Machine Learning
Homework 3

Due on May 10, 2018

1. Let $f : R^n \to R$ be a *strictly convex* function *i.e.*,

$$f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y).$$

   If x^* is a local minimizer then x^* is the unique global minimizer. (15 %)

2. Let

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} g_1(\mathbf{x}) \\ g_2(\mathbf{x}) \\ \vdots \\ g_m(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \mathbf{h}(\mathbf{x}) = \begin{bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \\ \vdots \\ h_k(\mathbf{x}) \end{bmatrix}$$

   where $g_i : R^n \to R$ be a *convex* function for all $i = 1, 2 \ldots, m$, and $h_j : R^n \to R$ be a *linear* function for all $j = 1, 2 \ldots, k$.

   Consider $\mathcal{F} = \{\mathbf{x} \mid \mathbf{g}(\mathbf{x}) \leq \mathbf{0},\ \mathbf{h}(\mathbf{x}) = \mathbf{0}\} \subset R^n$. Prove that $\mathcal{F}$ is a *convex* set. (15 %)

3. Prove that for any matrix $B \in R^{m\times n}$, either the system (**I**)

$$B\mathbf{x} < \mathbf{0}$$

or the system (**II**)

$$B^\top \alpha = \mathbf{0} \;, \alpha \geq \mathbf{0} \text{ and } \alpha \neq \mathbf{0}$$

has a solution but *never both*. (20 %)
Hint 1: $B\mathbf{x} < \mathbf{0}$ if and only if $B\mathbf{x} + \mathbf{1}z \leq \mathbf{0}, z > 0$.
Hint 2: Use Farkas' Lemma with a suitable $b \in R^{n+1}$ and $A \in R^{m\times(n+1)}$

4. **(a)** Solve

$$\min_{x\in R^2} \; \frac{1}{2}x^\top \begin{bmatrix} 1 & 0 \\ 0 & 900 \end{bmatrix} x$$

using the *steep descent with exact line search*. You are welcome to copy the MATLAB code from my slides. Start your code with the initial point $x_0 = [1000\ 1]^T$. Stop until $\|x_{n+1} - x_n\|_2 < 10^{-8}$. Report your solution and the number of iteration. (15 %)

(b) Implement the Newton's method for minimizing a quadratic function $f(x) = \frac{1}{2}x^\top Qx + p^\top x$ in MATLAB code. Apply your code to solve the minimization problem in **(a)**.(15 %)

5. Find an approximate solution using MATLAB to the following system by minimizing $\|Ax - b\|_p$ for $p = 1, 2, \infty$. Write down both the approximate solution, and the value of the $\|Ax - b\|_p$. Draw the solution points in R^2 and the four equations being solved.

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 2 \\ 2x_1 & - & x_2 & = & -2 \\ x_1 & + & x_2 & = & 1 \\ 4x_1 & - & x_2 & = & 0 \\ x_1 & - & x_2 & = & -2 \end{array}$$

(20 %)