Modeling Short Time Series

What “Including Prior Information” really looks like

Tim Radtke - satRday Berlin - June 15, 2019

minimizeregret.com

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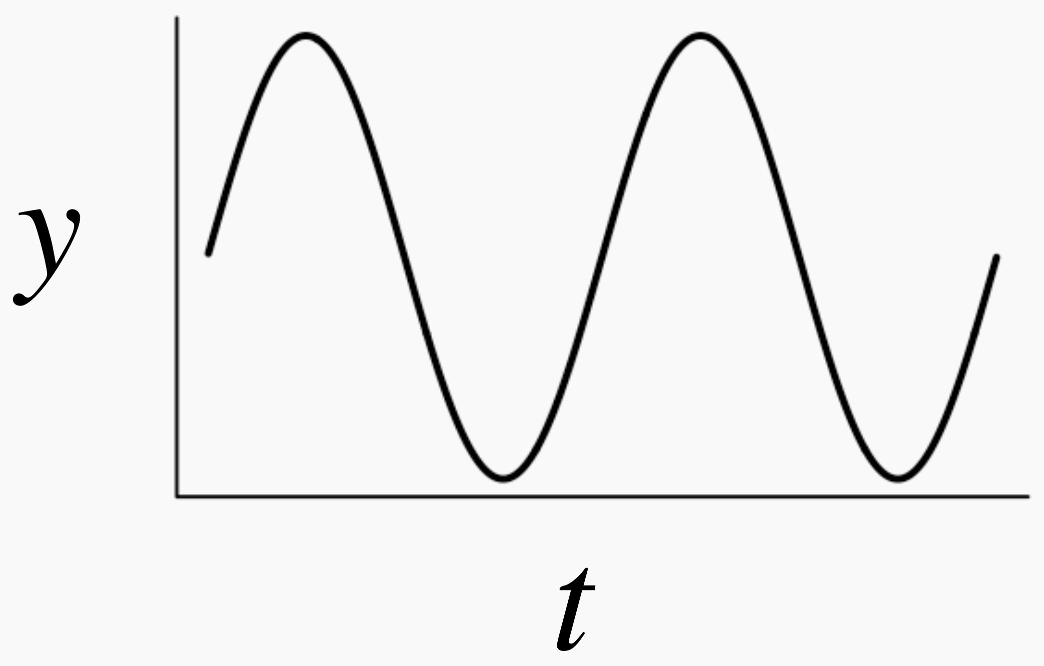
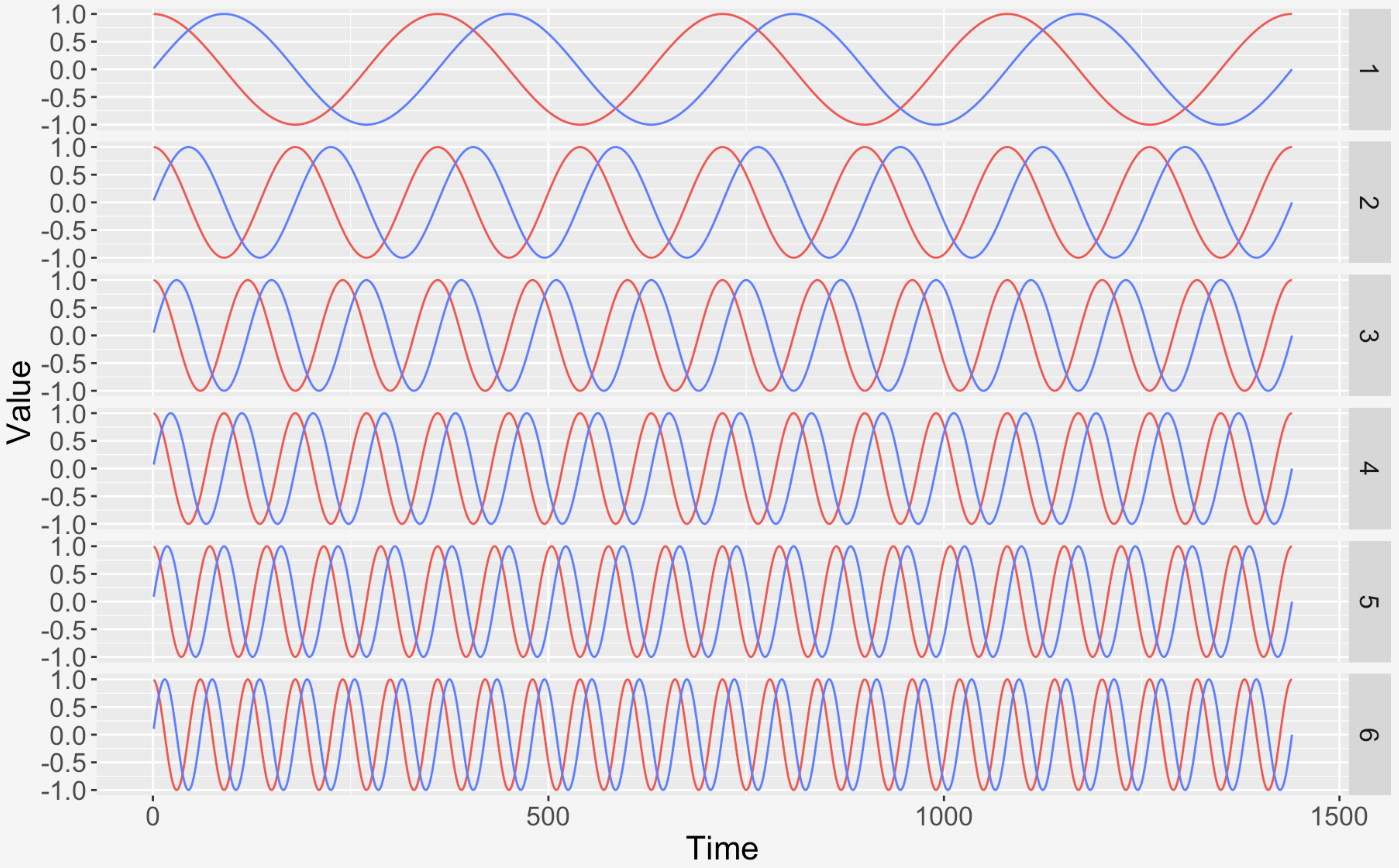
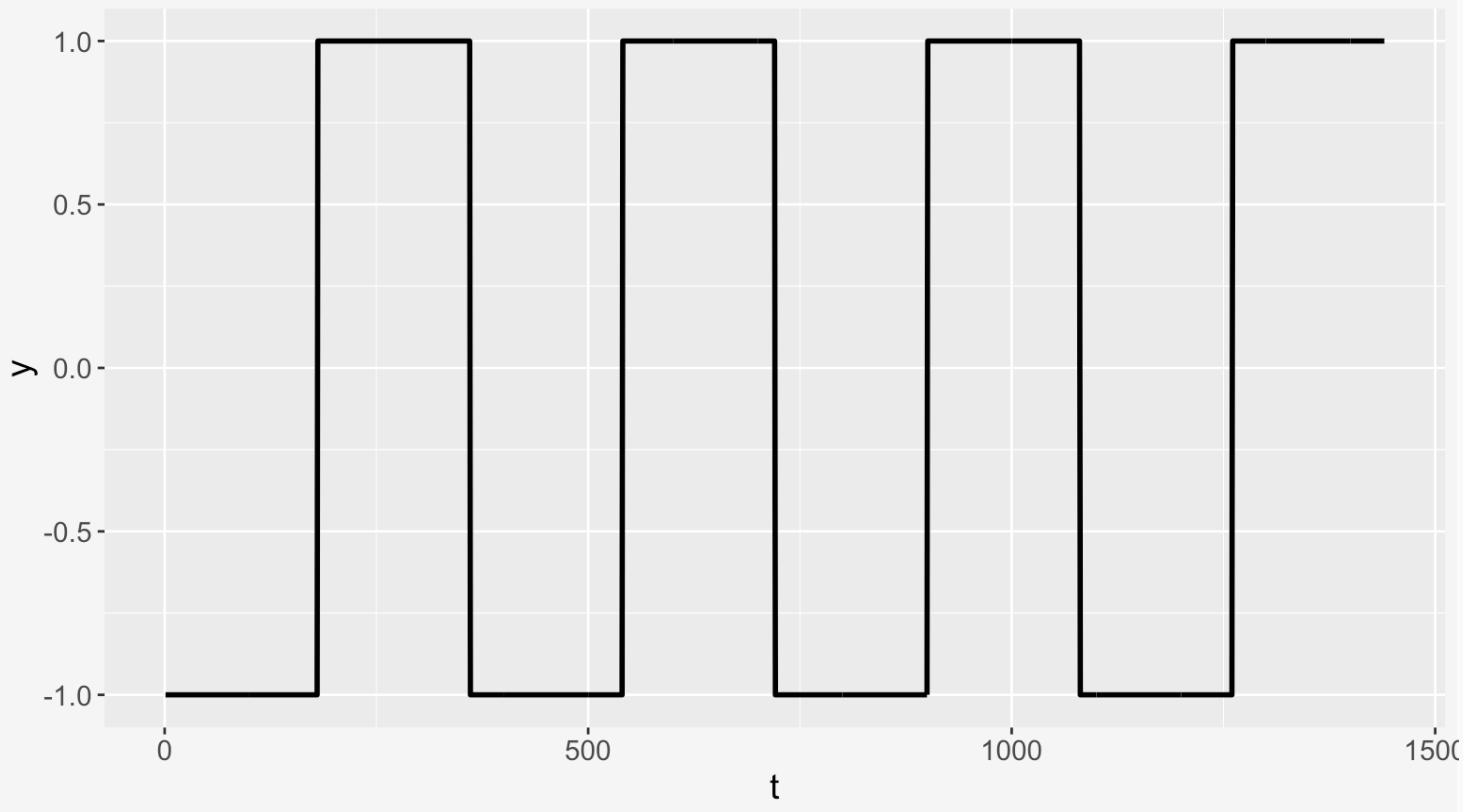
Fourier Terms

$$\sin(2\pi t) \qquad \cos(2\pi t)$$





Fourier terms at multiples of a period length of 360.



Step function with period length 360.

Fourier Terms

$$y(t) = \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi k t}{m}\right) + \gamma_k \cos\left(\frac{2\pi k t}{m}\right) \right)$$

Fourier Terms

with $K = 6$, these are 6 sine and 6 cosine curves

$$y(t) = \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi kt}{m}\right) + \gamma_k \cos\left(\frac{2\pi kt}{m}\right) \right)$$

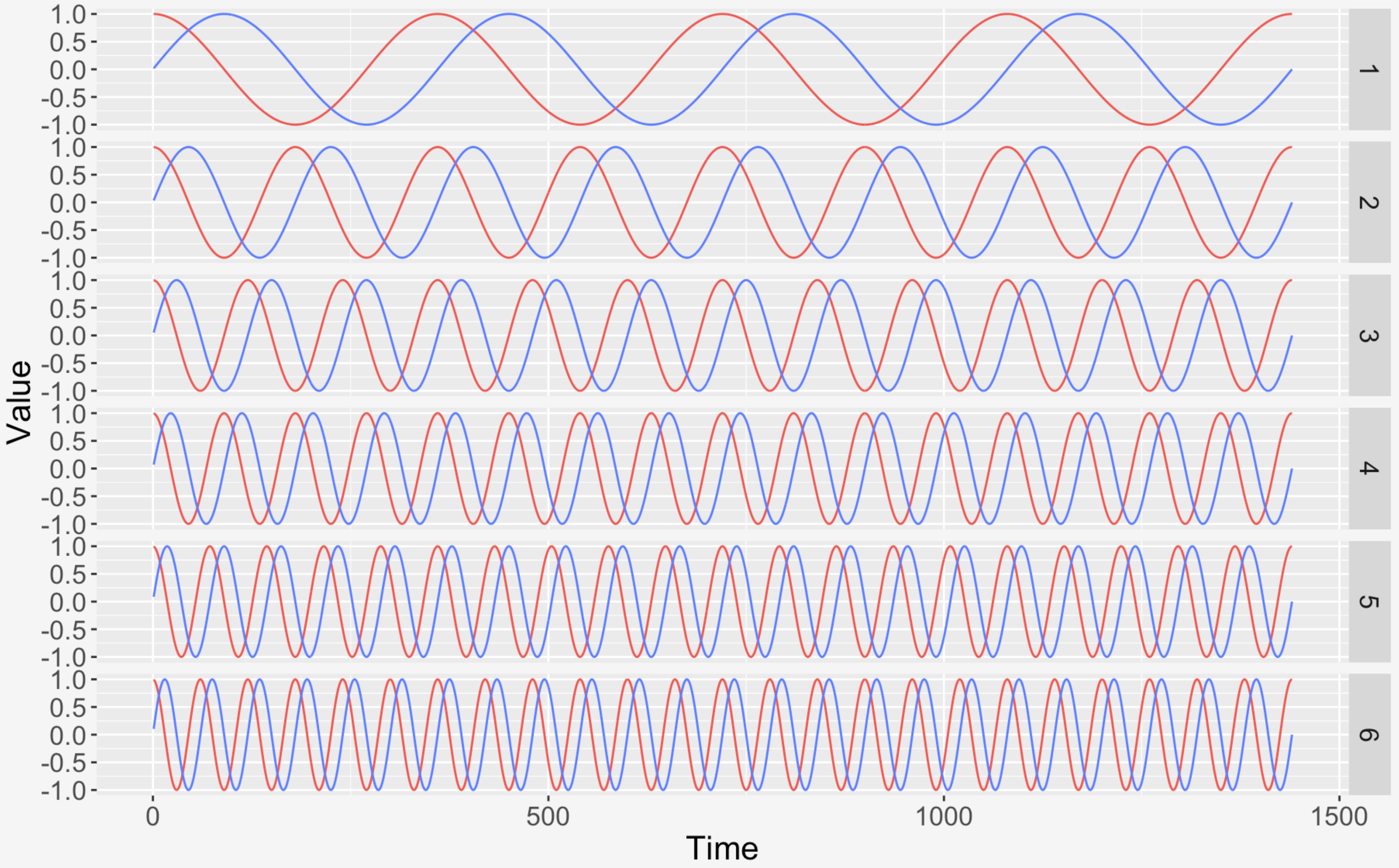
Fourier Terms

with $K = 6$, **these are** 6 **sine and** 6 **cosine curves**

$$y(t) = \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi k t}{m}\right) + \gamma_k \cos\left(\frac{2\pi k t}{m}\right) \right)$$

β_k **and** γ_k **are coefficients in a linear model**

```
> fouriers <- forecast::fourier(ts(1:(4*360),
+                                   frequency = 360), K = 6)
>
> fouriers[1:5, 1:4]
         S1-360    C1-360     S2-360    C2-360
[1,] 0.01745241 0.9998477 0.03489950 0.9993908
[2,] 0.03489950 0.9993908 0.06975647 0.9975641
[3,] 0.05233596 0.9986295 0.10452846 0.9945219
[4,] 0.06975647 0.9975641 0.13917310 0.9902681
[5,] 0.08715574 0.9961947 0.17364818 0.9848078
```

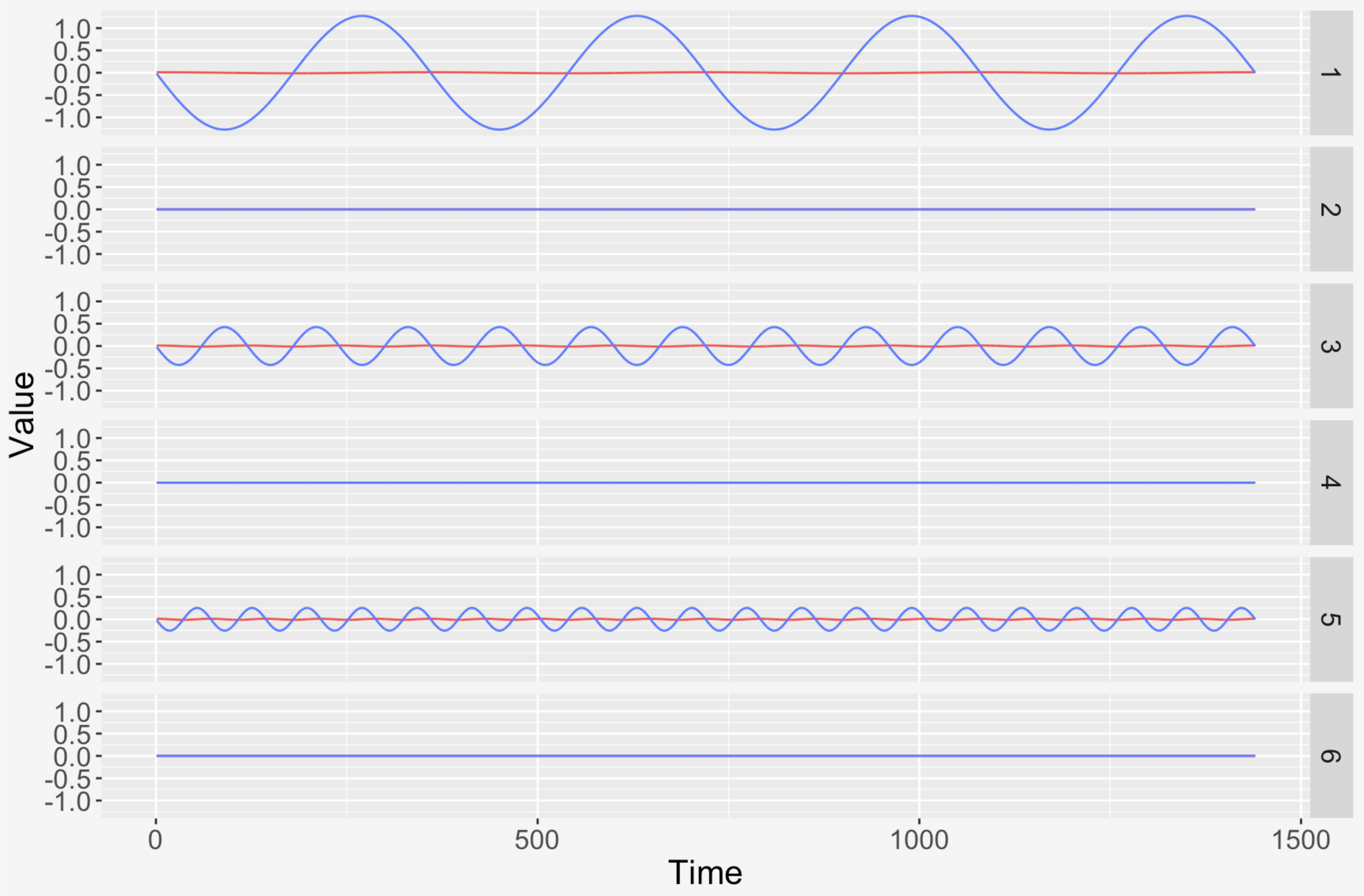


Fourier terms at multiples of a period length of 360.

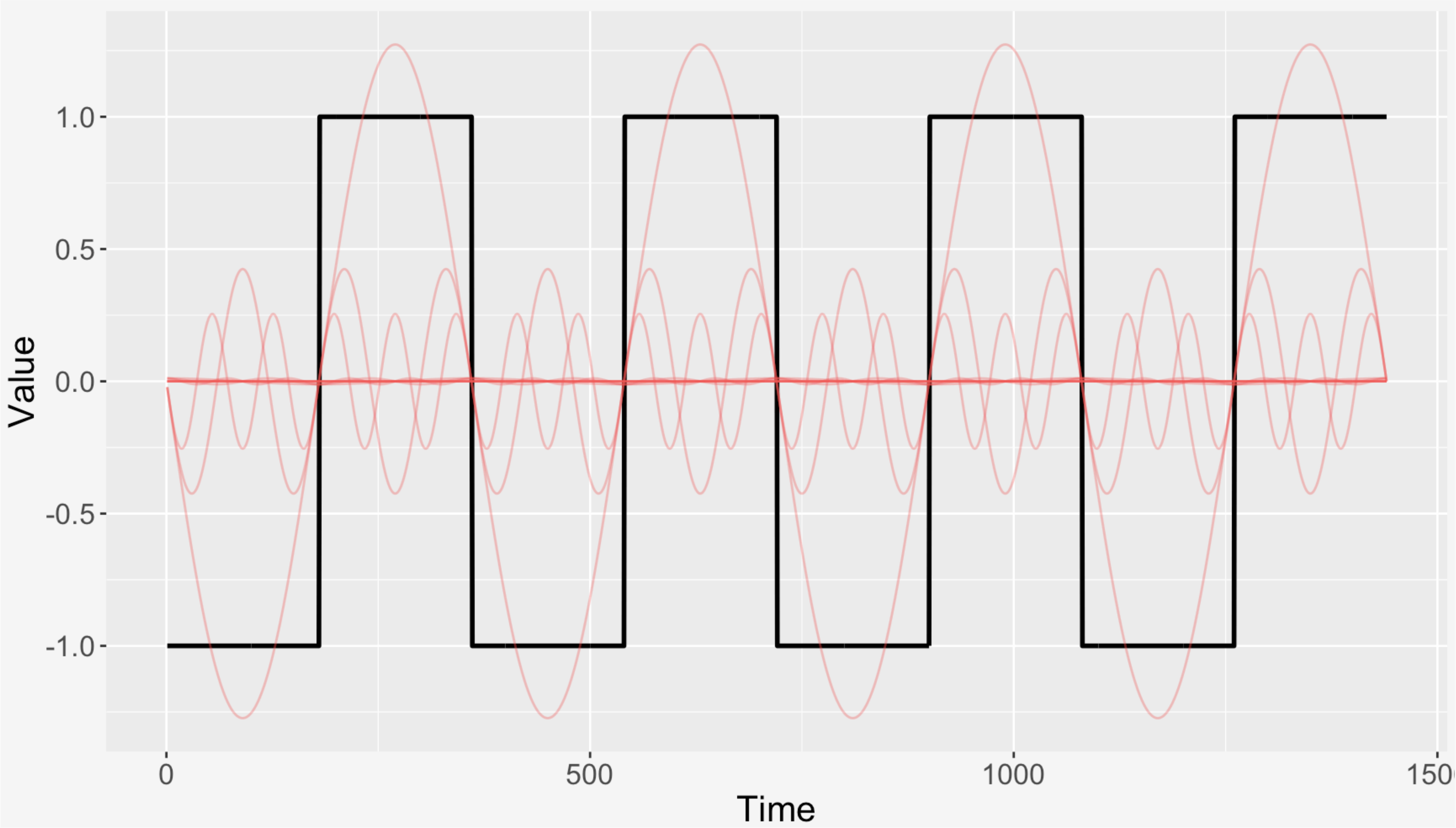
```r
> step_function <- rep(rep(c(-1,1), each = 180), 4)
>
> fouriers <- forecast::fourier(ts(step_function,
+                                   frequency = 360), K = 6)
>
> fourier_fit <- lm(step_function ~ -1 + fouriers)
```

```r
> step_function <- rep(rep(c(-1,1), each = 180), 4)
>
> fouriers <- forecast::fourier(ts(step_function,
+                                   frequency = 360), K = 6)
>
> fourier_fit <- lm(step_function ~ -1 + fouriers)
```

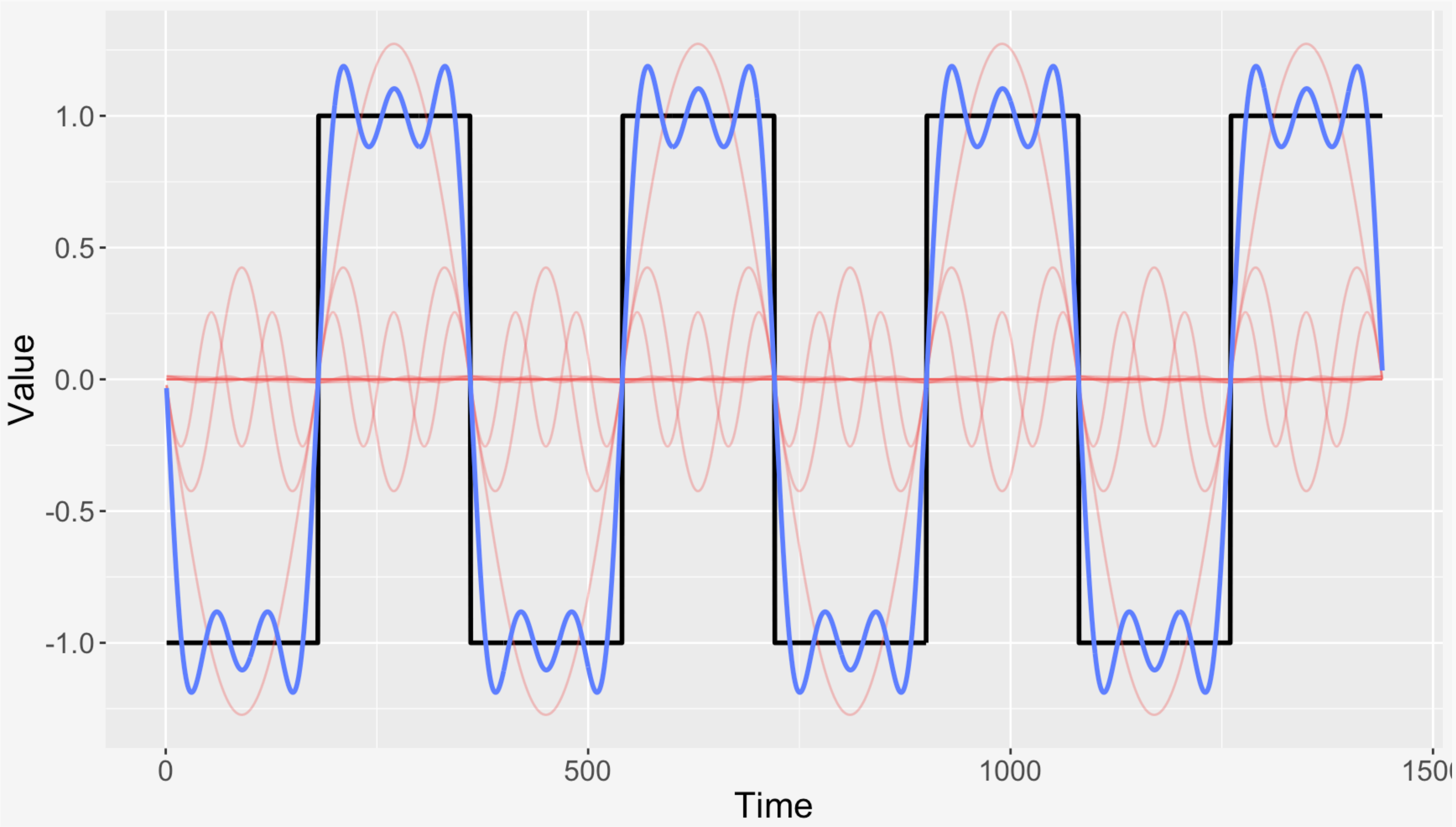
$$y(t) = \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi kt}{m}\right) + \gamma_k \cos\left(\frac{2\pi kt}{m}\right) \right)$$



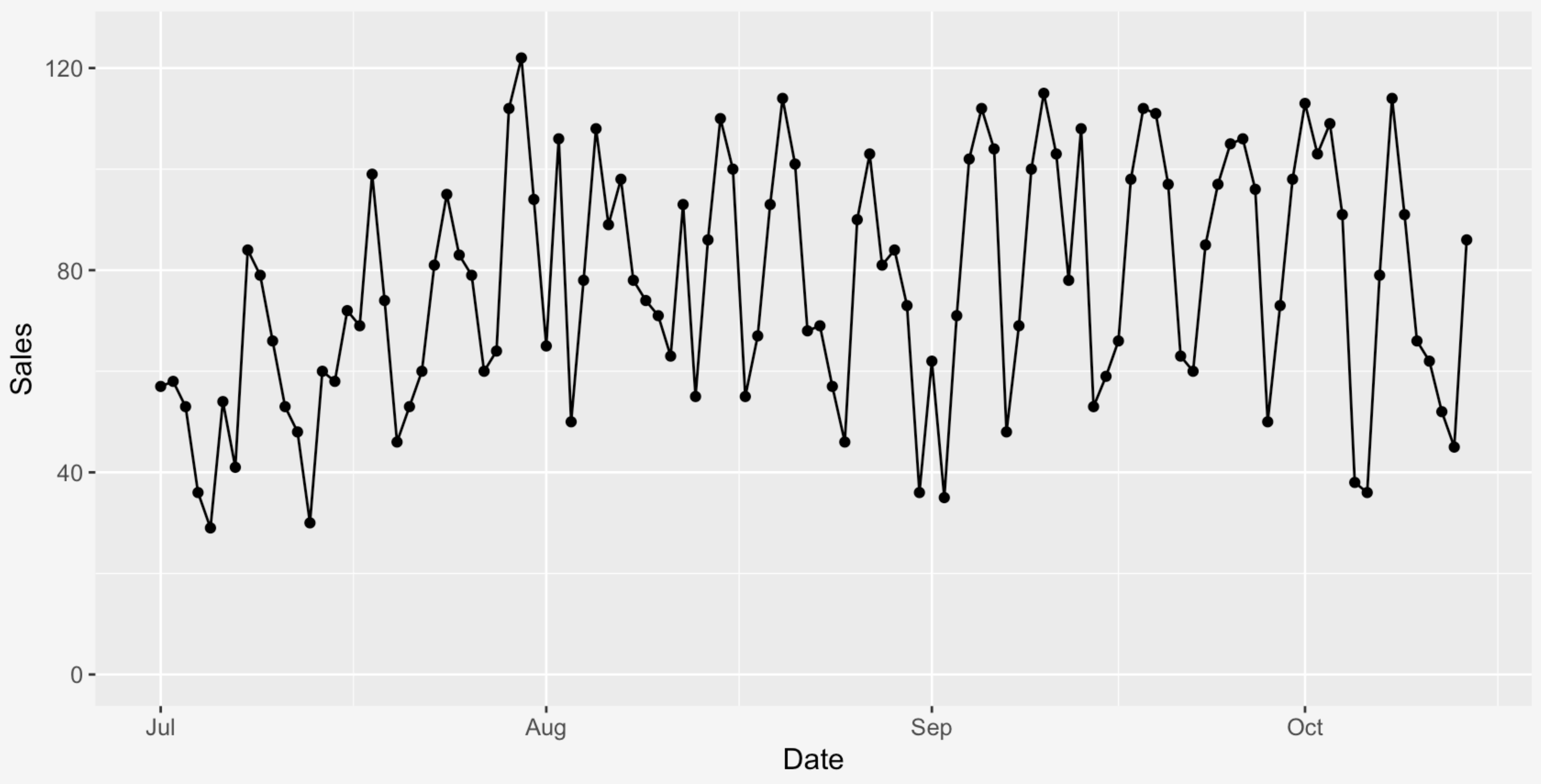
Each Fourier term weighted by its coefficient in linear model.



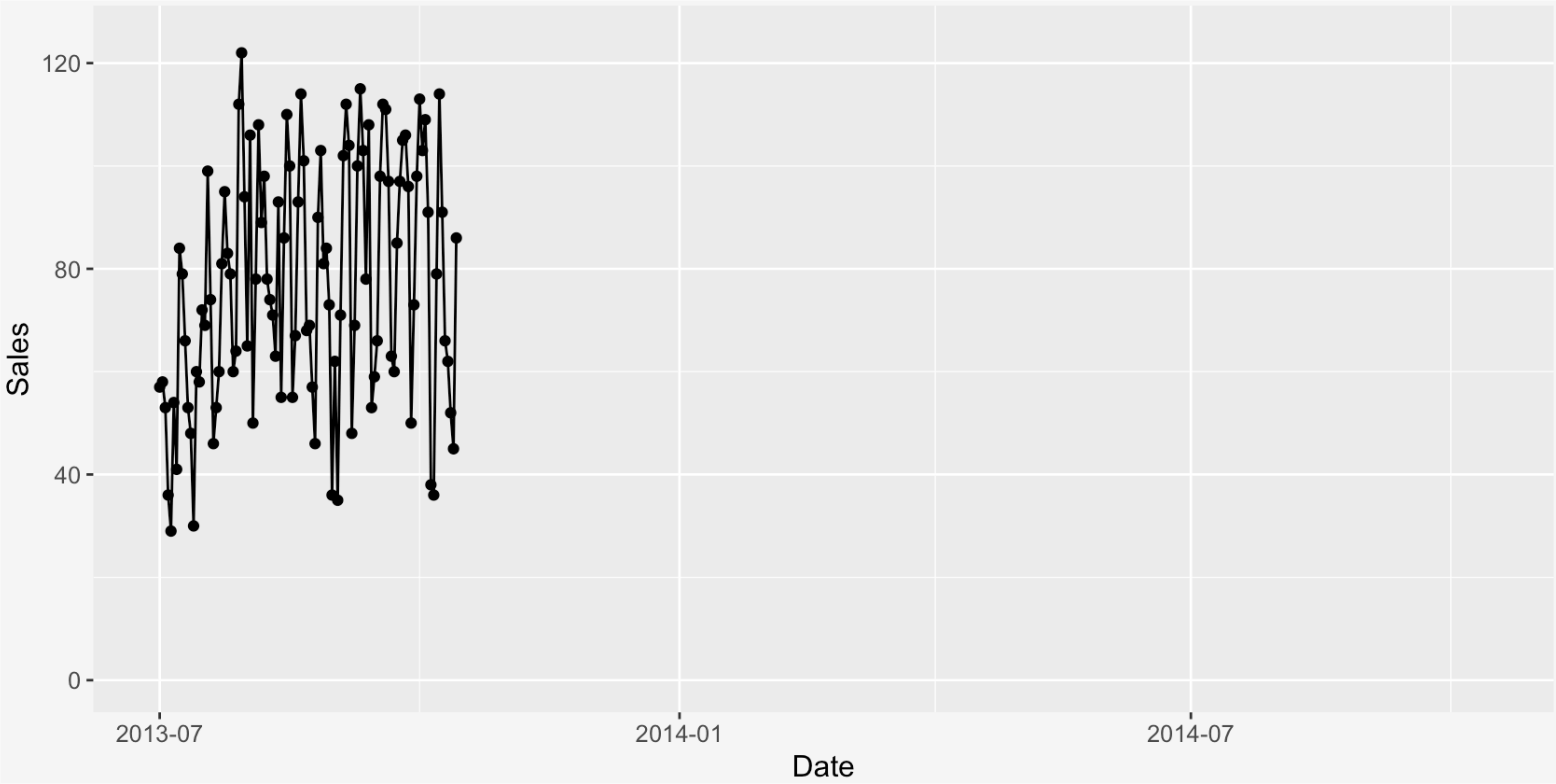
Step function and Fourier terms as weighted by their coefficient.



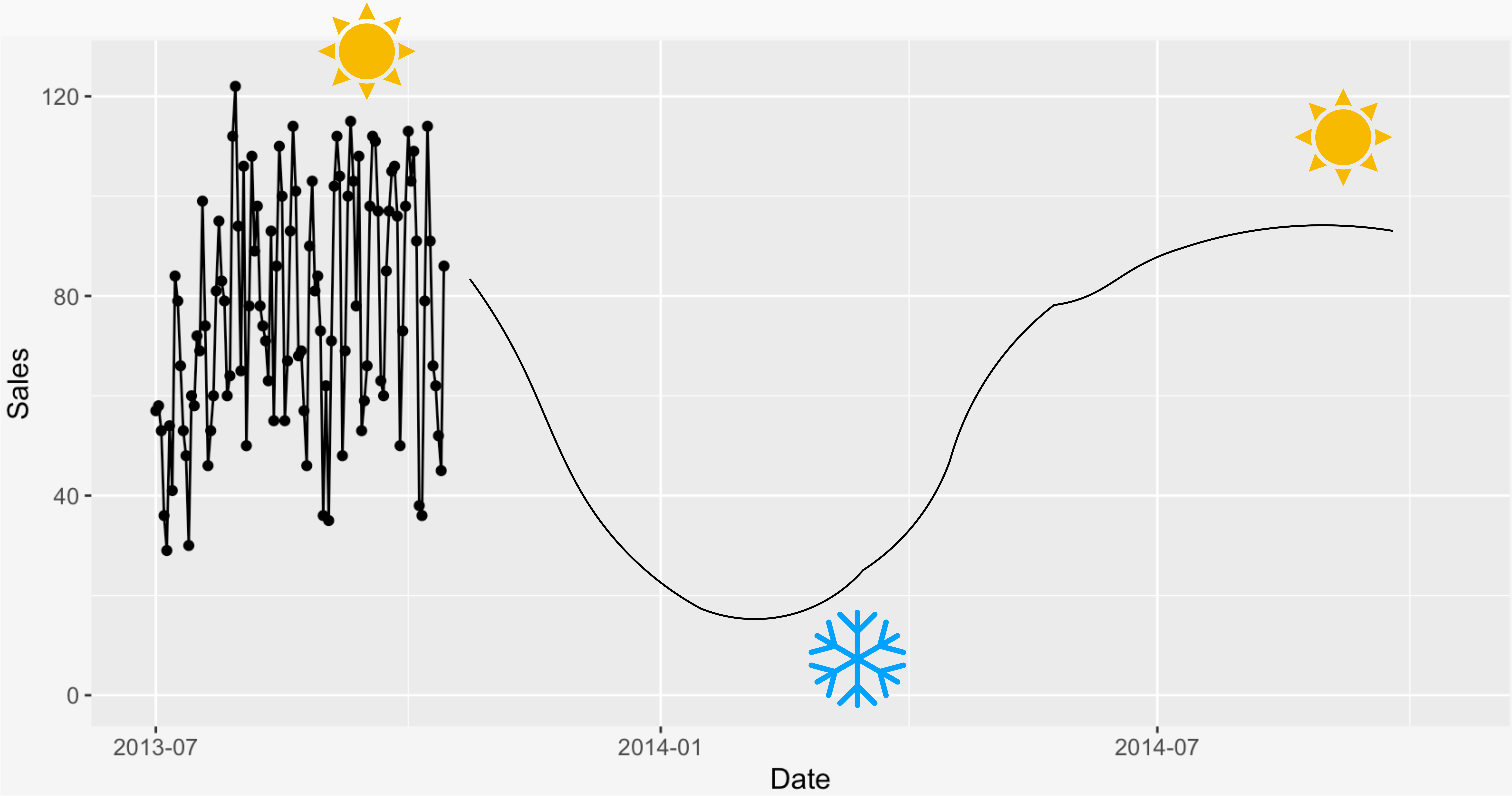
Sum of weighted Fourier terms gives approximation to step function.



Daily bike sales observed over 3.5 months in 2013 in a NYC shop.



Manager needs forecast for the upcoming year to order new bikes.



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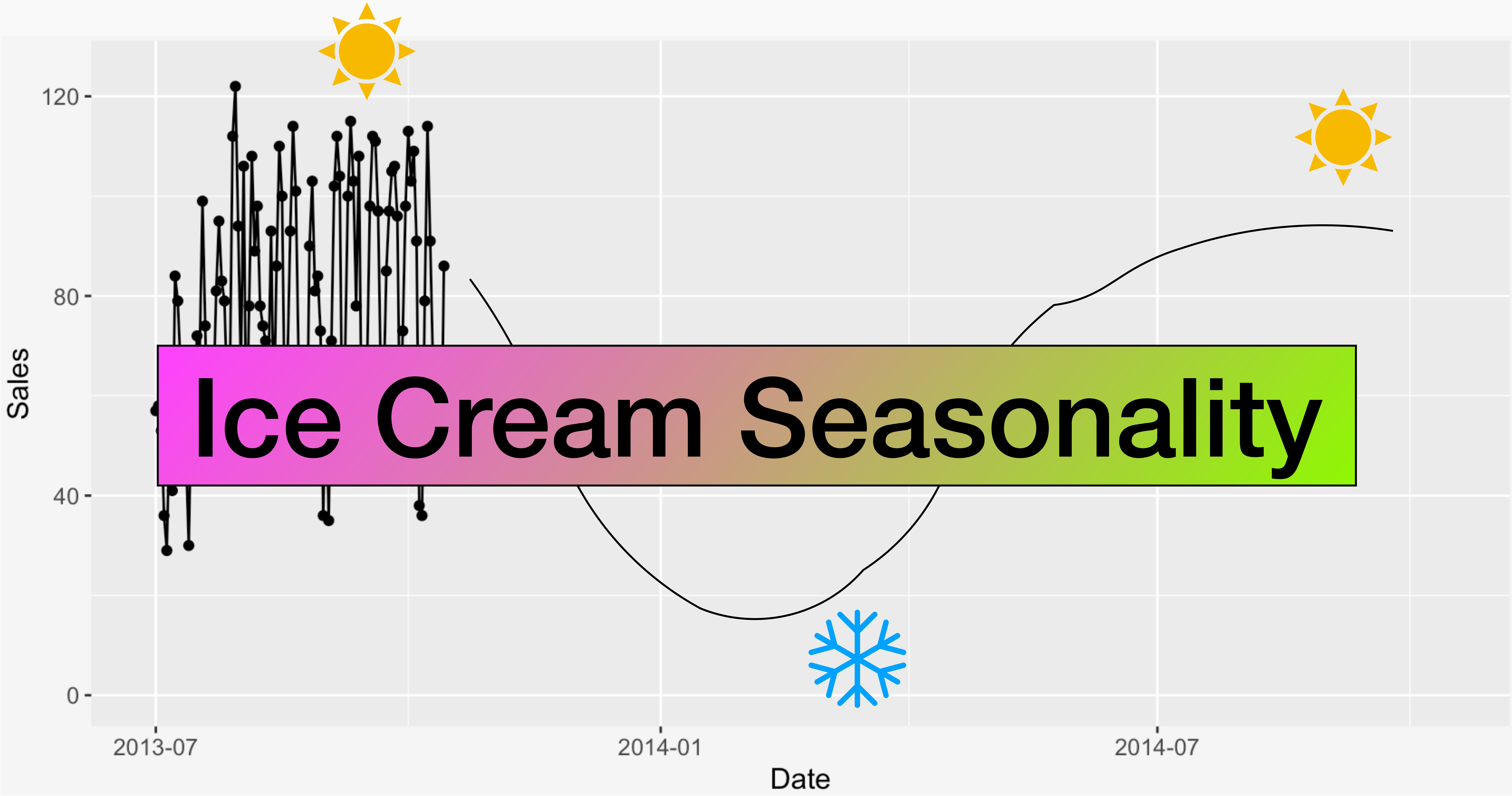
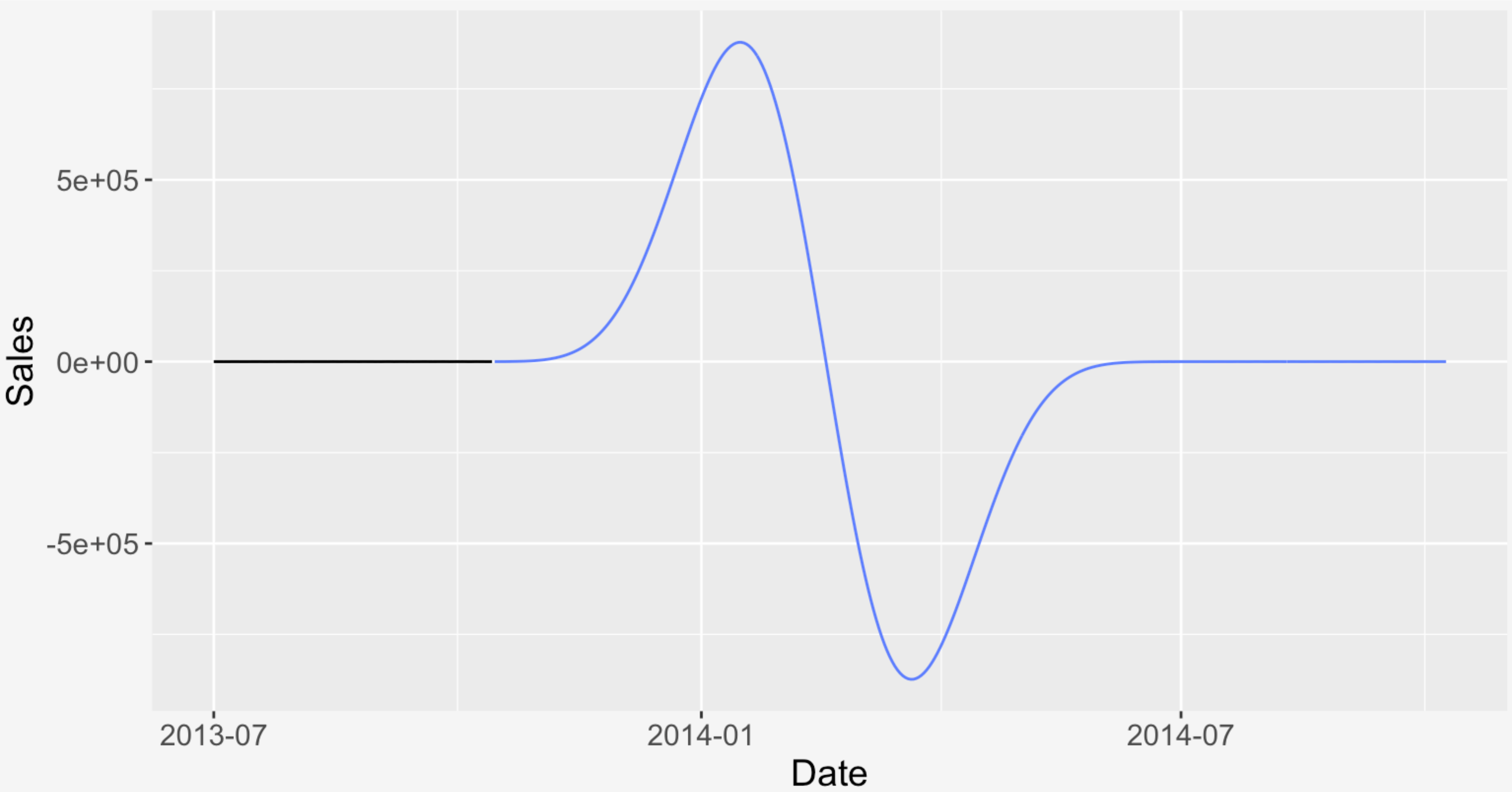
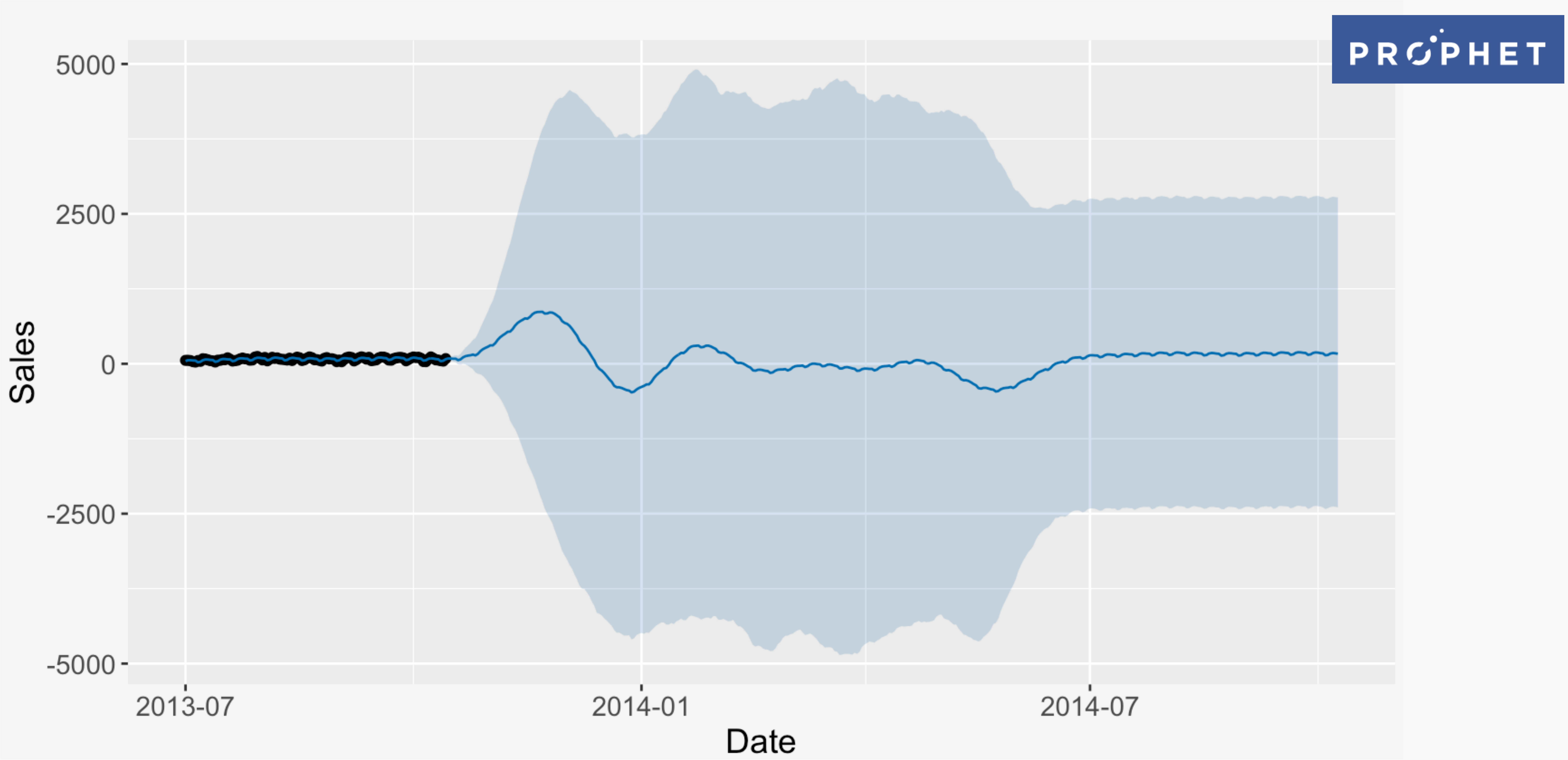


Manager needs forecast for the upcoming year to order new bikes.



Forecast from `forecast::auto.arima()` with Fourier terms for yearly seasonality.



Forecast from default `prophet` model with Fourier terms for yearly seasonality.

Data Generating Process

$Y_t \sim \textbf{NegBin}(\lambda_t, \phi)$ **where** $\lambda_t, \phi > 0, \quad E[Y_t] = \lambda_t$

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$$\lambda_t = (1 - \delta - \gamma) \cdot \mu_t + \delta \cdot \lambda_{t-1} + \gamma \cdot y_{t-1}$$

where $0 \le \delta \le 1$ **and** $0 \le \gamma \le 1 - \delta$

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$$\textbf{where} \quad 0 \leq \delta \leq 1 \quad \textbf{and} \quad 0 \leq \gamma \leq 1 - \delta$$

$$\log(\mu_t) = \alpha_0 + \sum_{i=1}^{6} \alpha_i D_i + \tau \frac{t}{m} + \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi k t}{m}\right) + \gamma_k \cos\left(\frac{2\pi k t}{m}\right) \right)$$

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prior distributions on β_k and γ_k define what seasonality is expected

Data Generating Process

$$\log(\mu_t) = \alpha_0 + \sum_{i=1}^{6} \alpha_i D_i + \tau \frac{t}{m} + \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi k t}{m}\right) + \gamma_k \cos\left(\frac{2\pi k t}{m}\right) \right)$$

$\beta_k, \gamma_k \sim N(0,\, 0.25)$ **for** $k = 1, \ldots, K$

```
> library(rstan)
> options(mc.cores = 4)
> rstan_options(auto_write = TRUE)
>
> negbin_prior_stan <- stan_model("negbin_prior.stan")
```

```stan
data {
  int<lower=1> N;
  int<lower=1> K;
  int<lower=1> F;
  int y[N];
  matrix[N, K] x;
  matrix[N, F] fouriers;
  matrix[N, 1] trend;
  real trend_loc;
  real trend_sd;
  real fourier_loc[F];
  real fourier_sd[F];
}
```

```
parameters {
  real<lower=0,upper=1> phi;
  real<lower=0,upper=1-phi> alpha;
  real delta_inv;
  vector[K] beta;
  vector[F] beta_fourier;
  vector[1] beta_trend;
  real beta_0;
}
```

```
transformed parameters {
  real<lower=0> delta;
  real<lower=0> mu_t[N];
  vector<lower=0>[N] seasonal;
  delta = 1/pow(delta_inv, 2);
  seasonal = exp(x * beta + fouriers * beta_fourier +
                 trend * beta_trend + beta_0);
  mu_t[1] = y[1];
  for (n in 2:N) {
    mu_t[n] = (1 - phi - alpha) * seasonal[n] +
               phi * mu_t[n-1] + alpha * y[n-1];
  }
}
```

```
model {
  phi ~ gamma(1,10);
  alpha ~ gamma(0.5,10);
  delta_inv ~ normal(0,0.5);
  beta_fourier ~ normal(fourier_loc, fourier_sd);
  beta_trend ~ normal(trend_loc, trend_sd);
  beta ~ normal(0,0.5);
  beta_0 ~ normal(2,1);
  for (n in 2:N) {
    target += neg_binomial_2_lpmf(y[n] | mu_t[n], delta);
  }
}
```

```
model {
  phi ~ gamma(1,10);
  alpha ~ gamma(0.5,10);
  delta_inv ~ normal(0,0.5);
  beta_fourier ~ normal(fourier_loc, fourier_sd);
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  }
}
```

$\beta_k, \gamma_k \sim N(0, 0.25)$ **for** $k = 1,\ldots,K$

```r
res_y <- res_mu <- matrix(ncol = 1000, nrow = N_hciti)
for(j in 1:1000) {
  delta_inv <- rnorm(1, 0, 0.5)
  delta <- 1/(delta_inv^2)
  phi <- rgamma(1,1,10)
  alpha <- rgamma(1, 0.5, 10)
  if(phi>1) phi <- 1
  if(alpha > 1-phi) alpha <- 1 - phi

  beta_fouriers <- rnorm(12, 0, 0.25)
  beta_trend <- rnorm(1, 0.03, 0.02)
  beta_0 <- rnorm(1, 4, 0.5)
  beta <- rnorm(6,0,0.1)

  seasonal <- exp(beta_0 + beta_trend * trend_hciti + as.numeric(xreg_hciti %*% beta) + as.numeric(fourier_hciti %*% beta_fouriers))

  mu_t <- y_t <- rep(NA, N_hciti)
  mu_t[1] <- y_hciti[1] #(1-phi-alpha) * seasonal[1]
  y_t[1] <- y_hciti[1] #rnbinom(1, mu = mu_t[1], size = delta)

  for(i in 2:N_hciti) {
    mu_t[i] <- (1 - phi - alpha) * seasonal[i] + phi * mu_t[i-1] + alpha * y_t[i-1]
    y_t[i] <- rnbinom(1, mu = mu_t[i], size = delta)
  }

  res_mu[,j] <- mu_t
  res_y[,j] <- y_t
}
```

```r
res_y <- matrix(ncol = 1000, nrow = N_hciti)
res_mu <- matrix(ncol = 1000, nrow = N_hciti)

for(j in 1:1000) {
  delta_inv <- rnorm(1, 0, 0.5)
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  phi <- rgamma(1, 1, 10)
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  beta_0 <- rnorm(1, 4, 0.5)
  beta <- rnorm(6, 0, 0.1)
```

```r
res_y <- res_mu <- matrix(ncol = 10      nrow
= N_hciti)
for(j in 1:1000) {
  delta_inv <- rnorm(1, 0, 0.5)
  delta <- 1/(delta_inv^2)
  phi <- rgamma(1,1,10)
  alpha <- rgamma(1, 0.5, 10)
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  beta_0 <- rnorm(1, 4, 0.5)
  beta <- rnorm(6,0,0.1)

  seasonal <- exp(beta_0 + beta_trend *
trend_hciti + as.numeric(xreg_hciti %*%
beta) + as.numeric(fourier_hciti %*%
beta_fouriers))

  mu_t <- y_t <- rep(NA, N_hciti)
  mu_t[1] <- y_hciti[1] #(1-phi-alpha) *
seasonal[1]
  y_t[1] <- y_hciti[1] #rnbinom(1, mu =
mu_t[1], size = delta)

  for(i in 2:N_hciti) {
    mu_t[i] <- (1 - phi - alpha) *
seasonal[i] + phi * mu_t[i-1] + alpha *
y_t[i-1]
    y_t[i] <- rnbinom(1, mu = mu_t[i], size
= delta)
  }

  res_mu[,j] <- mu_t
  res_y[,j] <- y_t
}
```

```r
res_y <- matrix(ncol = 1000, nrow = N_hciti)
res_mu <- matrix(ncol = 1000, nrow = N_hciti)

for(j in 1:1000) {
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```

$$\beta_k, \gamma_k \sim N(0, 0.25) \quad \textbf{for } k = 1, \ldots, K$$

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res_y <- res_mu <- matrix(ncol = 10    nro
= N_hciti)
for(j in 1:1000) {
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  beta_fouriers <- rnorm(12,0,0.25)
  beta_trend <- rnorm(1, 0.03, 0.02)
  beta_0 <- rnorm(1, 4, 0.5)
  beta <- rnorm(6,0,0.1)
  seasonal <- exp(beta_0 + beta_trend *
trend_hciti + as.numeric(xreg_hciti %*%
beta) + as.numeric(fourier_hciti %*%
beta_fouriers))

  mu_t <- y_t <- rep(NA, N_hciti)
  mu_t[1] <- y_hciti[1] #(1-phi-alpha) *
seasonal[1]
  y_t[1] <- y_hciti[1] #rnbinom(1, mu =
mu_t[1], size = delta)

  for(i in 2:N_hciti) {
    mu_t[i] <- (1 - phi - alpha) *
seasonal[i] + phi * mu_t[i-1] + alpha *
y_t[i-1]
    y_t[i] <- rnbinom(1, mu = mu_t[i], size
= delta)
  }

  res_mu[,j] <- mu_t
  res_y[,j] <- y_t
}
```

```r
seasonal <- exp(beta_0 +
                beta_trend * trend_hciti +
        as.numeric(xreg_hciti %*% beta) +
        as.numeric(fourier_hciti %*% beta_fouriers))

mu_t <- y_t <- rep(NA, N_hciti)

mu_t[1] <- y_hciti[1]
y_t[1] <- y_hciti[1]

for(i in 2:N_hciti) {
  mu_t[i] <- (1 - phi - alpha) * seasonal[i] +
               phi * mu_t[i-1] + alpha * y_t[i-1]
  y_t[i] <- rnbinom(1, mu = mu_t[i], size = delta)
}

res_mu[,j] <- mu_t
res_y[,j] <- y_t
}
```

```r
res_y <- res_mu <- matrix(ncol = 10     nrow
= N_hciti)
for(j in 1:1000) {
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```

```r
  seasonal <- exp(beta_0 + beta_trend *
trend_hciti + as.numeric(xreg_hciti %*%
beta) + as.numeric(fourier_hciti %*%
beta_fouriers))

  mu_t <- y_t <- rep(NA, N_hciti)
  mu_t[1] <- y_hciti[1] #(1-phi-alpha) *
seasonal[1]
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mu_t[1], size = delta)

  for(i in 2:N_hciti) {
    mu_t[i] <- (1 - phi - alpha) *
seasonal[i] + phi * mu_t[i-1] + alpha *
y_t[i-1]
    y_t[i] <- rnbinom(1, mu = mu_t[i], size
= delta)
  }

  res_mu[,j] <- mu_t
  res_y[,j] <- y_t
}
```
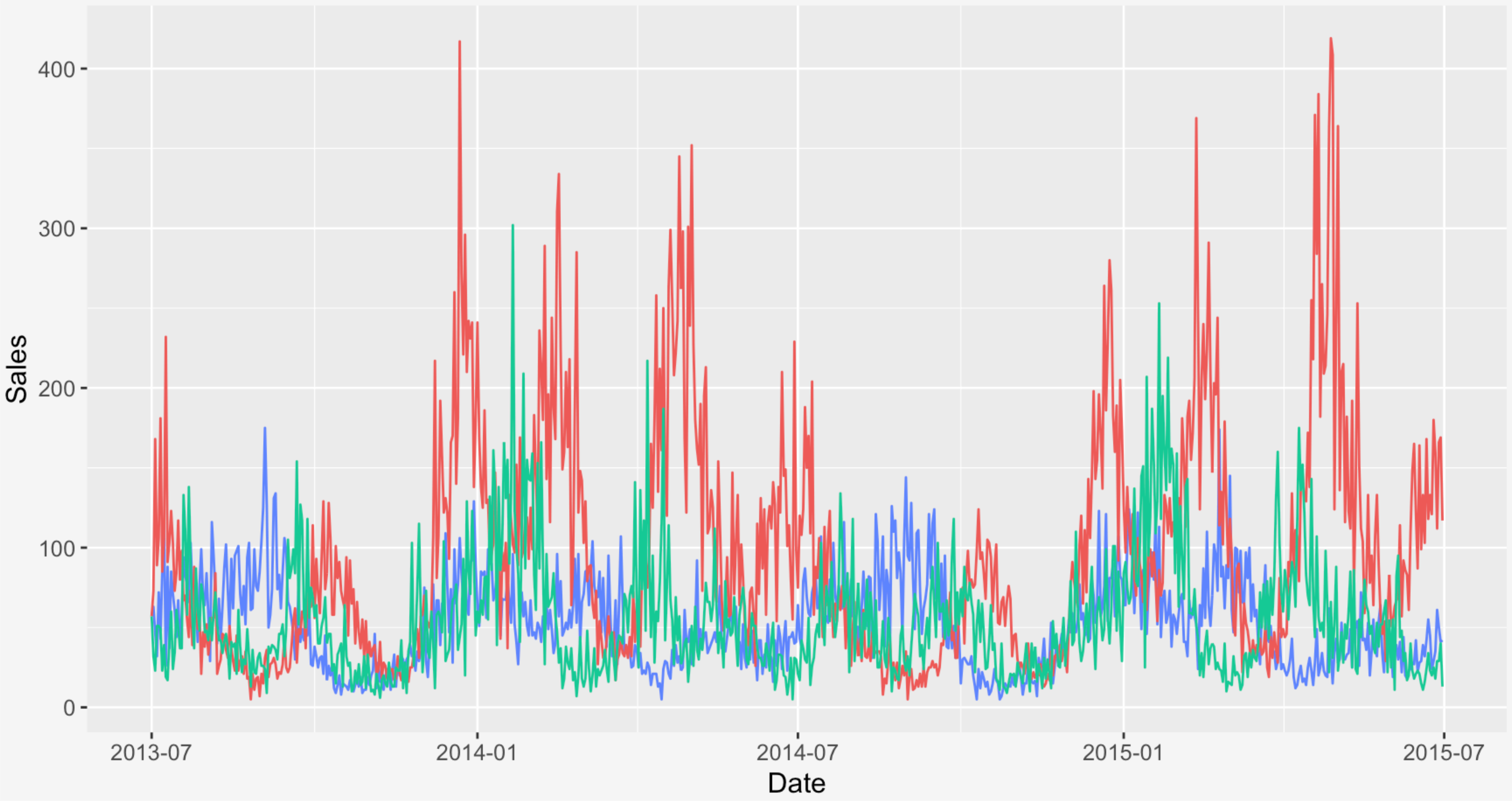
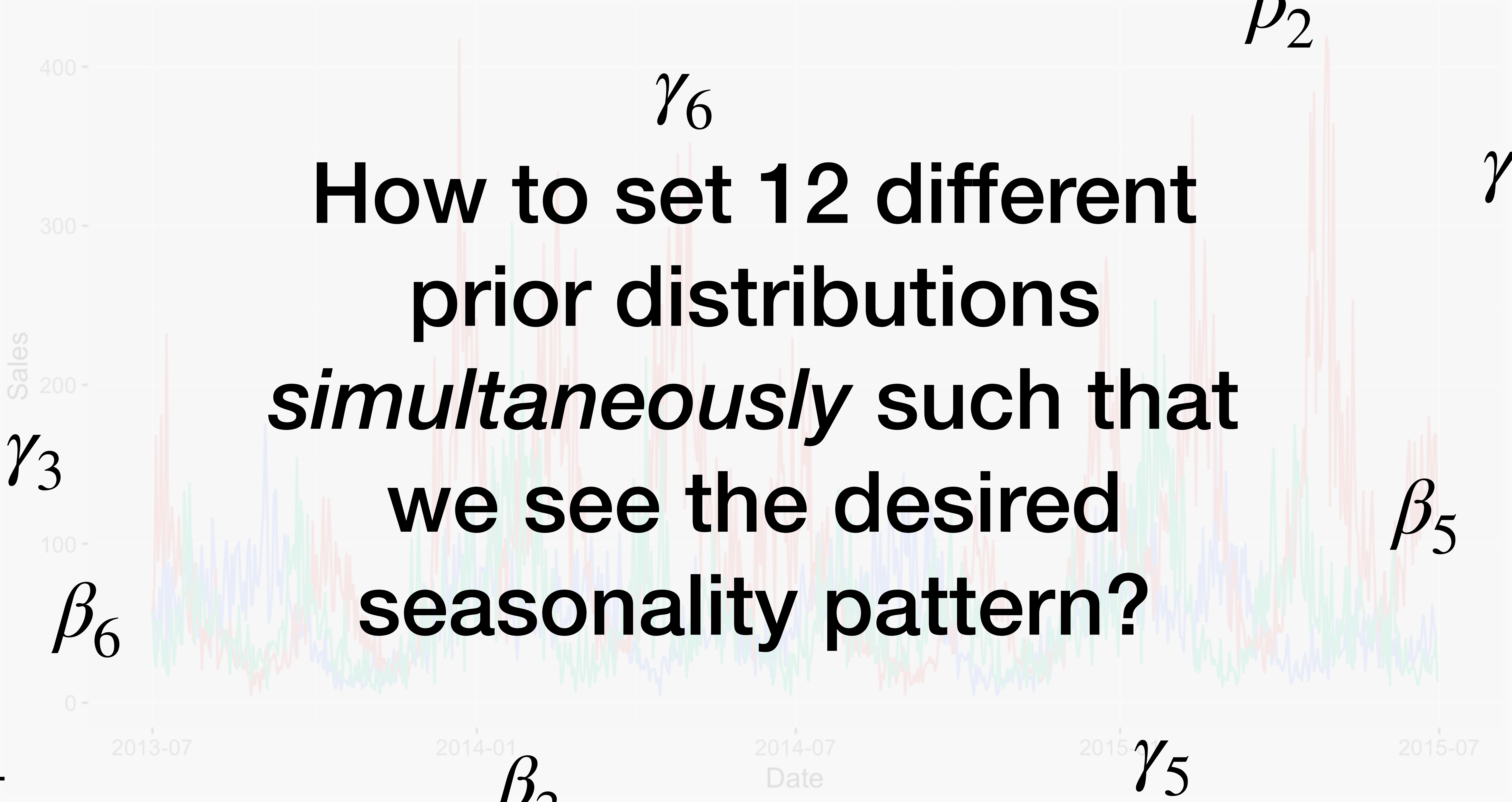



Prior predictive checks show how Fourier terms result in arbitrary seasonalities.

Since we have data for less than a year, good prior is necessary!

Prior predictive checks show how Fourier terms result in arbitrary seasonalities.

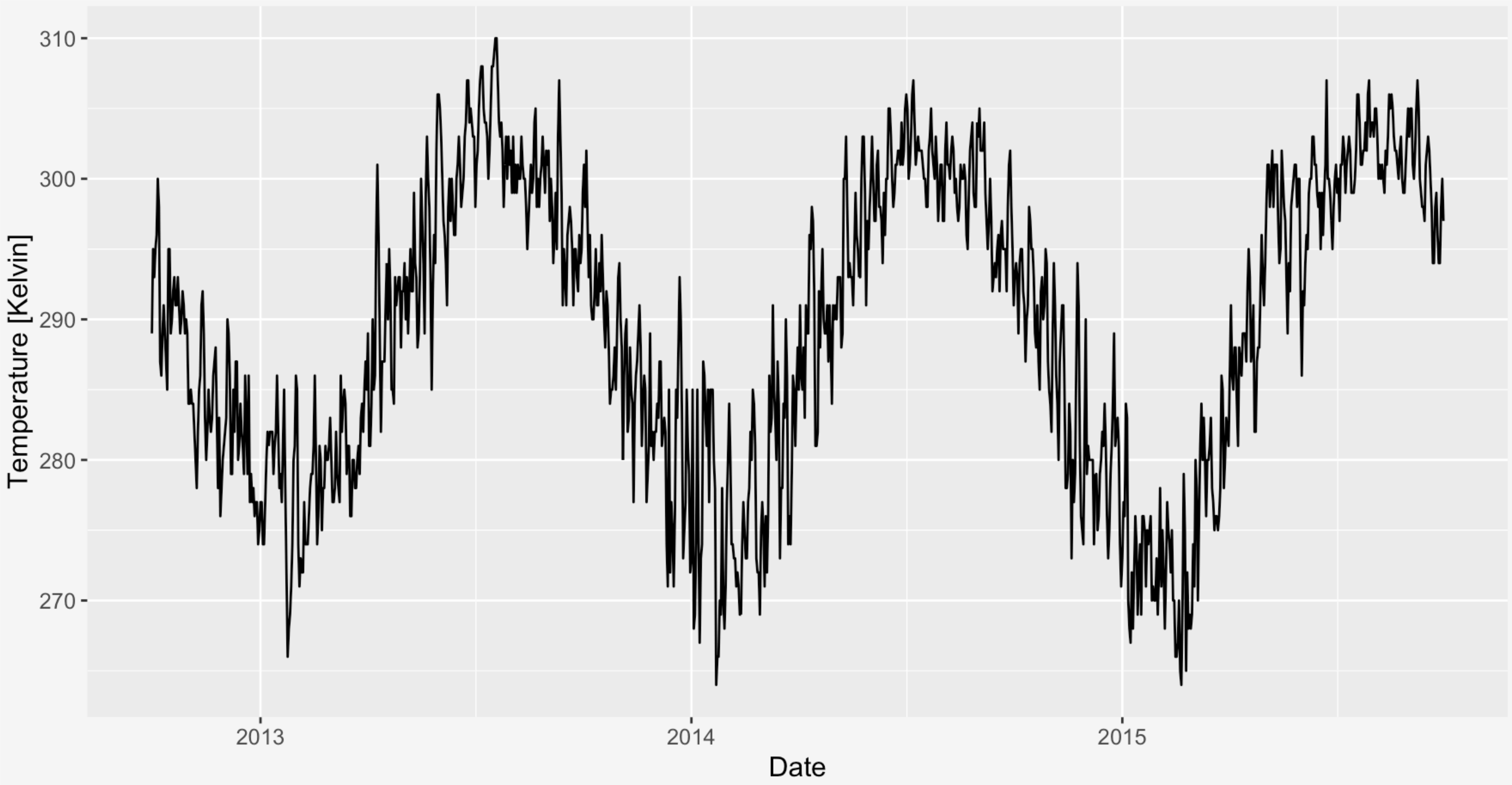
β_1 γ_6 β_2 γ_2



How to set 12 different prior distributions *simultaneously* such that we see the desired seasonality pattern?

γ_3 β_5 β_6 γ_4 β_3 γ_5 β_4 γ_1

Prior predictive checks show how Fourier terms result in arbitrary seasonalities.

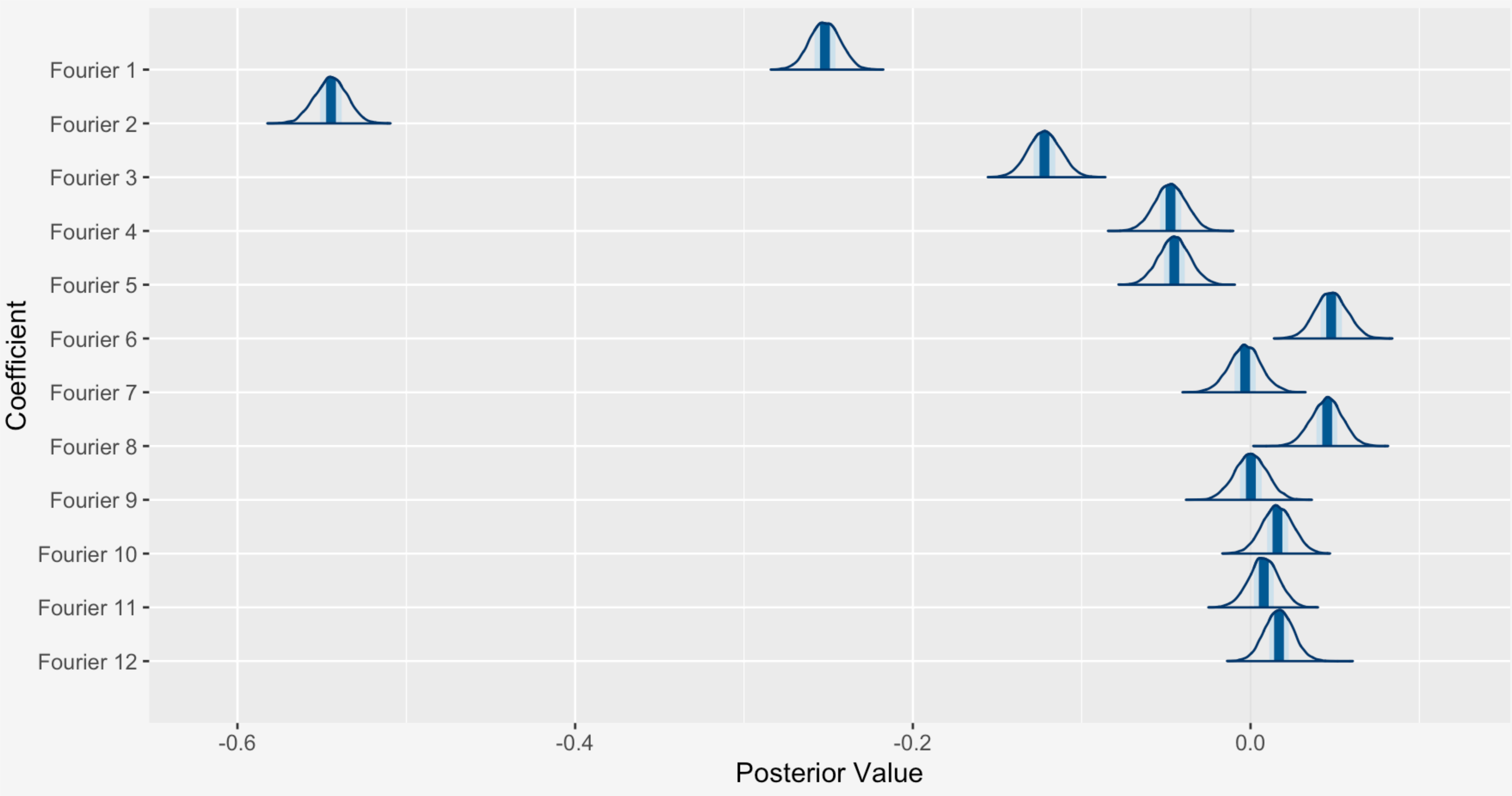


Daily temperature in New York City (available on Kaggle).

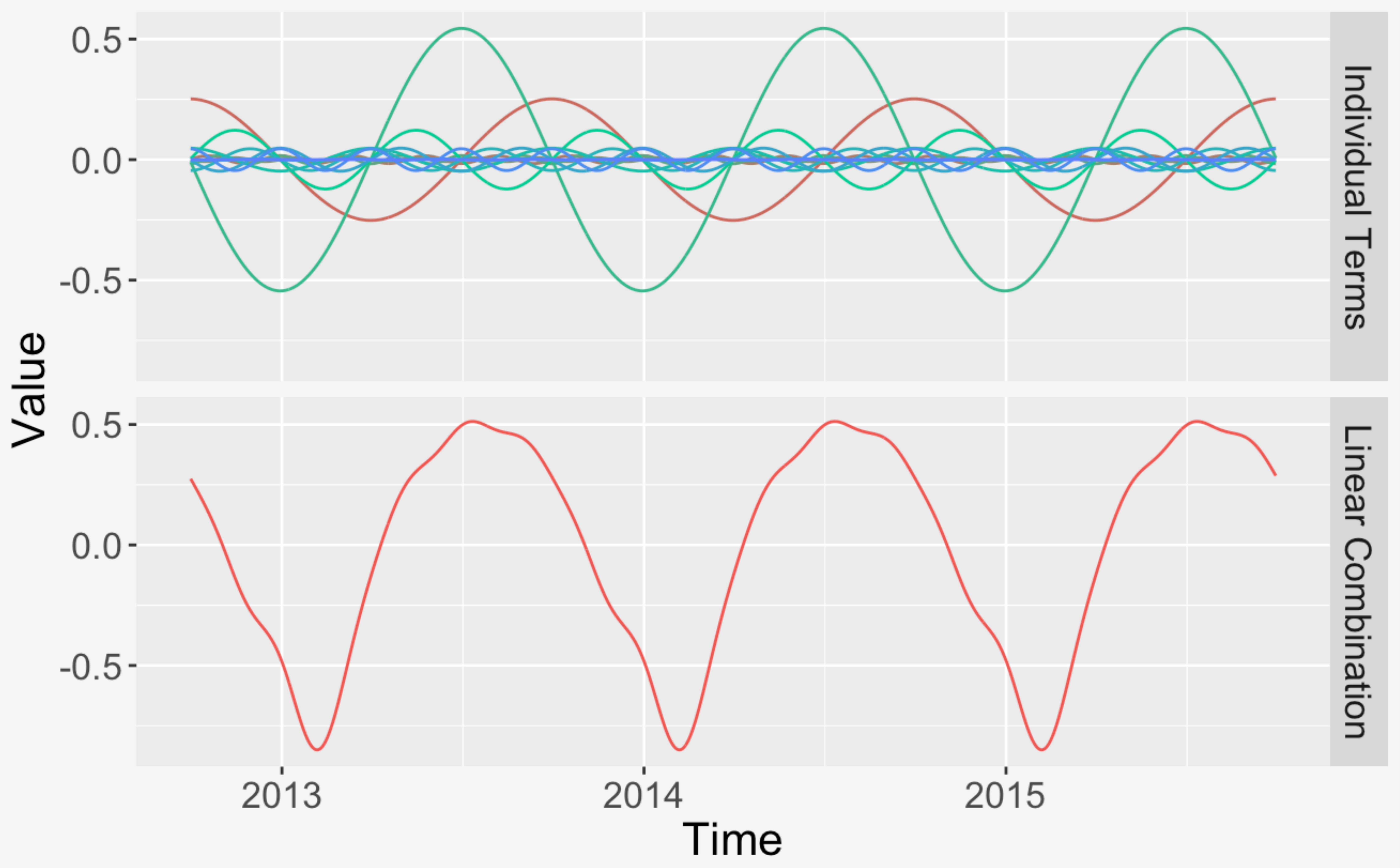
Transfer Posterior as Prior

Learn coefficients in model for which we have enough data.

$$P(\lambda \mid \tilde{T}) \propto P(\tilde{T} \mid \lambda) \cdot P(\lambda)$$



Posterior distributions of Fourier coefficients in temperature model.



Linear combination of Fourier terms weighted by the posterior mean coefficients.

Transfer Posterior as Prior

Learn coefficients in model for which we have enough data.

$$P(\lambda \mid \tilde{T}) \propto P(\tilde{T} \mid \lambda) \cdot P(\lambda)$$

$$\downarrow$$

$$P(\lambda \mid \tilde{Y}) \propto P(\tilde{Y} \mid \lambda) \cdot P(\lambda)$$

Transfer learned posterior as new prior distribution into original model where only few data is available.

Data Generating Process

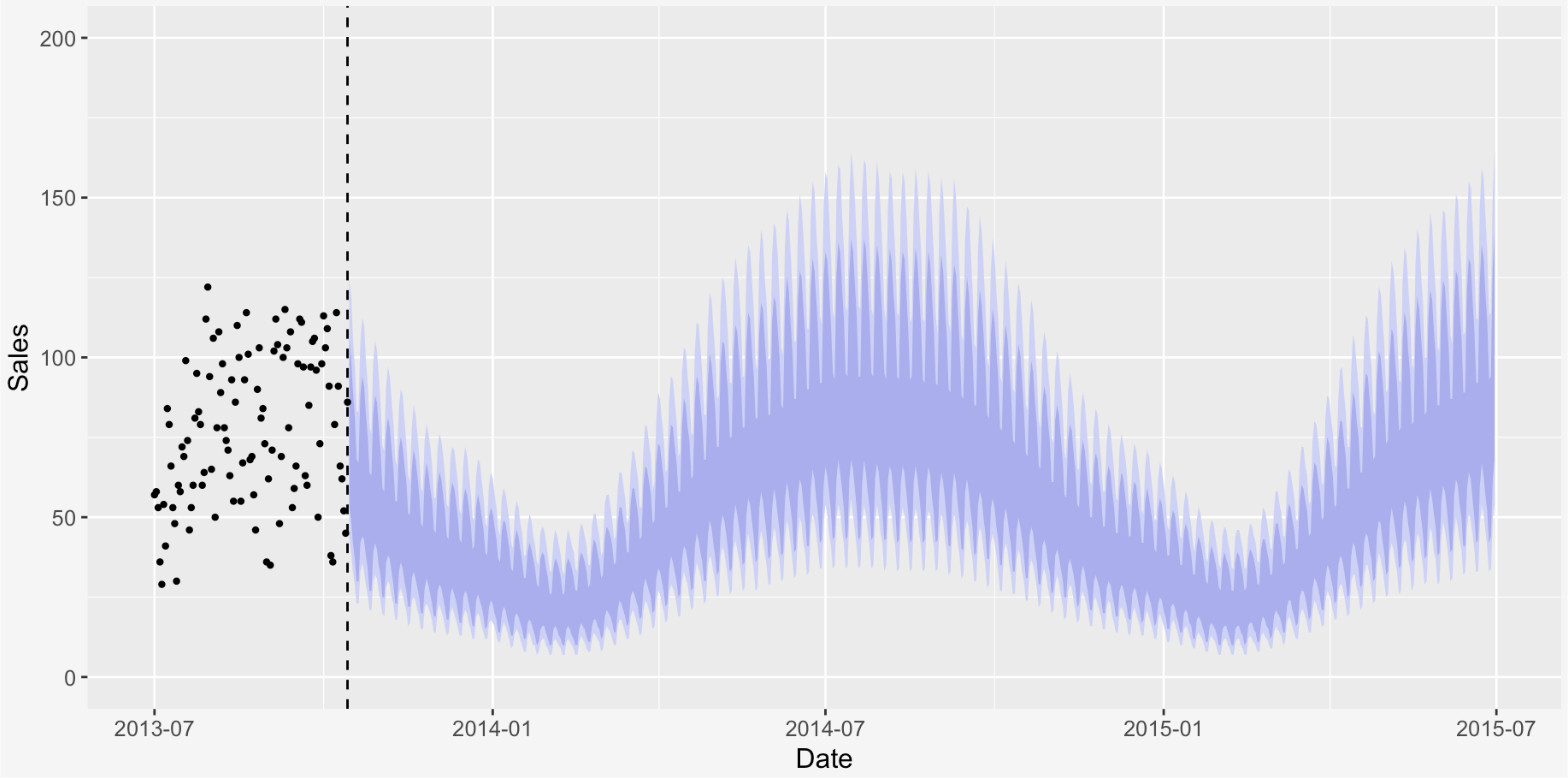
$$\log(\mu_t) = \alpha_0 + \sum_{i=1}^{6} \alpha_i D_i + \tau \frac{t}{m} + \sum_{k=1}^{K} \left(\beta_k \sin\left(\frac{2\pi kt}{m}\right) + \gamma_k \cos\left(\frac{2\pi kt}{m}\right) \right)$$

$$\beta_k \sim P(\beta_k \mid \tilde{T}) \quad \textbf{for } k = 1,\ldots,K$$

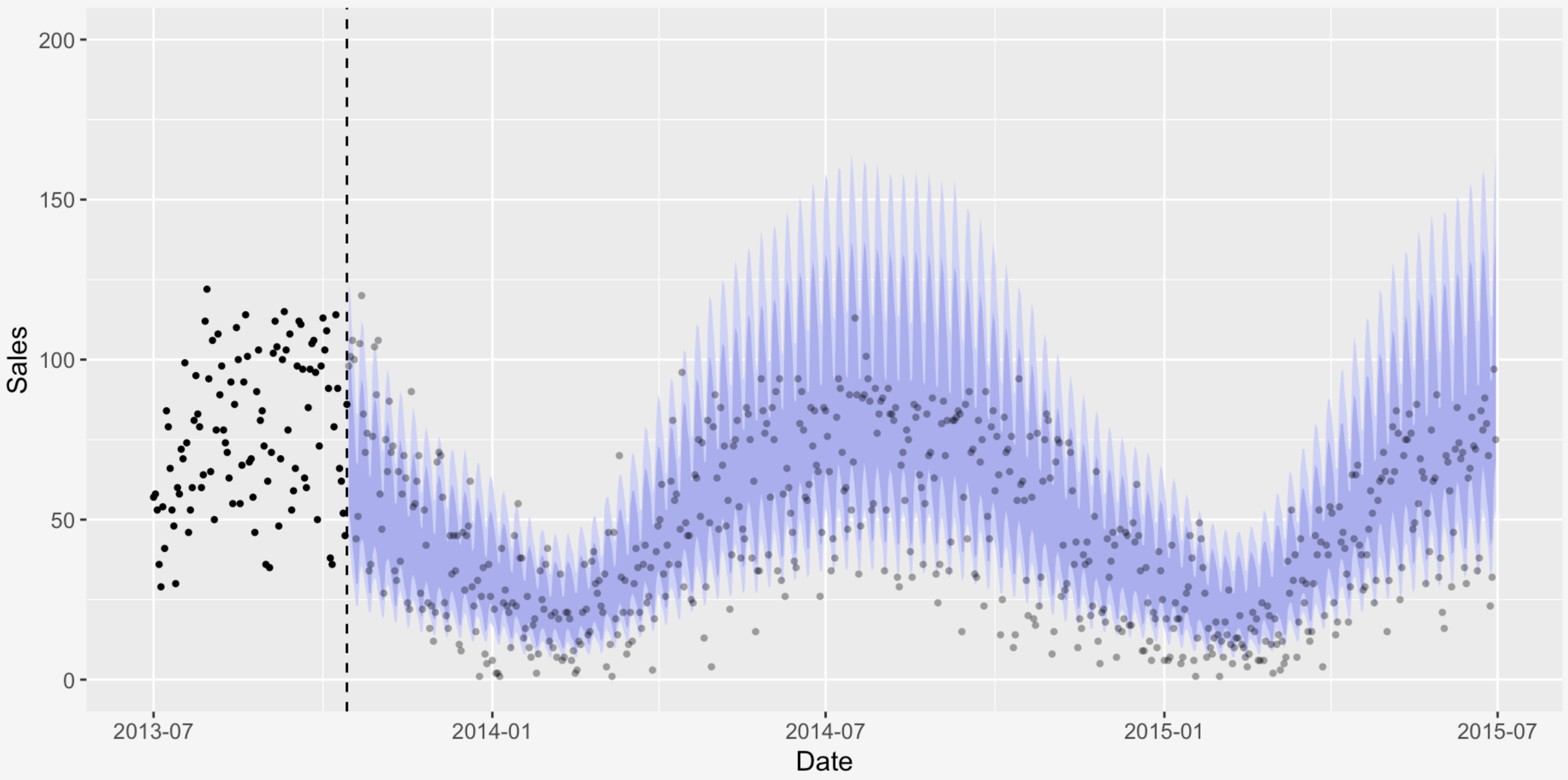
$$\gamma_k \sim P(\gamma_k \mid \tilde{T}) \quad \textbf{for } k = 1,\ldots,K$$

```
> beta_samples <- as.data.frame(temperature_fit)

> beta_means <- as.numeric(colMeans(beta_samples))
> beta_sds <- as.numeric(apply(beta_samples, 2, sd))
```

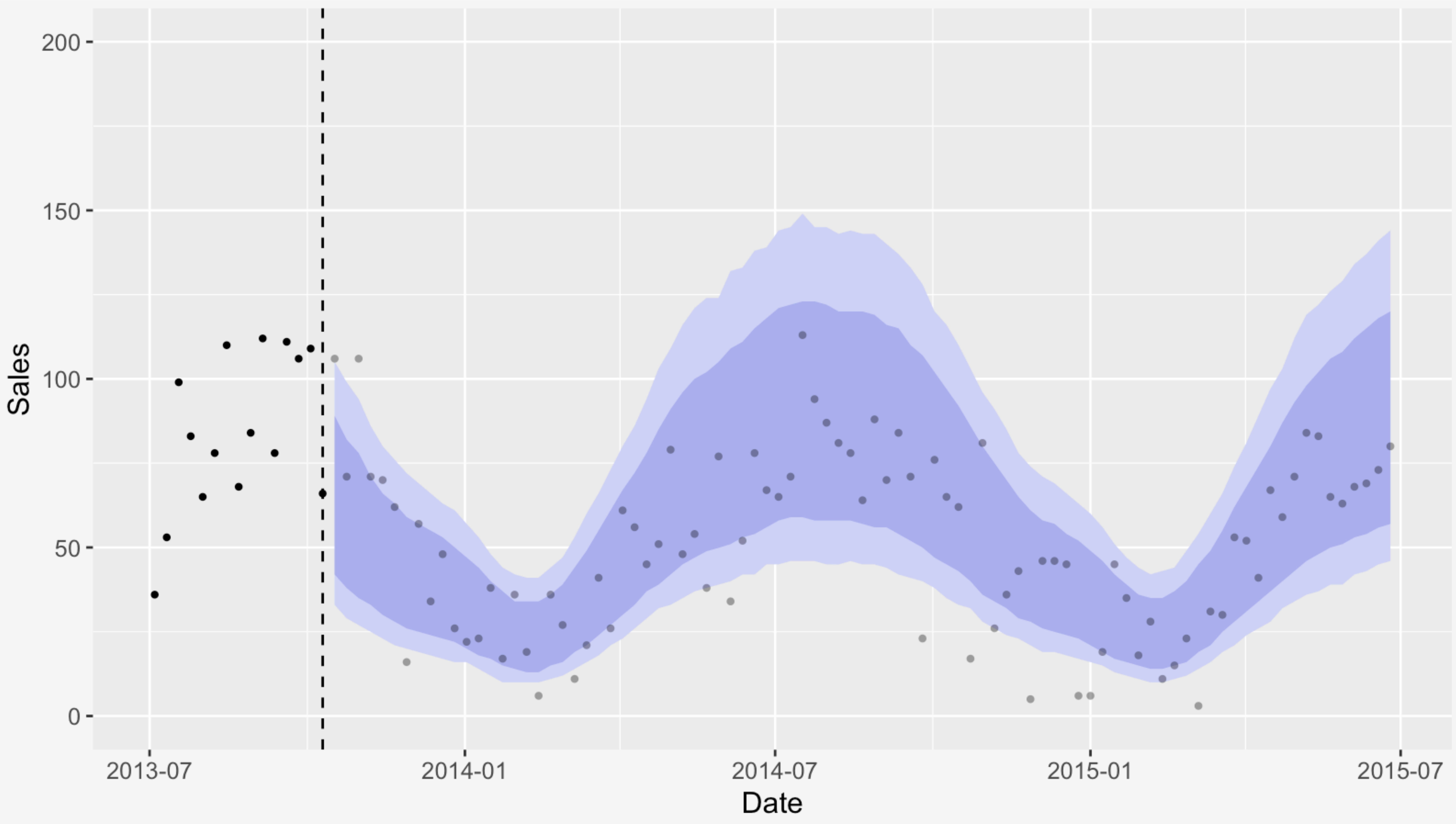
```
> sales_temp_fit <- sampling(negbin_prior_stan,
+                            data = list(y = sales,
+                                        x = xreg,
+                                        fouriers = fouriers,
+                                        trend = trend,
+                                        trend_loc = 0.03,
+                                        trend_sd = 0.02,
+                                        fourier_loc = beta_means,
+                                        fourier_sd = beta_sds,
+                                        N = N, K = K,
+                                        F = 12),
+                            iter = 4000, algorithm = "NUTS", seed = 512)
```



Posterior predictive distribution.

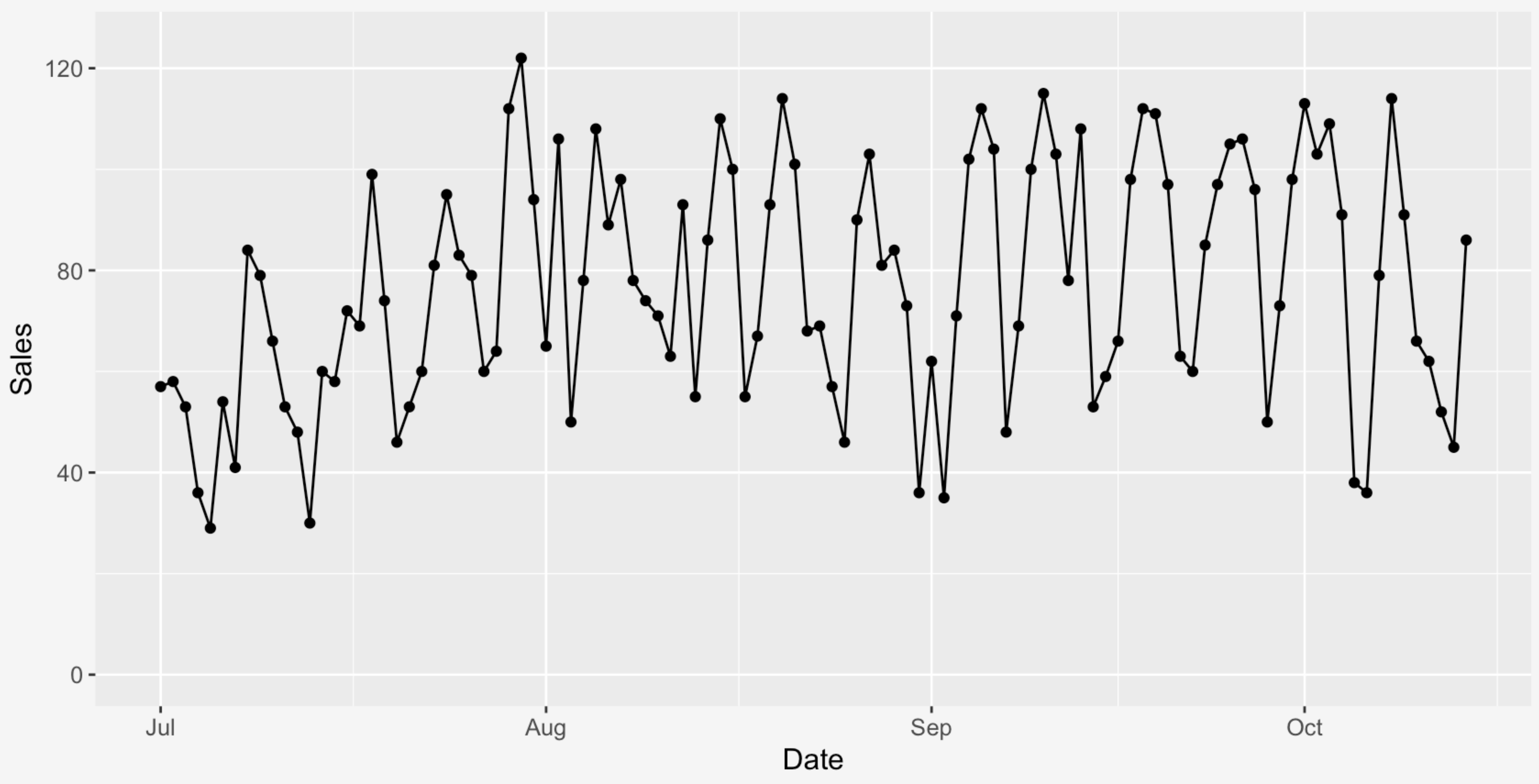


Posterior predictive distribution and actual future observations.



Posterior predictive distribution and future observations for Thursdays only.

github.com/timradtke/short-time-series



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Thank you!

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