \exists y_1, y_2, y_3, y_4.\quad (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_1 \vee \neg x_2 \vee y_1) \wedge$$
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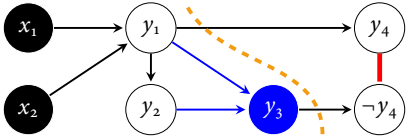


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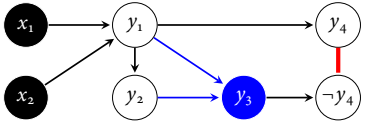
Find a cut separating the conflict from the universals **and the decision variables**.

The learnt clause is the negation of the conjunction of the origins of all cut edges.

Here: $(\neg y_1 \vee \neg y_3)$

# Termination

**Important property:**
For the same decisions and universal assignment, adding the conflict clause and removing all clauses right of the cut, would still result in a conflict.



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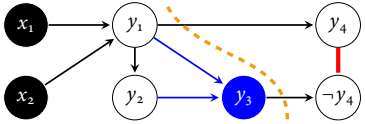


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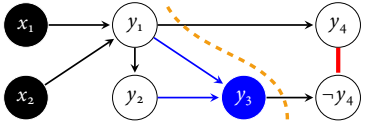


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**Termination:** As there are only exponentially many clauses, Incremental Determinization is guaranteed to terminate.

Incremental Determinization is an **EXPSPACE** algorithm, like CDCL and QCDCL.

# Termination (2)

1: **procedure** IncrementalDeterminization($\forall X.\exists Y.\varphi$)
2: $dlvl \leftarrow 0$; $D \leftarrow \varnothing$
3: **while** true **do**
4: $D, \varphi, conflict, \mathbf{x} \leftarrow$ Propagate($\forall X.\exists Y.\varphi, D, dlvl$)
5: **if** $conflict$ **then**
6: $c \leftarrow$ AnalyzeConflict($\forall X.\exists Y.\ \varphi, \mathbf{x}, D, dlvl$)
7: **if** $c$ only contains variables in $X$ **then**
8: **return** false
9: $dlvl \leftarrow$ (maximal decision level in $c$) $- 1$
10: $\varphi, D \leftarrow$ Backtrack($\varphi, D, dlvl$)
11: $\varphi \leftarrow \varphi \wedge c$
12: **else**
13: **if** $D = Y$ **then**
14: **return** true
15: $v \leftarrow$ PickVar($Y \setminus D$)
16: $dlvl \leftarrow dlvl + 1$
17: $\varphi \leftarrow \varphi \wedge$ Decision($v, \varphi, D$)

# Termination (2)

```
1:  procedure IncrementalDeterminization(∀X.∃Y.φ)
2:      dlvl ← 0;  D ← ∅
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```

**Invariant:** For every assignment to $X$ the clauses that contain only variables in $D$ admit exactly one assignment to $D$, which can be computed by Boolean constraint propagation.

Termination with *false*: clauses containing only universals are empty ($\equiv$ *false*), when we set the universals to the opposite values.

Termination with *true*: all variables are in $D$. The original clauses, decision clauses, and learnt clauses form a **functional relation**.

# Incremental Determinization (ID) vs CEGAR/QCDCL

- $\forall X \exists Y.\ X = Y$ is solved instantly by ID.
- Empirical performance of ID is good.
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- $\forall X \exists Y.\ X = Y$ is solved instantly by ID.
- Empirical performance of ID is good.
- ID does not need preprocessing.
- ID computes certificates without overhead!

# Overview

**QBF with one quantifier alternation**

A Different Take on QBF solving

**2QBF Certification and Functional Synthesis**

From Quantifiers to Functions

**Dependency QBF**

Principled Approaches to Reasoning about Functions

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# Recap: Certification

Given a QBF $\forall x\, \exists y.\ \varphi(x, y)$, a **Skolem function** $f$ is mapping assignments to $x$ to assignments to $y$ such that $\varphi(x, f(x))$ is valid.

A QBF is true if, and only if, it has a Skolem function.

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Examples

| Formula | Skolem functions |
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| $\forall x\ \exists y.\ \mathit{true}$ | $1, 0, x\%2, x+1, \ldots$ |

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Let's not consider counter-models (Herbrand functions) in this talk.

# How to Extract Skolem Functions?

QCDCL/CEGAR for 2QBF $\forall X \exists Y.\ \varphi(X, Y)$ with certification:

1. Start with formula $\psi := \mathit{true}$ over variables $X$.
2. Search a solution $x$ for $\psi$.
3. If there is no such assignment, terminate with *true*.
4. Search an assignment $y$ to $\varphi(x, Y)$.
5. If there is no such assignment, terminate with *false*.
6. **Remember $y$.**
7. Exclude all assignments to $X$ for which $\varphi(X, y)$; i.e. $\psi' := \psi \wedge \neg\varphi(X, y)$
8. Go back to step 2.

Certificate:

$$y = \begin{cases} y_1 & \text{if } \varphi(X, y_1) \\ y_2 & \text{if } \varphi(X, y_2) \\ \dots & \\ y_n & \text{if } \varphi(X, y_n) \end{cases}$$

# Certification in Incremental Determinization

ID's invariant guarantees that the final set of clauses forms a **functional relation**. That is, for every $x$ there is exactly one $y$ satisfying all clauses.

But how to turn the clauses into a circuit?

- The variables are ordered (implicitly) by the order they were added to $D$.
- Upon termination all clauses have a unique consequence; implication from "smaller" to "larger" variables.
- Define each variable as the conjunction of its unique consequence clauses in which it occurs positively.

# Certificate extraction in Incremental Determinization - Example

$$\forall x_1, x_2.\ \exists y_1, y_2, y_3, y_4.\ \underbrace{(x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_1 \vee \neg x_2 \vee y_1)}_{\text{unique consequences of } y_1} \wedge$$

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$$\underbrace{(\neg y_1 \vee y_4) \wedge (\neg y_3 \vee \neg y_4) \wedge (y_1 \vee \neg y_4)}_{\text{unique consequences of } y_4}$$

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$y_1 := x_1 \wedge x_2$ $y_2 := x_2 \vee y_1$ $y_3 := \bigvee \varnothing$ $y_4 := y_1$

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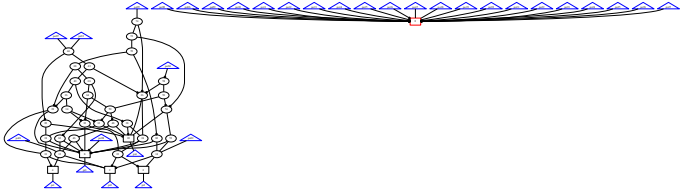
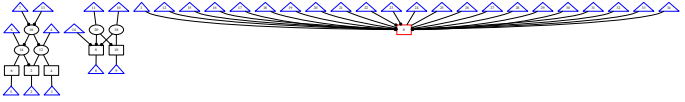
# ID's certificates can be much smaller - Example

DepQBF



CADET



(stmt6_13_14.qdimacs)

# Logical Problems and their Skolem functions

Consider a quantified Boolean formula (QBF) with universal variables $X = \{x_1, \ldots, x_m\}$ and existential variables $Y = \{y_1, \ldots, y_n\}$.

The formula is true if, and only if, there are Skolem functions $f_1 : X_1 \to \mathbb{B}$, …, $f_n : X_n \to \mathbb{B}$ for the existential variables $y_1$, …, $y_n$. Depending on which logical problem we consider there are additional restrictions on $X_i$:

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QBF with one quantifier alternation: $X = X_1 = \ldots = X_n = X$

Full QBF: $X = X_1 \supseteq \ldots \supseteq X_n$

Dependency QBF: $X_i$ arbitrary

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⇩ Functional Synthesis

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⇩

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# (Boolean) Functional Synthesis

**Functional Synthesis**:

Given a Boolean formula $\varphi$ over groups of variables $X$ and $Y$, the functional synthesis problem is to find a function $f$ mapping $X$ to $Y$ such that $\varphi(X, f(X))$ for all $X$ for which there is a $Y$ such that $\varphi(X, Y)$. [Kuncak et al.'11]

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Possible truth tables for $f$:

| | |
|---|---|
| $x_1, \neg x_2$ | 1 |
| $\neg x_1, x_2$ | 0 |
| $x_1, x_2$ | $\{0, 1\}$ |
| $\neg x_1, \neg x_2$ | $\{0, 1\}$ |

# How does functional synthesis compare to QBF?

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Consider this 3QBF: $\psi_3 := \forall X \exists Y.\ \varphi(X, Y) \ \vee \ (\forall Y'.\ \neg\varphi(X, Y'))$

- A Skolem function for $\psi_3$ is a solution to the functional synthesis problem.
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Compare to this 2QBF: $\psi_2 := \forall X \exists Y.\ \varphi(X, Y)$

- A Skolem function for $\psi_2$ is a solution to the functional synthesis problem.
- $\psi_2$ is not always true.
- **If the 2QBF $\psi_2$ is false, functional synthesis will still compute a function.**

# Functional Synthesis and Projection

**Functional Synthesis**:

Given a Boolean formula $\varphi$ over groups of variables $X$ and $Y$, the functional synthesis problem is to find a function $f$ mapping $X$ to $Y$ such that $\varphi(X, f(X))$ for all $X$ for which there is a $Y$ such that $\varphi(X, Y)$. [Kuncak et al.'11]

**Projection problem:**

Given $\varphi(X, Y)$ the projection on $X$ is a formula $\varphi'(X)$ such that $\varphi'(X)$ iff $\exists Y.\ \varphi(X, Y)$ for all $X$.

Projection via functional synthesis:

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$\forall X \exists Y.\ \varphi(X, Y)$ iff the projection of $\varphi(X, Y)$ to $X$ is true.

# Algorithms for Functional Synthesis

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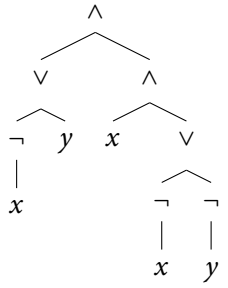
Recent approaches:

- Expansion, BDDs [Fried et al.'16],[Tabaja et al.'17]
- Compositional Functional Synthesis [Akshay et al.'16]
- Incremental Determinization

# Compositional Functional Synthesis [Akshay et al.'17]

Assume we can solve functional synthesis for small formulas, can we build the function for large formulas out of its components?

Consider the formula in negation normal form:




**Build functions bottom-up:** Exploit that function composition for disjunctions is easy. Conjunctions require to apply functional synthesis to the intersection of the two sub-expressions.

# Incremental Determinization for Functional Synthesis

Resolution

$$\frac{C_1 \cup \{l\} \qquad C_2 \cup \{\bar{l}\}}{C_1 \cup C_2}$$

Boolean constraint propagation (BCP)

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Different property needed functional synthesis:

For all universal assignments, BCP assigns all variables and **the minimal subsets of clauses needed to derive the empty clause, do not include the decision clauses**.

# Incremental Determinization for Functional Synthesis (cont'd)

Function table view of decisions:

| Condition | $\mathbf{y_4}$ |
|---|---|
| $\neg y_1 \wedge \neg y_2$ | 1 |
| | ?? |
| $y_2$ | 0 |
| $\neg y_3$ | 0 |

Functional: $?? \stackrel{?}{=} \varnothing$

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In general, we require in the global conflict check that one of the decisions must be used.

# Incremental Determinization for Functional Synthesis

1: **procedure** IncrementalDeterminization($\forall X.\exists Y.\varphi$)
2: $dlvl \leftarrow 0$; $D \leftarrow \varnothing$
3: **while** true **do**
4: $D, \varphi, conflict, \mathbf{x} \leftarrow$ Propagate($\forall X.\exists Y.\varphi, D, dlvl$)
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6: $c \leftarrow$ AnalyzeConflict($\forall X.\exists Y.\ \varphi, \mathbf{x}, D, dlvl$)
7: **if** $c$ only contains variables in $X$ **then**
8: **return** false
9: $dlvl \leftarrow$ (maximal decision level in $c$) $- 1$
10: $\varphi, D \leftarrow$ Backtrack($\varphi, D, dlvl$)
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12: **else**
13: **if** $D = Y$ **then**
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15: $v \leftarrow$ PickVar($Y \setminus D$)
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$\Leftarrow$ With the additional constraint learnt clauses must contain existential variables and Incremental Determinization cannot terminate with false anymore.

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**QBF with one quantifier alternation**

A Different Take on QBF solving

**2QBF Certification and Functional Synthesis**

From Quantifiers to Functions

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Grammar: DQBFs start with a list of Henkin quantifiers, followed by a propositional formula.

A DQBF is true, if for all existentials $y$ there are functions $f_y : \mathbb{B}^{dep(y)} \to \mathbb{B}$ such that $\varphi[y \,/\, f_y(dep(y))]$ is valid.

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Given a QBF: $\forall x_1.\exists y_1.\forall x_2.\exists y_2\ \ldots\ \forall x_k.\exists y_k.\ \varphi$

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Observations:

- Dependency sets of prenex QBFs are nested.
- The order of Henkin quantifiers does not matter.
- For general DQBF dependency sets may be **incomparable**.

# Functions in DQBF

Consider this example:

$$\exists f : \mathbb{B}^n \to \mathbb{B}.\ \forall X_1, X_2.\ f(X_1) \neq f(X_2)$$

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*Can we state pairwise equality with less than $k^2$ constraints?*

Exploit transitivity of equality: $\bigwedge_{k>1} [X_1 = X_k \to f_1 = f_k]$

# DQBF Examples

Synthesize an adder:

$$\begin{aligned}&\exists add : \mathbb{B}^{32} \times \mathbb{B}^{32} \to \mathbb{B}^{32}.\\ &\quad \forall x \in \mathbb{B}^{32}.\ add(x, 0) = x \ \wedge\\ &\quad \forall x, y \in \mathbb{B}^{32}.\ add(x, y+1) = add(x, y) + 1\end{aligned}$$

Synthesize an invariant:

$$\begin{aligned}&\exists inv : \mathbb{B}^{|s|} \to \mathbb{B}\\ &\quad \forall s.\quad I(s) \to inv(s) \wedge\\ &\quad \forall s.\quad inv(s) \wedge E(s) \to P(s) \wedge\\ &\quad \forall s, s'.\ inv(s) \wedge T(s, s') \to inv(s'),\end{aligned}$$

where $s$ is the program state, $T$ is the transition function of the loop body, $I$ is the initial condition, $E$ is the loop exit condition, and $P$ is the property to prove.

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Proof systems for DQBF:

| | |
|---|---|
| Resolution+Expansion/IR-calculus | [Beyersdorff et al.'16] |
| Fork-Resolution (for restricted DQBF) | [R.'17] |

# The Problem: Resolution is incomplete for DQBF

SAT: Build all resolvents; did we derive the empty clause?

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In CNF: $(y_1 \vee y_2 \vee x_1) \wedge (\neg y_1 \vee \neg y_2 \vee x_1) \wedge (y_1 \vee y_2 \vee x_2) \wedge (\neg y_1 \vee \neg y_2 \vee x_2) \wedge (y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2) \wedge (\neg y_1 \vee y_2 \vee \neg x_1 \vee \neg x_2)$

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There are 1.5 ways to approach this: Expansion and Fork Extension.

# Resolution+Expansion

Given a DQBF $\exists y_1 : X_1. \ \ldots \ \exists y_n : X_n : \ \varphi$, we can **expand** a universal variable $x$ as follows:

$$\exists y_1 : X_1 \setminus \{x\}. \ \ldots \ \exists y_n : X_n \setminus \{x\}. \exists y'_1 : X_1 \setminus \{x\}. \ \ldots \ \exists y'_n : X_n \setminus \{x\} : \\ \varphi[x/0] \wedge \varphi[x/1][y_i/y'_i \text{ for all } i]$$

Step by step:

- Introduce copies $y'_i$ for each existential $y_i$ (with $x \in dep(y_i)$).
- Remove $x$ from all dependency sets.
- Consider two copies of $\varphi$; replace $x$ by 0 in the first, and by 1 in the second.
- In the second copy of $\varphi$, replace $y_i$ by $y'_i$.

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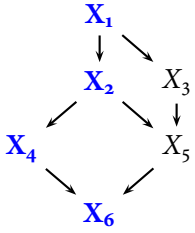
Given a DQBF, consider the inclusion order of its dependencies:




Still does not really work in practice.

# Towards an Alternative Proof System for DQBF

Example: $\exists y_1 : X_1.\, \exists y_2 : X_2.\, \varphi \wedge (y_1 \vee y_2)$

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Clauses containing vars with incomparable dependencies: **Information forks**

Concept loosely connected to distributed reactive synthesis. [Finkbeiner&Schewe'05]

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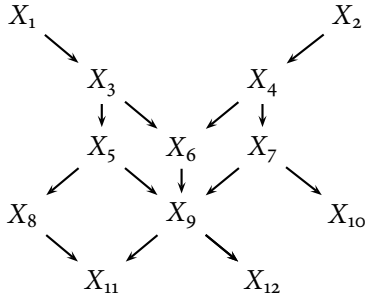
Universal Reduction

$$\frac{C \quad var(l) \notin dep(C \setminus \{l\}) \quad \bar{l} \notin C \quad var(l) \text{ is universal}}{C \setminus \{l\}}$$

# Fork Resolution

Consider the lattice of dependency sets of a DQBF.




**Algorithm sketch:** Pick a variable from a leaf element of the lattice; eliminate its information forks; eliminate the variable with resolution. Iterate.

# Completeness Guarantee

Fork resolution is complete for DQBF with two disjoint function applications.

For example, consider invariant synthesis:

$$\begin{aligned}&\exists inv : \mathbb{B}^{|s|} \to \mathbb{B}\\ &\quad \forall s. \quad I(s) \to inv(s) \wedge\\ &\quad \forall s. \quad inv(s) \wedge E(s) \to P(s) \wedge\\ &\quad \forall s, s'.\ inv(s) \wedge T(s, s') \to inv(s'),\end{aligned}$$

where $s$ is the program state, $T$ is the transition function of the loop body, $I$ is the initial condition, $E$ is the loop exit condition, and $P$ is the property to prove.

The function to synthesize, $inv$, is only applied to two disjoint arguments.

# Example

Quantifier prefix: $\exists y_1 : \{x_1\}.\ \exists y_2 : \{x_2\}.$

$$y_1 \vee y_2 \vee x_1 \tag{1}$$
$$\neg y_1 \vee \neg y_2 \vee x_1 \tag{2}$$
$$y_1 \vee y_2 \vee x_2 \tag{3}$$
$$\neg y_1 \vee \neg y_2 \vee x_2 \tag{4}$$
$$y_1 \vee \neg y_2 \vee \neg x_1 \vee \neg x_2 \tag{5}$$
$$\neg y_1 \vee y_2 \vee \neg x_1 \vee \neg x_2 \tag{6}$$

All clauses are information forks!

# Example

Quantifier prefix: $\exists y_1 : \{x_1\}.\ \exists y_2 : \{x_2\}.\ \exists t_1 : \varnothing.\ \ldots \exists t_6 : \varnothing.$

| | | | |
|---|---|---|---|
| $t_1 \vee y_1 \vee x_1$ | (1a) | $\neg t_1 \vee y_2$ | (1b) |
| $t_2 \vee \neg y_1 \vee x_1$ | (2a) | $\neg t_2 \vee \neg y_2$ | (2b) |
| $t_3 \vee y_1$ | (3a) | $\neg t_3 \vee y_2 \vee x_2$ | (3b) |
| $t_4 \vee \neg y_1$ | (4a) | $\neg t_4 \vee \neg y_2 \vee x_2$ | (4b) |
| $t_5 \vee y_1 \vee \neg x_1$ | (5a) | $\neg t_5 \vee \neg y_2 \vee \neg x_2$ | (5b) |
| $t_6 \vee \neg y_1 \vee \neg x_1$ | (6a) | $\neg t_6 \vee y_2 \vee \neg x_2$ | (6b) |

Split clauses with fork extension. Fresh variables $t$ are propositional.

# Example

Quantifier prefix: $\exists y_1 : \{x_1\}.\ \exists y_2 : \{x_2\}.\ \exists t_1 : \varnothing.\ \ldots \exists t_6 : \varnothing.$

| | | | |
|---|---|---|---|
| $t_1 \vee t_4$ | $(1a4a)$ | $\neg t_1 \vee \neg t_5$ | $(1b5b)$ |
| $t_3 \vee t_6$ | $(3a6a)$ | $\neg t_2 \vee \neg t_6$ | $(2b6b)$ |
| $t_4 \vee t_5$ | $(4a5a)$ | $\neg t_3 \vee \neg t_4$ | $(3b4b)$ |

Eliminate variables $y_1$ and $y_2$ through variable elimination.

Obtain propositional, unsatisfiable formula.

# Conclusion

- Synthesizing Boolean functions would solve many problems in verification and synthesis.
- Closely connected to certification for quantified Boolean logics.
- Incremental Determinization: an effective algorithm to compute 2QBF certificates.
- Functional synthesis as an extension of 2QBF certification. Incremental Determinization can be modified to solve functional synthesis.
- DQBF can express very general Boolean synthesis problems, but efficient algorithms are still elusive.