INTRODUCTION

In previous classes, you have learnt that composite numbers can be expressed as the product of prime numbers.

For example:

$$42 = 2 \times 3 \times 7$$

Here, 2, 3 and 7 are prime factors of 42.

Similarly, algebraic expressions can also be expressed as the product of irreducible factors. By an irreducible factor, we mean a factor which cannot be expressed further as the product of factors.

We know that the product of $3x + 7$ and $3x - 7 = (3x + 7)(3x - 7) = 9x^2 - 49$; we say that $3x + 7$ and $3x - 7$ are factors of $9x^2 - 49$. We write it as

$$9x^2 - 49 = (3x + 7)(3x - 7)$$

Similarly, the product of $2x + 1$ and $x - 3 = (2x + 1)(x - 3) = 2x^2 - 5x - 3$; we say that $2x + 1$ and $x - 3$ are factors of $2x^2 - 5x - 3$. We write it as

$$2x^2 - 5x - 3 = (2x + 1)(x - 3)$$

*Thus, when an algebraic expression can be written as the product of two or more algebraic expressions, then each of these expression is called a **factor** of the given expression.*

To find the factors of a given algebraic expression means to obtain two or more expressions whose product is the given expression.

*The process of finding two or more expressions whose product is the given expression is called **factorisation.***

Thus, factorisation is the reverse process of multiplication.

For example:

Product	Factors
(*i*) $(2x + 5)(2x - 5) = 4x^2 - 25$	$4x^2 - 25 = (2x + 5)(2x - 5)$
(*ii*) $(p + 3)(p - 7) = p^2 - 4p - 21$	$p^2 - 4p - 21 = (p + 3)(p - 7)$
(*iii*) $(2y + 3)(3y - 5) = 6y^2 - y - 15$	$6y^2 - y - 15 = (2y + 3)(3y - 5)$

However, in this book, we will deal only with some special types of expressions.

4.1 FACTORISING BY TAKING OUT COMMON FACTORS

If the different terms of a given polynomial have common factors, then the given polynomial can be factorised by the following procedure:

(*i*) Find the H.C.F. of all the terms of the given polynomial.

(*ii*) Divide each term of the given polynomial by H.C.F. Enclose the quotient within the brackets and keep the common factor outside the bracket.

Illustrative Examples

Example 1. Factorise the following:

(i) $24x^3 - 32x^2$ (ii) $15ab^2 - 21a^2b$

(iii) $6xy^2 + 9x^2y - 21xy$ (iv) $14x^2y^2 - 10x^2y + 8xy^2$

Solution. (i) H.C.F. of $24x^3$ and $32x^2$ is $8x^2$

$\therefore \quad 24x^3 - 32x^2 = 8x^2(3x - 4)$

Divide each term by $8x^2$ and keep $8x^2$ outside the bracket.

(ii) H.C.F. of $15ab^2$ and $21a^2b$ is $3ab$

$\therefore \quad 15ab^2 - 21a^2b = 3ab(5b - 7a)$

(iii) H.C.F. of $6xy^2$, $9x^2y$ and $21xy$ is $3xy$

$\therefore \quad 6xy^2 + 9x^2y - 21xy = 3xy(2y + 3x - 7)$

(iv) H.C.F. of $14x^2y^2$, $10x^2y$ and $8xy^2$ is $2xy$

$\therefore \quad 14x^2y^2 - 10x^2y + 8xy^2 = 2xy(7xy - 5x + 4y)$

Example 2. Factorise the following:

(i) $3x(y + 2z) + 5a(y + 2z)$

(ii) $10(p - 2q)^3 + 6(p - 2q)^2 - 20(p - 2q)$

Solution. (i) H.C.F. of the expressions $3x(y + 2z)$ and $5a(y + 2z)$ is $y + 2z$

$\therefore \quad 3x(y + 2z) + 5a(y + 2z) = (y + 2z)(3x + 5a)$

Divide each term by $y + 2z$ and keep $y + 2z$ outside the bracket.

(ii) H.C.F. of the expressions

$10(p - 2q)^3$, $6(p - 2q)^2$ and $20(p - 2q)$ is $2(p - 2q)$

$\therefore \quad 10(p - 2q)^3 + 6(p - 2q)^2 - 20(p - 2q) = 2(p - 2q)\,[5(p - 2q)^2 + 3(p - 2q) - 10]$

Exercise 4.1

Factorise the following (1 to 9):

1. (i) $8xy^3 + 12x^2y^2$ (ii) $15ax^3 - 9ax^2$
2. (i) $21py^2 - 56py$ (ii) $4x^3 - 6x^2$
3. (i) $2\pi r^2 - 4\pi r$ (ii) $18m + 16n$
4. (i) $25abc^2 - 15a^2b^2c$ (ii) $28p^2q^2r - 42pq^2r^2$
5. (i) $8x^3 - 6x^2 + 10x$ (ii) $14mn + 22m - 62p$
6. (i) $18p^2q^2 - 24pq^2 + 30p^2q$ (ii) $27a^3b^3 - 18a^2b^3 + 75a^3b^2$
7. (i) $15a(2p - 3q) - 10b(2p - 3q)$ (ii) $3a(x^2 + y^2) + 6b(x^2 + y^2)$
8. (i) $6(x + 2y)^3 + 8(x + 2y)^2$ (ii) $14(a - 3b)^3 - 21p(a - 3b)$
9. (i) $10a(2p + q)^3 - 15b(2p + q)^2 + 35(2p + q)$

   (ii) $x(x^2 + y^2 - z^2) + y(-x^2 - y^2 + z^2) - z(x^2 + y^2 - z^2)$

4.2 FACTORISING BY GROUPING OF TERMS

When the grouping of terms of the given polynomial gives rise to common factor, the given polynomial can be factorised by the following procedure:

(*i*) Arrange the terms of the given polynomial in groups in such a way that each has a common factor.

(*ii*) Factorise each group.

(*iii*) Take out the factor which is common to each group.

Illustrative Examples

Example 1. Factorise the following:

(*i*) $ax - ay + bx - by$ (*ii*) $4x^2 - 10xy - 6xz + 15yz$

Solution. (*i*) $ax - ay + bx - by = (ax - ay) + (bx - by)$
$= a(x - y) + b(x - y)$
$= (x - y)(a + b)$

(*ii*) $4x^2 - 10xy - 6xz + 15yz = (4x^2 - 10xy) - (6xz - 15yz)$
$= 2x(2x - 5y) - 3z(2x - 5y)$
$= (2x - 5y)(2x - 3z)$

Example 2. Factorise the following:

(*i*) $x^3 + 2x^2 + x + 2$ (*ii*) $1 + p + pq + p^2q$

Solution. (*i*) $x^3 + 2x^2 + x + 2 = (x^3 + 2x^2) + (x + 2)$
$= x^2(x + 2) + 1(x + 2)$
$= (x + 2)(x^2 + 1)$

(*ii*) $1 + p + pq + p^2q = (1 + p) + (pq + p^2q)$
$= 1(1 + p) + pq(1 + p)$
$= (1 + p)(1 + pq)$

Example 3. Factorise the following:

(*i*) $xy - pq + qy - px$ (*ii*) $ab(x^2 + y^2) + xy(a^2 + b^2)$

Solution. (*i*) Since xy and pq have nothing in common, we do not group the terms in the order in which the given expression is written. Hence, we interchange $-pq$ and $-px$

$\therefore$ $xy - pq + qy - px = (xy - px) + (qy - pq)$
$= x(y - p) + q(y - p)$
$= (y - p)(x + q)$

(*ii*) $ab(x^2 + y^2) + xy(a^2 + b^2) = abx^2 + aby^2 + a^2xy + b^2xy$
$= (abx^2 + a^2xy) + (aby^2 + b^2xy)$
$= ax(bx + ay) + by(ay + bx)$
$= (bx + ay)(ax + by)$

Example 4. Factorise the following:

(*i*) $a(a + b - c) - bc$ (*ii*) $a^2x^2 + (ax^2 + 1)x + a.$

Solution. (*i*) $a(a + b - c) - bc = a^2 + ab - ac - bc$
$= (a^2 + ab) - (ac + bc)$
$= a(a + b) - c(a + b)$
$= (a + b)(a - c)$

(ii) $a^2x^2 + (ax^2 + 1)x + a = a^2x^2 + ax^3 + x + a$

$= (ax^3 + a^2x^2) + (x + a) = ax^2(x + a) + 1(x + a)$

$= (x + a)\,(ax^2 + 1)$

Example 5. Factorise the following:

(i) $ax - 2by - az - 2bx + ay + 2bz$ (ii) $x^3 - x^2 + ax + x - a - 1$

Solution. (i) $ax - 2by - ax - 2bx + ay + 2bz$

$= ax + ay - az - 2bx - 2by + 2bz$

$= a\,(x + y - z) - 2b\,(x + y - z)$

$= (x + y - z)(a - 2b)$

(ii) $x^3 - x^2 + ax + x - a - 1 = (x^3 - x^2) + (ax - a) + (x - 1)$

$= x^2(x - 1) + a(x - 1) + 1(x - 1)$

$= (x - 1)\,(x^2 + a + 1)$

Example 6. Factorise the following:

(i) $p(x - y)^2 - qy + qx + 3x - 3y$ (ii) $ax - (ax + by)^2 + a^2x + aby + by$

Solution. (i) $p(x - y)^2 - qy + qx + 3x - 3y$

$= p(x - y)^2 + (qx - qy) + (3x - 3y)$

$= p(x - y)^2 + q(x - y) + 3(x - y)$

$= (x - y)\,[p(x - y) + q + 3]$

(ii) $ax - (ax + by)^2 + a^2x + aby + by$

$= (ax + a^2x) + (aby + by) - (ax + by)^2$

$= ax(1 + a) + by(a + 1) - (ax + by)^2$

$= (1 + a)\,(ax + by) - (ax + by)^2$

$= (ax + by)\,[1 + a - (ax + by)]$

$= (ax + by)\,(1 + a - ax - by)$

Example 7. Factorise the following:

(i) $a^3x + a^2(x - y) - a(y + z) - z$ (ii) $(x^2 - 2x)^2 - 5(x^2 - 2x) - y(x^2 - 2x) + 5y$

Solution. (i) $a^3x + a^2(x - y) - a(y + z) - z$

$= a^3x + a^2x - a^2y - ay - az - z$

$= (a^3x + a^2x) - (a^2y + ay) - (az + z)$

$= a^2x(a + 1) - ay(a + 1) - z(a + 1)$

$= (a + 1)\,(a^2x - ay - z)$

(ii) $(x^2 - 2x)^2 - 5(x^2 - 2x) - y(x^2 - 2x) + 5y$

$= ((x^2 - 2x)^2 - 5(x^2 - 2x)) - (y(x^2 - 2x) - 5y)$

$= (x^2 - 2x)\,(x^2 - 2x - 5) - y(x^2 - 2x - 5)$

$= (x^2 - 2x - 5)\,(x^2 - 2x - y)$

Example 8. Factorise: $a^2 + b^2 - 2(ab - ac + bc)$

Solution. $a^2 + b^2 - 2(ab - ac + bc) = (a^2 + b^2 - 2ab) - 2(-ac + bc)$

$= (a - b)^2 - 2(-c)\,(a - b) = (a - b)\,[(a - b) + 2c]$

$= (a - b)\,(a - b + 2c)$

Exercise 4.2

Factorise the following (1 to 14):

1. (i) $x^2 + xy - x - y$ (ii) $y^2 - yz - 5y + 5z$
2. (i) $5xy + 7y - 5y^2 - 7x$ (ii) $5p^2 - 8pq - 10p + 16q$
3. (i) $a^2b - ab^2 + 3a - 3b$ (ii) $x^3 - 3x^2 + x - 3$
4. (i) $6xy^2 - 3xy - 10y + 5$ (ii) $3ax - 6ay - 8by + 4bx$
5. (i) $5px - 8qy + 4qx - 10py$ (ii) $9a^2y - 9ay + 6a - 6$
6. (i) $1 - a - b + ab$ (ii) $a(a - 2b - c) + 2bc$
7. (i) $x^2 + xy(1 + y) + y^3$ (ii) $y^2 - xy(1 - x) - x^3$
8. (i) $ab^2 + (a - 1)b - 1$ (ii) $2a - 4b - xa + 2bx$
9. (i) $5ph - 10qk + 2rph - 4qrk$ (ii) $x^2 - x(a + 2b) + 2ab$
10. (i) $ab(x^2 + y^2) - xy(a^2 + b^2)$ (ii) $(ax + by)^2 + (bx - ay)^2$
11. (i) $a^3 + ab(1 - 2a) - 2b^2$ (ii) $3x^2y - 3xy + 12x - 12$
12. $a^2b + ab^2 - abc - b^2c + axy + bxy$
13. $ax^2 - bx^2 + ay^2 - by^2 + az^2 - bz^2$
14. $x - 1 - (x - 1)^2 + ax - a$

4.3 DIFFERENCE OF TWO SQUARES

We shall use the identity $a^2 - b^2 = (a + b)(a - b)$

Illustrative Examples

Example 1. Factorise the following:

(i) $4x^2 - 169y^2$ (ii) $1 - (b - c)^2$

(iii) $x^2 - 2y + xy - 4$ (iv) $a(a - 3) - b(b - 3)$

Solution. (i) $4x^2 - 169y^2 = (2x)^2 - (13y)^2$

$= (2x + 13y)(2x - 13y)$

(ii) $1 - (b - c)^2 = (1)^2 - (b - c)^2$

$= (1 + \overline{b - c})(1 - \overline{b - c})$

$= (1 + b - c)(1 - b + c)$

(iii) $x^2 - 2y + xy - 4 = (x^2 - 4) + (xy - 2y)$

$= (x + 2)(x - 2) + y(x - 2)$

$= (x - 2)(\overline{x + 2} + y)$

$= (x - 2)(x + y + 2)$

(iv) $a(a - 3) - b(b - 3) = a^2 - 3a - b^2 + 3b$

$= (a^2 - b^2) - 3a + 3b$

$= (a - b)(a + b) - 3(a - b)$

$= (a - b)(a + b - 3)$

Example 2. Factorise the following:

(i) $16y^3 - 4y$ (ii) $9x^2 - 4a^2 + 4ay - y^2$ (iii) $x^3 - 3x^2 - x + 3$

Solution. (i) $16y^3 - 4y = 4y(4y^2 - 1)$

$= 4y[(2y)^2 - (1)^2]$

$= 4y(2y + 1)(2y - 1)$

(ii) $9x^2 - 4a^2 + 4ay - y^2 = 9x^2 - (4a^2 - 4ay + y^2)$

$= (3x)^2 - (2a - y)^2$

$= (3x + \overline{2a - y})(3x - \overline{2a - y})$

$= (3x - y + 2a)(3x + y - 2a)$

(iii) $x^3 - 3x^2 - x + 3 = (x^3 - 3x^2) + (-x + 3)$

$= x^2(x - 3) - 1(x - 3)$

$= (x - 3)(x^2 - 1)$

$= (x - 3)(x^2 - 1^2)$

$= (x - 3)(x + 1)(x - 1)$

Example 3. Factorise the following:

(i) $3 - 12(a - b)^2$ (ii) $4a^2 - 9b^2 - 2a - 3b$

(iii) $(a + b + c)^2 - (a - b - c)^2 + 4b^2 - 4c^2$

Solution. (i) $3 - 12(a - b)^2 = 3[1 - 4(a - b)^2]$

$= 3[1^2 - (2(a - b))^2]$

$= 3(1 + 2(a - b))(1 - 2(a - b))$

$= 3(1 + 2a - 2b)(1 - 2a + 2b)$

(ii) $4a^2 - 9b^2 - 2a - 3b = ((2a)^2 - (3b)^2) - 2a - 3b$

$= (2a + 3b)(2a - 3b) - 1(2a + 3b)$

$= (2a + 3b)(2a - 3b - 1)$

(iii) $(a + b + c)^2 - (a - b - c)^2 + 4b^2 - 4c^2$

$= ((a + b + c) + (a - b - c))((a + b + c) - (a - b - c)) + 4(b^2 - c^2)$

$= 2a(2b + 2c) + 4(b + c)(b - c)$

$= 4a(b + c) + 4(b + c)(b - c)$

$= 4(b + c)(a + b - c)$

Example 4. Factorise the following: imp

(i) $3x^5 - 48x$ ~~(ii)~~ $2(ab + cd) - a^2 - b^2 + c^2 + d^2$

(iii) $(1 - x^2)(1 - y^2) + 4xy$ (iv) $x^4 + y^4 - 11x^2y^2$

Solution. (i) $3x^5 - 48x = 3x(x^4 - 16) = 3x[(x^2)^2 - (4)^2]$

$= 3x(x^2 + 4)(x^2 - 4)$

$= 3x(x^2 + 4)(x + 2)(x - 2)$

(ii) $2(ab + cd) - a^2 - b^2 + c^2 + d^2 = 2ab + 2cd - a^2 - b^2 + c^2 + d^2$

$= (c^2 + 2cd + d^2) - (a^2 - 2ab + b^2)$

$= (c + d)^2 - (a - b)^2$

$= (\overline{c + d} + \overline{a - b})(\overline{c + d} - \overline{a - b})$

$= (c + d + a - b)(c + d - a + b)$

(*iii*) $(1 - x^2)(1 - y^2) + 4xy = 1 - x^2 - y^2 + x^2y^2 + 4xy$

$$= x^2y^2 + 1 + 2xy - x^2 - y^2 + 2xy$$
$$= (x^2y^2 + 2xy + 1) - (x^2 - 2xy + y^2)$$
$$= (xy + 1)^2 - (x - y)^2$$
$$= (\overline{xy + 1} + \overline{x - y})(\overline{xy + 1} - \overline{x - y})$$
$$= (xy + x - y + 1)(xy - x + y + 1)$$

(*iv*) $x^4 + y^4 - 11x^2y^2 = (x^4 + y^4 - 2x^2y^2) - 9x^2y^2$

$$= (x^2 - y^2)^2 - (3xy)^2$$
$$= (x^2 - y^2 + 3xy)(x^2 - y^2 - 3xy)$$

Example 5. Factorise the following:

(*i*) $x^4 + 4$ (*ii*) $x^4 + x^2 + 1$ (*iii*) $x^4 + x^2y^2 + y^4$.

Solution. (*i*) $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2$ (Adding and subtracting $4x^2$)

$$= (x^2 + 2)^2 - (2x)^2$$
$$= (x^2 + 2 + 2x)(x^2 + 2 - 2x)$$
$$= (x^2 + 2x + 2)(x^2 - 2x + 2)$$

(*ii*) $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$ (Adding and subtracting x^2)

$$= (x^2 + 1)^2 - x^2$$
$$= (x^2 + 1 + x)(x^2 + 1 - x)$$
$$= (x^2 + x + 1)(x^2 - x + 1)$$

(*iii*) $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$ (Adding and subtracting x^2y^2)

$$= (x^2 + y^2)^2 - (xy)^2$$
$$= (x^2 + y^2 + xy)(x^2 + y^2 - xy)$$

Example 6. Factorise completely $(x^2 + y^2 - z^2)^2 - 4x^2y^2$

Solution. $(x^2 + y^2 - z^2)^2 - 4x^2y^2 = (x^2 + y^2 - z^2)^2 - (2xy)^2$

$$= (x^2 + y^2 - z^2 + 2xy)(x^2 + y^2 - z^2 - 2xy)$$
$$= (\overline{x^2 + 2xy + y^2} - z^2)(\overline{x^2 - 2xy + y^2} - z^2)$$
$$= [(x + y)^2 - z^2][(x - y)^2 - z^2]$$
$$= (x + y + z)(x + y - z)(x - y + z)(x - y - z)$$

Example 7. Express $(x^2 - 4x + 9)(x^2 + 4x - 9)$ as a difference of two squares.

Solution. $(x^2 - 4x + 9)(x^2 + 4x - 9) = (x^2 - \overline{4x - 9})(x^2 + \overline{4x - 9})$

(Expressing as $(a - b)(a + b)$)

$$= (x^2)^2 - (4x - 9)^2$$

Exercise 4.3

Factorise the following (1 to 18):

1. (*i*) $4x^2 - 25y^2$ (*ii*) $9x^2 - 1$
2. (*i*) $150 - 6a^2$ (*ii*) $32x^2 - 18y^2$
3. (*i*) $(x - y)^2 - 9$ (*ii*) $9(x + y)^2 - x^2$
4. (*i*) $20x^2 - 45y^2$ (*ii*) $9x^2 - 4(y + 2x)^2$

5. (i) $2(x - 2y)^2 - 50y^2$ (ii) $32 - 2(x - 4)^2$

6. (i) $108a^2 - 3(b - c)^2$ (ii) $\pi a^5 - \pi^3 ab^2$

7. (i) $50x^2 - 2(x - 2)^2$ (ii) $(x - 2)(x + 2) + 3$

8. (i) $x - 2y - x^2 + 4y^2$ (ii) $4a^2 - b^2 + 2a + b$

9. (i) $a(a - 2) - b(b - 2)$ (ii) $a(a - 1) - b(b - 1)$

10. (i) $9 - x^2 + 2xy - y^2$ (ii) $9x^4 - (x^2 + 2x + 1)$

11. (i) $9x^4 - x^2 - 12x - 36$ (ii) $x^3 - 5x^2 - x + 5$

12. (i) $a^4 - b^4 + 2b^2 - 1$ (ii) $x^3 - 25x$

13. (i) $2x^4 - 32$ (ii) $a^2(b + c) - (b + c)^3$

14. (i) $(a + b)^3 - a - b$ (ii) $x^2 - 2xy + y^2 - a^2 - 2ab - b^2$

15. (i) $(a^2 - b^2)(c^2 - d^2) - 4abcd$ (ii) $4x^2 - y^2 - 3xy + 2x - 2y$

16. (i) $x^2 + \frac{1}{x^2} - 11$ (ii) $x^4 + 5x^2 + 9$

17. (i) $x^2 + \frac{4}{x^2} - 5$ (ii) $x^4 + 12x^2 + 11$

18. (i) $a^4 + b^4 - 7a^2b^2$ (ii) $x^4 - 14x^2 + 1$

19. Express each of the following as the difference of two squares:
    (i) $(x^2 - 5x + 7)(x^2 + 5x + 7)$ (ii) $(x^2 - 5x + 7)(x^2 + 5x - 7)$
    (iii) $(x^2 + 5x - 7)(x^2 - 5x + 7)$

20. Evaluate the following by using factors:
    (i) $(979)^2 - (21)^2$ (ii) $(99{\cdot}9)^2 - (0{\cdot}1)^2$

4.4 FACTORISATION OF TRINOMIALS

In this section, we will learn the factorisation of trinomials of the form $ax^2 + bx + c$, where a, b and c are real numbers.

Rule to factorise trinomial $ax^2 + bx + c$, where a, b and c are real numbers:

Split b (the coefficient of x) into two real numbers such that the algebraic sum of these two numbers is b and their product is ac, then factorise by grouping method.

Remark

It is not always possible to factorise a trinomial $ax^2 + bx + c$ (*i.e.* a quadratic expression); the following rule can save lot of time:

For the expression $ax^2 + bx + c$, work out $b^2 - 4ac$. If it is a perfect square, then the given expression will factorise; otherwise, not.

Illustrative Examples

Example 1. Factorise the following trinomials:

(i) $x^2 + 9x + 18$ (ii) $y^2 - 3y - 54$

Solution. (i) To factorise $x^2 + 9x + 18$, we want to find two real numbers whose sum is 9 and product is 18. By trial, we see that $3 + 6 = 9$ and $3 \times 6 = 18$

$$\therefore \quad x^2 + 9x + 18 = x^2 + 3x + 6x + 18$$
$$= x(x + 3) + 6(x + 3)$$
$$= (x + 3)(x + 6)$$

(*ii*) To factorise $y^2 - 3y - 54$, we want to find two real numbers whose sum is -3 and product is -54. By trial, we see that $(-9) + 6 = -3$ and $(-9) \times 6 = -54$

$$\therefore \quad y^2 - 3y - 54 = y^2 - 9y + 6y - 54$$
$$= y(y - 9) + 6(y - 9)$$
$$= (y - 9)(y + 6)$$

Example 2. Factorise the following trinomials:

(*i*) $6x^2 + 17x + 5$ (*ii*) $12x^2 - 7x + 1$

(*iii*) $2x^2 - 7x - 15$ (*iv*) $84 - 2r - 2r^2$

Solution. (*i*) To factorise $6x^2 + 17x + 5$, we want to find two real numbers whose sum is 17 and product is 6×5 *i.e.* 30. By trial, we see that $2 + 15 = 17$ and $2 \times 15 = 30$

$$\therefore \quad 6x^2 + 17x + 5 = 6x^2 + 2x + 15x + 5$$
$$= 2x(3x + 1) + 5(3x + 1)$$
$$= (3x + 1)(2x + 5)$$

(*ii*) To factorise $12x^2 - 7x + 1$, we want to find two real numbers whose sum is -7 and product is 12×1 *i.e.* 12. By trial, we see that $(-3) + (-4) = -7$ and $(-3) \times (-4) = 12$

$$\therefore \quad 12x^2 - 7x + 1 = 12x^2 - 3x - 4x + 1$$
$$= 3x(4x - 1) - 1(4x - 1)$$
$$= (4x - 1)(3x - 1)$$

(*iii*) To factorise $2x^2 - 7x - 15$, we want to find two real numbers whose sum is -7 and product is $2 \times (-15)$ *i.e.* -30. By trial, we see that $(-10) + 3 = -7$ and $(-10) \times 3 = -30$

$$\therefore \quad 2x^2 - 7x - 15 = 2x^2 - 10x + 3x - 15$$
$$= 2x(x - 5) + 3(x - 5)$$
$$= (x - 5)(2x + 3)$$

(*iv*) We note that $84 - 2r - 2r^2 = 2(42 - r - r^2)$

To factorise $42 - r - r^2$, we want to find two real numbers whose sum = -1 and product is $42 \times (-1)$ *i.e.* -42. By trial, we see that $(-7) + 6 = -1$ and $(-7) \times 6 = -42$

$$\therefore \quad 84 - 2r - 2r^2 = 2(42 - r - r^2)$$
$$= 2(42 - 7r + 6r - r^2)$$
$$= 2[7(6 - r) + r(6 - r)]$$
$$= 2(6 - r)(7 + r)$$

Example 3. Factorise the following:

(*i*) $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$ (*ii*) $x^2 + \frac{1}{4}x - \frac{1}{8}$

Solution. (*i*) To factorise $7\sqrt{2}x^2 - 10x - 4\sqrt{2}$, we want to find two real numbers whose sum is -10 and product is $(7\sqrt{2}) \times (-4\sqrt{2})$ *i.e.* -56. By trial, we see that $(-14) + 4 = -10$ and $(-14) \times 4 = -56$

$$\therefore \quad 7\sqrt{2}x^2 - 10x - 4\sqrt{2} = 7\sqrt{2}x^2 - 14x + 4x - 4\sqrt{2}$$
$$= 7\sqrt{2}x(x - \sqrt{2}) + 4(x - \sqrt{2})$$
$$= (x - \sqrt{2})(7\sqrt{2}x + 4)$$

(ii) Since $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ and $\frac{1}{2}\left(-\frac{1}{4}\right) = -\frac{1}{8}$,

$$\therefore \quad x^2 + \frac{1}{4}x - \frac{1}{8} = x^2 + \frac{1}{2}x - \frac{1}{4}x - \frac{1}{8}$$

$$= x\left(x + \frac{1}{2}\right) - \frac{1}{4}\left(x + \frac{1}{2}\right)$$

$$= \left(x + \frac{1}{2}\right)\left(x - \frac{1}{4}\right)$$

Example 4. Factorise the following:

(i) $3x^2 - 5xy - 12y^2$ (ii) $2x^3 + 5x^2y - 12xy^2$

(iii) $8(a - 2b)^2 - 2a + 4b - 1$ (iv) $9x^2 - (x^2 - 4)^2$

Solution. (i) Since $-9 + 4 = -5$ and $(-9).4 = -36$,

$$\therefore \quad 3x^2 - 5xy - 12y^2 = 3x^2 - 9xy + 4xy - 12y^2$$

$$= 3x(x - 3y) + 4y\,(x - 3y)$$

$$= (x - 3y)\,(3x + 4y)$$

(ii) $2x^3 + 5x^2y - 12xy^2 = x(2x^2 + 5xy - 12y^2)$

$$= x(2x^2 + 8xy - 3xy - 12y^2)$$

[$\because$ $8 + (-3) = 5$ and $8.(-3) = -24$]

$$= x\,[2x\,(x + 4y) - 3y\,(x + 4y)]$$

$$= x\,(x + 4y)\,(2x - 3y)$$

(iii) $8(a - 2b)^2 - 2a + 4b - 1 = 8(a - 2b)^2 - 2(a - 2\,b) - 1$

$$= 8x^2 - 2x - 1 \text{ where } x = a - 2b$$

$$= 8x^2 - 4x + 2x - 1 \quad [\because -4 + 2 = -2 \text{ and } (-4).2 = -8]$$

$$= 4x(2x - 1) + 1\,(2x - 1)$$

$$= (2x - 1)\,(4\,x + 1)$$

$$= (2.\,\overline{a - 2b} - 1)\,(4.\,\overline{a - 2b} + 1)$$

[replacing back the value of x]

$$= (2a - 4b - 1)\,(4a - 8b + 1)$$

(iv) $9x^2 - (x^2 - 4)^2 = (3x)^2 - (x^2 - 4)^2$

$$= (3x + \overline{x^2 - 4})\,(3x - \overline{x^2 - 4})$$

$$= (x^2 + 3x - 4)\,(4 + 3x - x^2)$$

$$= (x^2 + 4x - x - 4)\,(4 + 4x - x - x^2)$$

$$= [x\,(x + 4) - 1\,(x + 4)]\,[4(1 + x) - x\,(1 + x)]$$

$$= (x + 4)\,(x - 1)\,(1 + x)\,(4 - x)$$

Example 5. Factorise the following:

(i) $(x^2 - 4x)\,(x^2 - 4x - 1) - 20$ (ii) $(x - y)^2 - 7(x^2 - y^2) + 12(x + y)^2$

Solution. (i) Let $x^2 - 4x = p$, then

$$(x^2 - 4x)\,(x^2 - 4x - 1) - 20 = p(p - 1) - 20$$

$$= p^2 - p - 20 = p^2 - 5p + 4p - 20$$

$$= p(p - 5) + 4\,(p - 5) = (p - 5)\,(p + 4)$$

$$= (x^2 - 4x - 5)\,(x^2 - 4x + 4)$$

Now $x^2 - 4x - 5 = x^2 - 5x + x - 5$

$$= x(x - 5) + 1(x - 5) = (x - 5)\,(x + 1)$$

and $x^2 - 4x + 4 = (x - 2)^2$

$\therefore (x^2 - 4x)(x^2 - 4x - 1) - 20 = (x - 5)(x + 1)(x - 2)(x - 2)$

(ii) $(x - y)^2 - 7(x^2 - y^2) + 12(x + y)^2$

$$= (x - y)^2 - 7(x - y)(x + y) + 12(x + y)^2$$

Let $x - y = p$ and $x + y = q$, then

$$\begin{aligned}(x - y)^2 - 7(x^2 - y^2) + 12(x + y)^2 &= p^2 - 7pq + 12q^2\\ &= p^2 - 4pq - 3pq + 12q^2\\ &= p(p - 4q) - 3q(p - 4q)\\ &= (p - 4q)(p - 3q)\\ &= ((x - y) - 4(x + y))((x - y) - 3(x + y))\\ &= (-3x - 5y)(-2x - 4y)\\ &= 2(3x + 5y)(x + 2y)\end{aligned}$$

Example 6. Factorise: $(x^2 - 3x)^2 - 8(x^2 - 3x) - 20$

Solution. Let $x^2 - 3x = y$, then

$$\begin{aligned}(x^2 - 3x)^2 - 8(x^2 - 3x) - 20 &= y^2 - 8y - 20\\ &= y^2 - 10y + 2y - 20\\ &= y(y - 10) + 2(y - 10)\\ &= (y - 10)(y + 2)\\ &= (x^2 - 3x - 10)(x^2 - 3x + 2)\\ &= (x^2 - 5x + 2x - 10)(x^2 - 2x - x + 2)\\ &= [x(x - 5) + 2(x - 5)][x(x - 2) - 1(x - 2)]\\ &= (x - 5)(x + 2)(x - 2)(x - 1)\end{aligned}$$

Example 7. Factorise: $5 - (3x^2 - 2x)(6 - 3x^2 + 2x)$

Solution.

$$\begin{aligned}5 - (3x^2 - 2x)(6 - 3x^2 + 2x) &= 5 - (3x^2 - 2x)(6 - \overline{3x^2 - 2x})\\ &= 5 - y(6 - y) \text{ where } y = 3x^2 - 2x\\ &= 5 - 6y + y^2 = 5 - 5y - y + y^2\\ &= 5(1 - y) - y(1 - y) = (1 - y)(5 - y)\\ &= (1 - 3x^2 + 2x)(5 - 3x^2 + 2x)\\ &= (1 + 3x - x - 3x^2)(5 + 5x - 3x - 3x^2)\\ &= [1(1 + 3x) - x(1 + 3x)][5(1 + x) - 3x(1 + x)]\\ &= (1 + 3x)(1 - x)(1 + x)(5 - 3x)\end{aligned}$$

Example 8. Factorise: $(x^2 + 3x)(3x^2 + 9x - 24) - 60$

Solution. $(x^2 + 3x)(3x^2 + 9x - 24) - 60$

$$\begin{aligned}&= (x^2 + 3x)\{3(x^2 + 3x) - 24\} - 60\\ &= y(3y - 24) - 60, \text{ where } y = x^2 + 3x\\ &= 3y(y - 8) - 60 = 3\{y^2 - 8y - 20\}\\ &= 3\{y^2 - 10y + 2y - 20\}\\ &= 3\{y(y - 10) + 2(y - 10)\}\\ &= 3(y + 2)(y - 10)\\ &= 3(x^2 + 3x + 2)(x^2 + 3x - 10)\\ &= 3(x^2 + 2x + x + 2)(x^2 + 5x - 2x - 10)\\ &= 3\{x(x + 2) + 1(x + 2)\}\{x(x + 5) - 2(x + 5)\}\\ &= 3(x + 2)(x + 1)(x + 5)(x - 2)\end{aligned}$$

Example 9. Factorise the following:

(i) $x^4 - 14x^2y^2 - 51y^4$ (ii) $(x^2 + x)^2 + 4(x^2 + x) - 12$

Solution. (i) Since $-17 + 3 = -14$ and $(-17) \cdot 3 = -51$,

$$\begin{aligned}\therefore \quad x^4 - 14x^2y^2 - 51y^2 &= x^4 - 17x^2y^2 + 3x^2y^2 - 51y^4\\ &= x^2(x^2 - 17y^2) + 3y^2(x^2 - 17y^2)\\ &= (x^2 - 17y^2)(x^2 + 3y^2)\\ &= (x^2 - (\sqrt{17}\,y)^2)(x^2 + 3y^2)\\ &= (x - \sqrt{17}\,y)(x + \sqrt{17}\,y)(x^2 + 3y^2)\end{aligned}$$

(ii)
$$\begin{aligned}(x^2 + x)^2 + 4(x^2 + x) - 12 &= y^2 + 4y - 12 \text{ where } y = x^2 + x\\ &= y^2 + 6y - 2y - 12\\ &= y(y + 6) - 2(y + 6)\\ &= (y + 6)(y - 2)\\ &= (x^2 + x + 6)(x^2 + x - 2)\\ &= (x^2 + x + 6)(x^2 + 2x - x - 2)\\ &= (x^2 + x + 6)[x(x + 2) - 1(x + 2)]\\ &= (x^2 + x + 6)(x + 2)(x - 1)\end{aligned}$$

Now compare $x^2 + x + 6$ with $ax^2 + bx + c$

Here $a = 1$, $b = 1$ and $c = 6$

$\therefore \quad b^2 - 4ac = 1^2 - 4 \cdot 1 \cdot 6 = -23$, which is not a perfect square.

Therefore, $x^2 + x + 6$ cannot be factorised.

Hence $(x^2 + x)^2 + 4(x^2 + x) - 12 = (x^2 + x + 6)(x + 2)(x - 1)$

Example 10. Factorise the following:

$12(x^2 + 7)^2 - 8(x^2 + 7)(2x - 1) - 15(2x - 1)^2$

Solution. Let $x^2 + 7 = p$ and $2x - 1 = q$, then

$$\begin{aligned}&12(x^2 + 7)^2 - 8(x^2 + 7)(2x - 1) - 15(2x - 1)^2\\ &= 12p^2 - 8pq - 15q^2\\ &= 12p^2 - 18pq + 10pq - 15q^2\\ &= 6p(2p - 3q) + 5q(2p - 3q)\\ &= (2p - 3q)(6p + 5q)\\ &= (2(x^2 + 7) - 3(2x - 1))(6(x^2 + 7) + 5(2x - 1))\\ &= (2x^2 - 6x + 17)(6x^2 + 10x + 37)\end{aligned}$$

Note that $2x^2 - 6x + 7$ and $6x^2 + 10x + 37$ can not be factorised.

Example 11. Factorise the following:

(i) $125a^3 - 27b^3 + 75a^2b - 45ab^2$ (ii) $x^4 + 2x^3y - 2xy^3 - y^4$

Solution. (i)
$$\begin{aligned}&125a^3 - 27b^3 + 75a^2b - 45ab^2\\ &= (125a^3 + 75a^2b) - (45ab^2 + 27b^3)\\ &= 25a^2(5a + 3b) - 9b^2(5a + 3b)\\ &= (5a + 3b)(25a^2 - 9b^2)\\ &= (5a + 3b)((5a)^2 - (3b)^2)\\ &= (5a + 3b)(5a + 3b)(5a - 3b)\end{aligned}$$

(ii)
$$\begin{aligned}x^4 + 2x^3y - 2xy^3 - y^4 &= ((x^2)^2 - (y^2)^2) + 2x^3y - 2xy^3\\ &= (x^2 - y^2)(x^2 + y^2) + 2xy(x^2 - y^2)\end{aligned}$$

$$= (x^2 - y^2)(x^2 + y^2 + 2xy)$$
$$= (x - y)(x + y)(x + y)^2$$
$$= (x - y)(x + y)(x + y)(x + y)$$

Exercise 4.4

Factorise the following (1 to 18):

1. (i) $x^2 + 5x + 6$ (ii) $x^2 - 8x + 7$
2. (i) $x^2 + 6x - 7$ (ii) $y^2 + 7y - 18$
3. (i) $y^2 - 7y - 18$ (ii) $a^2 - 3a - 54$
4. (i) $2x^2 - 7x + 6$ (ii) $6x^2 + 13x - 5$
5. (i) $6x^2 + 11x - 10$ (ii) $6x^2 - 7x - 3$
6. (i) $2x^2 - x - 6$ (ii) $1 - 18y - 63y^2$
7. (i) $2y^2 + y - 45$ (ii) $5 - 4x - 12x^2$
8. (i) $x(12x + 7) - 10$ (ii) $(4 - x)^2 - 2x$
9. (i) $60x^2 - 70x - 30$ (ii) $x^2 - 6xy - 7y^2$
10. (i) $2x^2 + 13xy - 24y^2$ (ii) $6x^2 - 5xy - 6y^2$
11. (i) $5x^2 + 17xy - 12y^2$ (ii) $x^2y^2 - 8xy - 48$
12. (i) $2a^2b^2 - 7ab - 30$ (ii) $a(2a - b) - b^2$
13. (i) $(x - y)^2 - 6(x - y) + 5$ (ii) $(2x - y)^2 - 11(2x - y) + 28$
14. (i) $4(a - 1)^2 - 4(a - 1) - 3$ (ii) $1 - 2a - 2b - 3(a + b)^2$
15. (i) $3 - 5a - 5b - 12(a + b)^2$ (ii) $a^4 - 11a^2 + 10$
16. (i) $(x + 4)^2 - 5xy - 20y - 6y^2$ (ii) $(x^2 - 2x)^2 - 23(x^2 - 2x) + 120$
17. $4(2a - 3)^2 - 3(2a - 3)(a - 1) - 7(a - 1)^2$
18. $(2x^2 + 5x)(2x^2 + 5x - 19) + 84$

4.5 SUM OR DIFFERENCE OF TWO CUBES

We will use the following identities:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Illustrative Examples

Example 1. Resolve the following into factors:

(i) $8x^3 + 125y^3$ (ii) $27x^3 - \frac{343}{x^3}$ (iii) $27x^4 - 8x$

Solution. (i) $8x^3 + 125y^3 = (2x)^3 + (5y)^3$

$$= (2x + 5y)[(2x)^2 - 2x.5y + (5y)^2]$$
$$= (2x + 5y)(4x^2 - 10xy + 25y^2)$$

(ii) $27x^3 - \frac{343}{x^3} = (3x)^3 - \left(\frac{7}{x}\right)^3$

$$= \left(3x - \frac{7}{x}\right)\left[(3x)^2 + 3x.\frac{7}{x} + \left(\frac{7}{x}\right)^2\right]$$

$$= \left(3x - \frac{7}{x}\right)\left(9x^2 + \frac{49}{x^2} + 21\right)$$

(iii) $27x^4 - 8x = x(27x^3 - 8) = x[(3x)^3 - (2)^3]$

$$= x(3x - 2)\ [(3x)^2 + 3x.2 + (2)^2]$$

$$= x(3x - 2)\ (9x^2 + 6x + 4)$$

Example 2. Factorise the following:

(i) $x^4 - 125xy^3$ (ii) $8x^3 - (2x - y)^3$

Solution. (i) $x^4 - 125xy^3 = x(x^3 - 125y^3) = x(x^3 - (5y)^3)$

$$= x(x - 5y)\ (x^2 + x.5y + (5y)^2)$$

$$= x(x - 5y)\ (x^2 + 5xy + 25y^2)$$

(ii) $8x^3 - (2x - y)^3 = (2x)^3 - (2x - y)^3$

$$= (2x - \overline{2x - y})\ ((2x)^2 + 2x(2x - y) + (2x - y)^2)$$

$$= y(4x^2 + 4x^2 - 2xy + 4x^2 - 4xy + y^2)$$

$$= y(12x^2 - 6xy + y^2)$$

Example 3. Factorise the following:

(i) $64 - a^3b^3 + 8 - 2ab$ (ii) $64a^6 - b^6$.

Solution. (i) $64 - a^3b^3 + 8 - 2ab = [(4)^3 - (ab)^3] + 2(4 - ab)$

$$= (4 - ab)\ (16 + 4.ab + a^2b^2) + 2\ (4 - ab)$$

$$= (4 - ab)\ (16 + 4ab + a^2b^2 + 2)$$

$$= (4 - ab)\ (18 + 4ab + a^2b^2)$$

(ii) $64a^6 - b^6 = (8a^3)^2 - (b^3)^2$

$$= (8a^3 + b^3)\ (8a^3 - b^3)$$

$$= [(2a)^3 + b^3]\ [(2a)^3 - b^3]$$

$$= (2a + b)\ (4a^2 - 2a.b + b^2)\ (2a - b)\ (4a^2 + 2a.b + b^2)$$

$$= (2a + b)\ (2a - b)\ (4a^2 - 2ab + b^2)\ (4a^2 + 2ab + b^2)$$

Example 4. Factorise the following:

(i) $a^7 - ab^6$ (ii) $27(x + y)^3 - 8(x - y)^3$

Solution. (i) $a^7 - ab^6 = a(a^6 - b^6) = a((a^3)^2 - (b^3)^2)$

$$= a(a^3 + b^3)\ (a^3 - b^3)$$

$$= a(a + b)\ (a^2 - ab + b^2)\ (a - b)\ (a^2 + ab + b^2)$$

(ii) $27(x + y)^3 - 8(x - y)^3 = (3(x + y))^3 - (2(x - y))^3$

$$= (3(x + y) - 2(x - y))\ [(3(x + y))^2 + 3(x + y).2(x - y) + (2(x - y))^2]$$

$$= (x + 5y)\ [9(x^2 + 2xy + y^2) + 6(x^2 - y^2) + 4(x^2 - 2xy + y^2)]$$

$$= (x + 5y)\ (19x^2 + 10xy + 7y^2)$$

Example 5. Factorise the following:

(*i*) $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$ (*ii*) $x^3 + 3x^2y + 3xy^2 + 2y^3$

Solution. (*i*) $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2 = p^2(x^3 - 8y^3) - 4q^2(x^3 - 8y^3)$

$= (x^3 - 8y^3)(p^2 - 4q^2)$

$= [x^3 - (2y)^3][p^2 - (2q)^2]$

$= (x - 2y)(x^2 + 2xy + 4y^2)(p + 2q)(p - 2q)$

(*ii*) $x^3 + 3x^2y + 3xy^2 + 2y^3 = (x^3 + 3x^2y + 3xy^2 + y^3) + y^3$ (Note this

$= (x + y)^3 + y^3$

$= p^3 + y^3$, where $p = x + y$

$= (p + y)(p^2 - py + y^2)$

$= (\overline{x + y} + y)[(x + y)^2 - (x + y)y + y^2]$

$= (x + 2y)(x^2 + 2xy + y^2 - xy - y^2 + y^2)$

$= (x + 2y)(x^2 + xy + y^2)$

Example 6. Factorise the following:

(*i*) $x^3 + 3x^2 + 3x - 7$ (*ii*) $x^3 - 3x^2 + 3x + 7$

Solution. (*i*) $x^3 + 3x^2 + 3x - 7 = (x^3 + 3x^2 + 3x + 1) - 8$ (Note this

$= (x + 1)^3 - (2)^3$

$= \{(x + 1) - 2\}\{(x + 1)^2 + 2(x + 1) + 2^2\}$

$= (x - 1)(x^2 + 2x + 1 + 2x + 2 + 4)$

$= (x - 1)(x^2 + 4x + 7)$

(*ii*) $x^3 - 3x^2 + 3x + 7 = (x^3 - 3x^2 + 3x - 1) + 8$ (Note this

$= (x^3 - 3x^2 + 3x - 1) + 8$

$= (x - 1)^3 + (2)^3$

$= \{(x - 1) + 2\}\{(x - 1)^2 - 2(x - 1) + 2^2\}$

$= (x + 1)(x^2 - 2x + 1 - 2x + 2 + 4)$

$= (x + 1)(x^2 - 4x + 7)$

Example 7. Factorise: $x^3 - 5x + 12$

Solution. $x^3 - 5x + 12 = x^3 - 5x + 27 - 15$ (Note this

$= x^3 + 27 - 5x - 15$

$= (x^3 + 3^3) - 5(x + 3)$

$= (x + 3)(x^2 - 3x + 9) - 5(x + 3)$

[Using $a^3 + b^3 = (a + b)(a^2 - ab +$

$= (x + 3)(x^2 - 3x + 9 - 5)$

$= (x + 3)(x^2 - 3x + 4)$

Example 8. Factorise: $x^6 - 26x^3 - 27$

Solution. $x^6 - 26x^3 - 27 = y^2 - 26y - 27$ where $y = x^3$

$= y^2 - 27y + y - 27$

$= y(y - 27) + 1(y - 27)$

$= (y - 27)(y + 1)$

$= (x^3 - 27)(x^3 + 1)$

$= [x^3 - (3)^3][x^3 + 1^3]$

$= (x - 3)(x^2 + 3x + 9)(x + 1)(x^2 - x + 1)$

Exercise 4.5

Factorise the following (1 to 13):

1. (i) $8x^3 + y^3$ (ii) $64x^3 - 125y^3$
2. (i) $64x^3 + 1$ (ii) $7a^3 + 56b^3$
3. (i) $\frac{x^6}{343} + \frac{343}{x^6}$ (ii) $8x^3 - \frac{1}{27y^3}$
4. (i) $x^2 + x^5$ (ii) $32x^4 - 500x$
5. (i) $27x^3y^3 - 8$ (ii) $27(x + y)^3 + 8(2x - y)^3$
6. (i) $a^3 + b^3 + a + b$ (ii) $a^3 - b^3 - a + b$
7. (i) $x^3 + x + 2$ (ii) $a^3 - a - 120$
   (iii) $a^3 - 2a - 115$
8. (i) $x^3 + 6x^2 + 12x + 16$ (ii) $a^3 - 3a^2b + 3ab^2 - 2b^3$
9. (i) $2a^3 + 16b^3 - 5a - 10b$ (ii) $a^3 - \frac{1}{a^3} - 2a + \frac{2}{a}$
10. (i) $a^6 - b^6$ (ii) $x^6 - 1$
11. (i) $64x^6 - 729y^6$ (ii) $x^2 - \frac{8}{x}$
12. (i) $250\ (a - b)^3 + 2$ (ii) $32a^2x^3 - 8b^2x^3 - 4a^2y^3 + b^2y^3$
13. (i) $x^9 + y^9$ (ii) $x^6 - 7x^3 - 8$

Multiple Choice Questions MCQs

Choose the correct answer from the given four options (1 to 18):

1. Factorisation of $12a^2b + 15ab^2$ is
   (a) $3a(4ab + 5b^2)$ (b) $3b(4a^2 + 5ab)$ (c) $3ab(4a + 5b)$ (d) none of these
2. Factorisation of $6xy - 4y + 6 - 9x$ is
   (a) $(3y - 2)(2x - 3)$ (b) $(3x - 2)(2y - 3)$ (c) $(2y - 3)(2 - 3x)$ (d) none of these
3. Factorisation of $49p^3q - 36pq$ is
   (a) $p(7p + 6q)(7p - 6q)$ (b) $q(7p - 6)(7p + 6)$
   (c) $pq(7p + 6)(7p - 6)$ (d) none of these
4. Factorisation of $y(y - z) + 9(z - y)$ is
   (a) $(y - z)(y + 9)$ (b) $(y - z)(y - 9)$ (c) $(z - y)(y + 9)$ (d) none of these
5. Factorisation of $(lm + l) + m + 1$ is
   (a) $(lm + 1)(m + l)$ (b) $(lm + m)(l + 1)$ (c) $l(m + 1)$ (d) $(l + 1)(m + 1)$
6. Factorisation of $63x^2 - 112y^2$ is
   (a) $63(x - 2y)(x + 2y)$ (b) $7(3x + 2y)(3x - 2y)$
   (c) $7(3x + 4y)(3x - 4y)$ (d) none of these

7. Factorisation of $p^4 - 81$ is
(a) $(p^2 - 9)(p^2 + 9)$
(b) $(p - 3)(p + 3)(p^2 + 9)$
(c) $(p - 3)^2 (p + 3)^2$
(d) none of these

8. One of the factors of $(25x^2 - 1) + (1 + 5x)^2$ is
(a) $5 + x$
(b) $5 - x$
(c) $5x - 1$
(d) $10x$

9. Factorisation of $x^2 - 4x - 12$ is
(a) $(x + 6)(x - 2)$
(b) $(x - 6)(x + 2)$
(c) $(x - 6)(x - 2)$
(d) $(x + 6)(x +$

10. Factorisation of $3x^2 + 7x - 6$ is
(a) $(3x - 2)(x + 3)$
(b) $(3x + 2)(x - 3)$
(c) $(3x - 2)(x - 3)$
(d) $(3x + 2)(x$

11. The factorisation of $4x^2 + 8x + 3$ is
(a) $(x + 1)(x + 3)$
(b) $(2x + 1)(2x + 3)$
(c) $(2x + 2)(2x + 5)$
(d) $(2x - 1)(2x - 3)$

12. Factorisation of $16x^2 + 40x + 25$ is
(a) $(4x + 5)(4x + 5)$
(b) $(4x + 5)(4x - 5)$
(c) $(4x - 5)(4x - 5)$
(d) $(4x + 5)(4x$

13. Factorisation of $x^2 - 4xy + 4y^2$ is
(a) $(x + 2y)(x - 2y)$
(b) $(x + 2y)(x + 2y)$
(c) $(x - 2y)(x - 2y)$
(d) $(2x - y)(2x$

14. Which of the following is a factor of $(x + y)^3 - (x^3 + y^3)$?
(a) $x^2 + xy + 2xy$
(b) $x^2 + y^2 - xy$
(c) xy^2
(d) $3xy$

15. If $\frac{x}{y} + \frac{y}{x} = -1$ $(x \neq 0, y \neq 0)$, then the value of $x^3 - y^3$ is
(a) 1
(b) –1
(c) 0
(d) $\frac{1}{2}$

16. If $a + b + c = 0$, then the value of $a^3 + b^3 + c^3$ is
(a) 0
(b) abc
(c) $2abc$
(d) $3abc$

17. If $x^{1/3} + y^{1/3} + z^{1/3} = 0$, then
(a) $x^3 + y^3 + z^3 = 0$
(b) $x^3 + y^3 + z^3 = 27xyz$
(c) $(x + y + z)^3 = 27xyz$
(d) $x + y + z = 3xyz$

18. Consider the following two statements:
Statement I: The factorisation of $x^2 + 2x + 1$ is $(x - 1)^2$
Statement II: $(a - b)^2 = a^2 + 2ab + b^2$
Which of the following is valid?
(a) Both the Statements are true.
(b) Both the Statements are false.
(c) Statement I is true, and Statement II is false.
(d) Statement I is false, and Statement II is true.

ASSERTION-REASON TYPE QUESTION (SOLVED)

In these examples and following questions, read the given statements carefully choose the correct option.

(a) Assertion (A) is true, Reason (R) is false.
(b) Assertion (A) is false, Reason (R) is true.
(c) Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct rea Assertion (A).
(d) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct (or explanation) for Assertion (A).

1. **Assertion (A):** $8x^3 - 27y^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$

Reason (R): $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Sol. We know Reason (R) to be true as an identity.

We will use this to find out if Assertion (A) is true.

$$8x^3 - 27y^3 = (2x)^3 - (3y)^3$$
$$= (2x - 3y)[(2x)^2 + (2x)(3y) + (3y)^2]$$
$$= (2x - 3y)(4x^2 + 6xy + 9y^2)$$

$\therefore$ Assertion (A) is true.

Thus both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason for Assertion (A).

$\therefore$ Correct answer is (*c*).

ASSERTION-REASON TYPE QUESTIONS (UNSOLVED)

1. **Assertion (A):** For the trinomial $2x^2 + 8x - 9$ cannot be factorised.
   Reason (R): For trinomial $ax^2 + bx + c$ to be factorised, $b^2 - 4ac$ must be a perfect square.
2. **Assertion (A):** Factorisation of $4x^2 + 9y^2$ is $(2x - 3y)(2x + 3y)$
   Reason (R): $a^2 - b^2 = (a - b)(a + b)$
3. **Assertion (A):** Factorisation of $x^3 - 27$ is $(x - 3)(x^2 + 3x + 9)$
   Reason (R): $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Summary

- Algebraic expressions can be expressed as the product of irreducible factors.
- When an algebraic expression is written as the product of two or more algebraic expressions, then each of these expression is called a **factor** of the given expression.
- The process of finding two or more algebraic expressions whose product is the given expression is called **factorisation.**
- Factorisation is the reverse process of multiplication.
- Different methods of factorisation are:
  (*i*) Factorisation by taking out common factors.
  (*ii*) Factorisation by grouping of terms.
  (*iii*) Factorisation by using identities:
  - $a^2 - b^2 = (a + b)(a - b)$
  - $a^2 + 2ab + b^2 = (a + b)(a + b)$
  - $a^2 - 2ab + b^2 = (a - b)(a - b)$

  (*iv*) Factorisation of trinomials.
  (*v*) Factorisation by using the identities:
  $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Chapter Test

Factorise the following (1 to 12):

1. (i) $15(2x-3)^3 - 10(2x-3)$ (ii) $a(b-c)(b+c) - d(c-b)$
2. (i) $2a^2x - bx + 2a^2 - b$ (ii) $p^2 - (a+2b)p + 2ab$
3. (i) $(x^2-y^2)z + (y^2-z^2)x$ (ii) $5a^4 - 5a^3 + 30a^2 - 30a$
4. (i) $b(c-d)^2 + a(d-c) + 3c - 3d$ (ii) $x^3 - x^2 - xy + x + y - 1$
5. (i) $x(x+z) - y(y+z)$ imp. (ii) $a^{12}x^4 - a^4x^{12}$
6. (i) $9x^2 + 12x + 4 - 16y^2$ imp. (ii) $x^4 + 3x^2 + 4$
7. (i) $21x^2 - 59xy + 40y^2$ (ii) $4x^3y - 44x^2y + 112xy$
8. (i) $x^2y^2 - xy - 72$ (ii) $9x^3y + 41x^2y^2 + 20xy^3$
9. (i) $(3a-2b)^2 + 3(3a-2b) - 10$ (ii) $(x^2-3x)(x^2-3x+7) + 10$
10. (i) $(x^2-x)(4x^2-4x-5) - 6$ imp. (ii) $x^4 + 9x^2y^2 + 81y^4$
11. (i) $\frac{8}{27}x^3 - \frac{1}{8}y^3$ imp. (ii) $x^6 + 63x^3 - 64$
12. (i) $x^3 + x^2 - \frac{1}{x^2} + \frac{1}{x^3}$ (ii) $(x+1)^6 - (x-1)^6$
13. Factorise $(x+1)(x-3) + (x+1)(x+4)$
14. imp. Show that $97^3 + 14^3$ is divisible by 111
15. imp. If $a + b = 8$ and $ab = 15$, find the value of $a^4 + a^2b^2 + b^4$

Simultaneous Linear Equations

5

INTRODUCTION

In previous classes, we have read that an equation of the form $ax + b = 0$, where a, b are real numbers and $a \neq 0$, is called a linear equation in one variable. We have also learnt that every linear equation in one variable has a unique solution.

In this chapter, we will extend our knowledge of linear equations in one variable to linear equations in two variables and we shall also learn the various methods of solving a pair or a system of two linear equations in two variables.

5.1 SIMULTANEOUS LINEAR EQUATIONS

Linear equation in two variables

An equation of the form $ax + by + c = 0$, where a, b and c are real numbers and a and b are non-zero, is called a ***general linear equation*** *in two variables x and y.*

For example, $x + y - 3 = 0$ is a linear equation in two variables (unknowns) x and y.

Solution of a linear equation in two variables

$x = \alpha$ and $y = \beta$ is a solution of the linear equation $ax + by + c = 0$ if and only if $a\alpha + b\beta + c = 0$, where α, β are real numbers.

Every linear equation in two variables has an unlimited number of solutions.

For example, $x = 0$, $y = 3$; $x = 1$, $y = 2$; $x = 2$, $y = 1$; $x = 3$, $y = 0$ and $x = 7$, $y = -4$ etc. are *all solutions* of the equation $x + y - 3 = 0$.

System of simultaneous linear equations

Let us consider two linear equations in two variables,

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0.$$

These two equations are said to form a **system of simultaneous linear equations.**

For example,

$$x + y - 3 = 0$$
$$2x - 5y + 1 = 0$$

is a system of two simultaneous linear equations in the two variables x and y.

A ***solution*** *to a system of two simultaneous linear equations in two variables is an ordered pair of numbers which satisfy both the equations.*

For the above example, $x = 2$, $y = 1$ is a solution to the system of simultaneous linear equations. We can check this by substituting $x = 2$, $y = 1$ into each of these two equations.

If there is *only one* such solution, then the system of linear equations is said to be **consistent** and **independent.** In this book, we will be dealing only with such a system of simultaneous linear equations.

The various methods of solving a pair or a system of linear equations are:

(*i*) Substitution method.

(*ii*) Elimination method.

(*iii*) Cross-multiplication method.

We will discuss these methods one by one.

5.2 SUBSTITUTION METHOD

Procedure:

(*i*) Solve one of the given equations for one of the variables, whichever is convenient.

(*ii*) Substitute that value of the variable in the *other* equation.

(*iii*) Solve the resulting single variable equation. Substitute this value into either of the two *original* equations, and solve it to find the value of the second variable.

Remark

The solution may be checked by substituting in both the original equations.

Illustrative Examples

Example 1. Solve the following system of linear equations:

$$4x - 3y = 8$$
$$x - 2y = -3$$

Solution. The given equations are

$$4x - 3y = 8$$
$$x - 2y = -3$$

We can solve either equation for either variable. But to avoid fractions, we solve the second equation for x,

$$x = 2y - 3$$

Substituting this value of x in equation (*i*), we get

$$4(2y - 3) - 3y = 8$$
$$\Rightarrow \quad 8y - 12 - 3y = 8$$
$$\Rightarrow \quad 5y = 20 \quad \Rightarrow \quad y = 4$$

Substituting this value of y in (*ii*), we get

$$x - 2 \times 4 = -3 \quad \Rightarrow \quad x - 8 = -3 \quad \Rightarrow \quad x = 5$$

Hence, the solution is $x = 5$, $y = 4$

Example 2. Solve the following system of linear equations:

$$5x + 3y = 7$$
$$x - 7y = -10$$

Solution. The given equations are

$$5x + 3y = 7$$
$$x - 7y = -10$$

From (ii), we have

$x = 7y - 10$ ...(iii)

Substituting this value of x in equation (i), we get

$5(7y - 10) + 3y = 7$

$\Rightarrow \quad 35y - 50 + 3y = 7$

$\Rightarrow \quad 38y = 57$

$\Rightarrow \quad y = \frac{57}{38} \Rightarrow y = \frac{3}{2}$

Substituting $y = \frac{3}{2}$ in (iii), we get

$x = 7 \times \frac{3}{2} - 10 = \frac{21 - 20}{2} = \frac{1}{2}$

Hence, the solution is $x = \frac{1}{2}, y = \frac{3}{2}$

Example 3. Solve the following system of linear equations:

$8x + 5y = 9$

$3x + 2y = 4$

Solution. The given system of simultaneous linear equations is

$8x + 5y = 9$ ...(i)

$3x + 2y = 4$ ...(ii)

From equation (ii), we get

$2y = 4 - 3x \Rightarrow y = \frac{4 - 3x}{2}$

Substituting this value of y in equation (i), we get

$8x + 5 \cdot \frac{4 - 3x}{2} = 9$

$\Rightarrow \quad 16x + 20 - 15x = 18 \Rightarrow x + 20 = 18$

$\Rightarrow \quad x = -2$

Substituting this value of x in equation (ii), we get

$3 \times (-2) + 2y = 4$

$\Rightarrow \quad -6 + 2y = 4 \Rightarrow 2y = 10 \Rightarrow y = 5$

Hence, the solution is $x = -2, y = 5$

Example 4. Solve the following pair of linear equations:

$\frac{3x}{2} - \frac{5y}{3} = -2$

$\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$

Solution. The given equations are:

$\frac{3x}{2} - \frac{5y}{3} = -2$ ...(i) and $\frac{x}{3} + \frac{y}{2} = \frac{13}{6}$ ...(ii)

Multiplying both sides of the equations (i) and (ii) by 6, we get

$9x - 10y = -12$ ...(iii) and $2x + 3y = 13$ ...(iv)

From equation (*iv*), we get

$$y = \frac{13-2x}{3}$$

Substituting this value of y in equation (*iii*), we get

$$9x - 10\left(\frac{13-2x}{3}\right) = -12$$

$\Rightarrow \quad 27x - 130 + 20x = -36$ (Multiplying both sides

$\Rightarrow \quad 47x = 94 \Rightarrow x = 2$

Substituting this value of x in equation (*v*), we get

$$y = \frac{13-2\times 2}{3} = \frac{13-4}{3} = \frac{9}{3} = 3$$

Hence, the solution is $x = 2$ and $y = 3$

Example 5. Solve the following pairs of linear equations by substitution method:

(*i*) $0{\cdot}2x + 0{\cdot}3y = 1{\cdot}3$
$0{\cdot}4x + 0{\cdot}5y = 2{\cdot}3$

(*ii*) $\sqrt{2}x + \sqrt{3}y = 0$
$\sqrt{3}x - \sqrt{8}y = 0$

Solution. (*i*) The given equations are:

$0{\cdot}2x + 0{\cdot}3y = 1{\cdot}3 \quad \ldots(1)$ and $0{\cdot}4x + 0{\cdot}5y = 2{\cdot}3$

Multiplying both sides of equations (1) and (2) by 10, we get

$2x + 3y = 13 \quad \ldots(3)$ $\quad 4x + 5y = 23$

From equation (3), we get $x = \frac{13-3y}{2}$

Substituting this value of x in equation (4), we get

$$4\left(\frac{13-3y}{2}\right) + 5y = 23$$

$\Rightarrow \quad 26 - 6y + 5y = 23$

$\Rightarrow \quad -y = -3 \Rightarrow y = 3$

Substituting this value of y in equation (5), we get

$$x = \frac{13-3\times 3}{2} = \frac{13-9}{2} = \frac{4}{2} = 2$$

Hence, the solution is $x = 2$ and $y = 3$

(*ii*) The given equations are:

$\sqrt{2}x + \sqrt{3}y = 0 \quad \ldots(1)$ and $\sqrt{3}x - \sqrt{8}y = 0$

From equation (2), $x = \frac{\sqrt{8}}{\sqrt{3}}y$

Substituting this value of x in equation (1), we get

$$\sqrt{2}.\frac{\sqrt{8}}{\sqrt{3}}y + \sqrt{3}y = 0$$

$\Rightarrow \quad \sqrt{16}y + 3y = 0$ (Multiplying both sides by

$\Rightarrow \quad 4y + 3y = 0 \Rightarrow 7y = 0 \Rightarrow y = 0$

Substituting this value of y in equation (3), we get

$$x = \frac{\sqrt{8}}{\sqrt{3}} \times 0 \Rightarrow x = 0$$

Hence, the solution is $x = 0$, $y = 0$

Example 6. Solve the following pair of linear equations for x and y:

$$\frac{b}{a}x + \frac{a}{b}y = a^2 + b^2 \text{ and } x + y = 2ab$$

Solution. The given pair of linear equations is

$$\frac{b}{a}x + \frac{a}{b}y = a^2 + b^2 \quad \ldots(i) \qquad \text{and} \qquad x + y = 2ab \quad \ldots(ii)$$

From (ii), $y = 2ab - x$ ...(iii)

Substituting this value of y in (i), we get

$$\frac{b}{a}x + \frac{a}{b}(2ab - x) = a^2 + b^2$$

$$\Rightarrow \quad b^2x + a^2(2ab - x) = ab(a^2 + b^2) \quad \text{(Multiplying both sides by } ab\text{)}$$

$$\Rightarrow \quad b^2x + 2a^3b - a^2x = a^3b + ab^3$$

$$\Rightarrow \quad (b^2 - a^2)x = ab^3 - a^3b$$

$$\Rightarrow \quad (b^2 - a^2)x = ab(b^2 - a^2) \Rightarrow x = ab$$

From (iii), $y = 2ab - ab = ab$

Hence, the solution is $x = ab$ and $y = ab$

Example 7. Solve $2x + 3y = 11$ and $2x - 4y = -24$. Hence, find the value of 'm' for which $y = mx + 3$

Solution. The given equations are

$$2x + 3y = 11 \quad \ldots(i) \qquad \text{and} \qquad 2x - 4y = -24 \quad \ldots(ii)$$

From equation (ii), we get $x = \frac{4y - 24}{2}$

$\Rightarrow \quad x = 2y - 12$ ...(iii)

Substituting this value of x in equation (i), we get

$$2(2y - 12) + 3y = 11$$

$$\Rightarrow \quad 4y + 3y - 24 = 11 \Rightarrow 7y = 35 \Rightarrow y = 5$$

Substituting this value of y in equation (iii), we get

$$x = 2 \times 5 - 12 = 10 - 12 = -2$$

Hence, the solution is $x = -2$ and $y = 5$

Now, $y = mx + 3$ ...(iv)

Putting $x = -2$ and $y = 5$ in (iv), we get

$$5 = m(-2) + 3 \Rightarrow 2m = 3 - 5$$

$$\Rightarrow \quad 2m = -2 \Rightarrow m = -1$$

Hence, the value of m is -1

Exercise 5.1

Solve the following systems of simultaneous linear equations by the substitution method (1 to 4):

1. (i) $x + y = 14$, $x - y = 4$
   (ii) $s - t = 3$, $\frac{s}{3} + \frac{t}{2} = 6$
   (iii) $2x + 3y = 9$, $3x + 4y = 5$
   (iv) $3x - 5y = 4$, $9x - 2y = 7$
2. (i) $3x - 5y = -2$, $7x - 3y = -9$
   (ii) $5x + 4y - 4 = 0$, $x - 20 = 12y$

3. (i) $2x - \frac{3y}{4} = 3$

$5x - 2y - 7 = 0$

(ii) $2x + 3y = 23$

$5x - 20 = 8y$

4. (i) $mx - ny = m^2 + n^2$

$x + y = 2m$

(ii) $\frac{2x}{a} + \frac{y}{b} = 2$

$\frac{x}{a} - \frac{y}{b} = 4$

5. Solve $2x + y = 35$, $3x + 4y = 65$. Hence, find the value of $\frac{x}{y}$

6. Solve the simultaneous equations $3x - y = 5$, $4x - 3y = -1$. Hence, find p, if $y = px$...

5.3 ELIMINATION METHOD

Now let us consider another method of eliminating (removing) one variable. This is sometimes more convenient than substitution method.

Procedure:

(i) Multiply one or both equations (if necessary) by a suitable number(s) to tra... them so that addition or subtraction will eliminate one variable.

(ii) Solve the resulting single variable equation and substitute this value into ei... the two original equations, and solve it to find the value of the second vari...

Remark

In particular, if the coefficient of x in the first equation is numerically equal to the coef... of y in the second equation and the coefficient of y in the first equation is numerically... to the coefficient of x in the second equation, then add and subtract the given equation... gives the values of $x + y$ and $x - y$. Then find the values of x and y. (See example...

Illustrative Examples

Example 1. Solve the following system of simultaneous linear equations:

(i) $x + y = 5$

$2x - 3y = 4$

(ii) $\frac{x}{2} + \frac{2y}{3} = -1$

$x - \frac{y}{3} = 3$

Solution. (i) The given equations are:

$x + y = 5$ ...(1) $2x - 3y = 4$

Multiplying both sides of equation (1) by 3, we get

$3x + 3y = 15$ ...(3)

On adding equations (2) and (3), we get

$$5x = 19 \Rightarrow x = \frac{19}{5}$$

Substituting this value of x in (1), we get

$$\frac{19}{5} + y = 5 \Rightarrow y = 5 - \frac{19}{5} \Rightarrow y = \frac{6}{5}$$

Hence, the solution is $x = \frac{19}{5}$ and $y = \frac{6}{5}$

(ii) The given equations are:

$$\frac{x}{2}+\frac{2y}{3}=-1 \quad ...(1) \quad \text{and} \quad x-\frac{y}{3}=3 \quad ...(2)$$

Multiplying both sides of equation (2) by 2, we get

$$2x-\frac{2y}{3}=6 \quad ...(3)$$

On adding equations (1) and (3), we get

$$\frac{x}{2}+2x=5 \Rightarrow \frac{5}{2}x=5 \Rightarrow x=2$$

Substituting this value of x in (2), we get

$$2-\frac{y}{3}=3 \Rightarrow -\frac{y}{3}=1 \Rightarrow y=-3$$

Hence, the solution is $x = 2$ and $y = -3$

Example 2. Solve the following system of simultaneous linear equations:

(i) $\frac{x}{3}+\frac{y}{4}=4$

$\frac{5x}{6}-\frac{y}{8}=4$

(ii) $4x+\frac{6}{y}=15$

$6x-\frac{8}{y}=14, y \neq 0$

Solution. (i) The given equations are:

$$\frac{x}{3}+\frac{y}{4}=4 \quad ...(1) \quad \text{and} \quad \frac{5x}{6}-\frac{y}{8}=4 \quad ...(2)$$

Multiplying equation (1) by 12 and equation (2) by 24, we get

$$4x+3y=48 \quad ...(3) \quad \text{and} \quad 20x-3y=96 \quad ...(4)$$

On adding equations (3) and (4), we get

$$24x=144 \Rightarrow x=6$$

Substituting this value of x in (3), we get

$$4 \times 6+3y=48 \Rightarrow 3y=24 \Rightarrow y=8$$

Hence, the solution is $x = 6$ and $y = 8$

(ii) The given equation are:

$$4x+\frac{6}{y}=15 \quad ...(1) \quad \text{and} \quad 6x-\frac{8}{y}=14 \quad ...(2)$$

Multiplying equation (1) by 4 and equation (2) by 3, we get

$$16x+\frac{24}{y}=60 \quad ...(3) \quad \text{and} \quad 18x-\frac{24}{y}=42 \quad ...(4)$$

On adding equations (3) and (4), we get

$$34x=102 \Rightarrow x=3$$

Substituting this value of x in equation (1), we get

$$4 \times 3+\frac{6}{y}=15 \Rightarrow \frac{6}{y}=15-12$$

$$\Rightarrow \frac{6}{y}=3 \Rightarrow 3y=6 \Rightarrow y=2$$

Hence, the solution is $x = 3$ and $y = 2$

Example 3. Solve the following pairs of linear equations:

(i) $x+y=3.3$

$\frac{0.6}{3x-2y}=-1, 3x-2y \neq 0$

(ii) $\frac{x}{a}+\frac{y}{b}=a+b$

$\frac{x}{a^2}+\frac{y}{b^2}=2, a \neq 0, b \neq 0$

Solution. (*i*) The given equations are:

$x + y = 3.3$ ...(1) $\frac{0.6}{3x - 2y} = -1$

The equation (2) can be written as

$3x - 2y = -0.6$

Multiplying equation (1) by 2, we get

$2x + 2y = 6.6$

On adding equations (3) and (4), we get

$5x = 6 \Rightarrow x = \frac{6}{5} \Rightarrow x = 1.2$

Substituting this value of x in equation (1), we get

$1.2 + y = 3.3 \Rightarrow y = 2.1$

Hence, the solution is $x = 1.2$ and $y = 2.1$

(*ii*) The given equations are:

$\frac{x}{a} + \frac{y}{b} = a + b$ ...(1) and $\frac{x}{a^2} + \frac{y}{b^2} = 2$

On multiplying equation (2) by b, we get

$$\frac{bx}{a^2} + \frac{y}{b} = 2b$$

On subtracting equation (3) from (1), we get

$$\left(\frac{1}{a} - \frac{b}{a^2}\right) x = a - b$$

$$\Rightarrow \quad \frac{a-b}{a^2} x = a - b \Rightarrow x = a^2$$

Substituting this value of x in equation (2), we get

$$\frac{a^2}{a^2} + \frac{y}{b^2} = 2 \Rightarrow 1 + \frac{y^2}{b^2} = 2 \Rightarrow \frac{y^2}{b^2} = 1 \Rightarrow y = b^2$$

Hence, the solution is $x = a^2$ and $y = b^2$

Example 4. Solve the following pairs of linear equations:

$(a - b)x + (a + b)y = a^2 - 2ab - b^2$

$(a + b)(x + y) = a^2 + b^2$

Solution. The given equations are:

$(a - b)x + (a + b)y = a^2 - 2ab - b^2$

$(a + b)(x + y) = a^2 + b^2$

Equation (*ii*) can be written as

$(a + b)x + (a + b)y = a^2 + b^2$

Subtracting equation (*i*) from equation (*iii*), we get

$2bx = 2ab + 2b^2 \Rightarrow x = a + b$

Substituting $x = a + b$ in equation (*iii*), we get

$(a + b)(a + b) + (a + b)y = a^2 + b^2$

$\Rightarrow \quad a^2 + b^2 + 2ab + (a + b)y = a^2 + b^2$

$\Rightarrow \quad (a + b)y = -2ab \Rightarrow y = -\frac{2ab}{a+b}$

Hence, the solution is $x = a + b$ and $y = -\frac{2ab}{a+b}$

Example 5. Solve: $83x - 67y = 383$
$67x - 83y = 367$

Solution. Given
$$83x - 67y = 383 \quad \ldots(i)$$
$$67x - 83y = 367 \quad \ldots(ii)$$

Adding (*i*) and (*ii*), we get
$$150x - 150y = 750$$
$$\Rightarrow \quad x - y = 5 \quad \ldots(iii)$$

Subtracting (*ii*) from (*i*), we get
$$16x + 16y = 16$$
$$\Rightarrow \quad x + y = 1 \quad \ldots(iv)$$

On adding (*iii*) and (*iv*), we get
$$2x = 6 \Rightarrow x = 3$$

Substituting $x = 3$ in equation (*iv*), we get
$$3 + y = 1 \Rightarrow y = -2$$

Hence, the solution is $x = 3$, $y = -2$

Example 6. Solve : $\frac{3x-7}{2} - \frac{2y-8}{3} = -1$, $\frac{5-x}{3} - \frac{3-2y}{7} = 1$

Solution. Given
$$\frac{3x-7}{2} - \frac{2y-8}{3} = -1 \quad \ldots(i)$$
$$\frac{5-x}{3} - \frac{3-2y}{7} = 1 \quad \ldots(ii)$$

To clear fractions, multiplying equation (*i*) by 6 and equation (*ii*) by 21, we get
$$3(3x-7) - 2(2y-8) = -6 \Rightarrow 9x - 4y + 1 = 0 \quad \ldots(iii)$$
$$7(5-x) - 3(3-2y) = 21 \Rightarrow -7x + 6y + 5 = 0 \quad \ldots(iv)$$

On multiplying equation (*iii*) by 3 and equation (*iv*) by 2, we get
$$27x - 12y + 3 = 0 \quad \ldots(v)$$
$$-14x + 12y + 10 = 0 \quad \ldots(vi)$$

On adding (*v*) and (*vi*), we get
$$13x + 13 = 0 \Rightarrow 13x = -13 \Rightarrow x = -1$$

Substituting $x = -1$ in (*iv*), we get
$$-7(-1) + 6y + 5 = 0 \Rightarrow 6y = -12 \Rightarrow y = -2$$

Hence, the solution is $x = -1$, $y = -2$

Example 7. Can the following equations hold simultaneously?

$$\frac{x}{2} + \frac{5y}{3} = 12, \frac{5x}{4} - \frac{y}{6} = 4 \text{ and } 7x - 3y = 10$$

If so, find x and y.

Solution. The given equations are
$$\frac{x}{2} + \frac{5y}{3} = 12 \quad \ldots(i) \qquad \frac{5x}{4} - \frac{y}{6} = 4 \quad \ldots(ii)$$
$$7x - 3y = 10 \quad \ldots(iii)$$

Let us solve the first two equations simultaneously. To clear fractions, multiplying equation (*i*) by 6 and equation (*ii*) by 12, we get
$$3x + 10y = 72 \quad \ldots(iv)$$
$$15x - 2y = 48 \quad \ldots(v)$$

Example 5. Solve: $83x - 67y = 383$
$67x - 83y = 367$

Solution. Given
$$83x - 67y = 383 \quad \ldots(i)$$
$$67x - 83y = 367 \quad \ldots(ii)$$

Adding (i) and (ii), we get
$$150x - 150y = 750$$
$$\Rightarrow \quad x - y = 5 \quad \ldots(iii)$$

Subtracting (ii) from (i), we get
$$16x + 16y = 16$$
$$\Rightarrow \quad x + y = 1 \quad \ldots(iv)$$

On adding (iii) and (iv), we get
$$2x = 6 \Rightarrow x = 3$$

Substituting $x = 3$ in equation (iv), we get
$$3 + y = 1 \Rightarrow y = -2$$

Hence, the solution is $x = 3$, $y = -2$

Example 6. Solve : $\frac{3x-7}{2} - \frac{2y-8}{3} = -1$, $\frac{5-x}{3} - \frac{3-2y}{7} = 1$

Solution. Given
$$\frac{3x-7}{2} - \frac{2y-8}{3} = -1 \quad \ldots(i)$$
$$\frac{5-x}{3} - \frac{3-2y}{7} = 1 \quad \ldots(ii)$$

To clear fractions, multiplying equation (i) by 6 and equation (ii) by 21, we get
$$3(3x-7) - 2(2y-8) = -6 \Rightarrow 9x - 4y + 1 = 0 \quad \ldots(iii)$$
$$7(5-x) - 3(3-2y) = 21 \Rightarrow -7x + 6y + 5 = 0 \quad \ldots(iv)$$

On multiplying equation (iii) by 3 and equation (iv) by 2, we get
$$27x - 12y + 3 = 0 \quad \ldots(v)$$
$$-14x + 12y + 10 = 0 \quad \ldots(vi)$$

On adding (v) and (vi), we get
$$13x + 13 = 0 \Rightarrow 13x = -13 \Rightarrow x = -1$$

Substituting $x = -1$ in (iv), we get
$$-7(-1) + 6y + 5 = 0 \Rightarrow 6y = -12 \Rightarrow y = -2$$

Hence, the solution is $x = -1$, $y = -2$

Example 7. Can the following equations hold simultaneously?

$\frac{x}{2} + \frac{5y}{3} = 12$, $\frac{5x}{4} - \frac{y}{6} = 4$ and $7x - 3y = 10$

If so, find x and y.

Solution. The given equations are
$$\frac{x}{2} + \frac{5y}{3} = 12 \quad \ldots(i) \qquad \frac{5x}{4} - \frac{y}{6} = 4 \quad \ldots(ii)$$
$$7x - 3y = 10 \quad \ldots(iii)$$

Let us solve the first two equations simultaneously. To clear fractions, multiplying equation (i) by 6 and equation (ii) by 12, we get
$$3x + 10y = 72 \quad \ldots(iv)$$
$$15x - 2y = 48 \quad \ldots(v)$$

Exercise 5.2

Solve the following systems of simultaneous linear equations by the elimination method (

1. (i) $3x + 4y = 10$
   $2x - 2y = 2$

   (ii) $2x = 5y + 4$
   $3x - 2y + 16 = 0$

2. (i) $\frac{3}{4}x - \frac{2}{3}y = 1$
   $\frac{3}{8}x - \frac{1}{6}y = 1$

   (ii) $2x - 3y - 3 = 0$
   $\frac{2x}{3} + 4y + \frac{1}{2} = 0$

3. (i) $15x - 14y = 117$
   $14x - 15y = 115$

   (ii) $41x + 53y = 135$
   $53x + 41y = 147$

4. (i) $\frac{x}{6} = y - 6$
   $\frac{3x}{4} = 1 + y$

   (ii) $x - \frac{2}{3}y = \frac{8}{3}$
   $\frac{2x}{5} - y = \frac{7}{5}$

5. (i) $9 - (x - 4) = y + 7$
   $2(x + y) = 4 - 3y$

   (ii) $2x + \frac{x - y}{6} = 2$
   $x - \frac{2x + y}{3} = 1$

6. $x - 3y = 3x - 1 = 2x - y$

7. (i) $4x + \frac{x - y}{8} = 17$
   $2y + x - \frac{5y + 2}{3} = 2$

   (ii) $\frac{x + 1}{2} + \frac{y - 1}{3} = 8$
   $\frac{x - 1}{3} + \frac{y + 1}{2} = 9$

8. (i) $\frac{3}{x} + 4y = 7$
   $\frac{5}{x} + 6y = 13$

   (ii) $5x - 9 = \frac{1}{y}$
   $x + \frac{1}{y} = 3$

9. (i) $px + qy = p - q$
$qx - py = p + q$

(ii) $\frac{x}{a} - \frac{y}{b} = 0$
$ax + by = a^2 + b^2$

10. Solve $2x + y = 23$, $4x - y = 19$. Hence, find the values of $x - 3y$ and $5y - 2x$

11. The expression $ax + by$ has value 7 when $x = 2$, $y = 1$. When $x = -1$, $y = 1$, it has value 1, find a and b

12. Can the following equations hold simultaneously?

$3x - 7y = 7$
$11x + 5y = 87$
$5x + 4y = 43$

If so, find x and y

5.4 CROSS-MULTIPLICATION METHOD

Let the system of simultaneous linear equations be

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

To solve this system of linear equation by cross-multiplication method:

Write the coefficients of the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

as

$$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

The arrows between the two numbers indicate that they are to be multiplied. The down arrows (↘) show the term with a plus sign and up arrows (↗) show the term with a negative sign.

The solution is given by

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Illustrative Examples

Example 1. Solve the following pairs of linear equations by cross-multiplication method

(i) $2x + y = 5$
$3x + 2y = 8$

(ii) $x - 3y - 7 = 0$
$3x - 3y = 15$

Solution. (i) The given equations can be written as

$2x + y - 5 = 0$ and $3x + 2y - 8 = 0$

To solve the given pair of equations by cross-multiplication method, write coefficients of the pair of these linear equations as

On multiplying equation (v) by 5, we get

$$75x - 10y = 240$$

On adding (iv) and (vi), we get

$$78x = 312 \Rightarrow x = 4$$

Substituting $x = 4$ in (iv), we get

$$3 \times 4 + 10y = 72 \Rightarrow 10y = 60 \Rightarrow y = 6$$

Thus $x = 4$ and $y = 6$ is the solution of (i) and (ii)

Putting $x = 4$ and $y = 6$ in equation (iii), we get

$$7 \times 4 - 3 \times 6 = 10 \Rightarrow 28 - 18 = 10 \Rightarrow 10 = 10,$$ which is true.

Therefore, the three given equations can hold simultaneously *i.e.* they are consist[ent]
the solution is $x = 4$ and $y = 6$

Note

If the values of x and y obtained from two equations do not satisfy the third, th[e]
three equations cannot hold simultaneously and we conclude that the three equati[ons]
inconsistent.

Exercise 5.2

Solve the following systems of simultaneous linear equations by the elimination method (1 ...

1. (i) $3x + 4y = 10$, $2x - 2y = 2$ (ii) $2x = 5y + 4$, $3x - 2y + 16 = 0$

2. (i) $\frac{3}{4}x - \frac{2}{3}y = 1$, $\frac{3}{8}x - \frac{1}{6}y = 1$ (ii) $2x - 3y - 3 = 0$, $\frac{2x}{3} + 4y + \frac{1}{2} = 0$

3. (i) $15x - 14y = 117$, $14x - 15y = 115$ (ii) $41x + 53y = 135$, $53x + 41y = 147$

4. (i) $\frac{x}{6} = y - 6$, $\frac{3x}{4} = 1 + y$ (ii) $x - \frac{2}{3}y = \frac{8}{3}$, $\frac{2x}{5} - y = \frac{7}{5}$

5. (i) $9 - (x - 4) = y + 7$, $2(x + y) = 4 - 3y$ (ii) $2x + \frac{x - y}{6} = 2$, $x - \frac{2x + y}{3} = 1$

6. $x - 3y = 3x - 1 = 2x - y$

7. (i) $4x + \frac{x - y}{8} = 17$, $2y + x - \frac{5y + 2}{3} = 2$ (ii) $\frac{x + 1}{2} + \frac{y - 1}{3} = 8$, $\frac{x - 1}{3} + \frac{y + 1}{2} = 9$

8. (i) $\frac{3}{x} + 4y = 7$, $\frac{5}{x} + 6y = 13$ (ii) $5x - 9 = \frac{1}{y}$, $x + \frac{1}{y} = 3$

9. (i) $px + qy = p - q$

$qx - py = p + q$

(ii) $\frac{x}{a} - \frac{y}{b} = 0$

$ax + by = a^2 + b^2$

10. Solve $2x + y = 23$, $4x - y = 19$. Hence, find the values of $x - 3y$ and $5y - 2x$

11. The expression $ax + by$ has value 7 when $x = 2$, $y = 1$. When $x = -1$, $y = 1$, it has value 1, find a and b

12. Can the following equations hold simultaneously?

$3x - 7y = 7$

$11x + 5y = 87$

$5x + 4y = 43$

If so, find x and y

5.4 CROSS-MULTIPLICATION METHOD

Let the system of simultaneous linear equations be

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

To solve this system of linear equation by cross-multiplication method:

Write the coefficients of the pair of linear equations

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

as

$$\begin{array}{cccc} b_1 & c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 & b_2 \end{array}$$

The arrows between the two numbers indicate that they are to be multiplied. The down arrows (↘) show the term with a plus sign and up arrows (↗) show the term with a negative sign.

The solution is given by

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\Rightarrow \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Illustrative Examples

Example 1. Solve the following pairs of linear equations by cross-multiplication method:

(i) $2x + y = 5$

$3x + 2y = 8$

(ii) $x - 3y - 7 = 0$

$3x - 3y = 15$

Solution. (i) The given equations can be written as

$2x + y - 5 = 0$ and $3x + 2y - 8 = 0$

To solve the given pair of equations by cross-multiplication method, write the coefficients of the pair of these linear equations as

$$\begin{array}{cccc} 1 & -5 & 2 & 1 \\ 2 & -8 & 3 & 2 \end{array}$$

$$\therefore \quad \frac{x}{1\times(-8)-2\times(-5)}=\frac{y}{(-5)\times 3-(-8)\times 2}=\frac{1}{2\times 2-3\times 1}$$

$$\Rightarrow \quad \frac{x}{-8+10}=\frac{y}{-15+16}=\frac{1}{4-3}$$

$$\Rightarrow \quad \frac{x}{2}=\frac{y}{1}=\frac{1}{1} \Rightarrow x = 2 \text{ and } y = 1$$

Hence, the solution is $x = 2$ and $y = 1$

(ii) The given equations can be written as

$x - 3y - 7 = 0$ and $3x - 3y - 15 = 0$

To solve the given pair of linear equations by cross-multiplication method the coefficients of these equations as

$$\begin{array}{cccc} -3 & -7 & 1 & -3 \\ -3 & -15 & 3 & -3 \end{array}$$

$$\therefore \quad \frac{x}{(-3)(-15)-(-3)(-7)}=\frac{y}{(-7)\times 3-(-15)\times 1}=\frac{1}{1\times(-3)-3\times(-3)}$$

$$\Rightarrow \quad \frac{x}{45-21}=\frac{y}{-21+15}=\frac{1}{-3+9}$$

$$\Rightarrow \quad \frac{x}{24}=\frac{y}{-6}=\frac{1}{6}$$

$$\Rightarrow \quad x = \frac{24}{6} = 4 \text{ and } y = \frac{-6}{6} = -1$$

Hence, the solution is $x = 4$ and $y = -1$

Example 2. Solve the following system of linear equations by cross-multiplication method:

$$2(ax - by) + (a + 4b) = 0$$
$$2(bx + ay) + (b - 4a) = 0, \; a^2 + b^2 \neq 0$$

Solution. The given linear equations are

$$2ax - 2by + (a + 4b) = 0$$
$$2bx + 2ay + (b - 4a) = 0$$

Write the coefficients of the pair of these linear equations as

$$\begin{array}{cccc} -2b & a+4b & 2a & -2b \\ 2a & b-4a & 2b & 2a \end{array}$$

By cross-multiplication method, the solution is given by

$$\frac{x}{-2b(b-4a)-2a(a+4b)}=\frac{y}{(a+4b)2b-(b-4a)2a}=\frac{1}{2a.2a-2b.(-2b)}$$

$$\Rightarrow \quad \frac{x}{-2b^2-2a^2}=\frac{y}{8b^2+8a^2}=\frac{1}{4a^2+4b^2}$$

$$\Rightarrow \quad \frac{x}{-2(a^2+b^2)}=\frac{y}{8(a^2+b^2)}=\frac{1}{4(a^2+b^2)}$$

$$\Rightarrow \quad \frac{x}{-2}=\frac{y}{8}=\frac{1}{4}$$

$$\Rightarrow \quad x = -\frac{2}{4} \text{ and } y = \frac{8}{4} \Rightarrow x = -\frac{1}{2} \text{ and } y = 2$$

Hence, the solution is $x = -\frac{1}{2}$, $y = 2$

Exercise 5.3

Solve the following systems of simultaneous linear equations by cross-multiplication method:

1. (i) $3x + 2y = 4$
$8x + 5y = 9$

(ii) $3x - 7y + 10 = 0$
$y - 2x = 3$

2. (i) $2x - 5y = -1$
$3x + y = 7$

(ii) $x + 3y + 4 = 0$
$3x - y = -2$

3. (i) $x - y = a + b$
$ax + by = a^2 - b^2$

(ii) $2bx + ay = 2ab$
$bx - ay = 4ab$

5.5 EQUATIONS REDUCIBLE TO PAIR OF LINEAR EQUATIONS

In this section, we will find solutions of such pairs of equations in two variables which are not linear but can be reduced to linear equations in two variables by making some suitable substitutions.

Illustrative Examples

Example 1. Solve the following pairs of equations by reducing them to pairs of linear equations:

(i) $\frac{2}{x} + \frac{3}{y} = 13$
$\frac{5}{x} - \frac{4}{y} = -2$

(ii) $\frac{1}{2x} + \frac{1}{3y} = 2$
$\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6}$

Solution. (i) Substituting $\frac{1}{x} = p$ and $\frac{1}{y} = q$ in the given equations, we get

$2p + 3q = 13$ ...(1) and $5p - 4q = -2$ ...(2)

Multiplying equation (1) by 4 and equation (2) by 3, we get

$8p + 12q = 52$ ...(3) and $15p - 12q = -6$ ...(4)

On adding equations (3) and (4), we get

$23p = 46 \Rightarrow p = 2$

Substituting this value of p in equation (1), we get

$2 \times 2 + 3q = 13 \Rightarrow 3q = 9 \Rightarrow q = 3$

$\therefore \quad \frac{1}{x} = 2$ and $\frac{1}{y} = 3 \Rightarrow x = \frac{1}{2}$ and $y = \frac{1}{3}$

Hence, the solution of the given pair of equations is

$$x = \frac{1}{2} \text{ and } y = \frac{1}{3}$$

(ii) Substituting $\frac{1}{x} = u$ and $\frac{1}{y} = v$ in the given equations, we get

$\frac{1}{2}u + \frac{1}{3}v = 2$ *i.e.* $3u + 2v = 12$ ...(1)

and $\frac{1}{3}u + \frac{1}{2}v = \frac{13}{6}$ *i.e.* $2u + 3v = 13$ ...(2)

On adding equations (3) and (4), we get

$$5u + 5v = 25 \Rightarrow u + v = 5$$

On subtracting equation (2) from equation (1), we get

$$u - v = -1$$

On adding equations (3) and (4), we get

$$2u = 4 \Rightarrow u = 2.$$

On substituting this value of u in (3), we get

$$2 + v = 5 \Rightarrow v = 3$$

$$\therefore \quad \frac{1}{x} = 2 \text{ and } \frac{1}{y} = 3 \Rightarrow x = \frac{1}{2} \text{ and } y = \frac{1}{3}$$

Hence, the solution of the given pair of equations is $x = \frac{1}{2}$ and $y = \frac{1}{3}$

Example 2. Solve: $\frac{2}{x}+\frac{5}{y} = 1$, $\frac{60}{x}-\frac{20}{y} = 13$. Hence, find the value of k if $y = kx - 2$

Solution. Given $\frac{2}{x}+\frac{5}{y} = 1$

$$\frac{60}{x}-\frac{20}{y} = 13$$

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$, then the given equations become

$$2a + 5b = 1$$

$$60a - 20b = 13$$

Multiplying (*iii*) by 4, we get

$$8a + 20b = 4$$

Adding (*iv*) and (*v*), we get

$$68a = 17 \Rightarrow a = \frac{1}{4}$$

Substituting this value of a in (*iii*), we get

$$2 \times \frac{1}{4} + 5b = 1 \Rightarrow \frac{1}{2} + 5b = 1 \Rightarrow 5b = 1 - \frac{1}{2}$$

$$\Rightarrow \quad 5b = \frac{1}{2} \Rightarrow b = \frac{1}{10}$$

$$\therefore \quad \frac{1}{x} = \frac{1}{4} \text{ and } \frac{1}{y} = \frac{1}{10} \Rightarrow x = 4 \text{ and } y = 10$$

Hence, the solution is $x = 4$, $y = 10$

To find k

Putting $x = 4$ and $y = 10$ in $y = kx - 2$, we get

$$10 = 4k - 2 \Rightarrow 4k = 12 \Rightarrow k = 3$$

Example 3. Solve the following pairs of equations:

(*i*) $\frac{2}{\sqrt{x}}+\frac{3}{\sqrt{y}} = 2$

$\frac{4}{\sqrt{x}}-\frac{9}{\sqrt{y}} = -1$

(*ii*) $\frac{5}{x-1}+\frac{1}{y-2} = 2$

$\frac{6}{x-1}-\frac{3}{y-2} = 1$

Solution. (*i*) Substituting $\frac{1}{\sqrt{x}} = a$ and $\frac{1}{\sqrt{y}} = b$, in the given pair of equations, we get

$$2a + 3b = 2 \quad \text{...(1)} \quad \text{and} \quad 4a - 9b = -1$$

Multiplying (1) by 3, we get

$$6a + 9b = 6 \quad ...(3)$$

On adding (2) and (3), we get

$$10a = 5 \Rightarrow a = \frac{1}{2}$$

Putting $a = \frac{1}{2}$ in (1), we get

$$2 \cdot \frac{1}{2} + 3b = 2 \Rightarrow 3b = 1 \Rightarrow b = \frac{1}{3}$$

$$\therefore \quad \frac{1}{\sqrt{x}} = \frac{1}{2} \text{ and } \frac{1}{\sqrt{y}} = \frac{1}{3} \Rightarrow x = 4 \text{ and } y = 9$$

Hence, the solution of the given pair of equations is $x = 4$, $y = 9$

(*ii*) Substituting $\frac{1}{x-1} = p$ and $\frac{1}{y-2} = q$ in the given equations, we get

$$5p + q = 2 \quad ...(1) \qquad \text{and} \qquad 6p - 3q = 1 \quad ...(2)$$

From (1), $q = 2 - 5p$ ...(3)

Substituting this value of q in equation (2), we get

$$6p - 3(2 - 5p) = 1$$

$$\Rightarrow \quad 6p - 6 + 15p = 1 \Rightarrow 21p = 7 \Rightarrow p = \frac{1}{3}$$

From (3), $q = 2 - 5 \times \frac{1}{3} = 2 - \frac{5}{3} = \frac{1}{3}$

$$\therefore \quad \frac{1}{x-1} = \frac{1}{3} \text{ and } \frac{1}{y-2} = \frac{1}{3}$$

$$\Rightarrow \quad x - 1 = 3 \text{ and } y - 2 = 3$$

$$\Rightarrow \quad x = 4 \text{ and } y = 5$$

Hence, the solution of the given pair of equations is $x = 4$ and $y = 5$

Example 4. Solve: $4x + 9y = 30xy$, $5y - 3x = xy$

Solution. The given system of simultaneous equations is

$$4x + 9y = 30xy \quad ...(i)$$

$$5x - 3y = xy \quad ...(ii)$$

First, we note that $x = 0$, $y = 0$ is a solution of the given system of equations.

Now, when $x \neq 0$, $y \neq 0$, then dividing both sides of each equation by xy, we get

$$\frac{4}{y} + \frac{9}{x} = 30 \quad ...(iii)$$

$$\frac{5}{x} - \frac{3}{y} = 1 \quad ...(iv)$$

Let $\frac{1}{x} = a$ and $\frac{1}{y} = b$, then equations (*iii*) and (*iv*) become

$$9a + 4b = 30 \quad ...(v)$$

$$5a - 3b = 1 \quad ...(vi)$$

On multiplying (*v*) by 3 and (*vi*) by 4, we get

$$27a + 12b = 90 \quad ...(vii)$$

$$20a - 12b = 4 \quad ...(viii)$$

Adding (*vii*) and (*viii*), we get

$$47a = 94 \Rightarrow a = 2$$

Substituting $a = 2$ in (v), we get

$$9 \times 2 + 4b = 30 \Rightarrow 4b = 12 \Rightarrow b = 3$$

$$\therefore \quad \frac{1}{x} = 2 \text{ and } \frac{1}{y} = 3 \Rightarrow x = \frac{1}{2} \text{ and } y = \frac{1}{3}$$

Hence, the solutions of the given system of equations are

$$x = 0, y = 0; x = \frac{1}{2}, y = \frac{1}{3}$$

Example 5. Solve: $\frac{20}{x+y} + \frac{3}{x-y} = 7, \frac{8}{x-y} - \frac{15}{x+y} = 5$

Solution. Let $\frac{1}{x+y} = a$ and $\frac{1}{x-y} = b$, then the given equations become

$$20a + 3b = 7 \quad ...(i) \quad \text{and} \quad 8b - 15a = 5$$

Multiplying (i) by 3 and (ii) by 4, we get

$$60a + 9b = 21 \quad ...(iii) \quad \text{and} \quad -60a + 32b = 20$$

Adding (iii) and (iv), we get

$$41b = 41 \Rightarrow b = 1$$

Substituting $b = 1$ in (i), we get

$$20a + 3 \times 1 = 7 \Rightarrow 20a = 4 \Rightarrow a = \frac{1}{5}$$

$$\therefore \quad \frac{1}{x+y} = \frac{1}{5} \text{ and } \frac{1}{x-y} = 1$$

$$\Rightarrow \quad x + y = 5, x - y = 1$$

Adding these equations, we get

$$2x = 6 \Rightarrow x = 3$$

$$\therefore \quad 3 + y = 5 \Rightarrow y = 2$$

Hence, the solution of the given linear equations is $x = 3, y = 2$

Example 6. Solve the following pairs of equations:

$$\frac{1}{3x+y} + \frac{1}{3x-y} = \frac{3}{4}, \quad \frac{1}{2(3x+y)} - \frac{1}{2(3x-y)} = -\frac{1}{8}$$

Solution. Substituting $\frac{1}{3x+y} = p$ and $\frac{1}{3x-y} = q$ in the given equations, we get

$$p + q = \frac{3}{4} \quad ...(i) \quad \frac{p}{2} - \frac{q}{2} = -\frac{1}{8} \text{ i.e. } p - q = -\frac{1}{4}$$

On adding equations (i) and (ii), we get

$$2p = \frac{3}{4} - \frac{1}{4} \Rightarrow 2p = \frac{1}{2} \Rightarrow p = \frac{1}{4}$$

Substituting this value of p in equation (i), we get

$$\frac{1}{4} + q = \frac{3}{4} \Rightarrow q = \frac{3}{4} - \frac{1}{4} \Rightarrow q = \frac{1}{2}$$

$$\therefore \quad \frac{1}{3x+y} = \frac{1}{4} \text{ and } \frac{1}{3x-y} = \frac{1}{2}$$

$$\Rightarrow \quad 3x + y = 4 \quad ...(iii) \quad \text{and} \quad 3x - y = 2$$

On adding equations (iii) and (iv), we get

$$6x = 6 \Rightarrow x = 1$$

Substituting this value of x in equation (*iii*), we get

$$3 \times 1 + y = 4 \Rightarrow y = 1$$

Hence, the solution is $x = 1$ and $y = 1$

Exercise 5.4

Solve the following pairs of equations (1 to 5):

1. (*i*) $\frac{2}{x} + \frac{2}{3y} = \frac{1}{3}$, $\frac{2}{x} - \frac{1}{y} = 2$

   (*ii*) $\frac{3}{2x} + \frac{2}{3y} = 5$, $\frac{5}{x} - \frac{3}{y} = 1$

2. (*i*) $\frac{7x - 2y}{xy} = 5$, $\frac{8x + 7y}{xy} = 15$

   (*ii*) $99x + 101y = 499xy$, $101x + 99y = 501xy$

3. (*i*) $3x + 14y = 5xy$, $21y - x = 2xy$

   (*ii*) $3x + 5y = 4xy$, $2y - x = xy$

4. (*i*) $\frac{20}{x+1} + \frac{4}{y-1} = 5$, $\frac{10}{x+1} - \frac{4}{y-1} = 1$

   (*ii*) $\frac{3}{x+y} + \frac{2}{x-y} = 3$, $\frac{2}{x+y} + \frac{3}{x-y} = \frac{11}{3}$

5. (*i*) $\frac{1}{2(2x+3y)} + \frac{12}{7(3x-2y)} = \frac{1}{2}$, $\frac{7}{2x+3y} + \frac{4}{3x-2y} = 2$

   (*ii*) $\frac{1}{2(x+2y)} + \frac{5}{3(3x-2y)} = -\frac{3}{2}$, $\frac{5}{4(x+2y)} - \frac{3}{5(3x-2y)} = \frac{61}{60}$

Multiple Choice Questions MCQs

Choose the correct answer from the given four options (1 to 6):

1. If $x = 3$, $y = k$ is a solution of the equation $3x - 4y + 7 = 0$, then the value of k is
   (*a*) 16 (*b*) -16 (*c*) 4 (*d*) -4

2. The solution of the pair of linear equations $2x - y = 5$ and $5x - y = 11$ is
   (*a*) $x = -1, y = 2$ (*b*) $x = 2, y = -1$ (*c*) $x = 0, y = -5$ (*d*) $x = \frac{5}{2}, y = 0$

3. If $x = a$, $y = b$ is the solution of the equations $x - y = 2$ and $x + y = 4$, then the values of a and b are, respectively
   (*a*) 3 and 5 (*b*) 5 and 3 (*c*) 3 and 1 (*d*) -1 and -3

4. The solution of the system of equations $\frac{4}{x} + 5y = 7$ and $\frac{3}{x} + 4y = 5$ is
   (*a*) $x = \frac{1}{3}, y = -1$ (*b*) $x = -\frac{1}{3}, y = 1$ (*c*) $x = 3, y = -1$ (*d*) $x = -3, y = 1$

5. A pair of linear equations which has a unique solution $x = 2$, $y = -3$ is
   (*a*) $x + y = -1$, $2x - 3y = -5$
   (*b*) $2x + 5y = -11$, $4x + 10y = -22$
   (*c*) $2x - y = 1$, $3x + 2y = 0$
   (*d*) $x - 4y - 14 = 0$, $5x - y - 13 = 0$

6. Consider the following two statements:

Statement I: A solution to linear equation $5x - 2y = 1$ is $x = 3, y = 7$

Statement II: The linear equation $5x - 2y = 1$ has a unique solution.

Which of the following is valid?

(*a*) Both the Statements are true.

(*b*) Both the Statements are false.

(*c*) Statement I is true, and Statement II is false.

(*d*) Statement I is false, and Statement II is true.

ASSERTION-REASON TYPE QUESTIONS (SOLVED)

In these examples and following questions, read the given statements carefully a[nd] choose the correct option.

(*a*) Assertion (A) is true, Reason (R) is false.

(*b*) Assertion (A) is false, Reason (R) is true.

(*c*) Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason f[or] Assertion (A).

(*d*) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct reaso[n] (or explanation) for Assertion (A).

1. **Assertion (A):** Solving two linear equations $4x - 5y = 1$, $x + y = 7$ simultaneously, we get a unique solution $x = 4$, $y = 3$

Reason (R): One of the methods of solving a pair of linear equations is substitution.

Sol. We have learnt that Reason (R) is true. Let us use the method of substitution to solv[e] Assertion (A).

From $x + y = 7$, we get $y = 7 - x$

Let us substitute this value of y in first equation

$4x - 5y = 1 \Rightarrow 4x - 5(7 - x) = 1$

$\Rightarrow \quad 4x - 35 + 5x = 1 \Rightarrow 9x = 36 \Rightarrow x = 4$

$\therefore y = 7 - x = 7 - 4 = 3$

Hence, $x = 4$, $y = 3$ is the unique solution.

$\therefore$ Assertion (A) is true.

Thus both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason of Assertion (A).

$\therefore$ Correct answer is (*c*).

ASSERTION-REASON TYPE QUESTIONS (UNSOLVED)

1. **Assertion (A):** A solution of $x - y = 1, 2x + y = \frac{7}{2}$ is $x = \frac{3}{2}, y = \frac{1}{2}$

Reason (R): One of the methods of solving a pair of linear equations is elimination method.

2. **Assertion (A):** Solving $\sqrt{2}x - \sqrt{3}y = 0$, $\sqrt{3}x + \sqrt{2}y = 5$ yields $x = \sqrt{3}$, $y = \sqrt{2}$

Reason (R): We can use cross-multiplication method to solve a pair of linear equations

Summary

- An equation of the form $ax + by + c = 0$, where a, b and c are real numbers and a, b are non-zero is called a general linear equation in the two variables x and y.
- $x = \alpha$ and $y = \beta$ is a solution of the linear equation $ax + by + c = 0$ if and only if $a\alpha + b\beta + c = 0$
- Every linear equation in two variables has infinitely many solutions.

Algebraic methods

The various methods of solving a pair of linear equations in two variables are:

(i) *Substitution method*

(ii) *Elimination method*

(iii) *Cross-multiplication method.*

Chapter Test

Solve the following pairs of equations (1 to 5):

1. (i) $2x - \frac{3}{4}y = 3$, $5x - 2y = 7$ (ii) $2(x - 4) = 9y + 2$, $x - 6y = 2$

2. (i) $97x + 53y = 177$, $53x + 97y = 573$ (ii) $67x - 58y = 192$, $58x - 67y = 183$

3. (i) $x + y = 7xy$, $2x - 3y + xy = 0$ (ii) $\frac{30}{x-y} + \frac{44}{x+y} = 10$, $\frac{40}{x-y} + \frac{55}{x+y} = 13$

4. (i) $ax + by = a - b$, $bx - ay = a + b$ (ii) $3x + 2y = 2xy$, $\frac{1}{x} + \frac{2}{y} = 1\frac{1}{6}$

5. $\frac{2}{3(2x-y)} + \frac{1}{2(x+2y)} = \frac{5}{12}$, $\frac{1}{2x-y} - \frac{2}{x+2y} = \frac{1}{6}$

6. Solve $2x - \frac{3}{y} = 9$, $3x + \frac{7}{y} = 2$. Hence, find the value of k if $x = ky + 5$.

7. Solve $\frac{1}{x+y} - \frac{1}{2x} = \frac{1}{30}, \frac{5}{x+y} + \frac{1}{x} = \frac{4}{3}$. Hence, find the value of $2x^2 - y^2$

8. Can x, y be found to satisfy the following equations simultaneously?

$$\frac{2}{y} + \frac{5}{x} = 19,\quad \frac{5}{y} - \frac{3}{x} = 1,\quad 3x + 8y = 5$$

If so, find them.

Problems on Simultaneous Linear Equations

6

INTRODUCTION

In our day-to-day life, we come across many situations where we have to solve a pair of linear equations in two variables. In the previous chapter, we learnt about linear equations in two variables and also learnt the various methods of solving a pair of linear equations in two variables. In this chapter, we will learn the applications of a pair of linear equations in two variables in solving word problems.

6.1 WORD PROBLEMS

Problems stated in words are called ***word*** *or* ***applied problems.***

Success with word (or applied) problems comes with practice and knowing some simple techniques of translating. Solving word problems involves two steps. First, translating the words of the problem into algebraic equations. Second, solving the resulting equations.

6.1.1 Solving word problems

Due to the wide variety of word (or applied) problems, there is no single technique that works in all cases. However, the following general suggestions will prove helpful:

(*i*) Read and reread the statement of the problem carefully, and determine what quantities must be found.

(*ii*) Represent the unknown quantities by letters.

(*iii*) Determine which expressions are equal and write equations.

(*iv*) Solve the resulting equations.

Remark

Check the answer (or answers) obtained, by determining whether or not they fulfil the condition(s) of the original problem.

6.2 PROBLEMS ON SIMULTANEOUS LINEAR EQUATIONS

Illustrative Examples

Example 1. If four times a number is subtracted from 3 times another number, the result is 7. Find the two numbers if their sum is 28.

Solution. Let the numbers be x and y.

According to the question,

$$3y - 4x = 7 \qquad \ldots(i)$$

and $x + y = 28$ ...(ii)

From (ii), $y = 28 - x$

Substitute the value of y in (i)

$3(28 - x) - 4x = 7$

$\Rightarrow$ $84 - 7x = 7$

$\Rightarrow$ $-7x = 7 - 84 = -77 \Rightarrow x = 11$

$\therefore$ $y = 28 - 11 = 17$

Hence, the numbers are 11 and 17

Example 2. Once a mule and a donkey were talking. The mule said, "I am carrying more sacks than you. In fact, if you give me one of your sacks, then I would have twice as many as you. If I give you a sack, our loads would be equal." How many sacks was each animal carrying?

Solution. Let mule and donkey carry x sacks and y sacks respectively. Then the two statements mean:

If you give me one of your sacks, then I would have twice as many as you (*i.e.* if the mule had one more and the donkey one less, the mule would have twice as many as donkey).

$\therefore$ $x + 1 = 2(y - 1) \Rightarrow x - 2y + 3 = 0$ ...(i)

If I give you a sack, our loads would be equal (*i.e.* if the mule had one less and the donkey one more, they would have the same number of sacks).

$\therefore$ $x - 1 = y + 1 \Rightarrow x - y - 2 = 0$ ...(ii)

Subtracting (i) from (ii), we get

$y - 5 = 0 \Rightarrow y = 5$

Substituting $y = 5$ in (ii), we get

$x - 5 - 2 = 0 \Rightarrow x = 7$

Hence, the mule was carrying 7 sacks and the donkey 5 sacks.

Example 3. A man buys postage stamps of denominations 25 paise and 50 paise for ₹10. He buys 28 stamps in all. Find the number of 25 paise stamps bought by him.

Solution. Let the number of 25 paise stamps be x and the number of 50 paise stamps be y.

According to the problem,

$x + y = 28$ ...(i)

and $25x + 50y = 1000$ ($\because$ ₹10 = 1000 paise)

i.e. $x + 2y = 40$ ...(ii)

Subtracting (i) from (ii), we get $y = 12$

Substituting this value of y in (i), we get

$x + 12 = 28 \Rightarrow x = 28 - 12 \Rightarrow x = 16$

Hence, the number of 25 paise stamps = 16

Example 4. The sum of the digits of a two digit number is 5. The digit obtained by increasing the digit in ten's place by unity is one-eighth of the number. Find the number.

Solution. Let x be the digit at ten's place and y be the digit at unit's place.

$\therefore$ The digit by increasing the digit in ten's place by unity = $x + 1$

The number is $10x + y$

According to the problem, $x + y = 5$ ...(i) and $$x + 1 = \frac{1}{8}(10x + y)$$

$$\Rightarrow \quad 8(x + 1) = 10x + y$$

$$\Rightarrow \quad 8x + 8 = 10x + y$$

$$\Rightarrow \quad 2x + y = 8$$

Subtracting (*i*) from (*ii*), we get $x = 3$

On substituting this value of x in (*i*), we get

$$3 + y = 5 \Rightarrow y = 5 - 3 \Rightarrow y = 2$$

Hence, the required number is 32

Example 5. A two digit number is seven times the sum of its digits. The number fo[rmed] by reversing the digits is 18 less than the original number. Find the number.

Solution. Let x be the digit at ten's place and y be the digit at unit's place.

Then the number is $10x + y$

According to the first condition of the problem,

$$10x + y = 7(x + y)$$

$$\Rightarrow \quad 10x + y = 7x + 7y$$

$$\Rightarrow \quad 3x = 6y$$

$$\Rightarrow \quad x = 2y$$

The number formed by reversing the digits is $10y + x$

According to the second condition of the problem,

$$10y + x = (10x + y) - 18$$

$$\Rightarrow \quad 10y - y = 10x - x - 18$$

$$\Rightarrow \quad 9y = 9x - 18$$

$$\Rightarrow \quad y = x - 2$$

$$\Rightarrow \quad y = 2y - 2$$

$$\Rightarrow \quad y = 2. \qquad \text{(using}$$

From (*i*), $x = 2 \times 2 = 4$

Hence, the required number is 42

Example 6. The sum of a two digit number and the number obtained by interchanging t[he] digits is 132. If the two digits differ by 2, find the number(s).

Solution. Let the ten's and unit's digits in the number be x and y respectively. Then th[e] number $= 10x + y$ and the number obtained on reversing the digits $= 10y + x$.

According to given conditions, we have

$$(10x + y) + (10y + x) = 132 \text{ i.e. } 11x + 11y = 132$$

$$\Rightarrow \quad x + y = 12 \qquad \ldots$$

and $x - y = 2$ ...(*ii*) or $y - x = 2$ ...(*i*

If $x - y = 2$, then solving (*i*) and (*ii*), we get $x = 7, y = 5$

In this case, the number is 75

If $y - x = 2$, then solving (*i*) and (*iii*), we get $y = 7, x = 5$

In this case, the number is 57

Hence, there are two numbers 75 and 57 satisfying the given conditions.

Example 7. A fraction becomes $\frac{9}{11}$, if 2 is added to both numerator and denominator. If 3 is added to both the numerator and denominator, it becomes $\frac{5}{6}$. Find the fraction.

Solution. Let the fraction be $\frac{x}{y}$. Then, according to given, we have

$$\frac{x+2}{y+2} = \frac{9}{11} \text{ and } \frac{x+3}{y+3} = \frac{5}{6}$$

$\Rightarrow \quad 11x + 22 = 9y + 18$ and $6x + 18 = 5y + 15$

$\Rightarrow \quad 11x - 9y + 4 = 0 \quad \ldots(i) \quad$ and $\quad 6x - 5y + 3 = 0 \quad \ldots(ii)$

From (*ii*), $x = \frac{5y-3}{6} \quad \ldots(iii)$

Substituting this value of x in (*i*), we get

$$11\left(\frac{5y-3}{6}\right) - 9y + 4 = 0$$

$\Rightarrow \quad 55y - 33 - 54y + 24 = 0$

$\Rightarrow \quad y = 9.$

From (*iii*), $x = \frac{5 \times 9 - 3}{6} = \frac{42}{6} = 7$

Hence, the fraction is $\frac{7}{9}$

Example 8. If the numerator and denominator of a fraction are increased by 2 and 1 respectively, it becomes $\frac{3}{4}$. If the numerator and denominator are decreased by 2 and 1 respectively, it becomes $\frac{1}{2}$. Find the fraction.

Solution. Let the fraction be $\frac{x}{y}$

Since on increasing the numerator and denominator by 2 and 1 respectively it becomes $\frac{3}{4}$,

$\therefore \quad \frac{x+2}{y+1} = \frac{3}{4}$

$\Rightarrow \quad 4x + 8 = 3y + 3$

$\Rightarrow \quad 4x - 3y + 5 = 0 \quad \ldots(i)$

On decreasing the numerator and denominator by 2 and 1 respectively, it becomes $\frac{1}{2}$,

$\therefore \quad \frac{x-2}{y-1} = \frac{1}{2} \Rightarrow 2x - 4 = y - 1$

$\Rightarrow \quad 2x - y - 3 = 0 \quad \ldots(ii)$

Multiplying both sides of (*ii*) by 2, we get

$4x - 2y - 6 = 0 \quad \ldots(iii)$

Subtracting (*i*) from (*iii*), we get

$y - 11 = 0 \Rightarrow y = 11$

On substituting this value of y in (*ii*), we get

$2x - 11 - 3 = 0 \Rightarrow 2x = 14 \Rightarrow x = 7$

Hence, the required fraction is $\frac{7}{11}$

Example 9. The sum of the numerator and denominator of a fraction is 4 more than twice the numerator. If the numerator and denominator each are increased by 3, they are in the ratio 2 : 3. Determine the fraction.

Solution. Let the fraction be $\frac{x}{y}$. Then, according to given, we have

$$x + y = 2x + 4 \Rightarrow y = x + 4$$

and $\frac{x+3}{y+3} = \frac{2}{3} \Rightarrow 3x + 9 = 2y + 6$

$\Rightarrow \quad 3x + 3 = 2y$

Substituting the value of y from (*i*) in (*ii*), we get

$$3x + 3 = 2(x + 4) \Rightarrow 3x + 3 = 2x + 8$$

$\Rightarrow \quad 3x - 2x = 8 - 3 \Rightarrow x = 5$

From (*i*), $y = 5 + 4 = 9$

Hence, the fraction is $\frac{5}{9}$

Example 10. Half the perimeter of a rectangular garden, whose length is 4 m more than its width, is 36 m. Find the dimensions of the garden.

Solution. Let the length and the breadth of the rectangular garden be x metres and y metres respectively.

Then, according to given, we have

$x + y = 36 \quad \ldots(i) \quad$ and $\quad x = y + 4$

Substituting the value of x from (*ii*) in (*i*), we get

$$(y + 4) + y = 36 \Rightarrow 2y + 4 = 36$$

$\Rightarrow \quad 2y = 32 \Rightarrow y = 16$

From (*ii*), $x = y + 4 = 16 + 4 = 20$

Hence, the length of garden = 20 m and its breadth = 16 m

Example 11. Six years hence a man's age will be three times his son's age, and three years ago he was nine times as old as his son. Find their present ages.

Solution. Let the present age of the man be x years and the present age of his son be y years.

6 years hence, their ages will be $(x + 6)$ years and $(y + 6)$ years.

According to the problem,

$$x + 6 = 3(y + 6) \Rightarrow x + 6 = 3y + 18$$

$\Rightarrow \quad x - 3y = 12$

3 years ago, their ages were $(x - 3)$ years and $(y - 3)$ years.

According to the problem,

$$x - 3 = 9(y - 3) \Rightarrow x - 3 = 9y - 27$$

$\Rightarrow \quad x - 9y = -24$

Subtracting (*ii*) from (*i*), we get

$$6y = 36 \Rightarrow y = 6$$

Substituting this value of y in (*i*), we get

$$x - 3 \times 6 = 12 \Rightarrow x - 18 = 12 \Rightarrow x = 12 + 18 = 30$$

Hence, the present age of the man is 30 years and that of his son is 6 years.

Example 12. The age of the father is twice the sum of ages of his two children. After 20 years, his age will be equal to the sum of the ages of his children. Find the age of the father.

Solution. Let the present age of the father be x years and the sum of the present ages of his two children be y years. Then

$$x = 2y \quad \ldots(i)$$

After 20 years

age of father $(x + 20)$ years,

sum of ages of his two children $= (y + 2 \times 20)$ years $= (y + 40)$ years

According to given,

$$x + 20 = y + 40 \Rightarrow x = y + 20 \quad \ldots(ii)$$

Eliminating x from (i) and from (ii), we get

$$2y = y + 20 \Rightarrow y = 20$$

Substituting this value y in (i), we get

$$x = 2 \times 20 = 40$$

Hence, the present age of the father = 40 years.

Example 13. At a certain time in a deer park, the number of heads and the number of legs of deer and human visitors were counted and it was found that there were 41 heads and 136 legs. Find the number of deer and human visitors in the park.

Solution. Let there be x deer and y human visitors in the park at the given time.

As a deer and a human each has one head, we get

$$x + y = 41 \quad \ldots(i)$$

As a deer has 4 legs and a human has 2 legs, we get

$$4x + 2y = 136 \Rightarrow 2x + y = 68 \quad \ldots(ii)$$

Subtracting (i) from (ii), we get

$$x = 27$$

Substituting $x = 27$ in (i), we get

$$27 + y = 41 \Rightarrow y = 14$$

Hence, there were 27 deer and 14 human visitors in the park.

Example 14. Ten percent of the red balls were added to twenty percent of the blue balls and the total was 24. Yet three times the number of red balls exceeds the number of blue balls by 20. How many were red balls and how many were blue balls?

Solution. Let there be x red balls and y blue balls.

According to the problem,

$$10\% \text{ of } x + 20\% \text{ of } y = 24 \Rightarrow \frac{10}{100}x + \frac{20}{100}y = 24$$

$$\Rightarrow \quad x + 2y = 240 \quad \ldots(i)$$

Also $\quad 3x = y + 20 \Rightarrow 3x - y = 20 \quad \ldots(ii)$

Multiplying (ii) by 2, we get

$$6x - 2y = 40 \quad \ldots(iii)$$

Adding (i) and (iii), we get

$$7x = 280 \Rightarrow x = 40$$

Substituting $\quad x = 40$ in (i), we get

$$40 + 2y = 240 \Rightarrow 2y = 200 \Rightarrow y = 100$$

Hence, there were 40 red balls and 100 blue balls.

Example 15. The taxi charges in a city consist of a fixed charge together with the charge for the distance covered. For a distance of 10 km, the charge paid is ₹105 and for a journey of 15 km, the charge paid is ₹155. What are the fixed charges and the charge per km? How much does a person have to pay for travelling a distance of 25 km?

Solution. Let the fixed charges be ₹ x and charges per km be ₹ y

Then, according to given, we have

$x + 10y = 105$ ...(1) and $x + 15y = 155$ ...(2)

From (1), $x = 105 - 10y$ ...(3)

Substituting this value of x in (2), we get

$$(105 - 10y) + 15y = 155$$

$$\Rightarrow \quad -10y + 15y = 155 - 105$$

$$\Rightarrow \quad 5y = 50 \Rightarrow y = 10$$

From (3), $x = 105 - 10 \times 10 = 105 - 100 = 5$

Hence, the fixed charges are ₹ 5 and the charges per km are ₹ 10

Taxi charges for a distance of 25 km

= fixed charges + charges for travelling 25 km

= ₹ 5 + ₹ (25 × 10) = ₹ 5 + ₹ 250 = ₹ 255

Example 16. The ratio of incomes of two persons is 9 : 7 and the ratio of their expenditures is 4 : 3. If each of them manages to save ₹ 2000 per month, find their monthly incomes.

Solution. Let the incomes per month of two persons be ₹ x and ₹ y respectively. As each person saves ₹ 2000 per month, so their expenditures are ₹ $(x - 2000)$ and ₹ $(y - 2000)$ respectively.

According to given, we have

$$\frac{x}{y} = \frac{9}{7} \text{ i.e. } 7x - 9y = 0 \qquad ...(i)$$

and $$\frac{x - 2000}{y - 2000} = \frac{4}{3} \text{ i.e. } 3x - 4y + 2000 = 0$$

Multiplying equation (*i*) by 3 and equation (*ii*) by 7, we get

$21x - 27y = 0$ ...(*iii*) and $21x - 28y + 14000 = 0$

Subtracting equation (*iv*) from equation (*iii*), we get

$$y - 14000 = 0 \Rightarrow y = 14000$$

Substituting this value of y in (*i*), we get

$$7x - 9 \times 14000 = 0 \Rightarrow x = 18000$$

Hence, the monthly incomes of the two persons are ₹ 18000 and ₹ 14000 respectively.

Example 17. A railway half-ticket costs half the full fare but the reservation charges are the same of a half-ticket as on a full ticket. One reserved first class ticket from station A to station B costs ₹2530. Also, one reserved first class ticket and one reserved half first class ticket from station A to station B costs ₹3810. Find the full fare from station A to B and also the reservation charges for a ticket.

Solution. Let the reservation charges for one railway ticket be ₹ x and the fare for one first class ticket be ₹ y, then

Cost of one reserved first class ticket = ₹ $(x + y)$

As the reservation charges for half-ticket are same as the full ticket, so the cost of reserved half-ticket = ₹ $\left(x + \frac{1}{2}y\right)$

According to given, $x + y = 2530$ ...(1)

and $(x + y) + \left(x + \frac{1}{2}y\right) = 3810$

$\Rightarrow$ $2x + \frac{3}{2}y = 3810$

$\Rightarrow$ $4x + 3y = 7620$ ...(2)

Multiplying equation (1) by 3, we get

$3x + 3y = 7590$ ...(3)

Subtracting equation (3) from equation (2), we get

$x = 30$

Substituting this value of x in (1), we get

$30 + y = 2530 \Rightarrow y = 2500$

Hence, the full fare for a ticket is ₹ 2500 and the reservation charges are ₹ 30

Example 18. Vijay had some bananas, and he divided them into two lots A and B. He sold the first lot at the rate of ₹ 2 for 3 bananas and the second lot at the rate of ₹ 1 per banana, and got a total of ₹ 400. If he had sold the first lot at ₹ 1 per banana and the second lot at the rate of ₹ 4 for 5 bananas, his total collection would have been ₹ 460. Find the total number of bananas he had.

Solution. Let Vijay had x bananas in lot A and y bananas in lot B. Then, according to given, we have:

$\frac{2}{3}x + y = 400$ *i.e.* $2x + 3y = 1200$ ...(*i*)

and $x + \frac{4}{5}y = 460$ *i.e.* $5x + 4y = 2300$ ...(*ii*)

Multiplying equation (*i*) by 5 and equation (*ii*) by 2, we get

$10x + 15y = 6000$ ...(*iii*) $10x + 8y = 4600$ ...(*iv*)

Subtracting equation (*iv*) from equation (*iii*), we get

$7y = 1400 \Rightarrow y = 200.$

Substituting this value of y in (*i*), we get

$2x + 3 \times 200 = 1200 \Rightarrow 2x = 600 \Rightarrow x = 300$

Hence, the total number of bananas that Vijay had

$= x + y = 300 + 200 = 500$

Example 19. The area of a rectangle gets reduced by 9 sq. units, if its length is reduced by 5 units and the breadth is increased by 3 units. If we increase the length by 3 units and the breadth by 2 units, then the area is increased by 67 sq. units. Find the dimensions of the rectangle.

Solution. Let the length and the breadth of the rectangle be x units and y units respectively. Then the area of rectangle $= xy$ sq. units

According to given conditions, we have

$(x - 5)(y + 3) = xy - 9$ and $(x + 3)(y + 2) = xy + 67$

$\Rightarrow$ $xy + 3x - 5y - 15 = xy - 9$

and $xy + 2x + 3y + 6 = xy + 67$

$\Rightarrow$ $3x - 5y - 6 = 0$ ...(*i*)

and $2x + 3y - 61 = 0$ ...(*ii*)

Multiplying equation (*i*) by 2 and equation (*ii*) by 3, we get

$6x - 10y - 12 = 0$ ...(*iii*) and $6x + 9y - 183 = 0$ ...(*iv*)

Subtracting equation (*iii*) from equation (*iv*), we get

$19y - 171 = 0 \Rightarrow y = 9$

Substituting this value of y in equation (*ii*), we get

$2x + 3 \times 9 - 61 = 0 \Rightarrow 9x - 34 = 0 \Rightarrow x = 17$

Hence, the length of rectangle = 17 units and breadth = 9 units

Example 20. A chemist has one solution which is 50% acid and a second which is 25% acid. How much of each should be mixed to make 10 litres of a 40% acid solution?

Solution. Let x litres of 50% and y litres of 25% acid solutions be mixed. Then, according to given conditions, we have

$x + y = 10$ ...(*i*)

and 50% of x + 25% of y = 40% of 10

$\Rightarrow \frac{50}{100}x + \frac{25}{100}y = \frac{40}{100} \times 10$

$\Rightarrow \frac{1}{2}x + \frac{1}{4}y = 4$

$\Rightarrow 2x + y = 16$ ...(*ii*)

Subtracting equation (*i*) from equation (*ii*), we get $x = 6$

Substituting this value of x in equation (*i*), we get

$6 + y = 10 \Rightarrow y = 4$

Hence, 6 litres of 50% and 4 litres of 25% acid solutions be mixed to get 10 litres of 40% acid solution.

Example 21. Reena deposited certain amount of her pocket money in two saving banks A and B, which offer interest at the rate of 7% p.a., and 8% p.a. respectively. She receives ₹134 as annual interest. However, had she interchanged the amount invested in two banks, she would have received ₹2 more as annual interest. How much did she invest in each bank?

Solution. Let Reena invest ₹x in bank A and ₹y in bank B.

According to the conditions,

$$\frac{x \times 7 \times 1}{100} + \frac{y \times 8 \times 1}{100} = 134$$

and $$\frac{x \times 8 \times 1}{100} + \frac{y \times 7 \times 1}{100} = 134 + 2$$

$\Rightarrow 7x + 8y = 13400$ ...(*i*)

and $8x + 7y = 13600$ ...(*ii*)

Adding (*i*) and (*ii*), we get

$15x + 15y = 27000$

$\Rightarrow x + y = 1800$ ...(*iii*)

Subtracting (*i*) from (*ii*), we get

$x - y = 200$ ...(*iv*)

Using (*iii*) and (*iv*), we get

$2x = 2000$

$x = 1000$

and $y = 800$

$\therefore$ Money invested by her in bank A = ₹1000 and in bank B = ₹800

Example 22. Some amount is distributed equally among students. If there are 8 students less, every one will get ₹10 more. If there are 16 students more, every one will get ₹10 less. What is the number of students and how much does each get? What is the total amount distributed?

Solution. Let the number of students be x and let each student get ₹y

Then the amount distributed = ₹xy

In the first case, if there are 8 students less, every one will get ₹10 more

$$(x - 8) \times (y + 10) = xy$$

$$\therefore \quad xy + 10x - 8y - 80 = xy \Rightarrow 10x - 8y = 80$$

$$\Rightarrow \quad 5x - 4y = 40 \quad \ldots(i)$$

$\Rightarrow$ In the second case, if there are 16 students more, every one will get ₹10 less,

$$(x + 16) \times (y - 10) = xy$$

$$\therefore \quad xy - 10x + 16y - 160 = xy \Rightarrow -10x + 16y = 160$$

$$\Rightarrow \quad -5x + 8y = 80 \quad \ldots(ii)$$

$\Rightarrow$ On adding (*i*) and (*ii*), we get

$$4y = 120 \Rightarrow y = 30$$

Putting $y = 30$ in (*i*), we get

$$5x - 4 \times 30 = 40 \Rightarrow 5x = 40 + 120 \Rightarrow 5x = 160 \Rightarrow x = 32$$

Hence, the number of students = 32 and each student gets ₹30

Amount distributed = ₹(32 × 30) = ₹960

Example 23. In a triangle ABC, $\angle C = 3\angle B = 2(\angle A + \angle B)$. Find all the angles in degrees.

Solution. Let $\angle A = x$ and $\angle B = y$

Given $\angle C = 3\angle B = 2(\angle A + \angle B)$

$\Rightarrow \angle C = 3\angle B$ and $3\angle B = 2(\angle A + \angle B)$

$\Rightarrow \angle C = 3y$ and $3y = 2(x + y)$

$\Rightarrow \angle C = 3y$ and $2x - y = 0$ ...(*i*)

In ΔABC, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow x + y + 3y = 180° \Rightarrow x + 4y = 180°$ ...(*ii*)

Multiplying equation (*i*) by 4, we get

$8x - 4y = 0$ ...(*iii*)

On adding equations (*ii*) and (*iii*), we get

$9x = 180° \Rightarrow x = 20°$

Substituting this value of x in equation (*i*), we get

$(2 \times 20)° - y = 0 \Rightarrow y = 40°$

$\therefore \angle C = 3y = (3 \times 40)° = 120°$

Hence, $\angle A = 20°$, $\angle B = 40°$ and $\angle C = 120°$

Example 24. A train leaves New Delhi for Ludhiana, 324 km away, at 9 a.m. One hour later, another train leaves Ludhiana for New Delhi. They meet at noon. If the second train had started at 9 a.m. and the first train at 10.30 a.m., they would still have met at noon. Find the speed of each train.

Solution. Let the speed of the first train be x km/h and that of the second train be y km/h. The two trains meet at noon *i.e.* 12 noon.

Look at the following chart:

	Time	*Rate*	*Distance*
Train 1	3	x	$3x$
Train 2	2	y	$2y$
Train 1	1·5	x	$1{\cdot}5x$
Train 2	3	y	$3y$

When the two trains meet, sum of distances covered by them is 324 km,

$\therefore$ $\quad 3x + 2y = 324 \quad$ ...(i)

and $\quad 1{\cdot}5x + 3y = 324 \quad$ ...(ii)

Multiplying (*ii*) by 2, we get

$$3x + 6y = 648 \quad \text{...(iii)}$$

Subtracting (*i*) from (*ii*), we get

$$4y = 324 \Rightarrow y = 81$$

Substituting $\quad y = 81$ in (*i*), we get

$$3x + 2 \times 81 = 324 \Rightarrow 3x = 324 - 162$$

$\Rightarrow \quad 3x = 162 \Rightarrow x = 54$

Hence, the speed of the first train is 54 km/h and that of the second train is 81 km/h.

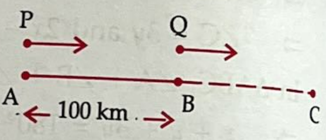
Example 25. Places A and B are 100 km apart on a highway. One car starts from A and another from B at the same time. If the cars travel in the same direction at different speeds, they meet in 5 hours. If they travel towards each other, they meet in one hour. What are the speeds of the two cars?

Solution. Name the cars at places A and B as P and Q respectively.

Let the speeds of the cars P and Q be x km/h and y km/h respectively, $x > y$.

When the cars move in the same direction:

Let the cars meet at place C after 5 hours (as shown in figure)



Distance covered by car P in 5 hours = AC = $5x$ km

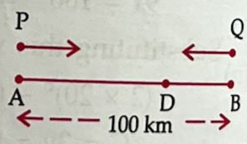
and distance covered by car Q in 5 hours = BC = $5y$ km.

From figure, AC – BC = AB $\Rightarrow 5x - 5y = 100$

$\Rightarrow \quad x - y = 20 \quad$ ...(*i*)

When the cars move in opposite directions:

Let the cars meet at the place D after 1 hour
s shown in figure)



Distance covered by car P in 1 hour = AD = x km,

and distance covered by car Q in 1 hour = BD = y km.

From figure, AD + BD = AB

$\Rightarrow \quad x + y = 100 \quad$ ...(*ii*)

Adding equations (*i*) and (*ii*), we get

$$2x = 120 \Rightarrow x = 60$$

ubstituting this value of x in equation (*ii*), we get

$$60 + y = 100 \Rightarrow y = 40$$

ence, the speeds of the two cars are 60 km/h and 40 km/h.

Example 26. Ankita travels 14 km to her home partly by rickshaw and partly by bus. She takes half an hour if she travels 2 km by rickshaw and the remaining distance by bus. On the other hand, if she travels 4 km by rickshaw and the remaining distance by bus, she takes 9 minutes longer. Find the speed of rickshaw and of the bus.

Solution. Let the speeds of the rickshaw and the bus be x km/h and y km/h.

As Ankita has to travel a total distance of 14 km and she travels 2 km by the rickshaw, so the remaining distance of 12 km has to be travelled by the bus.

Time taken to cover 2 km by rickshaw $= \frac{2}{x}$ h and time taken to cover 12 km by bus $= \frac{12}{y}$ h

Since the total time taken is half an hour *i.e.* $\frac{1}{2}$ hour,

$$\therefore \quad \frac{2}{x}+\frac{12}{y}=\frac{1}{2} \qquad ...(i)$$

On the other hand, if Ankita travels 4 km by rickshaw then the remaining distance of 10 km has to be covered by bus.

In this situation, the total time taken is 9 minutes more than the previous time, so the total time taken is 39 minutes *i.e.* $\frac{39}{60}$ hours *i.e.* $\frac{13}{20}$ h

$$\therefore \quad \frac{4}{x}+\frac{10}{y}=\frac{13}{20} \qquad ...(ii)$$

Substituting $\frac{1}{x} = p$ and $\frac{1}{y} = q$ in equations (*i*) and (*ii*), we get

$$2p + 12q = \frac{1}{2} \quad ...(iii) \quad \text{and} \quad 4p + 10q = \frac{13}{20} \quad ...(iv)$$

Multiplying equation (*iii*) by 2, we get

$$4p + 24q = 1 \qquad ...(v)$$

Subtracting equation (*iv*) from equation (*v*), we get

$$14q = 1 - \frac{13}{20} \Rightarrow 14q = \frac{7}{20} \Rightarrow q = \frac{1}{40}$$

Substituting this value of q in equation (*iii*), we get

$$2p + 12 \times \frac{1}{40} = \frac{1}{2} \Rightarrow 2p = \frac{1}{2} - \frac{3}{10}$$

$$\Rightarrow \quad 2p = \frac{2}{10} \Rightarrow p = \frac{1}{10}$$

$$\therefore \quad \frac{1}{x}=\frac{1}{10} \text{ and } \frac{1}{y}=\frac{1}{40} \Rightarrow x = 10 \text{ and } y = 40$$

Hence, the speed of the rickshaw is 10 km/h and that of the bus is 40 km/h.

Example 27. 3 men and 4 boys can do a piece of work in 14 days, while 4 men and 6 boys can do it in 10 days. How long would it take 1 boy to finish the work?

Solution. Suppose 1 man can finish the work in x days and 1 boy can do it in y days, then

1 man's one day work $= \frac{1}{x}$ and 1 boy's one day work $= \frac{1}{y}$

Given 3 men and 4 boys can do the work in 14 days,

$\therefore$ 3 men's one day work + 4 boy's one day work $= \frac{1}{14}$

$$\Rightarrow \quad 3.\frac{1}{x}+4.\frac{1}{y}=\frac{1}{14} \quad \ldots(i)$$

Also 4 men and 6 boys can do the work in 10 days.

$\therefore$ 4 men's one day work + 6 boy's one day work = $\frac{1}{10}$

$$\Rightarrow \quad 4.\frac{1}{x}+6.\frac{1}{y}=\frac{1}{10} \quad \ldots(ii)$$

Multiplying (*i*) by 4 and (*ii*) by 3, we get

$$\frac{12}{x}+\frac{16}{y} = \frac{2}{7} \quad \ldots(iii)$$

$$\frac{12}{x}+\frac{18}{y} = \frac{3}{10} \quad \ldots(iv)$$

Subtracting (*iii*) from (*iv*), we get

$$\frac{18-16}{y} = \frac{3}{10}-\frac{2}{7} \Rightarrow \frac{2}{y}=\frac{21-20}{70}$$

$$\Rightarrow \quad \frac{2}{y} = \frac{1}{70} \Rightarrow y = 140$$

Hence, one boy can finish the work in 140 days.

Example 28. It takes 12 hours to fill a swimming pool using two pipes. If the pipe of larger diameter is used for 4 hours and the pipe of smaller diameter for 9 hours, only half the pool can be filled. How long would it take for each pipe to fill the pool separately?

Solution. Let the pipes of larger diameter take x hours and the pipe of smaller diameter take y hours separately to fill the pool. Then,

the part of the pool filled by the larger pipe in one hour = $\frac{1}{x}$,

and the part of the pool filled by the smaller pipe in one hour = $\frac{1}{y}$

Since it takes 12 hours to fill the swimming pool by using the two pipes together,

$$\therefore \quad \frac{12}{x}+\frac{12}{y} = 1 \quad \ldots(i)$$

If the larger pipe is used for 4 hours and the smaller pipe for 9 hours, only half the pool is filled,

$$\therefore \quad \frac{4}{x}+\frac{9}{y}=\frac{1}{2} \quad \ldots(ii)$$

Substituting $\frac{1}{x}$ = p and $\frac{1}{y}$ = q in these equations, we get

$12p + 12q = 1$ ...(*iii*) and $4p + 9y = \frac{1}{2}$

Multiplying equation (*iv*) by 3, we get

$$12p + 27q = \frac{3}{2}$$

Subtracting equation (*iii*) from equation (*v*), we get

$$15q = \frac{3}{2} - 1 \Rightarrow 15q = \frac{1}{2} \Rightarrow q = \frac{1}{30}$$

Substituting this value of q in equation (*iv*), we get

$$4p + 9 \times \frac{1}{30}=\frac{1}{2} \Rightarrow 4p = \frac{1}{2}-\frac{3}{10}=\frac{2}{10} \Rightarrow p = \frac{1}{20}$$

$\therefore \quad \frac{1}{x} = \frac{1}{20}$ and $\frac{1}{y} = \frac{1}{30} \quad \Rightarrow x = 20$ and $y = 30$

Hence, the pipe of larger diameter alone can fill the pool in 20 hours and the pipe of smaller diameter alone can fill the pool in 30 hours.

Example 29. When a motor boat goes 39 km upstream and 46 km downstream, it takes 5 hours. It taken 7 hours when distance travelled upstream is 52 km and that of down stream is 69 km. Find the speed of the stream and that of the boat in still water.

Solution. Let the speed of the boat in still water be x km/h and the speed of the stream be y km/h.

$\therefore$ Speed of the boat upstream $= (x - y)$ km/h

and speed of the boat downstream $(x + y)$ km/h

Since time taken by the boat in going 39 km upstream and 46 km downstream is 5 hours

$$\therefore \quad \frac{39}{x-y} + \frac{46}{x+y} = 5 \quad \text{...}(i)$$

Similarly for the second case

$$\frac{52}{x-y} + \frac{69}{x+y} = 7 \quad \text{...}(ii)$$

Substituting, $x - y = \frac{1}{a}$ and $x + y = \frac{1}{b}$ in (*i*) and (*ii*), we get

$$39a + 46b = 5 \quad \text{...}(iii)$$

$$52a + 69b = 7 \quad \text{...}(iv)$$

Multiplying (*iii*) by 4 and (*iv*) by 3 we get,

$$156a + 184b = 20 \quad \text{...}(v)$$

$$156a + 207b = 21 \quad \text{...}(vi)$$

Subtracting (*v*) from (*vi*), we get

$$23b = 1$$

$$\Rightarrow \quad b = \frac{1}{23} \Rightarrow x + y = 23 \quad \text{...}(vii)$$

From (*iii*),

$$39a + \frac{46}{23} = 5 \Rightarrow 39a = 5 - 2$$

$$\Rightarrow \quad a = \frac{1}{13} \Rightarrow x - y = 13 \quad \text{...}(viii)$$

Solving (*vii*) and (*viii*), we get

$$x = 18 \text{ and } y = 5$$

$\therefore$ The speed of boat in still water is 18 km/h and speed of the stream is 5 km/h.

Exercise 6

1. The sum of two numbers is 50 and their difference is 16. Find the numbers.
2. The sum of two numbers is 2. If their difference is 20, find the numbers.

3. The sum of two numbers is 43. If the larger is doubled and the smaller is tripled, the difference is 36. Find the two numbers.

4. The cost of 5 kg of sugar and 7 kg of rice is ₹765, and the cost of 7 kg of sugar and 5 kg of rice is ₹735. Find the cost of 6 kg of sugar and 10 kg of rice.

5. The Class IX students of a certain public school wanted to give a farewell party to the outgoing students of Class X. They decided to purchase two kinds of sweets, one costing ₹350 per kg and the other costing ₹440 per kg. They estimated that 36 kg of sweets were needed. If the total money spent on sweets was ₹14000, find how much sweets of each kind they purchased.

6. If from twice the greater of two numbers 16 is subtracted, the result is half the other number. If from half the greater number 1 is subtracted, the result is still half the other number. What are the numbers?

7. There are 38 coins in a collection of 20 paise coins and 25 paise coins. If the total value of the collection is ₹8·50, how many of each are there?

8. A man has certain notes of denominations ₹200 and ₹50 which amount to ₹3800. If the number of notes of each kind is interchanged, they amount to ₹600 less as before. Find the number of notes of each denomination.

9. The ratio of two numbers is $\frac{2}{3}$. If 2 is subtracted from the first and 8 from the second, the ratio becomes the reciprocal of the original ratio. Find the numbers.

10. If 1 is added to the numerator of a fraction, it becomes $\frac{1}{5}$; if 1 is taken from the denominator, it becomes $\frac{1}{7}$, find the fraction.

11. If the numerator of a certain fraction is increased by 2 and the denominator by 1, the fraction becomes equal to $\frac{5}{8}$ and if the numerator and denominator are each diminished by 1, the fraction becomes equal to $\frac{1}{2}$; find the fraction.

12. Find the fraction which becomes $\frac{1}{2}$ when the denominator is increased by 4 and is equal to $\frac{1}{8}$ when the numerator is diminished by 5.

13. In a two digit number the sum of the digits is 7. If the number with the order of the digits reversed is 28 greater than twice the unit's digit of the original number, find the number.

14. A number of two digits exceeds four times the sum of its digits by 6 and it is increased by 9 on reversing the digits. Find the number.

15. When a two digit number is divided by the sum of its digits the quotient is 8. If the ten's digit is diminished by three times the unit's digit, the remainder is 1. What is the number?

16. The result of dividing a number of two digits by the number with digits reversed is $1\frac{3}{4}$. If the sum of digits is 12, find the number.

17. The result of dividing a number of two digits by the number with the digits reversed is $\frac{5}{6}$. If the difference of digits is 1, find the number.

18. A number of three digits has the hundred digit 4 times the unit digit and the sum of three digits is 14. If the three digits are written in the reverse order, the value of the number is decreased by 594. Find the number.

19. Four years ago Marina was three times old as her daughter. Six years from now the mother will be twice as old as her daughter. Find their present ages.

20. On selling a tea set at 5% loss and a lemon set at 15% gain, a shopkeeper gains ₹70. If he sells the tea set at 5% gain and lemon set at 10% gain, he gains ₹130. Find the cost price of the lemon set.

21. A person invested some money at 12% simple interest and some other amount at 10% simple interest. He received yearly interest of ₹1300. If he had interchanged the amounts, he would have received ₹40 more as yearly interest. How much did he invest at different rates?

22. A shopkeeper sells a plastic table at 8% profit and a chair at 10% discount, thereby receiving ₹1008 as the total selling price. If he had sold the table at 10% profit and chair at 8% discount, he would have got ₹20 more. Find the cost price of the table and the list price of the chair.

23. A and B have some money with them. A said to B, 'if you give me ₹100, my money will become 75% of the money left with you'. "B said to A" instead if you give me ₹100, your money will become 40% of my money. How much money did A and B have originally?

24. The students of a class are made to stand in (complete) rows. If one student is extra in a row, there would be 2 rows less, and if one student is less in a row, there would be 3 rows more. Find the number of students in the class.

25. A jeweller has bars of 18-carat gold and 12-carat gold. How much of each must be melted together to obtain a bar of 16-carat gold weighing 120 grams? (Pure gold is 24-carat).

26. A and B together can do a piece of work in 15 days. If A's one day work is $1\frac{1}{2}$ times the one day's work of B, find in how many days can each do the work.

27. 2 men and 5 women can do a piece of work in 4 days, while one man and one woman can finish it in 12 days. How long would it take for 1 man to do the work?

28. A train covered a certain distance at a uniform speed. If the train had been 30 km/h faster, it would have taken 2 hours less than the scheduled time. If the train were slower by 15 km/h, it would have taken 2 hours more than the scheduled time. Find the length of the journey.

29. A boat takes 2 hours to go 40 km down the stream and it returns in 4 hours. Find the speed of the boat in still water and the speed of the stream.

30. A boat sails a distance of 44 km in 4 hours with the current. It takes 4 hours 48 minutes longer to cover the same distance against the current. Find the speed of the boat in still water and the speed of the current.

31. An aeroplane flies 1680 km with a head wind in 3·5 hours. On the return trip with same wind blowing, the plane takes 3 hours. Find the plane's air speed and the wind speed.

32. A part of monthly hostel charges is fixed and the remaining depends on the number of days one has taken food in the mess. When Bhawana takes food for 20 days, she has to pay ₹2600 as hostel charges; whereas when Divya takes food for 26 days, she pays ₹3020 as hostel charges. Find the fixed charges and the cost of food per day.

Multiple Choice Questions MCQs

Choose the correct answer from the given four options (1 to 9):

1. Sum of digits of a two digit number is 8. If the number obtained by reversing the digits is 18 more than the original number, then the original number is
(a) 35 (b) 53 (c) 26 (d) 62

2. The sum of two natural numbers is 25 and their difference is 7. The numbers are
(a) 17 and 8 (b) 16 and 9 (c) 18 and 7 (d) 15 and 10

3. The sum of two natural numbers is 240 and their ratio is 3 : 5. Then the greater number is
(a) 180 (b) 160 (c) 150 (d) 90

4. The sum of the digits of a two digit number is 9. If 27 is added to it, the digits of the number get reversed. The number is
(a) 27 (b) 72 (c) 63 (d) 36

5. The sum of the digits of a two digit number is 12. If the number is decreased by 18, its digits get reversed. The number is
(a) 48 (b) 84 (c) 57 (d) 75

6. Aruna has only ₹ 1 and ₹ 2 coins with her. If the total number of coins that she has is 50 and the amount of the money with her is ₹ 75, then the number of ₹ 1 and ₹ 2 coins are, respectively
(a) 35 and 15 (b) 35 and 20 (c) 15 and 75 (d) 25 and 25

7. The age of a woman is four times the age of her daughter. Five years hence, the age of the woman will be three times the age of her daughter. The present age of the daughter is
(a) 40 years (b) 20 years (c) 15 years (d) 10 years

8. Father's age is six times his son's age. Four years hence, the age of the father will be four times his son's age. The present age in years of the son and the father are, respectively
(a) 4 and 24 (b) 5 and 30 (c) 6 and 36 (d) 3 and 24

9. Consider the following two statements:
Statement I: A husband is 2 years older than his wife, and sum of their ages is 52 years. Then the wife is 25 years old.
Statement II: A father is twice as old as his daughter, and difference of their ages is 26 years. Then the father is 50 years old.
Which of the following is valid?
(a) Both the Statements are true.
(b) Both the Statements are false.
(c) Statement I is true, and Statement II is false.
(d) Statement I is false, and Statement II is true.

ASSERTION-REASON TYPE QUESTION (SOLVED)

In these examples and following questions, read the given statements carefully and choose the correct option.

(a) Assertion (A) is true, Reason (R) is false.
(b) Assertion (A) is false, Reason (R) is true.
(c) Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason for Assertion (A).
(d) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct reason (or explanation) for Assertion (A).

1. **Assertion (A):** Sum of digits of a two digit number is 16. The number obtained by reversing the two digits is the highest two digit prime number. Then the original number is 79.

Reason (R): If sum of digits of a two digit number is 8, then the number is 35.

Sol. Let us first consider Assertion (A).

Assume that ten's and unit's digits in original number be x, y. Then $x + y = 16$

Number obtained by reversing digits is $10y + x$

A little calculation shows that 97 is the highest two digit prime number.

Solving $x + y = 16$, $10y + x = 97$, we get $x = 7$, $y = 9$

$\therefore$ Original number is 79.

Hence, Assertion (A) is true.

Now, let us consider Reason (R).

We know that 17, 71, 26, 62, 35, 53, 44 are two digit numbers whose sum of digits is 8.

$\therefore$ Reason (R) is false.

Thus Assertion (A) is true, Reason (R) is false.

$\therefore$ Correct answer is (*a*).

ASSERTION-REASON TYPE QUESTIONS (UNSOLVED)

1. **Assertion (A):** Difference between ages of two brothers is 5 years, while sum of their ages is 25 years. Then the younger is 10 years old.

   Reason (R): The difference between age of two brothers remains constant, even when they grow older.

2. **Assertion (A):** Perimeter of a garden is 24 cm, while the difference between its length and width is 2 units. Then its area is 35 sq. cm.

   Reason (R): If length of a rectangle is doubled, while width remains the same, then the perimeter also gets doubled.

Summary

- Problems stated in words are called **word problems**.
- Solving word problems involves two steps. First translating the words of the problem into algebraic equations. Second, solving the resulting equations.

Solving word problems

Due to the wide variety of word (or applied) problems, there is no single technique that works in all cases. However, the following general suggestions should prove helpful.

(*i*) *Read and reread the statement of the problem carefully, and determine what quantities must be found.*

(*ii*) *Represent the unknown quantities by letters.*

(*iii*) *Determine which expressions are equal and write equations.*

(*iv*) *Solve the resulting equations.*

Chapter Test

1. A 700 g dry fruit pack costs ₹432. It contains some almonds and the rest cashew kernel. If almonds cost ₹576 per kg and cashew kernel cost ₹672 per kg, what are the quantities of the two dry fruits separately?
2. Drawing pencils cost ₹4 each and coloured pencils cost ₹5·50 each. If altogether two dozen pencils cost ₹108, how many coloured pencils are there?
3. Shikha works in a factory. In one week she earned ₹3900 for working 47 hours, of which 7 hours were overtime. The next week she earned ₹4160 for working 50 hours, of which 8 hours were overtime. What is Shikha's hourly earning rate?
4. The sum of the digits of a two digit number is 7. If the digits are reversed, the new number increased by 3 equals 4 times the original number. Find the number.
5. Three years hence a man's age will be three times his son's age, and 7 years ago he was seven times as old as his son. How old are they now?
6. Rectangles are drawn on line segments of fixed lengths. When the breadths are 6 m and 5 m respectively the sum of the areas of the rectangles is 83 m^2. But if the breadths are 5 m and 4 m respectively the sum of the areas is 68 m^2. Find the sum of the areas of the squares drawn on the line segments.
7. If the length and the breadth of a room are increased by 1 metre each, the area is increased by 21 square metres. If the length is decreased by 1 metre and the breadth is increased by 2 metres, the area is increased by 14 square metres. Find the perimeter of the room.
8. The lengths (in metres) of the sides of a triangle are $2x+\frac{y}{2}$, $\frac{5x}{3}+y+\frac{1}{2}$ and $\frac{2}{3}x+2y+\frac{5}{2}$. If the triangle is equilateral, find its perimeter.
9. On Diwali eve, two candles, one of which is 3 cm longer than the other, are lighted. The longer one is lighted at 5.30 p.m. and the shorter at 7 p.m. At 9.30 p.m. they both are of the same length. The longer one burns out at 11.30 p.m. and the shorter one at 11 p.m. How long was each candle originally?

Indices

7

INTRODUCTION

In previous classes, we have read that if a is any real number and n is a natural number, then

$$a^n = a \times a \times a.... n \text{ times}$$

where a is called the **base**, n is called the **exponent** or **index** and a^n is the **exponential** form. We defined

$$a^0 = 1 \text{ and } a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}, a \neq 0$$

For example:

(i) $3^0 = 1, \quad 3^5 = 3.3.3.3.3 = 243, \quad 3^{-5} = \frac{1}{3^5} = \frac{1}{243}$

(ii) $\left(\frac{2}{5}\right)^0 = 1, \left(\frac{2}{5}\right)^3 = \frac{2}{5}.\frac{2}{5}.\frac{2}{5} = \frac{8}{125}, \left(\frac{2}{5}\right)^{-3} = \frac{1}{\left(\frac{2}{5}\right)^3} = \frac{1}{\frac{8}{125}} = \frac{125}{8}$

(iii) $(-2)^0 = 1, (-2)^3 = (-2)(-2)(-2) = -8, (-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8}$

Also, we learnt the laws of exponents.

If a, b are rational numbers and m, n are integers, then the following results hold:

(i) $a^m . a^n = a^{m+n}$ (ii) $(a^m)^n = a^{mn}$ (iii) $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$

(iv) $a^m . b^m = (ab)^m$ (v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$ (vi) $a^0 = 1, a \neq 0$

(vii) $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}, a \neq 0$ (viii) $a^n = b^n, n \neq 0 \Rightarrow a = b$

(ix) $a^m = a^n, a \neq 1 \Rightarrow m = n$

In this chapter, we will extend these laws when the base is a positive real number and the exponents are rational numbers.

7.1 FRACTIONAL INDICES (OR SURDS)

We know that the square root of a positive real number a is that number which when multiplied by itself gives a as the product.

Thus, if b is the square root a, then $b \times b = a$ i.e. $b^2 = a$

The square root of the number a is denoted by $\sqrt{a}$, so $b = \sqrt{a}$

The concept of square root can be extended to cube root, fourth root, ..., nth root, where n is a natural number.

Let a be a positive real number and n be a natural number, then $\sqrt[n]{a} = b$ if and only if $b^n = a$, $b > 0$.

For example: $\sqrt[3]{8} = 2$ because $2^3 = 8$; $\sqrt[5]{243} = 3$ because $3^5 = 243$. Note that the symbol $\sqrt{\ }$ used in $\sqrt{5}, \sqrt[3]{8}, \sqrt[5]{243}$ is called the **radical sign.**

In the language of exponents, we write $\sqrt[n]{a} = a^{\frac{1}{n}}$

We write $\sqrt[3]{8} = 8^{\frac{1}{3}}, \sqrt[5]{243} = (243)^{\frac{1}{5}}$ etc.

Now let us try to understand what is $8^{\frac{2}{3}}$?

There are two ways:

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4; \qquad 8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

This leads us to the following definition:

If $a > 0$ is a real number and m, n are integers, $n > 0$, m, n have no common factors except 1, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

7.2 LAWS OF EXPONENTS FOR REAL NUMBERS

Laws of exponents for real numbers are:

If a, b are positive real numbers and m, n are rational numbers, then the following results hold:

(i) $a^m \cdot a^n = a^{m+n}$ (ii) $(a^m)^n = a^{mn}$ (iii) $\frac{a^m}{a^n} = a^{m-n}$

(iv) $a^m \cdot b^m = (ab)^m$ (v) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ (vi) $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$

(vii) $a^n = b^n, n \neq 0 \Rightarrow a = b$ (viii) $a^m = a^n \Rightarrow m = n$ provided $a \neq 1$

Note

If p and q are different positive prime integers, then $p^m q^n = p^l q^k \Rightarrow m = l$ and $n = k$

Illustrative Examples

Example 1. Simplify the following:

(i) $(5x^4y^5)(21x^3y^2)$ (ii) $\frac{5x^4y^5}{20x^3y^2}$

(iii) $\left(\frac{-3x^2y}{z^3}\right)^3$ (iv) $\sqrt[3]{125^{-2}}$

Solution. (i) $(5x^4y^5)(21x^3y^2) = 5 \times 21x^4 \cdot x^3 \cdot y^5 \cdot y^2 = 105x^7y^7$

(ii) $\frac{5x^4y^5}{20x^3y^2} = \frac{5}{20} \times \frac{x^4}{x^3} \times \frac{y^5}{y^2} = \frac{1}{4}x^{4-3} \times y^{5-2} = \frac{1}{4}xy^3$

(iii) $\left(\frac{-3x^2y}{z^3}\right)^3 = \frac{(-3)^3(x^2)^3(y)^3}{(z^3)^3} = \frac{-27x^6y^3}{z^9}$

(iv) $\sqrt[3]{125^{-2}} = (125^{-2})^{1/3} = (125)^{-2/3} = (5^3)^{-2/3}$

$= 5^{3 \times (-2/3)} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

Example 2. Simplify the following:

(i) $\left[\left((625)^{-\frac{1}{2}}\right)^{-\frac{1}{4}}\right]^2$ (ii) $\frac{9^{\frac{1}{3}} \times 27^{-\frac{1}{2}}}{3^{\frac{1}{6}} \times 3^{-\frac{2}{3}}}$

(iii) $(256)^{-\left(4^{-\frac{3}{2}}\right)}$ (iv) $\sqrt[4]{28} \div \sqrt[3]{7}$

Solution. (i) $\left[\left((625)^{-\frac{1}{2}}\right)^{-\frac{1}{4}}\right]^2 = \left[(625)^{\left(-\frac{1}{2}\right)\times\left(-\frac{1}{4}\right)}\right]^2 = \left[(625)^{\frac{1}{8}}\right]^2$

$= (625)^{\frac{1}{8}\times 2} = (625)^{\frac{1}{4}} = (5^4)^{\frac{1}{4}} = 5^{4\times\frac{1}{4}} = 5^1 = 5.$

(ii) $\frac{9^{\frac{1}{3}} \times 27^{-\frac{1}{2}}}{3^{\frac{1}{6}} \times 3^{-\frac{2}{3}}} = \frac{(3^2)^{\frac{1}{3}} \times (3^3)^{-\frac{1}{2}}}{3^{\frac{1}{6}} \times 3^{-\frac{2}{3}}} = \frac{3^{\frac{2}{3}} \times 3^{-\frac{3}{2}}}{3^{\frac{1}{6}} \times 3^{-\frac{2}{3}}}$

$= 3^{\frac{2}{3}+\left(-\frac{3}{2}\right)-\frac{1}{6}-\left(-\frac{2}{3}\right)} = 3^{\frac{2}{3}-\frac{3}{2}-\frac{1}{6}+\frac{2}{3}}$

$= 3^{\frac{4-9-1+4}{6}} = 3^{-\frac{2}{6}} = 3^{-\frac{1}{3}}$

(iii) Note that $4^{-\frac{3}{2}} = (2^2)^{-\frac{3}{2}} = 2^{2\times\left(-\frac{3}{2}\right)} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$\therefore \ (256)^{-\left(4^{-\frac{3}{2}}\right)} = (256)^{-\frac{1}{8}} = (2^8)^{-\frac{1}{8}} = 2^{8\times\left(-\frac{1}{8}\right)} = 2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

(iv) $\sqrt[4]{28} \div \sqrt[3]{7} = (28)^{\frac{1}{4}} \div (7)^{\frac{1}{3}}$

L.C.M. of 4 and 3 = 12, so write each number with exponent $\frac{1}{12}$

$(28)^{\frac{1}{4}} = ((28)^3)^{\frac{1}{12}}$ and $7^{\frac{1}{3}} = (7^4)^{\frac{1}{12}}$

$\therefore \ \sqrt[4]{28} \div \sqrt[3]{7} = \frac{((28)^3)^{\frac{1}{12}}}{(7^4)^{\frac{1}{12}}} = \left(\frac{(28)^3}{7^4}\right)^{\frac{1}{12}} = \left(\frac{28 \times 28 \times 28}{7 \times 7 \times 7 \times 7}\right)^{\frac{1}{12}}$

$= \left(\frac{4 \times 4 \times 4}{7}\right)^{\frac{1}{12}} = \left(\frac{64}{7}\right)^{\frac{1}{12}} = \sqrt[12]{\frac{64}{7}}$

Example 3. Simplify the following:

(i) $\left[5\left\{\left(\frac{1}{8}\right)^{-\frac{1}{3}} + \left(\frac{1}{27}\right)^{-\frac{1}{3}}\right\}\right]^{-\frac{1}{2}}$ (ii) $\sqrt[4]{81} - 8\sqrt[3]{216} + 15\sqrt[5]{32} + 2\sqrt{225}$

(iii) $4\sqrt[4]{80} - 3\sqrt[4]{405} - 2\sqrt[4]{5}$ (iv) $\frac{(\sqrt{206} - \sqrt{125})^{\frac{1}{4}}}{(\sqrt{206} + \sqrt{125})^{-\frac{1}{4}}}$

Solution. (*i*) $\left[5\left\{\left\{\frac{1}{8}\right\}^{-\frac{1}{3}}+\left(\frac{1}{27}\right)^{-\frac{1}{3}}\right\}\right]^{-\frac{1}{2}} = \left[5\left\{8^{\frac{1}{3}}+27^{\frac{1}{3}}\right\}\right]^{-\frac{1}{2}}$

$$= \left[5\left\{(2^3)^{\frac{1}{3}}+(3^3)^{\frac{1}{3}}\right\}\right]^{-\frac{1}{2}} = \left[5(2^{3\times\frac{1}{3}}+3^{3\times\frac{1}{3}})\right]^{-\frac{1}{2}}$$

$$= \{5(2+3)\}^{-\frac{1}{2}} = \{5\times5\}^{-\frac{1}{2}} = (5^2)^{-\frac{1}{2}}$$

$$= 5^{2\times\left(-\frac{1}{2}\right)} = 5^{-1} = \frac{1}{5}$$

(*ii*) $\sqrt[4]{81} - 8\sqrt[3]{216} + 15\sqrt[5]{32} + 2\sqrt{225}$

$$= (3^4)^{\frac{1}{4}} - 8(6^3)^{\frac{1}{3}} + 15(2^5)^{\frac{1}{5}} + 2(15^2)^{\frac{1}{2}}$$

$$= 3^{4\times\frac{1}{4}} - 8\left(6^{3\times\frac{1}{3}}\right) + 15\left(2^{5\times\frac{1}{5}}\right) + 2\left(15^{2\times\frac{1}{2}}\right)$$

$$= 3^1 - 8\times6^1 + 15\times2^1 + 2\times15^1$$

$$= 3 - 8\times6 + 15\times2 + 2\times15$$

$$= 3 - 48 + 30 + 30 = 15$$

(*iii*) $4\times\sqrt[4]{80} - 3\times\sqrt[4]{405} - 2\times\sqrt[4]{5} = 4\times(2^4\times5)^{\frac{1}{4}} - 3\times(3^4\times5)^{\frac{1}{4}} - 2\times(5)^{\frac{1}{4}}$

$$= 4\times2^{4\times\frac{1}{4}}\times5^{\frac{1}{4}} - 3\times3^{4\times\frac{1}{4}}\times5^{\frac{1}{4}} - 2\times5^{\frac{1}{4}}$$

$$= \{4\times2 - 3\times3 - 2\}\;5^{\frac{1}{4}} = -3\times5^{\frac{1}{4}} = -3\sqrt[4]{5}$$

(*iv*) $\dfrac{(\sqrt{206}-\sqrt{125})^{\frac{1}{4}}}{(\sqrt{206}+\sqrt{125})^{-\frac{1}{4}}} = (\sqrt{206}-\sqrt{125})^{\frac{1}{4}}(\sqrt{206}+\sqrt{125})^{\frac{1}{4}}$

$$= \{(\sqrt{206}-\sqrt{125})(\sqrt{206}+\sqrt{125})\}^{\frac{1}{4}} = \{(\sqrt{206})^2-(\sqrt{125})^2\}^{\frac{1}{4}}$$

$$= (206-125)^{\frac{1}{4}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$$

Example 4. Simplify: $\left(\frac{81}{16}\right)^{-\frac{3}{4}}\times\left[\left(\frac{25}{9}\right)^{-\frac{3}{2}}\div\left(\frac{5}{2}\right)^{-3}\right]$

Solution. $\left(\frac{81}{16}\right)^{-\frac{3}{4}}\times\left[\left(\frac{25}{9}\right)^{-\frac{3}{2}}\div\left(\frac{5}{2}\right)^{-3}\right]$

$$= \left(\left(\frac{3}{2}\right)^4\right)^{-\frac{3}{4}}\times\left[\left(\left(\frac{5}{3}\right)^2\right)^{-\frac{3}{2}}\div\left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{4\times\left(-\frac{3}{4}\right)}\times\left[\left(\frac{5}{3}\right)^{2\times\left(-\frac{3}{2}\right)}\div\left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{-3} \times \left[\left(\frac{5}{3}\right)^{-3} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{2}{3}\right)^{3} \times \left[\left(\frac{3}{5}\right)^{3} \div \left(\frac{2}{5}\right)^{3}\right] = \frac{2^3}{3^3} \times \left[\frac{3^3}{5^3} \div \frac{2^3}{5^3}\right] \qquad \left[\because a^{-n} = \left(\frac{1}{a}\right)^n\right]$$

$$= \frac{2^3}{3^3} \times \left(\frac{3^3}{5^3} \times \frac{5^3}{2^3}\right) = \frac{2^3}{3^3} \times \frac{3^3}{2^3} = 1$$

Example 5. Find the value of $\dfrac{4}{(216)^{-\frac{2}{3}}} + \dfrac{1}{(256)^{-\frac{3}{4}}} + \dfrac{2}{(243)^{-\frac{1}{5}}}$

Solution. $\dfrac{4}{(216)^{-\frac{2}{3}}} + \dfrac{1}{(256)^{-\frac{3}{4}}} + \dfrac{2}{(243)^{-\frac{1}{5}}}$

$$= 4(216)^{\frac{2}{3}} + (256)^{\frac{3}{4}} + 2(243)^{\frac{1}{5}}$$

$$= 4(6^3)^{\frac{2}{3}} + (4^4)^{\frac{3}{4}} + 2(3^5)^{\frac{1}{5}} = 4\left(6^{3\times\frac{2}{3}}\right) + 4^{4\times\frac{3}{4}} + 2\left(3^{5\times\frac{1}{5}}\right)$$

$$= 4(6^2) + 4^3 + 2(3^1) = 4 \times 36 + 64 + 6$$

$$= 144 + 64 + 6 = 214$$

Example 6. Evaluate: $\sqrt{\frac{1}{4}} + (0{\cdot}01)^{-1/2} - (27)^{2/3}$. Leave your answer as a fraction.

Solution. $\sqrt{\frac{1}{4}} + (0{\cdot}01)^{-1/2} - (27)^{2/3} = \frac{1}{2} + \left(\frac{1}{100}\right)^{-1/2} - (3^3)^{2/3}$

$$= \frac{1}{2} + (100)^{1/2} - 3^2 \qquad \left[\because a^{-n} = \left(\frac{1}{a}\right)^n\right]$$

$$= \frac{1}{2} + ((10)^2)^{1/2} - 9$$

$$= \frac{1}{2} + 10 - 9 = \frac{1}{2} + 1 = 1\frac{1}{2}$$

Example 7. Simplify: $\left(\frac{1}{4}\right)^{-2} - 3(8)^{2/3}(4)^0 + \left(\frac{9}{16}\right)^{-1/2}$

Solution. $\left(\frac{1}{4}\right)^{-2} - 3(8)^{2/3}(4)^0 + \left(\frac{9}{16}\right)^{-1/2} = (4)^2 - 3.(2^3)^{2/3}.1 + \left(\frac{16}{9}\right)^{1/2}$

$$\left[\because a^{-n} = \left(\frac{1}{a}\right)^n \text{ and } a^0 = 1\right]$$

$$= 16 - 3 \times 2^2 \times 1 + \left(\left(\frac{4}{3}\right)^2\right)^{1/2}$$

$$= 16 - 3 \times 4 + \frac{4}{3} = 16 - 12 + \frac{4}{3}$$

$$= 4 + \frac{4}{3} = \frac{16}{3} = 5\frac{1}{3}$$

Example 8. Simplify the following:

(i) $\dfrac{5^{n+2} - 6.5^{n+1}}{13.5^n - 2.5^{n+1}}$ (ii) $\left(\dfrac{x^m}{x^n}\right)^{m+n}\left(\dfrac{x^n}{x^l}\right)^{n+l}\left(\dfrac{x^l}{x^m}\right)^{l+m}$

(iii) $\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)\left(x^{\frac{2}{3}} - 1 + x^{-\frac{2}{3}}\right)$

Solution. (*i*) $\dfrac{5^{n+2} - 6.5^{n+1}}{13.5^n - 2.5^{n+1}} = \dfrac{5^n.5^2 - 6.5^n.5^1}{13.5^n - 2.5^1.5^n}$

$$= \frac{5^n(5^2 - 6 \times 5)}{5^n(13 - 2 \times 5)} = \frac{25 - 30}{13 - 10} = \frac{-5}{3} = -\frac{5}{3}$$

(*ii*) $\left(\dfrac{x^m}{x^n}\right)^{m+n}\left(\dfrac{x^n}{x^l}\right)^{n+l}\left(\dfrac{x^l}{x^m}\right)^{l+m}$

$$= (x^{m-n})^{m+n}.(x^{n-l})^{n+l}.(x^{l-m})^{l+m}$$
$$= x^{m^2-n^2}.x^{n^2-l^2}.x^{l^2-m^2}$$
$$= x^{m^2-n^2+n^2-l^2+l^2-m^2} = x^0 = 1$$

(*iii*) $\left(x^{\frac{1}{3}} + x^{-\frac{1}{3}}\right)\left(x^{\frac{2}{3}} - 1 + x^{-\frac{2}{3}}\right) = x^{\frac{1}{3}}.x^{\frac{2}{3}} - x^{\frac{1}{3}} + x^{\frac{1}{3}}.x^{-\frac{2}{3}} + x^{-\frac{1}{3}}.x^{\frac{2}{3}} - x^{-\frac{1}{3}} + x^{-\frac{1}{3}}.x^{-\frac{2}{3}}$

$$= x^1 - x^{\frac{1}{3}} + x^{-\frac{1}{3}} + x^{\frac{1}{3}} - x^{-\frac{1}{3}} + x^{-1}$$
$$= x + x^{-1} = x + \frac{1}{x}$$

Example 9. If $a = b^{2x}$, $b = c^{2y}$ and $c = a^{2z}$, prove that $xyz = \dfrac{1}{8}$

Solution. Given $\quad a = b^{2x} \quad \ldots(i) \qquad b = c^{2y} \quad \ldots(ii) \qquad c = a^{2z} \quad \ldots(iii)$

Substituting the value of b from (*ii*) in (*i*), we get

$$a = (c^{2y})^{2x} = c^{4xy} \quad \ldots(iv)$$

Substituting the value of c from (*iii*) in (*iv*), we get

$$a = (a^{2z})^{4xy} = a^{8xyz}$$
$$\Rightarrow \quad a^1 = a^{8xyz} \Rightarrow 1 = 8xyz \qquad \text{(Assume } a > 0, a \neq 1)$$
$$\Rightarrow \quad xyz = \frac{1}{8}.$$

Example 10. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that $y = \dfrac{2xz}{z + x}$

Solution. Let $a^x = b^y = c^z = k$ (say), then

$$a = k^{\frac{1}{x}},\ b = k^{\frac{1}{y}} \text{ and } c = k^{\frac{1}{z}}$$

Given $\quad b^2 = ac \Rightarrow \left(k^{\frac{1}{y}}\right)^2 = k^{\frac{1}{x}}.k^{\frac{1}{z}}$

$$\Rightarrow \quad k^{\frac{2}{y}} = k^{\frac{1}{x}+\frac{1}{z}} \Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$
$$\Rightarrow \quad \frac{2}{y} = \frac{z + x}{xz} \Rightarrow y = \frac{2xz}{z + x}$$

Example 11. If $2^x = 98^y = 7^z$, prove that, $z = \dfrac{2xy}{x - y}$

Solution. Let $2^x = 98^y = 7^z = k \Rightarrow 2^x = k,\ 98^y = k,\ 7^z = k$

$$\Rightarrow \quad 2 = k^{\frac{1}{x}},\ 98 = k^{\frac{1}{y}},\ 7 = k^{\frac{1}{z}}$$

Now, $98 = 2 \times 7^2$

$$\Rightarrow \quad k^{\frac{1}{y}} = \left(k^{\frac{1}{x}}\right) \times \left(k^{\frac{1}{z}}\right)^2 \qquad \text{...(From (}i\text{))}$$

$$\Rightarrow \quad k^{\frac{1}{y}} = k^{\frac{1}{x}} \times k^{\frac{2}{z}} \Rightarrow k^{\frac{1}{y}} = k^{\frac{1}{x}+\frac{2}{z}}$$

$$\Rightarrow \quad \frac{1}{y} = \frac{1}{x} + \frac{2}{z} \Rightarrow \frac{1}{y} - \frac{1}{x} = \frac{2}{z}$$

$$\Rightarrow \quad \frac{2}{z} = \frac{x-y}{xy} \Rightarrow z = \frac{2xy}{x-y}$$

Example 12. Prove that $\dfrac{a^{-1}}{a^{-1}+b^{-1}} + \dfrac{a^{-1}}{a^{-1}-b^{-1}} = \dfrac{2b^2}{b^2-a^2}$

Solution.
$$\frac{a^{-1}}{a^{-1}+b^{-1}} + \frac{a^{-1}}{a^{-1}-b^{-1}} = \frac{\frac{1}{a}}{\frac{1}{a}+\frac{1}{b}} + \frac{\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}$$

$$= \frac{\frac{1}{a}}{\frac{b+a}{ab}} + \frac{\frac{1}{a}}{\frac{b-a}{ab}} = \frac{1}{a} \times \frac{ab}{b+a} + \frac{1}{a} \times \frac{ab}{b-a}$$

$$= \frac{b}{b+a} + \frac{b}{b-a} = b\left(\frac{1}{b+a} + \frac{1}{b-a}\right)$$

$$= b\left(\frac{b-a+b+a}{b^2-a^2}\right) = \frac{2b^2}{b^2-a^2}$$

Example 13. If $abc = 1$, show that

$$\frac{1}{1+a+b^{-1}} + \frac{1}{1+b+c^{-1}} + \frac{1}{1+c+a^{-1}} = 1$$

Solution.
$$\frac{1}{1+a+b^{-1}} + \frac{1}{1+b+c^{-1}} + \frac{1}{1+c+a^{-1}}$$

$$= \frac{1}{1+a+\frac{1}{b}} + \frac{1}{1+b+\frac{1}{c}} + \frac{1}{1+c+\frac{1}{a}}$$

$$= \frac{b}{b+ab+1} + \frac{1}{1+b+ab} + \frac{1}{1+\frac{1}{ab}+\frac{1}{a}} \qquad \left(\because abc = 1 \Rightarrow \frac{1}{c} = ab \text{ and } c = \frac{1}{ab}\right)$$

$$= \frac{b}{1+b+ab} + \frac{1}{1+b+ab} + \frac{ab}{ab+1+b} = \frac{b+1+ab}{1+b+ab} = 1$$

Example 14. If $\dfrac{9^{n+1}\left(3^{-\frac{n}{2}}\right)^{-2} - 27^n}{(2\times 3^m)^3} = \dfrac{1}{729}$, then prove that $m - n = 2$

Solution. Given $\dfrac{9^{n+1}\left(3^{-\frac{n}{2}}\right)^{-2} - 27^n}{(2\times 3^m)^3} = \dfrac{1}{729}$

$$\Rightarrow \quad \frac{(3^2)^{n+1}\left(3^{\left(-\frac{n}{2}\right)(-2)}\right) - (3^3)^n}{2^3 \times 3^{m\times 3}} = \frac{1}{729}$$

$$\Rightarrow \quad \frac{3^{2n+2} \times 3^n - 3^{3\times n}}{2^3 \times 3^{3m}} = \frac{1}{3^6}$$

$$\Rightarrow \quad \frac{3^{3n}.3^2 - 3^{3n}}{2^3 \times 3^{3m}} = \frac{1}{3^6} \Rightarrow \frac{3^{3n}(3^2 - 1)}{8 \times 3^{3m}} = \frac{1}{3^6}$$

$$\Rightarrow \quad \frac{3^{3n} \times 8}{8 \times 3^{3m}} = \frac{1}{3^6} \Rightarrow 3^{3n-3m} = 3^{-6}$$

$$\Rightarrow \quad 3n - 3m = -6 \Rightarrow n - m = -2$$

$$\Rightarrow \quad m - n = 2$$

Example 15. If $x = \sqrt[3]{28}$ and $y = \sqrt[3]{27}$, find the value of $x + y - \frac{1}{x^2 + xy + y^2}$

Solution. $x + y - \frac{1}{x^2 + xy + y^2} = x + y - \frac{x - y}{(x - y)(x^2 + xy + y^2)}$ (Note this step)

$$= x + y - \frac{x - y}{x^3 - y^3}$$

$$= x + y - \frac{x - y}{((28)^{1/3})^3 - ((27)^{1/3})^3}$$

$$= x + y - \frac{x - y}{28 - 27} = x + y - \frac{x - y}{1}$$

$$= x + y - (x - y) = 2y$$

$$= 2 \times \sqrt[3]{27} = 2 \times (27)^{1/3}$$

$$= 2 \times (3^3)^{\frac{1}{3}} = 2 \times 3^1 = 6$$

Example 16. Given $1176 = 2^p \,.\, 3^q \,.\, 7^r$, find

(*i*) the numerical values of p, q and r (*ii*) the value of $2^p.3^q.7^{-r}$ as a fraction.

Solution. (*i*) Given $1176 = 2^p.3^q.7^r$

$$\Rightarrow \quad 2 \times 2 \times 2 \times 3 \times 7 \times 7 = 2^p.3^q.7^r$$

$$\Rightarrow \quad 2^3.3^1.7^2 = 2^p.3^q.7^r$$

$$\Rightarrow \quad p = 3,\ q = 1 \text{ and } r = 2$$

(*ii*) $2^p.3^q.7^{-r} = 2^3.3^1.7^{-2} = \frac{8 \times 3}{7^2} = \frac{24}{49}$

Example 17. If $\left(\frac{p^{-1}q^2}{p^3q^{-2}}\right)^{1/3} \div \left(\frac{p^6q^{-3}}{p^{-2}q^3}\right)^{1/2} = p^aq^b$, prove that $a + b + 1 = 0$, where p and q are different positive primes.

Solution.

$$\left(\frac{p^{-1}q^2}{p^3q^{-2}}\right)^{1/3} = (p^{-1-3}q^{2-(-2)})^{1/3} = (p^{-4}q^4)^{1/3} = p^{-\frac{4}{3}}q^{\frac{4}{3}},$$

$$\left(\frac{p^6q^{-3}}{p^{-2}q^3}\right)^{1/2} = (p^{6-(-2)}q^{-3\,-3})^{1/2} = (p^8q^{-6})^{1/2} = p^4q^{-3}$$

$$\therefore \quad \left(\frac{p^{-1}q^2}{p^3q^{-2}}\right)^{1/3} \div \left(\frac{p^6q^{-3}}{p^{-2}q^3}\right)^{1/2} = p^aq^b$$

$$\Rightarrow \quad \frac{p^{-\frac{4}{3}}q^{\frac{4}{3}}}{p^4q^{-3}} \Rightarrow p^aq^b \Rightarrow p^{-\frac{4}{3}-4}\, q^{\frac{4}{3}-(-3)} = p^aq^b$$

$\Rightarrow p^{-\frac{16}{3}} q^{\frac{13}{3}} = p^a q^b$

$\Rightarrow -\frac{16}{3} = a$ and $\frac{13}{3} = b$ (∵ p, q are different positive primes)

$a + b + 1 = -\frac{16}{3} + \frac{13}{3} + 1 = \frac{-16+13+3}{3} = 0$

Example 18. Solve the following equations for x:

(i) $4^{2x} = \frac{1}{32}$ (ii) $\sqrt{\left(\frac{3}{5}\right)^{1-2x}} = 4\frac{17}{27}$

Solution. (i) Given $4^{2x} = \frac{1}{32} \Rightarrow (2^2)^{2x} = \left(\frac{1}{2}\right)^5$

$\Rightarrow 2^{4x} = 2^{-5}$ $\left[\because \left(\frac{1}{a}\right)^n = a^{-n}\right]$

$\Rightarrow 4x = -5 \Rightarrow x = -\frac{5}{4}$

(ii) Given $\sqrt{\left(\frac{3}{5}\right)^{1-2x}} = 4\frac{17}{27} \Rightarrow \left(\left(\frac{3}{5}\right)^{1-2x}\right)^{1/2} = \frac{125}{27}$

$\Rightarrow \left(\frac{3}{5}\right)^{\frac{1-2x}{2}} = \left(\frac{5}{3}\right)^3 \Rightarrow \left(\frac{3}{5}\right)^{\frac{1-2x}{2}} = \left(\frac{3}{5}\right)^{-3}$ $\left[\because \left(\frac{1}{a}\right)^n = a^{-n}\right]$

$\Rightarrow \frac{1-2x}{2} = -3 \Rightarrow 1 - 2x = -6$

$\Rightarrow -2x = -6 - 1 \Rightarrow -2x = -7 \Rightarrow x = \frac{7}{2}$

Example 19. Solve the following equations for x:

(i) $\sqrt{\left(8^0 + \frac{2}{3}\right)} = (0.6)^{2-3x}$ (ii) $2^3(5^0 + 3^{2x}) = 8\frac{8}{27}$

Solution. (i) Given $\sqrt{\left(8^0 + \frac{2}{3}\right)} = (0.6)^{2-3x}$

$\Rightarrow \sqrt{\left(1 + \frac{2}{3}\right)} = \left(\frac{3}{5}\right)^{2-3x} \Rightarrow \left(\frac{5}{3}\right)^{\frac{1}{2}} = \left(\frac{3}{5}\right)^{2-3x}$

$\Rightarrow \left(\frac{3}{5}\right)^{-\frac{1}{2}} = \left(\frac{3}{5}\right)^{2-3x}$ $\left[\because \left(\frac{1}{a}\right)^n = a^{-n}\right]$

$\Rightarrow -\frac{1}{2} = 2 - 3x$

$\Rightarrow 3x = 2 + \frac{1}{2} \Rightarrow 3x = \frac{5}{2} \Rightarrow x = \frac{5}{6}$

(ii) Given $2^3(5^0 + 3^{2x}) = 8\frac{8}{27}$

$\Rightarrow 8(1 + 3^{2x}) = 8 + \frac{8}{27} \Rightarrow 8 + 8 \times 3^{2x} = 8 + \frac{8}{27}$

$\Rightarrow 8 \times 3^{2x} = \frac{8}{27} \Rightarrow 3^{2x} = \frac{1}{27}$

$\Rightarrow 3^{2x} = \frac{1}{3^3} \Rightarrow 3^{2x} = 3^{-3}$

$\Rightarrow 2x = -3 \Rightarrow x = -\frac{3}{2}$

Example 20. If $a = \dfrac{2^{x-1}}{2^{x-2}}, b = \dfrac{2^{-x}}{2^{x+1}}$ and $a - b = 0$, then find the value of x.

Solution. $a - b = 0 \Rightarrow a = b \Rightarrow \dfrac{2^{x-1}}{2^{x-2}} = \dfrac{2^{-x}}{2^{x+1}}$

$\Rightarrow \quad 2^{(x-1)-(x-2)} = 2^{-x-(x+1)}$

$\Rightarrow \quad 2^1 = 2^{-2x-1} \Rightarrow 1 = -2x - 1$

$\Rightarrow \quad 2x = -2 \Rightarrow x = -1$

Example 21. If $5^{2x-1} = 25^{x-1} + 100$, find the value of 3^{1+x}

Solution. Given $5^{2x-1} = 25^{x-1} + 100$

$\Rightarrow \quad 5^{2x-1} = (5^2)^{x-1} + 100 \Rightarrow 5^{2x-1} - 5^{2x-2} = 100$

$\Rightarrow \quad 5^{2x-2} . 5^1 - 5^{2x-2} = 100 \Rightarrow 5^{2x-2}(5-1) = 100$

$\Rightarrow \quad 5^{2x-2} \times 4 = 100 \Rightarrow 5^{2x-2} = 25$

$\Rightarrow \quad 5^{2x-2} = 5^2 \Rightarrow 2x - 2 = 2$

$\Rightarrow \quad 2x = 4 \Rightarrow x = 2.$

$\therefore \quad 3^{1+x} = 3^{1+2} = 3^3 = 27$

Example 22. Determine $(8x)^x$ if $9^{x+2} = 240 + 9^x$

Solution. Given $9^{x+2} = 240 + 9^x$

$\Rightarrow \quad 9^x.9^2 - 9^x = 240 \Rightarrow (9^2 - 1)\,9^x = 240$

$\Rightarrow \quad (81 - 1)\,9^x = 240 \Rightarrow 80 \times (3^2)^x = 240$

$\Rightarrow \quad 3^{2x} = \dfrac{240}{80} = 3 = 3^1 \Rightarrow 2x = 1 \Rightarrow x = \dfrac{1}{2}$

$\therefore \quad (8x)^x = \left(8 \times \dfrac{1}{2}\right)^{\frac{1}{2}} = 4^{\frac{1}{2}} = (2^2)^{\frac{1}{2}} = 2^{2\times\frac{1}{2}} = 2^1 = 2$

Example 23. Solve for x and y:

$$(\sqrt{32})^x \div 2^{y+1} = 1, \quad 16^{4-\frac{x}{2}} - 8^y = 0$$

Solution. Given $(\sqrt{32})^x \div 2^{y+1} = 1$

$\Rightarrow \quad \left(\left(2^5\right)^{\frac{1}{2}}\right)^x \div 2^{y+1} = 1$

$\Rightarrow \quad \dfrac{\left(2^{\frac{5}{2}}\right)^x}{2^{y+1}} = 1 \Rightarrow 2^{\frac{5x}{2}-y-1} = 2^0$

$\Rightarrow \quad \dfrac{5x}{2} - y - 1 = 0 \Rightarrow 5x - 2y - 2 = 0$

Also $\quad 16^{4-\frac{x}{2}} - 8^y = 0 \Rightarrow (2^4)^{4-\frac{x}{2}} = (2^3)^y$

$\Rightarrow \quad 2^{16-2x} = 2^{3y} \Rightarrow 16 - 2x = 3y$

$\Rightarrow \quad 2x + 3y - 16 = 0$ ...(*ii*)

Multiplying (*i*) by 3 and (*ii*) by 2, we get

$15x - 6y - 6 = 0$ ...(*iii*)

and $\quad 4x + 6y - 32 = 0$ ...(*iv*)

Adding (*iii*) and (*iv*), we get

$$19x - 38 = 0 \Rightarrow x = 2$$

Substituting $x = 2$ in (*ii*), we get

$$2 \times 2 + 3y - 16 = 0 \Rightarrow 3y = 12 \Rightarrow y = 4$$

Hence, the solution is $x = 2,\ y = 4$

Exercise 7

Simplify the following (1 to 20):

1. (*i*) $\left(-\frac{243}{32}\right)^{-\frac{3}{5}}$ (*ii*) $\left(5\frac{23}{64}\right)^{-\frac{2}{3}}$

2. (*i*) $(2a^{-3}b^2)^3$ (*ii*) $\frac{a^{-1}+b^{-1}}{(ab)^{-1}}$

3. (*i*) $\frac{x^{-1}y^{-1}}{x^{-1}+y^{-1}}$ (*ii*) $\frac{(4 \times 10^7)(6 \times 10^{-5})}{8 \times 10^{10}}$

4. (*i*) $\frac{3a}{b^{-1}}+\frac{2b}{a^{-1}}$ (*ii*) $5^0 \times 4^{-1} + 8^{1/3}$

5. (*i*) $\left(\frac{8}{125}\right)^{-\frac{1}{3}}$ (*ii*) $(0{\cdot}027)^{-\frac{1}{3}}$

6. (*i*) $\left(-\frac{1}{27}\right)^{-\frac{2}{3}}$ (*ii*) $(64)^{-\frac{2}{3}} \div 9^{-\frac{3}{2}}$

7. (*i*) $\frac{(27)^{\frac{2n}{3}} \times (8)^{-\frac{n}{6}}}{(18)^{-\frac{n}{2}}}$ (*ii*) $\frac{5.(25)^{n+1} - 25.(5)^{2n}}{5.(5)^{2n+3} - (25)^{n+1}}$

8. (*i*) $\left[8^{-\frac{4}{3}} \div 2^{-2}\right]^{1/2}$ (*ii*) $\left(\frac{27}{8}\right)^{2/3} - \left(\frac{1}{4}\right)^{-2} + 5^0$

9. (*i*) $(3x^2)^{-3} \times (x^9)^{2/3}$ (*ii*) $(8x^4)^{1/3} \div x^{1/3}$

10. (*i*) $(3^2)^0 + 3^{-4} \times 3^6 + \left(\frac{1}{3}\right)^{-2}$ (*ii*) $9^{5/2} - 3.(5)^0 - \left(\frac{1}{81}\right)^{-1/2}$

11. (*i*) $16^{3/4} + 2\left(\frac{1}{2}\right)^{-1}(3)^0$ (*ii*) $(81)^{3/4} - \left(\frac{1}{32}\right)^{-2/5} + (8)^{1/3}\left(\frac{1}{2}\right)^{-1}(2)^0$

12. (*i*) $\left(\frac{64}{125}\right)^{-\frac{2}{3}} + \frac{1}{\left(\frac{256}{625}\right)^{\frac{1}{4}}} + \left(\frac{\sqrt{25}}{\sqrt[3]{64}}\right)^0$ (*ii*) $\frac{5^{n+3} - 6 \times 5^{n+1}}{9 \times 5^n - 2^2 \times 5^n}$

13. (*i*) $\left[(64)^{\frac{2}{3}}2^{-2} \div 8^0\right]^{-1/2}$ (*ii*) $3^n \times 9^{n+1} \div (3^{n-1} \times 9^{n-1})$

14. (*i*) $\frac{\sqrt{2^2} \times \sqrt[4]{256}}{\sqrt[3]{64}} - \left(\frac{1}{2}\right)^{-2}$ (*ii*) $\frac{3^{-\frac{6}{7}} \times 4^{-\frac{3}{7}} \times 9^{\frac{3}{7}} \times 2^{\frac{6}{7}}}{2^2 + 2^0 + 2^{-2}}$

Adding (*iii*) and (*iv*), we get

$$19x - 38 = 0 \Rightarrow x = 2$$

Substituting $x = 2$ in (*ii*), we get

$$2 \times 2 + 3y - 16 = 0 \Rightarrow 3y = 12 \Rightarrow y = 4$$

Hence, the solution is $x = 2, y = 4$

Exercise 7

Simplify the following (1 to 20):

1. (*i*) $\left(-\frac{243}{32}\right)^{-\frac{3}{5}}$ (*ii*) $\left(5\frac{23}{64}\right)^{-\frac{2}{3}}$

2. (*i*) $(2a^{-3}b^2)^3$ (*ii*) $\frac{a^{-1}+b^{-1}}{(ab)^{-1}}$

3. (*i*) $\frac{x^{-1}y^{-1}}{x^{-1}+y^{-1}}$ (*ii*) $\frac{(4\times10^7)(6\times10^{-5})}{8\times10^{10}}$

4. (*i*) $\frac{3a}{b^{-1}}+\frac{2b}{a^{-1}}$ (*ii*) $5^0\times4^{-1}+8^{1/3}$

5. (*i*) $\left(\frac{8}{125}\right)^{-\frac{1}{3}}$ (*ii*) $(0{\cdot}027)^{-\frac{1}{3}}$

6. (*i*) $\left(-\frac{1}{27}\right)^{-\frac{2}{3}}$ (*ii*) $(64)^{-\frac{2}{3}}\div9^{-\frac{3}{2}}$

7. (*i*) $\frac{(27)^{\frac{2n}{3}}\times(8)^{-\frac{n}{6}}}{(18)^{-\frac{n}{2}}}$ (*ii*) $\frac{5.(25)^{n+1}-25.(5)^{2n}}{5.(5)^{2n+3}-(25)^{n+1}}$

8. (*i*) $\left[8^{-\frac{4}{3}}\div2^{-2}\right]^{1/2}$ (*ii*) $\left(\frac{27}{8}\right)^{2/3}-\left(\frac{1}{4}\right)^{-2}+5^0$

9. (*i*) $(3x^2)^{-3}\times(x^9)^{2/3}$ (*ii*) $(8x^4)^{1/3}\div x^{1/3}$

10. (*i*) $(3^2)^0+3^{-4}\times3^6+\left(\frac{1}{3}\right)^{-2}$ (*ii*) $9^{5/2}-3.(5)^0-\left(\frac{1}{81}\right)^{-1/2}$

11. (*i*) $16^{3/4}+2\left(\frac{1}{2}\right)^{-1}(3)^0$ (*ii*) $(81)^{3/4}-\left(\frac{1}{32}\right)^{-2/5}+(8)^{1/3}\left(\frac{1}{2}\right)^{-1}(2)^0$

12. (*i*) $\left(\frac{64}{125}\right)^{-\frac{2}{3}}+\frac{1}{\left(\frac{256}{625}\right)^{\frac{1}{4}}}+\left(\frac{\sqrt{25}}{\sqrt[3]{64}}\right)^0$ (*ii*) $\frac{5^{n+3}-6\times5^{n+1}}{9\times5^n-2^2\times5^n}$

13. (*i*) $\left[(64)^{\frac{2}{3}}2^{-2}\div8^0\right]^{-1/2}$ (*ii*) $3^n\times9^{n+1}\div(3^{n-1}\times9^{n-1})$

14. (*i*) $\frac{\sqrt{2^2}\times\sqrt[4]{256}}{\sqrt[3]{64}}-\left(\frac{1}{2}\right)^{-2}$ (*ii*) $\frac{3^{-\frac{6}{7}}\times4^{-\frac{3}{7}}\times9^{\frac{3}{7}}\times2^{\frac{6}{7}}}{2^2+2^0+2^{-2}}$

15. (i) $\frac{(32)^{\frac{2}{5}} \times (4)^{-\frac{1}{2}} \times (8)^{\frac{1}{3}}}{2^{-2} + (64)^{-1/3}}$ (ii) $\frac{5^{2(x+6)} \times (25)^{-7+2x}}{(125)^{2x}}$

16. (i) $\frac{7^{2n+3} - (49)^{n+2}}{((343)^{n+1})^{2/3}}$ (ii) $(27)^{4/3} + (32)^{0.8} + (0.8)^{-1}$

17. (i) $(\sqrt{32} - \sqrt{5})^{\frac{1}{3}}(\sqrt{32} + \sqrt{5})^{\frac{1}{3}}$ (ii) $\left(x^{\frac{1}{3}} - x^{-\frac{1}{3}}\right)\left(x^{\frac{2}{3}} + 1 + x^{-\frac{2}{3}}\right)$

18. (i) $\left(\frac{x^m}{x^n}\right)^l \cdot \left(\frac{x^n}{x^l}\right)^m \cdot \left(\frac{x^l}{x^m}\right)^n$ (ii) $\left(\frac{x^{a+b}}{x^c}\right)^{a-b} \cdot \left(\frac{x^{b+c}}{x^a}\right)^{b-c} \cdot \left(\frac{x^{c+a}}{x^b}\right)^{c-a}$

19. (i) $\sqrt[lm]{\frac{x^l}{x^m}} \cdot \sqrt[mn]{\frac{x^m}{x^n}} \cdot \sqrt[nl]{\frac{x^n}{x^l}}$ (ii) $\left(\frac{x^a}{x^b}\right)^{a^2+ab+b^2} \cdot \left(\frac{x^b}{x^c}\right)^{b^2+bc+c^2} \left(\frac{x^c}{x^a}\right)^{c^2+ac+a^2}$

(iii) $\left(\frac{x^a}{x^{-b}}\right)^{a^2-ab+b^2} \cdot \left(\frac{x^b}{x^{-c}}\right)^{b^2-bc+c^2} \left(\frac{x^c}{x^{-a}}\right)^{c^2-ca+a^2}$

20. (i) $(a^{-1} + b^{-1}) \div (a^{-2} - b^{-2})$ (ii) $\frac{1}{1 + a^{m-n}} + \frac{1}{1 + a^{n-m}}$

21. Prove the following:

(i) $(a + b)^{-1} \ (a^{-1} + b^{-1}) = \frac{1}{ab}$ (ii) $\frac{x + y + z}{x^{-1}y^{-1} + y^{-1}z^{-1} + z^{-1}x^{-1}} = xyz$

22. If $a = c^z$, $b = a^x$ and $c = b^y$, prove that $xyz = 1$

23. If $a = xy^{p-1}$, $b = xy^{q-1}$ and $c = xy^{r-1}$, prove that $a^{q-r} . b^{r-p} . c^{p-q} = 1$

24. If $2^x = 3^y = 6^{-z}$, prove that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

25. If $2^x = 3^y = 12^z$, prove that $x = \frac{2yz}{y - z}$

26. Simplify and express with positive exponents:

$(3x^2)^0$, $(xy)^{-2}$, $(-27a^9)^{2/3}$

27. If $a = 3$ and $b = -2$, find the values of:

(i) $a^a + b^b$ (ii) $a^b + b^a$

28. If $x = 10^3 \times 0.0099$, $y = 10^{-2} \times 110$, find the value of $\sqrt{\frac{x}{y}}$

29. Evaluate $x^{1/2} . y^{-1} . z^{2/3}$ when $x = 9$, $y = 2$ and $z = 8$

30. If $x^4y^2z^3 = 49392$, find the values of x, y and z, where x, y and z are different p primes.

31. If $\sqrt[3]{a^6b^{-4}} = a^x . b^{2y}$, find x and y, where a, b are different positive primes.

32. If $(p + q)^{-1} (p^{-1} + q^{-1}) = p^a q^b$, prove that $a + b + 2 = 0$, where p and q are different positive primes.

33. If $\left(\frac{p^{-1}q^2}{p^2q^{-4}}\right)^7 \div \left(\frac{p^3q^{-5}}{p^{-2}q^3}\right)^{-5} = p^x q^y$, find $x + y$, where p and q are different positive primes.

34. Solve the following equations for x:

(i) $5^{2x+3} = 1$ (ii) $(13)^{\sqrt{x}} = 4^4 - 3^4 - 6$

(iii) $\left(\sqrt{\frac{3}{5}}\right)^{x+1} = \frac{125}{27}$ (iv) $(\sqrt[3]{4})^{2x+\frac{1}{2}} = \frac{1}{32}$

35. Solve the following equations for x:

(i) $\sqrt{\frac{p}{q}} = \left(\frac{q}{p}\right)^{1-2x}$ (ii) $4^{x-1} \times (0{\cdot}5)^{3-2x} = \left(\frac{1}{8}\right)^x$.

36. If $5^{3x} = 125$ and $(10)^y = 0{\cdot}001$, find x and y.

37. If $\frac{9^n . 3^2 . 3^n - (27)^n}{3^{3m} . 2^3} = \frac{1}{27}$, prove that $m = 1 + n$

38. If $3^{4x} = (81)^{-1}$ and $(10)^{1/y} = 0{\cdot}0001$, find the value of $2^{-x} . (16)^y$

39. If $3^{x+1} = 9^{x-2}$, find the value of 2^{1+x}

40. Solve the following equations:

(i) $3(2^x + 1) - 2^{x+2} + 5 = 0$ (ii) $3^x = 9.3^y, \quad 8.2^y = 4^x$

Multiple Choice Questions MCQs

Choose the correct answer from the given four options (1 to 7):

1. The value of $\left(5\frac{1}{16}\right)^{-\frac{3}{4}}$ is

(a) $\frac{4}{9}$ (b) $\frac{9}{4}$ (c) $\frac{27}{8}$ (d) $\frac{8}{27}$

2. $\sqrt[4]{\sqrt[3]{2^2}}$ is equal to

(a) $2^{-\frac{1}{6}}$ (b) 2^{-6} (c) $2^{\frac{1}{6}}$ (d) 2^6

3. The product $\sqrt[3]{2} . \sqrt[4]{2} . \sqrt[12]{32}$ equals

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt[12]{2}$ (d) $\sqrt[12]{32}$

4. The value of $\sqrt[4]{(81)^{-2}}$ is

(a) $\frac{1}{9}$ (b) $\frac{1}{3}$ (c) 9 (d) $\frac{1}{81}$

5. Value of $(256)^{0{\cdot}16} \times (256)^{0{\cdot}09}$ is

(a) 4 (b) 16 (c) 64 (d) 256·25

6. Which of the following is equal to x?

(a) $x^{\frac{12}{7}} - x^{\frac{5}{7}}$ (b) $\sqrt[12]{(x^4)^{\frac{1}{3}}}$ (c) $(\sqrt{x^3})^{\frac{2}{3}}$ (d) $x^{\frac{12}{7}} \times x^{\frac{7}{12}}$

7. Consider the following two statements:

Statement I: $3^m + 2^n = 5^{m+n}$

Statement II: $a^m + b^n = (a + b)^{m+n}$, when a, b, m, n are positive integers.

Which of the following is valid?

(a) Both the Statements are true.

(b) Both the Statements are false.

(c) Statement I is true, and Statement II is false.

(d) Statement I is false, and Statement II is true.

ASSERTION-REASON TYPE QUESTION (SOLVED)

In these examples and following questions, read the given statements carefully and choose the correct option.

(*a*) Assertion (A) is true, Reason (R) is false.

(*b*) Assertion (A) is false, Reason (R) is true.

(*c*) Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason for Assertion (A).

(*d*) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct reason (or explanation) for Assertion (A).

1. Assertion (A): If $5^{3x+1} = 1$, then $x = -\frac{1}{3}$

Reason (R): $a^m = a^n \Rightarrow m = n$ provided $a \neq 1$

Sol. We know Reason (R) to be true as a given identity.

Let us consider the Assertion (A).

$5^{3x+1} = 1 = 5^0 \Rightarrow 3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$

$\therefore$ Assertion (A) is true.

Thus both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason for Assertion (A).

$\therefore$ Correct answer is (*c*).

ASSERTION-REASON TYPE QUESTIONS (UNSOLVED)

1. Assertion (A): If $2^x \,.\, 3^y \,.\, 5^z = 16200$, then $x = 3, y = 4, z = 2$

Reason (R): If p, q are different prime numbers, then $p^m \,.\, q^n = p^l \,.\, q^k \Rightarrow m = l$ and

2. Assertion (A): If $x = 9$ and $y = 2$, then $x^y = y^x$

Reason (R): $a^{-n} = \left(\frac{1}{a}\right)^n$ when a is a real positive number and n is a rational number.

3. Assertion (A): If $3^x = 9\sqrt{3}$, then $x = \frac{5}{2}$

Reason (R): If a is a real positive number and n is positive integer, then $\sqrt[n]{a}$ is also written as $a^{\frac{1}{n}}$

Summary

- If $a > 0$ is a real number and n is a positive integer, then $\sqrt[n]{a} = b$ if and only if $b^n = a$, $b > 0$. $\sqrt[n]{a}$ is also written as $a^{1/n}$.
- If $a > 0$ is a real number and m, n are integers, $n > 0$, m, n have no common factors except 1, then

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$$

Laws of exponents for real numbers are:

- If a, b are positive real numbers and m, n are rational numbers, then the following results hold:

Chapter Test

1. If $2^x.3^y.5^z = 2160$, find the values of x, y and z. Hence, compute the value of $3^x.2^{-y}.5^{-z}$

2. If $x = 2$ and $y = -3$, find the values of (i) $x^x + y^y$ (ii) $x^y + y^x$

3. If $p = x^{m+n}.y^l$, $q = x^{n+l}.y^m$ and $r = x^{l+m}.y^n$, prove that $p^{m-n}.q^{n-l}.r^{l-m} = 1$.

4. If $x = a^{m+n}$, $y = a^{n+l}$ and $z = a^{l+m}$, prove that $x^m y^n z^l = x^n y^l z^m$

5. Show that $$\frac{\left(p+\frac{1}{q}\right)^m \times \left(p-\frac{1}{q}\right)^n}{\left(q+\frac{1}{p}\right)^m \times \left(q-\frac{1}{p}\right)^n} = \left(\frac{p}{q}\right)^{m+n}$$

6. If x is a positive real number and exponents are rational numbers, then simplify the following:

(i) $$\frac{(x^{(a+b)})^2 (x^{(b+c)})^2 (x^{(c+a)})^2}{(x^a x^b x^c)^4}$$

(ii) $$\left(\frac{x^{a^2}}{x^{b^2}}\right)^{\frac{1}{a+b}} \left(\frac{x^{b^2}}{x^{c^2}}\right)^{\frac{1}{b+c}} \left(\frac{x^{c^2}}{x^{a^2}}\right)^{\frac{1}{c+a}}$$

(iii) $$\left(\frac{x^b}{x^c}\right)^{b+c-a} \left(\frac{x^c}{x^a}\right)^{c+a-b} \left(\frac{x^a}{x^b}\right)^{a+b-c}$$

7. Show that $$\frac{1}{1+a^{y-x}+a^{z-x}} + \frac{1}{1+a^{z-y}+a^{x-y}} + \frac{1}{1+a^{x-z}+a^{y-z}} = 1$$

8. If $3^x = 5^y = (75)^z$, show that $z = \dfrac{xy}{2x+y}$

9. Solve the following equations for x:

(i) $\sqrt{3^{x+1}} = 27$

(ii) $3^{x+1} = 27.3^4$

(iii) $4^{2x} = (\sqrt[3]{16})^{-\frac{6}{y}} = (\sqrt{8})^2$

(iv) $3^{x-1} \times 5^{2y-3} = 225$

(v) $8^{x+1} = 16^{y+2}$, $\left(\frac{1}{2}\right)^{3+x} = \left(\frac{1}{4}\right)^{3y}$

INTRODUCTION

Logarithms were developed for making complicated calculations simple. However, with the advent of computers and hand calculators, doing calculations with the use of logarithms is no longer necessary. But still, logarithmic and exponential equations and functions are very common in mathematics.

8.1 LOGARITHMS

To learn the concept of *logarithm*, consider the equality $2^3 = 8$, another way of writing this is $\log_2 8 = 3$

It is read as "*logarithm* (abbreviated 'log') of 8 to the *base* 2 is equal to 3". Thus, $2^3 = 8$ is equivalent to $\log_2 8 = 3$. In general, we have:

Definition. *If a is any positive real number (except 1), n is any rational number and $a^n = b$, then n is called* ***logarithm*** *of b to the base a. It is written as $\log_a b$ (read as log of b to the base a). Thus,*

$a^n = b$ if and only if $\log_a b = n$

$a^n = b$ is called the ***exponential form*** and $\log_a b = n$ is called the ***logarithmic form***.

For example:

(i)	$3^2 = 9,$	$\therefore \log_3 9 = 2$
(ii)	$5^4 = 625,$	$\therefore \log_5 625 = 4$
(iii)	$7^0 = 1,$	$\therefore \log_7 1 = 0$
(iv)	$2^{-3} = \frac{1}{2^3} = \frac{1}{8},$	$\therefore \log_2 \frac{1}{8} = -3$
(v)	$(10)^{-2} = \frac{1}{100} = 0{\cdot}01,$	$\therefore \log_{10}(0{\cdot}01) = -2$

Remarks

- Since a is any positive real number (except 1), a^n is always a positive real number for every rational number n *i.e.* b is always a positive real number, therefore, logarithm of only positive real numbers are defined.
- Since $a^0 = 1$, $\log_a 1 = 0$ and $a^1 = a$, $\log_a a = 1$

  Thus, remember that

  (i) $\log_a 1 = 0$ (ii) $\log_a a = 1$

  where a is any positive real number (except 1).
- If $\log_a x = \log_a y = n$ (say), then $x = a^n$ and $y = a^n$, so $x = y$

  Thus, remember that

  $\log_a x = \log_a y \Rightarrow x = y$

- Logarithms to the base 10 are called **common logarithms.**
- If no base is given, the base is always taken as 10.
  For example, $\log 2 = \log_{10} 2$

Illustrative Examples

Example 1. Convert the following to logarithmic form:

(*i*) $(10)^4 = 10000$ (*ii*) $3^{-5} = x$ (*iii*) $(0{\cdot}3)^3 = 0{\cdot}027$

Solution. (*i*) $(10)^4 = 10000 \Rightarrow \log_{10} 10000 = 4$

(*ii*) $3^{-5} = x \Rightarrow \log_3 x = -5$

(*iii*) $(0{\cdot}3)^3 = 0{\cdot}027 \Rightarrow \log_{0{\cdot}3}(0{\cdot}027) = 3$

Example 2. Convert the following to exponential form:

(*i*) $\log_3 81 = 4$ (*ii*) $\log_8 32 = \frac{5}{3}$ (*iii*) $\log_{10}(0{\cdot}1) = -1$

Solution. (*i*) $\log_3 81 = 4 \Rightarrow 3^4 = 81$

(*ii*) $\log_8 32 = \frac{5}{3} \Rightarrow (8)^{5/3} = 32$

(*iii*) $\log_{10}(0{\cdot}1) = -1 \Rightarrow (10)^{-1} = 0{\cdot}1$

Example 3. Find the value of the following (by converting to exponential form):

(*i*) $\log_2 16$ (*ii*) $\log_{16} 2$ (*iii*) $\log_3 \frac{1}{3}$ (*iv*) $\log_{\sqrt{2}} 8$ (*v*) $\log_5(0{\cdot}008)$

Solution. (*i*) Let $\log_2 16 = x \Rightarrow 2^x = 16 \Rightarrow 2^x = (2)^4 \Rightarrow x = 4,$

$\therefore \quad \log_2 16 = 4$

(*ii*) Let $\log_{16} 2 = x \Rightarrow 16^x = 2 \Rightarrow (2^4)^x = 2$

$\Rightarrow \quad 2^{4x} = 2^1 \Rightarrow 4x = 1 \Rightarrow x = \frac{1}{4},$

$\therefore \quad \log_{16} 2 = \frac{1}{4}$

(*iii*) Let $\log_3 \frac{1}{3} = x \Rightarrow 3^x = \frac{1}{3}$

$\Rightarrow \quad 3^x = (3)^{-1} \Rightarrow x = -1,$

$\therefore \quad \log_3 \frac{1}{3} = -1$

(*iv*) Let $\log_{\sqrt{2}} 8 = x \Rightarrow (\sqrt{2})^x = 8 \Rightarrow (2^{1/2})^x = 2^3$

$\Rightarrow \quad 2^{x/2} = 2^3 \Rightarrow \frac{x}{2} = 3 \Rightarrow x = 6,$

$\therefore \quad \log_{\sqrt{2}} 8 = 6$

(*v*) Let $\log_5(0{\cdot}008) = x \Rightarrow 5^x = 0{\cdot}008$

$\Rightarrow \quad 5^x = \frac{8}{1000} \Rightarrow 5^x = \frac{1}{125}$

$\Rightarrow \quad 5^x = (5)^{-3} \Rightarrow x = -3,$

$\therefore \quad \log_5(0{\cdot}008) = -3$

Example 4. Find the value of $\log_{2\sqrt{3}} 1728$

Solution. Let $\log_{2\sqrt{3}} 1728 = x \Rightarrow (2\sqrt{3})^x = 1728$

$\Rightarrow \quad (2\sqrt{3})^x = (12)^3 \Rightarrow (2\sqrt{3})^x = ((2\sqrt{3})^2)^3$

$\Rightarrow \quad (2\sqrt{3})^x = (2\sqrt{3})^6 \Rightarrow x = 6$

$\therefore \quad \log_{2\sqrt{3}} 1728 = 6$

Example 5. Find the value of x in each of the following:

(*i*) $\log_2 x = 5$ (*ii*) $\log_4 x = 2{\cdot}5$ (*iii*) $\log_{64} x = \frac{2}{3}$ (*iv*) $\log_{\sqrt{3}} x = 4$

(*v*) $\log_{0\cdot5}\left(\frac{x}{4}\right) = 4$

Solution. (*i*) $\log_2 x = 5 \Rightarrow x = 2^5 \Rightarrow x = 32$

(*ii*) $\log_4 x = 2{\cdot}5 \Rightarrow x = 4^{2\cdot5} \Rightarrow x = (2^2)^{5/2}$

$\Rightarrow \quad x = 2^{2\times\frac{5}{2}} \Rightarrow x = 2^5 \Rightarrow x = 32$

(*iii*) $\log_{64} x = \frac{2}{3} \Rightarrow x = (64)^{2/3} \Rightarrow x = (4^3)^{2/3}$

$\Rightarrow \quad x = 4^{3\times\frac{2}{3}} \Rightarrow x = 4^2 \Rightarrow x = 16$

(*iv*) $\log_{\sqrt{3}} x = 4 \Rightarrow x = (\sqrt{3})^4 \Rightarrow x = (3^{1/2})^4$

$\Rightarrow \quad x = 3^2 \Rightarrow x = 9$

(*v*) $\log_{0\cdot5}\left(\frac{x}{4}\right) = 4 \Rightarrow (0{\cdot}5)^4 = \frac{x}{4} \Rightarrow \left(\frac{1}{2}\right)^4 = \frac{x}{4}$

$\Rightarrow \quad \frac{1}{2^4} = \frac{x}{4} \Rightarrow \frac{1}{16} = \frac{x}{4} \Rightarrow x = \frac{1}{4}$

Example 6. Solve for x:

(*i*) $\log_x 243 = -5$ (*ii*) $\log_x 16 = 2$ (*iii*) $\log_9 27 = 2x + 3$

(*iv*) $\log (3x - 2) = 2$ (*v*) $\log_x 64 = \frac{3}{2}$ (*vi*) $\log_2 (x^2 - 4) = 5$

Solution. (*i*) Given $\log_x 243 = -5 \Rightarrow x^{-5} = 243$

$\Rightarrow \quad x^{-5} = 3^5 \Rightarrow \left(\frac{1}{x}\right)^5 = 3^5$ $\left[\because a^{-n} = \left(\frac{1}{a}\right)^n\right]$

$\Rightarrow \quad \frac{1}{x} = 3 \Rightarrow x = \frac{1}{3}$

(*ii*) Given $\log_x 16 = 2 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

But the base of a logarithm cannot be negative, so $x = -4$ is rejected.

$\therefore$ The solution of the given equation is $x = 4$

(*iii*) Given $\log_9 27 = 2x + 3 \Rightarrow 9^{2x+3} = 27 \Rightarrow (3^2)^{2x+3} = 3^3$

$\Rightarrow \quad 3^{2(2x+3)} = 3^3 \Rightarrow 2(2x + 3) = 3 \Rightarrow 4x + 6 = 3$

$\Rightarrow \quad 4x = -3 \Rightarrow x = -\frac{3}{4}$

(*iv*) Given $\log (3x - 2) = 2 \Rightarrow \log_{10}(3x - 2) = 2$ [If no base is given, we take it as 10.]

$\Rightarrow \quad 3x - 2 = 10^2 \Rightarrow 3x - 2 = 100$

$\Rightarrow \quad 3x = 102 \Rightarrow x = 34$

(v) Given $\log_x 64 = \frac{3}{2} \Rightarrow x^{3/2} = 64$

$\Rightarrow \quad x = (64)^{2/3} = (2^6)^{2/3} = 2^{6 \times 2/3} = 2^4$

$\Rightarrow \quad x = 16$

(vi) Given $\log_2 (x^2 - 4) = 5 \Rightarrow x^2 - 4 = 2^5$

$\Rightarrow \quad x^2 - 4 = 32 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

Example 7. Given $\log_{10} x = a$, $\log_{10} y = b$,

(i) write down 10^{a-1} in terms of x.

(ii) write down 10^{2b} in terms of y.

(iii) if $\log_{10} P = 2a - b$, express P in terms of x and y.

Solution. Given $\log_{10} x = a \Rightarrow 10^a = x$, and $\log_{10} y = b \Rightarrow 10^b = y$

(i) $10^{a-1} = 10^a \cdot 10^{-1} = 10^a \cdot \frac{1}{10} = \frac{x}{10}$

(ii) $10^{2b} = (10^b)^2 = y^2$

(iii) $\log_{10} P = 2a - b \Rightarrow 10^{2a-b} = P$

$\therefore \quad P = 10^{2a} \cdot 10^{-b} = (10^a)^2 \cdot \frac{1}{10^b} = \frac{x^2}{y}$

Example 8. If $\log_3 x = a$, find 81^{a-1} in terms of x.

Solution. Given $\log_3 x = a \Rightarrow 3^a = x$

$\therefore \quad 81^{a-1} = (3^4)^{a-1} = 3^{4a-4} = 3^{4a} \times 3^{-4}$ ...(i)

$= \frac{(3^a)^4}{3^4} = \frac{x^4}{81}$ (using (i))

Example 9. If $\log_2 x = a$ and $\log_3 y = a$, find 12^{2a-1} in terms of x and y.

Solution. Given $\log_2 x = a$ and $\log_3 y = a$

$\Rightarrow \quad 2^a = x$ and $3^a = y$ ...(i)

$\therefore \quad 12^{2a-1} = (2^2 \times 3)^{2a-1} = 2^{2(2a-1)} \times 3^{2a-1}$

$= 2^{4a-2} \times 3^{2a-1} = 2^{4a} \times 2^{-2} \times 3^{2a} \times 3^{-1}$

$= \frac{(2^a)^4 \times (3^a)^2}{2^2 \times 3^1} = \frac{x^4 y^2}{12}$ (using (i))

Exercise 8.1

1. Convert the following to logarithmic form:

(i) $5^2 = 25$ (ii) $a^5 = 64$ (iii) $7^x = 100$ (iv) $9^0 = 1$

(v) $6^1 = 6$ (vi) $3^{-2} = \frac{1}{9}$ (vii) $10^{-2} = 0.01$ (viii) $(81)^{3/4} = 27$

2. Convert the following into exponential form:

(i) $\log_2 32 = 5$ (ii) $\log_3 81 = 4$ (iii) $\log_3 \frac{1}{3} = -1$

(iv) $\log_8 4 = \frac{2}{3}$ (v) $\log_8 32 = \frac{5}{3}$ (vi) $\log_{10}(0.001) = -3$

(vii) $\log_2 0.25 = -2$ (viii) $\log_a \left(\frac{1}{a}\right) = -1$

3. By converting to exponential form, find the values of:

(i) $\log_2 16$ (ii) $\log_5 125$ (iii) $\log_4 8$ (iv) $\log_9 27$

(v) $\log_{10}(0.01)$ (vi) $\log_7 \frac{1}{7}$ (vii) $\log_{0.5} 256$ (viii) $\log_2 0.25$

4. Solve the following equations for x:

(i) $\log_3 x = 2$ (ii) $\log_x 25 = 2$ (iii) $\log_{10} x = -2$

(iv) $\log_4 x = \frac{1}{2}$ (v) $\log_x 11 = 1$ (vi) $\log_x \frac{1}{4} = -1$

(vii) $\log_{81} x = \frac{3}{2}$ (viii) $\log_9 x = 2{\cdot}5$ (ix) $\log_4 x = -1{\cdot}5$

(x) $\log_{\sqrt{5}} x = 2$ (xi) $\log_x 0{\cdot}001 = -3$ (xii) $\log_{\sqrt{3}} (x+1) = 2$

(xiii) $\log_4 (2x+3) = \frac{3}{2}$ (xiv) $\log_{3\sqrt{2}} x = 3$ (xv) $\log_2 (x^2 - 1) = 3$

(xvi) $\log x = -1$ (xvii) $\log (2x-3) = 1$ (xviii) $\log x = -2, 0, \frac{1}{3}$

5. Given $\log_{10} a = b$, express 10^{2b-3} in terms of a.

6. Given $\log_{10} x = a$, $\log_{10} y = b$ and $\log_{10} z = c$,

(i) write down 10^{2a-3} in terms of x.

(ii) write down 10^{3b-1} in terms of y.

(iii) if $\log_{10} \text{P} = 2a + \frac{b}{2} - 3c$, express P in terms of x, y and z.

7. If $\log_{10} x = a$ and $\log_{10} y = b$, find the value of xy.

8. Given $\log_{10} a = m$ and $\log_{10} b = n$, express $\frac{a^3}{b^2}$ in terms of m and n.

9. Given $\log_{10} x = 2a$ and $\log_{10} y = \frac{b}{2}$,

(i) write 10^a in terms of x.

(ii) write 10^{2b+1} in terms of y.

(iii) if $\log_{10} \text{P} = 3a - 2b$, express P in terms of x and y.

10. If $\log_2 y = x$ and $\log_3 z = x$, find 72^x in terms of y and z.

11. If $\log_2 x = a$ and $\log_5 y = a$, write 100^{2a-1} in terms of x and y.

8.2 THREE STANDARD LAWS OF LOGARITHMS

- $\log_a mn = \log_a m + \log_a n$ (Product Law)

  The above result is capable of extension *i.e.*

  $\log_a (mnp\ldots) = \log_a m + \log_a n + \log_a p + \ldots$

- $\log_a \frac{m}{n} = \log_a m - \log_a n$ (Quotient Law)

- $\log_a m^n = n \log_a m$ (Power Law)

Deductions

1. $\log_a a^x = x$ 2. $a^{\log_a x} = x$

8.2.1 Base changing formula

$$\log_a m = \frac{\log_b m}{\log_b a},\ m > 0,\ a, b > 0,\ a \neq 1,\ b \neq 1$$

Deductions

1. $\log_b m = \log_a m \times \log_b a$

2. $\log_b a \times \log_a b = 1$ (Put $m = b$ in 1)

3. $\log_b a = \frac{1}{\log_a b}$ (Reciprocal formula)

Illustrative Examples

Example 1. Express $\log_{10} \frac{a^2c}{\sqrt{b}}$ in terms of $\log_{10}a$, $\log_{10}b$, $\log_{10}c$

Solution. $\log_{10} \frac{a^2c}{\sqrt{b}} = \log_{10} a^2c - \log_{10} \sqrt{b}$ (Quotient Law)

$= \log_{10} a^2 + \log_{10} c - \log_{10} (b)^{\frac{1}{2}}$ (Product Law)

$= 2 \log_{10} a + \log_{10} c - \frac{1}{2} \log_{10} b$ (Power Law)

Example 2. Evaluate the following:

(i) $3 + \log_{10} (10^{-2})$ (ii) $5 + \log_{10} (0{\cdot}001)$

Solution. (i) $3 + \log_{10}(10^{-2}) = 3 + (-2) \log_{10} 10$ (Power Law)

$= 3 + (-2).1$ ($\because \log_{10} 10 = 1$)

$= 3 - 2 = 1$

(ii) $5 + \log_{10} (0{\cdot}001) = 5 + \log_{10} \left(\frac{1}{1000}\right)$

$= 5 + \log_{10} (10^{-3})$

$= 5 + (-3) \log_{10}10$

$= 5 + (-3) \times 1$

$= 5 - 3 = 2$

Example 3. Evaluate the following:

(i) $\frac{\log 125}{\log \sqrt{5}}$ (ii) $\log_6 72 - \log_6 2$ (iii) $\log_4 8 - \log_8 32$

Solution. (i) $\frac{\log 125}{\log \sqrt{5}} = \frac{\log 5^3}{\log 5^{1/2}} = \frac{3 \log 5}{\frac{1}{2}\log 5} = 6$

(ii) $\log_6 72 - \log_6 2 = \log_6 \frac{72}{2} = \log_6 36 = \log_6 6^2$

$= 2 \log_6 6 = 2 \times 1$ ($\because \log_a a = 1$)

$= 2$

(iii) $\log_4 8 - \log_8 32 = \log_4 2^3 - \log_8 2^5 = \log_4 (2^2)^{3/2} - \log_8 (2^3)^{5/3}$

$= \log_4 4^{3/2} - \log_8 8^{5/3} = \frac{3}{2} \log_4 4 - \frac{5}{3} \log_8 8$

$= \frac{3}{2} \times 1 - \frac{5}{3} \times 1$ ($\because \log_a a = 1$)

$= \frac{3}{2} - \frac{5}{3} = \frac{9-10}{6} = -\frac{1}{6}$

Example 4. Express as a single logarithm: $2 + \frac{1}{2} \log_{10}9 - 2 \log_{10}5$

Solution. $2 + \frac{1}{2} \log_{10}9 - 2 \log_{10}5 = 2.1 + \frac{1}{2} \log_{10}9 - 2 \log_{10}5$

$= 2 \log_{10}10 + \log_{10} (9)^{1/2} - \log_{10}(5)^2$ ($\because \log_{10}10 = 1$)

$= \log_{10}(10)^2 + \log_{10}3 - \log_{10}25$

$= \log_{10} \frac{(10)^2 \times 3}{25} = \log_{10} \frac{100 \times 3}{25}$

$= \log_{10} 12$

Example 5. Prove that $16^{\log 3} = 9^{\log 4}$

Solution. Let $16^{\log 3} = x$ and $9^{\log 4} = y$, then it is sufficient to prove that $x = y$.

Now, $x = 16^{\log 3}$

$\Rightarrow \quad \log x = \log (16^{\log 3})$ (Taking log of both sides)

$\Rightarrow \quad \log x = \log 3 \times \log 16 = \log 3 \times \log 2^4$

$= \log 3 \times 4 \log 2$

$\therefore \quad \log x = 4 (\log 3) \times (\log 2)$ ...(i)

Similarly, $y = 9^{\log 4}$

$\Rightarrow \quad \log y = \log (9^{\log 4}) = \log 4 \log 9$

$= (\log 2^2) (\log 3^2) = (2 \log 2) (2 \log 3)$

$\therefore \quad \log y = 4(\log 2) \times (\log 3)$ ...(ii)

From (i) and (ii), we get

$\log x = \log y \Rightarrow x = y$

Example 6. Prove that:

(i) $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$

(ii) $\frac{1}{2} \log 9 + 2 \log 6 + \frac{1}{4} \log 81 - \log 12 = 3 \log 3$

Solution. (i) $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}$

$$= \log \left(\frac{16}{15}\right)^7 + \log \left(\frac{25}{24}\right)^5 + \log \left(\frac{81}{80}\right)^3$$

$$= \log \left\{\left(\frac{16}{15}\right)^7 \times \left(\frac{25}{24}\right)^5 \times \left(\frac{81}{80}\right)^3\right\}$$

$$= \log \left\{\left(\frac{2^4}{3 \times 5}\right)^7 \times \left(\frac{5^2}{2^3 \times 3}\right)^5 \times \left(\frac{3^4}{2^4 \times 5}\right)^3\right\}$$

$$= \log \left(\frac{2^{28}}{3^7 \times 5^7} \times \frac{5^{10}}{2^{15} \times 3^5} \times \frac{3^{12}}{2^{12} \times 5^3}\right)$$

$$= \log(2^{28-15-12} \times 5^{10-7-3} \times 3^{12-7-5})$$

$$= \log(2^1 \times 5^0 \times 3^0) = \log(2 \times 1 \times 1) = \log 2$$

(ii) $\frac{1}{2} \log 9 + 2 \log 6 + \frac{1}{4} \log 81 - \log 12$

$$= \frac{1}{2} \log 3^2 + 2 \log 6 + \frac{1}{4} \log 3^4 - \log 12$$

$$= \log (3^2)^{\frac{1}{2}} + \log 6^2 + \log (3^4)^{\frac{1}{4}} - \log 12$$

$$= \log 3 + \log 36 + \log 3 - \log 12$$

$$= \log \frac{3 \times 36 \times 3}{12} = \log 27 = \log 3^3 = 3 \log 3$$

Example 7. If $\log 7 - \log 2 + \log 16 - 2 \log 3 - \log \frac{7}{45} = 1 + \log n$, find n.

Solution. Given $\log 7 - \log 2 + \log 16 - 2 \log 3 - \log \frac{7}{45} = 1 + \log n$

$\Rightarrow \quad \log 7 - \log 2 + \log 16 - \log (3)^2 - \log \frac{7}{45} = \log 10 + \log n$ $(\because \log 10 = 1)$

$\Rightarrow \quad \log \dfrac{7 \times 16}{2 \times 3^2 \times \dfrac{7}{45}} = \log (10 \times n) \Rightarrow \log \dfrac{7 \times 16 \times 45}{2 \times 9 \times 7} = \log 10n$

$\Rightarrow \quad \log 40 = \log 10n \Rightarrow 40 = 10n \Rightarrow n = 4$

Example 8. If $3 \log \sqrt{m} + 2 \log \sqrt[3]{n} - 1 = 0$, find the value of $m^9 n^4$

Solution. Given $3 \log \sqrt{m} + 2 \log \sqrt[3]{n} - 1 = 0$

$\Rightarrow \quad \log (\sqrt{m})^3 + \log (\sqrt[3]{n})^2 = 1$

$\Rightarrow \quad \log (m^{3/2} \times n^{2/3}) = \log 10 \qquad (\because \log 10 = 1)$

$\Rightarrow \quad m^{3/2}.n^{2/3} = 10$, raising both sides to the power 6, we get

$\Rightarrow \quad (m^{3/2}.n^{2/3})^6 = 10^6$

$\Rightarrow \quad (m^{3/2})^6.(n^{2/3})^6 = 10^6 \Rightarrow m^9 n^4 = 10^6$

Example 9. Given $2 \log_{10} x + \dfrac{1}{2} \log_{10} y = 1$, express y in terms of x

Solution. Given $2 \log_{10} x + \dfrac{1}{2} \log_{10} y = 1 \Rightarrow \log_{10} x^2 + \log_{10} y^{\frac{1}{2}} = 1$

$\Rightarrow \quad \log_{10} x^2 y^{\frac{1}{2}} = 1 \Rightarrow x^2 y^{\frac{1}{2}} = 10^1 \qquad$ (squaring)

$\Rightarrow \quad x^4 y = 10^2 \Rightarrow y = \dfrac{100}{x^4}$

Example 10. Simplify the following:

(*i*) $\dfrac{\log_3 8}{\log_9 16 \log_4 10}$ (*ii*) $(\sqrt{x})^{4 \log_x a}$ (*iii*) $\log (\log x^2) - \log (\log x)$

(*iv*) $\log_b a \,.\, \log_c b \,.\, \log_a c$ (*v*) $\log_2 (\log_2 (\log_2 16))$ (*vi*) $3^{-\frac{1}{2}\log_3 9}$

Solution. (*i*) $\dfrac{\log_3 8}{\log_9 16 \log_4 10} = \dfrac{\log_{10} 8}{\log_{10} 3} . \dfrac{\log_{10} 9 \log_{10} 4}{\log_{10} 16 \log_{10} 10}$ (Changing all logs to base 10)

$= \dfrac{\log_{10} 2^3 . \log_{10} 3^2 . \log_{10} 2^2}{\log_{10} 3 . \log_{10} 2^4 . 1} = \dfrac{(3 \log_{10} 2)(2 \log_{10} 3)(2 \log_{10} 2)}{(\log_{10} 3)(4 \log_{10} 2)}$

$= 3 \log_{10} 2$

(*ii*) $(\sqrt{x})^{4 \log_x a} = x^{\frac{1}{2} \times 4 \log_x a} = x^{2 \log_x a} = x^{\log_x a^2} = a^2 \qquad (\because a^{\log_a x} = x)$

(*iii*) $\log (\log x^2) - \log (\log x) = \log (2 \log x) - \log (\log x) = \log \left(\dfrac{2 \log x}{\log x} \right) = \log 2$

(*iv*) $\log_b a . \log_c b . \log_a c = (\log_b a . \log_c b) . \log_a c = \log_c a . \log_a c = 1$

(*v*) $\log_2 (\log_2 (\log_2 16)) = \log_2 (\log_2 (\log_2 2^4)) = \log_2 (\log_2 (4))$

$= \log_2 (\log_2 2^2) = \log_2 (2) = 1$

(*vi*) $3^{-\frac{1}{2}\log_3 9} = 3^{\log_3 9^{-1/2}} = 9^{-1/2} = \dfrac{1}{9^{1/2}} = \dfrac{1}{3}$

Example 11. (*i*) If $\dfrac{\log a}{b - c} = \dfrac{\log b}{c - a} = \dfrac{\log c}{a - b}$, prove that $a^a . b^b . c^c = 1$

(*ii*) If $\dfrac{1}{\log_a n} + \dfrac{1}{\log_c n} = \dfrac{2}{\log_b n}$, prove that $b^2 = ac$

(*iii*) Show that $\dfrac{\log_a n}{\log_{ab} n} = 1 + \log_a b$

Solution. (*i*) Let $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b}=k$

$\Rightarrow \quad \log a = k(b-c);\ \log b = k(c-a);\ \log c = k(a-b)$

$\Rightarrow \quad a\log a + b\log b + c\log c = ka(b-c) + kb(c-a) + kc(a-b) = 0$

$\Rightarrow \quad \log a^a.b^b.c^c = 0 = \log 1 \Rightarrow a^a.b^b.c^c = 1$

(*ii*) Given $\frac{1}{\log_a n}+\frac{1}{\log_c n}=\frac{2}{\log_b n}$

(using reciprocal formula)

$\Rightarrow \quad \log_n a + \log_n c = 2\log_n b$

$\Rightarrow \quad \log_n ac = \log_n b^2$

$\Rightarrow \quad ac = b^2$, as required.

(*iii*) $\frac{\log_a n}{\log_{ab} n}=\frac{1/\log_n a}{1/\log_n ab}=\frac{\log_n ab}{\log_n a}=\log_a ab=\log_a a+\log_a b=1+\log_a b$

Example 12. If $a=\log_x yz$, $b=\log_y zx$ and $c=\log_z xy$, then prove that $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=1$

Solution. $\frac{1}{1+a}=\frac{1}{1+\log_x yz}=\frac{1}{\log_x x+\log_x yz}=\frac{1}{\log_x xyz}=\log_{xyz} x$

Similarly, $\frac{1}{1+b}=\log_{xyz} y$ and $\frac{1}{1+c}=\log_{xyz} z$

$\therefore \quad \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=\log_{xyz} x+\log_{xyz} y+\log_{xyz} z$

$= \log_{xyz} xyz = 1$

Example 13. Solve for x:

(*i*) $\log x = \frac{\log 125}{\log\frac{1}{5}}$ (*ii*) $\log_2(\log_3 x) = 4$

(*iii*) $\log_x 15\sqrt{5} = 2-\log_x 3\sqrt{5}$ (*iv*) $\log(5x-4)-\log(x+1)=\log 4$

(*v*) $\log(x+5)+\log(x-5)=2\log 3+4\log 2$

Solution. (*i*) $\log x = \frac{\log 125}{\log\frac{1}{5}}=\frac{\log(5)^3}{\log 5^{-1}}=\frac{3\log 5}{(-1)\log 5}=-3$

$\Rightarrow \quad x = 10^{-3} \Rightarrow x = \frac{1}{(10)^3}=\frac{1}{1000}=0{\cdot}001$

(*ii*) $\log_2(\log_3 x) = 4 \Rightarrow \log_3 x = 2^4 = 16 \Rightarrow x = 3^{16}$

(*iii*) Given $\log_x 15\sqrt{5} = 2-\log_x 3\sqrt{5} \Rightarrow \log_x 15\sqrt{5}+\log_x 3\sqrt{5}=2$

$\Rightarrow \quad \log_x(15\sqrt{5}\times 3\sqrt{5}) = 2 \Rightarrow \log_x 225 = 2$

$\Rightarrow \quad \log_x(15)^2 = 2 \Rightarrow 2\log_x 15 = 2$

$\Rightarrow \quad \log_x 15 = 1 \Rightarrow x^1 = 15$

$\Rightarrow \quad x = 15$

(*iv*) Given $\log(5x-4)-\log(x+1)=\log 4$

$\Rightarrow \quad \log\frac{5x-4}{x+1}=\log 4$

$\Rightarrow \quad \frac{5x-4}{x+1}=4 \Rightarrow 5x-4=4x+4 \Rightarrow x=8$

(v) Given $\log(x+5) + \log(x-5) = 2\log 3 + 4\log 2$

$\Rightarrow \log(x+5)(x-5) = \log 3^2 + \log 2^4$

$\Rightarrow \log(x^2-25) = \log(3^2 \times 2^4)$

$\Rightarrow x^2 - 25 = 3^2 \times 2^4 \Rightarrow x^2 - 25 = 144$

$\Rightarrow x^2 = 169 \Rightarrow x = \pm 13$

When $x = -13$, then $x+5$ and $x-5$ are both negative and the logarithm of a negative number is not defined, so $x = -13$ is rejected.

$\therefore$ The solution of the given equation is $x = 13$

Example 14. Find the value of x if $\log_{10} x - \log_{10}(2x-1) = 1$

Solution. Given $\log_{10} x - \log_{10}(2x-1) = 1$

$\Rightarrow \log_{10}\dfrac{x}{2x-1} = 1 \Rightarrow \dfrac{x}{2x-1} = 10^1$

$\Rightarrow \dfrac{x}{2x-1} = 10 \Rightarrow 20x - 10 = x$

$\Rightarrow 19x = 10 \Rightarrow x = \dfrac{10}{19}$

Example 15. Solve the following equations for x:

(i) $\log_x 25 - \log_x 5 + \log_x \dfrac{1}{125} = 2$

(ii) $\log_x(8x-3) - \log_x 4 = 2$

(iii) $3^{\log x} - 2^{\log x} = 2^{\log x + 1} - 3^{\log x - 1}$

Solution. (i) Given $\log_x 25 - \log_x 5 + \log_x \dfrac{1}{125} = 2$

$\Rightarrow \log_x \dfrac{25 \times \frac{1}{125}}{5} = 2 \Rightarrow \log_x \dfrac{1}{25} = 2$

$\Rightarrow \log_x \left(\dfrac{1}{5}\right)^2 = 2 \Rightarrow 2\log_x \dfrac{1}{5} = 2$

$\Rightarrow \log_x \dfrac{1}{5} = 1 \Rightarrow x^1 = \dfrac{1}{5}$

$\Rightarrow x = \dfrac{1}{5}$

(ii) Given $\log_x(8x-3) - \log_x 4 = 2$

$\Rightarrow \log_x \dfrac{8x-3}{4} = 2 \Rightarrow x^2 = \dfrac{8x-3}{4} \Rightarrow 4x^2 = 8x - 3$

$\Rightarrow 4x^2 - 8x + 3 = 0 \Rightarrow 4x^2 - 6x - 2x + 3 = 0$

$\Rightarrow 2x(2x-3) - 1(2x-3) = 0 \Rightarrow (2x-3)(2x-1) = 0$

$\Rightarrow 2x - 3 = 0, 2x - 1 = 0$

$\Rightarrow x = \dfrac{3}{2}, \dfrac{1}{2}$

(iii) Given $3^{\log x} - 2^{\log x} = 2^{\log x + 1} - 3^{\log x - 1}$

$\Rightarrow 3^{\log x} + 3^{\log x - 1} = 2^{\log x + 1} + 2^{\log x}$

$\Rightarrow 3^{\log x} + 3^{\log x} \times 3^{-1} = 2^{\log x} \times 2^1 + 2^{\log x}$

$\Rightarrow \left(1 + \dfrac{1}{3}\right) 3^{\log x} = (2+1)\, 2^{\log x}$

$$\Rightarrow \frac{4}{3} \times 3^{\log x} = 3 \times 2^{\log x}$$

$$\Rightarrow \frac{3^{\log x}}{2^{\log x}} = \frac{9}{4} \Rightarrow \left(\frac{3}{2}\right)^{\log x} = \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \log x = 2 \Rightarrow x = 10^2$$

$$\Rightarrow x = 100$$

Example 16. Solve for x: $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

Solution. Given $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

$$\Rightarrow \frac{1}{\log_x 2} + \frac{1}{\log_x 4} + \frac{1}{\log_x 16} = \frac{21}{4}$$

$$\Rightarrow \frac{1}{\log_x 2} + \frac{1}{\log_x 2^2} + \frac{1}{\log_x 2^4} = \frac{21}{4} \Rightarrow \frac{1}{\log_x 2} + \frac{1}{2\log_x 2} + \frac{1}{4\log_x 2} = \frac{21}{4}$$

$$\Rightarrow \frac{1}{\log_x 2}\left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{21}{4} \Rightarrow \frac{7}{4} \cdot \frac{1}{\log_x 2} = \frac{21}{4}$$

$$\Rightarrow \log_x 2 = \frac{7}{4} \cdot \frac{4}{21} = \frac{1}{3}$$

$$\Rightarrow x^{1/3} = 2 \Rightarrow x = 2^3 \Rightarrow x = 8$$

Example 17. Solve the following equations for x :

(i) $\log_x 2 \times \log_{x/16} 2 = \log_{x/64} 2$

(ii) $\log_5(5^{1/x} + 125) = \log_5 6 + 1 + \frac{1}{2x}$

Solution. (i) Given $\log_x 2 \times \log_{x/16} 2 = \log_{x/64} 2$

$$\Rightarrow \frac{1}{\log_2 x} \times \frac{1}{\log_2\left(\frac{x}{16}\right)} = \frac{1}{\log_2\left(\frac{x}{64}\right)}$$

$$\Rightarrow \log_2 (x/64) = \log_2 x \times \log_2 (x/16)$$

$$\Rightarrow \log_2 x - \log_2 64 = \log_2 x(\log_2 x - \log_2 16)$$

$$\Rightarrow \log_2 x - \log_2 2^6 = \log_2 x(\log_2 x - \log_2 2^4)$$

$$\Rightarrow \log_2 x - 6 \log_2 2 = \log_2 x(\log_2 x - 4 \log_2 2)$$

$$\Rightarrow \log_2 x - 6 \times 1 = \log_2 x(\log_2 x - 4 \times 1)$$

$$\Rightarrow (\log_2 x)^2 - 5 \log_2 x + 6 = 0$$

$$\Rightarrow y^2 - 5y + 6 = 0 \text{ where } y = \log_2 x$$

$$\Rightarrow (y - 3)(y - 2) = 0 \Rightarrow y = 3 \text{ or } y = 2$$

$$\Rightarrow \log_2 x = 3 \text{ or } \log_2 x = 2$$

$$\Rightarrow x = 2^3 \text{ or } x = 2^2 \Rightarrow x = 8 \text{ or } x = 4$$

Hence, the solutions of the given equation are 8, 4

(ii) Given $\log_5 (5^{1/x} + 125) = \log_5 6 + 1 + \frac{1}{2x}$

$$\Rightarrow \log_5 (5^{1/x} + 125) = \log_5 6 + \log_5 5 + \frac{1}{2x} \qquad (\because \log_5 5 = 1)$$

$$\Rightarrow \log_5 (5^{1/x} + 125) - \log_5 6 - \log_5 5 = \frac{1}{2x}$$

$$\Rightarrow \log_5\left(\frac{5^{1/x} + 125}{6 \times 5}\right) = \frac{1}{2x} \Rightarrow \frac{5^{1/x} + 125}{30} = 5^{1/2x}$$

$$5^{\frac{1}{x}} + 125 = 30 \times 5^{\frac{1}{2x}}$$

$\Rightarrow$ $(5^{\frac{1}{2x}})^2 - 30 \times 5^{\frac{1}{2x}} + 125 = 0$

$\Rightarrow$ $y^2 - 30y + 125 = 0$ where $y = 5^{\frac{1}{2x}}$

$\Rightarrow$ $(y - 25)(y - 5) = 0 \Rightarrow y = 25$ or $y = 5$

$\Rightarrow$ $5^{\frac{1}{2x}} = 25$ or $5^{\frac{1}{2x}} = 5$

$\Rightarrow$ $5^{\frac{1}{2x}} = 5^2$ or $5^{\frac{1}{2x}} = 5^1$

$\Rightarrow$ $\frac{1}{2x} = 2$ or $\frac{1}{2x} = 1 \Rightarrow x = \frac{1}{4}$ or $x = \frac{1}{2}$

Hence, the solutions of the given equation are $\frac{1}{4}, \frac{1}{2}$

Example 18. If $a = 1 + \log_{10} 2 - \log_{10} 5$, $b = 2 \log_{10} 3$ and $c = \log_{10} m - \log_{10} 5$, find the value of m given that $a + b = 2c$

Solution. Given $a + b = 2c$

$\Rightarrow$ $1 + \log_{10} 2 - \log_{10} 5 + 2 \log_{10} 3 = 2 (\log_{10} m - \log_{10} 5)$

$\Rightarrow$ $\log_{10} 10 + \log_{10} 2 - \log_{10} 5 + 2 \log_{10} 3 + 2 \log_{10} 5 = 2 \log_{10} m$ $\quad (\because \log_{10} 10 = 1)$

$\Rightarrow$ $\log_{10} 10 + \log_{10} 2 + \log_{10} 5 + \log_{10} 3^2 = 2 \log_{10} m$

$\Rightarrow$ $\log_{10} (10 \times 2 \times 5 \times 3^2) = 2 \log_{10} m$

$\Rightarrow$ $\log_{10} 900 = 2 \log_{10} m \Rightarrow \log_{10} (30)^2 = 2 \log_{10} m$

$\Rightarrow$ $2 \log_{10} 30 = 2 \log_{10} m \Rightarrow \log_{10} 30 = \log_{10} m$

$\Rightarrow$ $30 = m$ *i.e.* $m = 30$

Example 19. If $a^2 + b^2 = 7ab$, prove that $2 \log (a + b) = \log 9 + \log a + \log b$

Solution. Given $a^2 + b^2 = 7ab$

Adding $2ab$ to both sides, we get

$a^2 + 2ab + b^2 = 9ab$

$\Rightarrow$ $(a + b)^2 = 9ab$, taking logs of both sides, we get

$\log (a + b)^2 = \log 9ab$

$\Rightarrow$ $2 \log (a + b) = \log 9 + \log a + \log b$

Example 20. If $\frac{\log(x+y)}{\log 2} = \frac{\log(x-y)}{\log 3} = \frac{\log 64}{\log 0{\cdot}125}$, find the values of x and y

Solution. $\frac{\log 64}{\log 0{\cdot}125} = \frac{\log 64}{\log \frac{1}{8}} = \frac{\log 8^2}{\log 8^{-1}} = \frac{2 \log 8}{(-1) \log 8} = -2$

$\therefore$ $\frac{\log(x+y)}{\log 2} = -2 \Rightarrow \log (x + y) = -2 \log 2 = \log 2^{-2} = \log \frac{1}{4}$

$\Rightarrow$ $x + y = \frac{1}{4}$ $\quad \ldots(i)$

Also, $\frac{\log(x-y)}{\log 3} = -2 \Rightarrow \log (x - y) = -2 \log 3 = \log 3^{-2} = \log \frac{1}{9}$

$\Rightarrow$ $x - y = \frac{1}{9}$ $\quad \ldots(ii)$

On adding (*i*) and (*ii*), we get

$$2x = \frac{1}{4} + \frac{1}{9} = \frac{9+4}{36} = \frac{13}{36} \Rightarrow x = \frac{13}{72}$$

Subtracting (*ii*) from (*i*), we get

$$2y = \frac{1}{4} - \frac{1}{9} = \frac{9-4}{36} = \frac{5}{36} \Rightarrow y = \frac{5}{72}$$

Hence, $x = \frac{13}{72}$ and $y = \frac{5}{72}$.

Example 21. Solve the following equations for x and y:

$$\log_{10}(xy) = 2, \log_{10}\left(\frac{x}{y}\right) + 2\log_{10}2 = 2$$

Solution. $\log_{10}(xy) = 2 \Rightarrow xy = 10^2 \Rightarrow xy = 100$ ...(*i*)

$$\log_{10}\left(\frac{x}{y}\right) + 2\log_{10}2 = 2$$

$$\Rightarrow \log_{10}\left(\frac{x}{y}\right) = 2 - 2\log_{10}2 = 2(1 - \log_{10}2)$$

$$= 2(\log_{10}10 - \log_{10}2) = 2\log_{10}\frac{10}{2} = 2\log_{10}5$$

$$= \log_{10}5^2 = \log_{10}25$$

$$\Rightarrow \frac{x}{y} = 25$$

Multiplying (*i*) and (*ii*), we get

$$(xy) \times \left(\frac{x}{y}\right) = 100 \times 25 \Rightarrow x^2 = 2500 \Rightarrow x = \pm 50$$

When $x = 50$, from (*i*), $y = \frac{100}{x} = \frac{100}{50} = 2$;

when $x = -50$, from (*i*), $y = \frac{100}{x} = \frac{100}{-50} = -2$

Hence, the solutions are $x = 50, y = 2; x = -50, y = -2$

Note that both these solutions satisfy the given equations.

Exercise 8.2

1. Simplify the following:

(*i*) $\log a^3 - \log a^2$ (*ii*) $\log a^3 \div \log a^2$ (*iii*) $\frac{\log 4}{\log 2}$

(*iv*) $\frac{\log 8 \log 9}{\log 27}$ (*v*) $\frac{\log 27}{\log \sqrt{3}}$ (*vi*) $\frac{\log 9 - \log 3}{\log 27}$

2. Evaluate the following:

(*i*) $\log(10 \div \sqrt[3]{10})$ (*ii*) $2 + \frac{1}{2}\log(10^{-3})$

(*iii*) $2\log 5 + \log 8 - \frac{1}{2}\log 4$

(*iv*) $2\log 10^3 + 3\log 10^{-2} - \frac{1}{3}\log 5^{-3} + \frac{1}{2}\log 4$

(v) $2 \log 2 + \log 5 - \frac{1}{2} \log 36 - \log \frac{1}{30}$

(vi) $2 \log 5 + \log 3 + 3 \log 2 - \frac{1}{2} \log 36 - 2 \log 10$

(vii) $2 \log_{10} 5 + \log_{10} 8 - \frac{1}{2} \log_{10} 4$

3. Express each of the following as a single logarithm:

(i) $2 \log 3 - \frac{1}{2} \log 16 + \log 12$

(ii) $2 \log_{10} 5 - \log_{10} 2 + 3 \log_{10} 4 + 1$

(iii) $\frac{1}{2} \log 36 + 2 \log 8 - \log 1{\cdot}5$

(iv) $\frac{1}{2} \log 25 - 2 \log 3 + 1$

(v) $\frac{1}{2} \log 9 + 2 \log 3 - \log 6 + \log 2 - 2$

4. Prove the following:

(i) $\log_{10} 4 \div \log_{10} 2 = \log_3 9$

(ii) $\log_{10} 25 + \log_{10} 4 = \log_5 25$

5. If $x = (100)^a$, $y = (10000)^b$ and $z = (10)^c$, express $\log \frac{10\sqrt{y}}{x^2 z^3}$ in terms of a, b, c.

6. If $a = \log_{10} x$, find the following in terms of a:

(i) x (ii) $\log_{10} \sqrt[5]{x^2}$ (iii) $\log_{10} 3x$

7. If $a = \log \frac{2}{3}$, $b = \log \frac{3}{5}$ and $c = 2 \log \sqrt{\frac{5}{2}}$, find the value of

(i) $a + b + c$ (ii) 5^{a+b+c}

8. If $x = \log \frac{3}{5}$, $y = \log \frac{5}{4}$ and $z = 2 \log \frac{\sqrt{3}}{2}$, find the values of

(i) $x + y - z$ (ii) 3^{x+y-z}

9. If $x = \log_{10} 12$, $y = \log_4 2 \times \log_{10} 9$ and $z = \log_{10} 0{\cdot}4$, find the values of

(i) $x - y - z$ (ii) 7^{x-y-z}.

10. If $\log V + \log 3 = \log \pi + \log 4 + 3 \log r$, find V in terms of other quantities.

11. Given $3 (\log 5 - \log 3) - (\log 5 - 2 \log 6) = 2 - \log n$, find n.

12. Given that $\log_{10} y + 2 \log_{10} x = 2$, express y in terms of x.

13. Express $\log_{10} 2 + 1$ in the form $\log_{10} x$.

14. If $a^2 = \log_{10} x$, $b^3 = \log_{10} y$ and $\frac{a^2}{2} - \frac{b^3}{3} = \log_{10} z$, express z in terms of x and y.

15. Given that $\log m = x + y$ and $\log n = x - y$, express the value of $\log m^2 n$ in terms of x and y.

16. Given that $\log x = m + n$ and $\log y = m - n$, express the value of $\log \left(\frac{10x}{y^2}\right)$ in terms of m and n.

17. If $\frac{\log x}{2} = \frac{\log y}{3}$, find the value of $\frac{y^4}{x^6}$

18. Solve for x:

(i) $\log x + \log 5 = 2 \log 3$

(ii) $\log_3 x - \log_3 2 = 1$

(iii) $x = \frac{\log 125}{\log 25}$

(iv) $\frac{\log 8}{\log 2} \times \frac{\log 3}{\log \sqrt{3}} = 2 \log x$

19. Given $2 \log_{10} x + 1 = \log_{10} 250$, find (i) x (ii) $\log_{10} 2x$

20. If $\frac{\log x}{\log 5} = \frac{\log y^2}{\log 2} = \frac{\log 9}{\log \frac{1}{3}}$, find x and y.

21. Prove the following:

(i) $3^{\log 4} = 4^{\log 3}$ (ii) $27^{\log 2} = 8^{\log 3}$

22. Solve the following equations:

(i) $\log (2x + 3) = \log 7$

(ii) $\log (x + 1) + \log (x - 1) = \log 24$

(iii) $\log (10x + 5) - \log (x - 4) = 2$

(iv) $\log_{10} 5 + \log_{10} (5x + 1) = \log_{10} (x + 5) + 1$

(v) $\log (4y - 3) = \log (2y + 1) - \log 3$

(vi) $\log_{10} (x + 2) + \log_{10} (x - 2) = \log_{10} 3 + 3 \log_{10} 4$

(vii) $\log(3x + 2) + \log(3x - 2) = 5 \log 2$

23. Solve for x: $\log_3 (x + 1) - 1 = 3 + \log_3 (x - 1)$

24. Solve for x: $5^{\log x} + 3^{\log x} = 3^{\log x + 1} - 5^{\log x - 1}$

25. If $\log \frac{x - y}{2} = \frac{1}{2} (\log x + \log y)$, prove that $x^2 + y^2 = 6xy$

26. If $x^2 + y^2 = 23xy$, prove that $\log \frac{x + y}{5} = \frac{1}{2} (\log x + \log y)$

27. If $p = \log_{10} 20$ and $q = \log_{10} 25$, find the value of x if

$2 \log_{10}(x + 1) = 2p - q$

28. Show that:

(i) $\frac{1}{\log_2 42} + \frac{1}{\log_3 42} + \frac{1}{\log_7 42} = 1$ (ii) $\frac{1}{\log_8 36} + \frac{1}{\log_9 36} + \frac{1}{\log_{18} 36} = 2$

29. Prove the following identities:

(i) $\frac{1}{\log_a abc} + \frac{1}{\log_b abc} + \frac{1}{\log_c abc} = 1$ (ii) $\log_b a \cdot \log_c b \cdot \log_d c = \log_d a$

30. Given that $\log_a x = \frac{1}{\alpha}$, $\log_b x = \frac{1}{\beta}$, $\log_c x = \frac{1}{\gamma}$, find $\log_{abc} x$.

31. Solve for x:

(i) $\log_3 x + \log_9 x + \log_{81} x = \frac{7}{4}$ (ii) $\log_2 x + \log_8 x + \log_{32} x = \frac{23}{15}$

Multiple Choice Questions MCQs

Choose the correct answer from the given four options (1 to 8):

1. If $\log_{\sqrt{3}} 27 = x$, then the value of x is

(a) 3 (b) 4 (c) 6 (d) 9

2. If $\log_5 (0{\cdot}04) = x$, then the value of x is

(a) 2 (b) 4 (c) -4 (d) -2

3. If $\log_{0{\cdot}5} 64 = x$, then the value of x is

(a) -4 (b) -6 (c) 4 (d) 6

4. If $\log_{3\sqrt{5}} x = -3$, then the value of x is

(a) $\frac{1}{5}$ (b) $-\frac{1}{5}$ (c) -1 (d) 5

5. If $\log (3x + 1) = 2$, then the value of x is

(a) $\frac{1}{3}$ (b) 99 (c) 33 (d) $\frac{19}{3}$

6. The value of $2 + \log_{10} (0{\cdot}01)$ is

(a) 4 (b) 3 (c) 1 (d) 0

7. The value of $\frac{\log 8 - \log 2}{\log 32}$ is

(a) $\frac{2}{5}$ (b) $\frac{1}{4}$ (c) $-\frac{2}{5}$ (d) $\frac{1}{3}$

8. Consider the following two statements:

Statement I: $\log_5 150 = \log_5 25 + \log_5 125$

Statement II: $\log_a (b + c) = \log_a b + \log_a c$

Which of the following is valid?

(a) Both the Statements are true.

(b) Both the Statements are false.

(c) Statement I is true, and Statement II is false.

(d) Statement I is false, and Statement II is true.

ASSERTION-REASON TYPE QUESTION (SOLVED)

In these examples and following questions, read the given statements carefully and choose the correct option.

(a) Assertion (A) is true, Reason (R) is false.

(b) Assertion (A) is false, Reason (R) is true.

(c) Both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason for Assertion (A).

(d) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct reason (or explanation) for Assertion (A).

1. **Assertion (A):** $\log_{\sqrt{2}} 8 = 6$

Reason (R): $\log_a b = n \Rightarrow a^n = b$ when a is any positive real number except 1, and b is a real positive number.

Sol. We know Reason (R) to be true.

We will use it to solve Assertion (A).

Let $\log_{\sqrt{2}} 8 = n \Rightarrow (\sqrt{2})^n = 8 \Rightarrow \left(2^{\frac{1}{2}}\right)^n = 2^3$

$\Rightarrow 2^{\frac{n}{2}} = 2^3 \Rightarrow \frac{n}{2} = 3 \Rightarrow n = 6$

$\therefore \log_{\sqrt{2}} 8 = 6$

$\therefore$ Assertion (A) is true.

Thus both Assertion (A) and Reason (R) are true, and Reason (R) is the correct reason for Assertion (A).

∴ Correct answer is (c).

ASSERTION-REASON TYPE QUESTIONS (UNSOLVED)

1. **Assertion (A):** $\log_2 16 = 4$

   Reason (R): $\log_a (bc) = \log_a b + \log_a c$

2. **Assertion (A):** $\log_3 \left(\frac{1}{9}\right) = -2$

   Reason (R): $\log_a \left(\frac{1}{b}\right) = -\log_a b$

3. **Assertion (A):** $\log_{\sqrt{2}} 2^5 = 10$

   Reason (R): $\log_{a^m} b^n = \frac{n}{m} . \log_a b$

4. **Assertion (A):** If $\log x = \frac{\log 8}{\log 0{\cdot}25}$, then $x = -6$

   Reason (R): If $\log_a b = \log_a c$, then $b = c$.

Summary

- $a^n = b \Leftrightarrow \log_a b = n,\ a > 0$ and $a \neq 1$
- $\log_a 1 = 0,\ \log_a a = 1$
- $\log_a mn = \log_a m + \log_a n$
- $\log_a \frac{m}{n} = \log_a m - \log_a n$
- $\log_a m^n = n \log_a m$
- $\log_a m = \frac{\log_b m}{\log_b a},\ m > 0,\ a, b > 0,\ a \neq 1,\ b \neq 1$
- $\log_b m = \log_a m \times \log_b a$
- $\log_b a \times \log_a b = 1$
- $\log_b a = \frac{1}{\log_a b}$
- $a^{\log_a x} = x$

Chapter Test

1. Expand $\log_a \sqrt[3]{x^7 y^8 \div \sqrt[4]{z}}$
2. Find the value of $\log_{\sqrt{3}} 3\sqrt{3} - \log_5 (0{\cdot}04)$
3. Prove the following:

   (i) $(\log x)^2 - (\log y)^2 = \log \frac{x}{y} . \log xy$ (ii) $2 \log \frac{11}{13} + \log \frac{130}{77} - \log \frac{55}{91} = \log 2$
4. If $\log (m + n) = \log m + \log n$, show that $n = \frac{m}{m-1}$

5. If $\log \frac{x+y}{2} = \frac{1}{2}(\log x + \log y)$, prove that $x = y$

6. If a, b are positive real numbers, $a > b$ and $a^2 + b^2 = 27ab$, prove that

$$\log\left(\frac{a-b}{5}\right) = \frac{1}{2}(\log a + \log b)$$

7. Solve the following equations for x:

(i) $\log_x \frac{1}{49} = -2$

(ii) $\log_x \frac{1}{4\sqrt{2}} = -5$

(iii) $\log_x \frac{1}{243} = 10$

(iv) $\log_4 32 = x - 4$

(v) $\log_7 (2x^2 - 1) = 2$

(vi) $\log (x^2 - 21) = 2$

(vii) $\log_6 (x-2)(x+3) = 1$

(viii) $\log_6 (x-2) + \log_6 (x+3) = 1$

(ix) $\log (x+1) + \log (x-1) = \log 11 + 2 \log 3$

8. Solve for x and y:

$$\frac{\log x}{3} = \frac{\log y}{2} \text{ and } \log (xy) = 5$$

9. If $a = 1 + \log_x yz$, $b = 1 + \log_y zx$ and $c = 1 + \log_z xy$, then show that $ab + bc + ca = abc$.

10. If $\frac{1}{\log_a x} + \frac{1}{\log_b x} = \frac{2}{\log_c x}$, prove that $c^2 = ab$

9

Triangles

INTRODUCTION

In previous classes, we have studied about triangles and various types of triangl the basis of sides and on the basis of angles. We have also studied the following

(*i*) angles sum property of a triangle and exterior angle property

(*ii*) congruence of triangles

(*iii*) inequalities in a triangle.

In this chapter, we will review these and shall study about congruence of tria in detail, rules of congruency, some more properties of triangles and inequalities triangle.

9.1 TRIANGLE

A triangle is a closed curve formed by three line segments. 'Tri' means 'three'. A triangle has three sides, three angles and three vertices.

The adjoining figure shows a triangle ABC. The line segments AB, BC and CA are called its **sides**. The angles ∠A, ∠B and ∠C are called its **interior angles** or simply **angles**. The points A, B and C are called its **vertices**. Three sides and the three angles are called its six **elements**.

In the above figure, look at the vertex A. It is the point of intersection of the side and AC, BC is the remaining side. We say that vertex A and side BC are opposite to other. Also ∠A and side BC are opposite to each other.

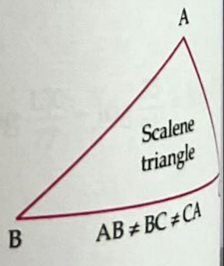
Similarly, vertex B and side CA are opposite to each other; ∠B and side CA are opp to each other. Same can be said about vertex C, ∠C and side AB.

9.1.1 Types of triangles

Types of triangles on the basis of sides.

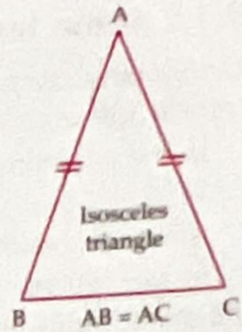
(*i*) **Scalene triangle.** *If all the sides of a triangle are unequal, it is called a* ***scalene triangle.***

In the adjoining diagram, AB ≠ BC ≠ CA, so ΔABC is a scalene triangle.



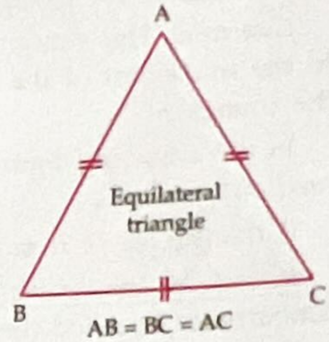
(*ii*) **Isosceles triangle.** *If any two sides of a triangle are equal, it is called an* ***isosceles triangle.***

In the adjoining diagram, AB = AC, so ΔABC is an isosceles triangle. Usually, equal sides are indicated by putting marks on each of them.



(iii) **Equilateral triangle.** *If all the three sides of a triangle are equal, it is called an* ***equilateral triangle.***

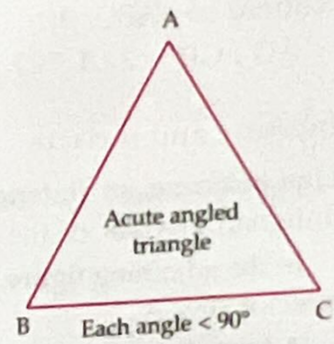
In the adjoining diagram, AB = BC = AC, so ΔABC is an equilateral triangle.



Types of triangles on the basis of angles.

(i) **Acute angled triangle.** *If all the three angles of a triangle are acute (less than 90°), it is called an* ***acute-angled triangle.***

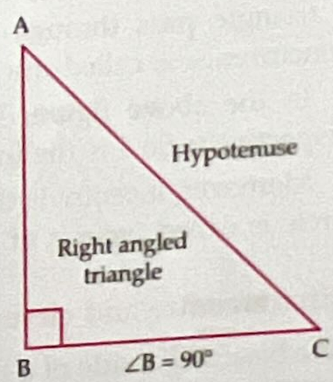
In the adjoining diagram, each angle is less than 90°, so ΔABC is an acute angled triangle.



(ii) **Right-angled triangle.** *If one angle of a triangle is a right angle (= 90°), it is called a* ***right-angled triangle.***

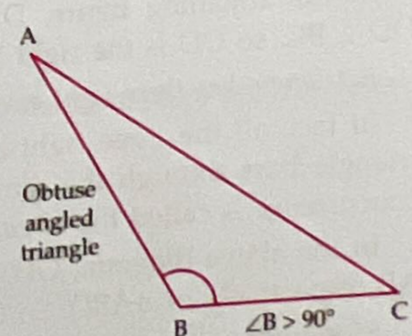
In a right angled triangle, the side opposite to right angle is called **hypotenuse.**

In the adjoining diagram, $\angle B = 90°$, so ΔABC is a right angled triangle and side AC is the hypotenuse.



(iii) **Obtuse angled triangle.** *If one angle of a triangle is obtuse (greater than 90°), it is called an* ***obtuse-angled triangle.***

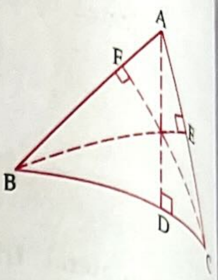
In the adjoining diagram, $\angle B$ is obtuse (greater than 90°), so ΔABC is an obtuse angled triangle.



9.1.2 Some terms connected with a triangle

Orthocentre. *Perpendicular from a vertex of a triangle to the opposite side is called an **altitude** of the triangle.*

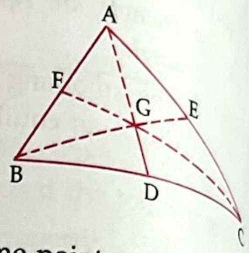
In the adjoining figure, AD ⊥ BC, so AD is an altitude of $\triangle$ABC.



A triangle has three altitudes.

In fact, all the three altitudes of a triangle pass through the same point and the point of concurrence is called the ***orthocentre*** of the triangle.

Centroid. The straight line joining a vertex of a triangle to the mid-point of the opposite side is called a ***median*** of the triangle.



In the adjoining figure, D is mid-point of BC, so AD is a median of $\triangle$ABC.

A triangle has three medians.

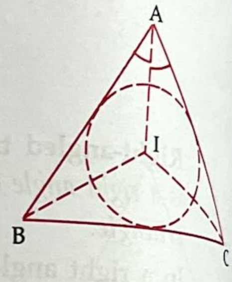
In fact, all the three medians of a triangle pass through the same point and the point of concurrence is called the ***centroid*** of the triangle.

The centroid of a triangle divides every median in the ratio of 2 : 1. Thus, if G is the centroid of $\triangle$ABC, then

AG : GD = 2 : 1, BG : GE = 2 : 1 and CG : GF = 2 : 1

Incentre and incircle

Line bisecting an (interior) angle of a triangle is called the (internal) ***bisector*** of the angle of the triangle.



In the adjoining figure, ∠BAI = ∠IAC, so AI is the (internal) bisector of ∠A.

A triangle has three internal bisectors of its angles.

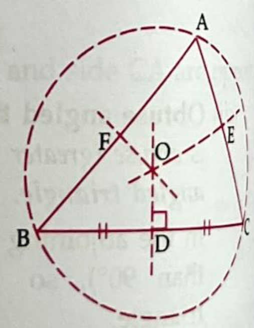
In fact, all the three (internal) bisectors of the angles of a triangle pass through the same point and the point of concurrence is called the ***incentre*** of the triangle.

In the above figure, IA, IB and IC are the (internal) bisectors of ∠A, ∠B and ∠C respectively. So I is the incentre of $\triangle$ABC.

Moreover, incentre is the centre of a circle which touches all the sides of $\triangle$ABC and this circle is called ***incircle*** of $\triangle$ABC.

Circumcentre and circumcircle

Line bisecting a side of a triangle and perpendicular to it is called the ***right bisector*** of the side of the triangle.



In the adjoining figure, D is mid-point of BC and OD ⊥ BC, so OD is the right bisector of the side BC.

A triangle has three right bisectors of its sides.

In fact, all the three right bisectors of the sides of a triangle pass through the same point and the point of concurrence is called the ***circumcentre*** of the triangle.

In the above diagram, OD, OE and OF are the right bisectors of the sides BC, CA and AB respectively of $\triangle$ABC. So O is the circumcentre of $\triangle$ABC.

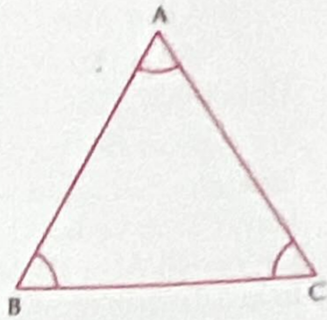
Moreover, circumcentre is the centre of a circle which passes through the vertices of $\triangle$ABC and this circle is called ***circumcircle*** of $\triangle$ABC.

9.1.3 Angles sum property of a triangle

The sum of angles of a triangle is 180°

In the adjoining figure, ABC is a triangle.

$\angle A + \angle B + \angle C = 180°$

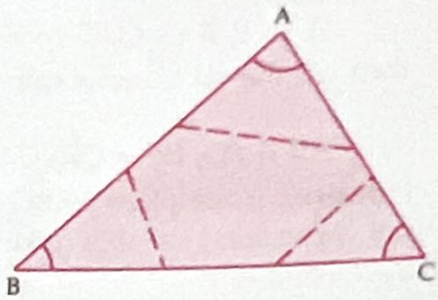


Activity

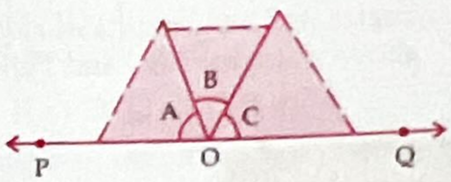
To verify that sum of angles of a triangle is 180° by cutting and pasting.

Steps

1. Take a sheet of paper and draw any triangle ABC on it and cut off the three angles *i.e.* $\angle A$, $\angle B$ and $\angle C$ along the dotted lines (as shown in the adjoining figure).



2. Draw any line PQ on the sheet of paper and mark a point O on it. Paste the cut outs of $\angle A$, $\angle B$ and $\angle C$ on the line PQ such that their vertices A, B and C all fall at the point O (as shown in the adjoining figure).



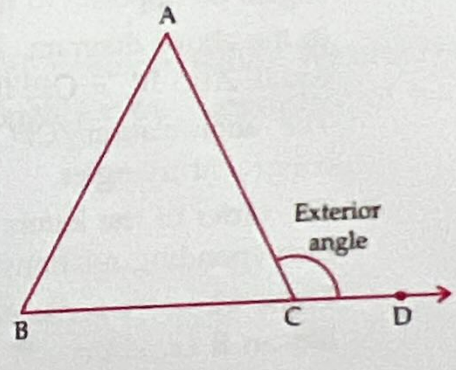
(as shown in the adjoining figure).
Note that the outer arms of $\angle A$ and $\angle C$ coincide with the line PQ. The three angles now constitute one angle *i.e.* $\angle POQ$.

But POQ is a straight angle,

$\therefore \quad \angle A + \angle B + \angle C = 180°$

9.1.4 An exterior angle property of a triangle

Let ABC be a triangle and its side BC be produced to D, then $\angle ACD$ is called an ***exterior angle*** at C. The two interior angles of the triangle that are opposite to the exterior $\angle ACD$ are called its ***interior opposite angles*** or ***remote interior angles***. Thus, $\angle ABC$ and $\angle BAC$ of ΔABC are interior opposite angles of the exterior $\angle ACD$.



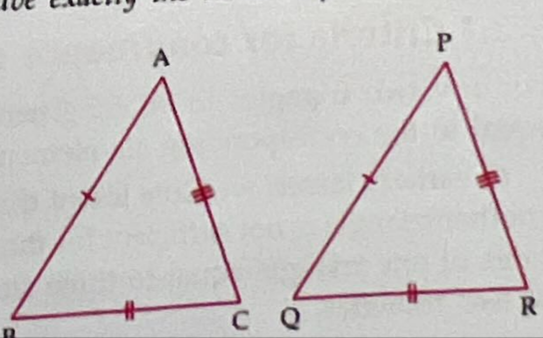
An exterior angle of a triangle is equal to sum of its interior opposite angles.

In the above figure, $\angle ACD = \angle A + \angle B$

9.2 CONGRUENCE OF TRIANGLES

Two triangles are called ***congruent*** *if and only if they have exactly the same shape and the same size.*

In the adjoining figure, two triangles ABC and PQR are congruent. It means that the sketch of one triangle can be slided onto the sketch of the other so that they fit each other exactly *i.e.* when one triangle is superimposed on the other, they cover each other exactly.



Notice that these triangles are such that

$$AB = PQ,\ BC = QR,\ AC = PR,\ \angle A = \angle P,\ \angle B = \angle Q \text{ and } \angle C = \angle R$$

Thus these triangles have the same shape and same size, so they are congruent. We express this as $\Delta ABC \cong \Delta PQR$.

It means that when we place a trace-copy of ΔABC on ΔPQR, vertex A falls on vertex P, vertex B on vertex Q and vertex C on vertex R. Then side AB falls on PQ, BC on QR and CA on RP. Also $\angle A$ falls on $\angle P$, $\angle B$ on $\angle Q$ and $\angle C$. Thus, the order in which the vertices match, automatically determines a correspondence between the sides and the angles of the two triangles. It follows that if the vertices of ΔABC match the vertices of ΔPQR in the order:

$$A \leftrightarrow P,\ B \leftrightarrow Q,\ C \leftrightarrow R$$

then all the six corresponding parts (3 sides and 3 angles) of two triangles are equal *i.e.*

$$AB = PQ,\ BC = QR,\ CA = RP,\ \angle A = \angle P,\ \angle B = \angle Q \text{ and } \angle C = \angle R$$

However, if we place ΔABC on ΔPQR such that A falls on Q, then other vertices may not correspond suitably. Take a trace-copy of ΔABC and place vertex A on vertex Q and try to find out!

This shows that while talking about congruence of triangles, not only the measures of angles and the lengths of sides matter, but also the matching of vertices matter. In the above triangles ABC and PQR, the correspondence is

$$A \leftrightarrow P,\ B \leftrightarrow Q,\ C \leftrightarrow R$$

Remarks

1. Congruent triangles are 'equal in all respects' *i.e.* they are the exact *duplicate* of each other.
2. If two triangles are congruent, then any one can be *superposed* on the other to cover it exactly.
3. In congruent triangles, the sides and the angles which coincide by superposition are called *corresponding sides* and *corresponding angles*.
4. The corresponding sides lie opposite to the equal angles and the corresponding angles lie opposite to the equal sides.

   In the above diagram, $\angle A = \angle P$, therefore, the corresponding sides BC and QR are equal. Also BC = QR, therefore, the corresponding angles A and P are equal.

   The abbreviation 'CPCT' or 'c.p.c.t.' will be used for corresponding parts of congruent triangles.
5. The order of the letters in the names of congruent triangles displays the corresponding relationship between the two triangles.

   Thus, when we write $\Delta ABC \cong \Delta PQR$, it means that A lies on P, B lies on Q and C lies on R *i.e.* $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$ and BC = QR, CA = RP, AB = PQ.

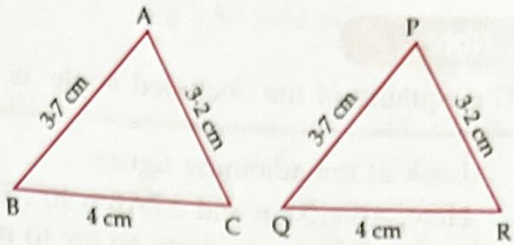
   Writing any other correspondence *i.e.* $\Delta ABC \cong \Delta PRQ$, $\Delta ABC \cong \Delta RPQ$ etc. will be incorrect.

9.2.1 Criteria for congruence of triangles

For any two triangles to be congruent, the *six elements* of one triangle *need not be proved* equal to the corresponding six elements of the other triangle.

In earlier classes, we have learnt that three angles of one triangle equal to three angles of another triangle is not sufficient for the congruence of two triangles. Let us see whether three sides of one triangle equal to three sides of another triangle is enough for the congruence of two triangles.

Draw two triangles ABC and PQR such that $AB = PQ = 3{\cdot}7$ cm, $BC = QR = 4$ cm and $CA = RP = 3{\cdot}2$ cm



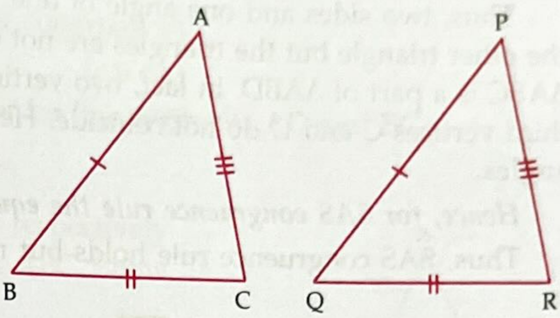
Make a trace-copy of ΔABC and place it over ΔPQR. We observe that the two triangles cover each other exactly and so they are congruent.

Repeat this activity with more pairs of triangles satisfying these conditions. We observe that the equality of three sides of two triangles is sufficient for the congruence of two triangles. We record it as:

- **SSS congruence rule**

Two triangles are congruent if the three sides of one triangle are equal to the three sides of the other triangle.

In the adjoining figure,



$$AB = PQ,\ BC = QR$$

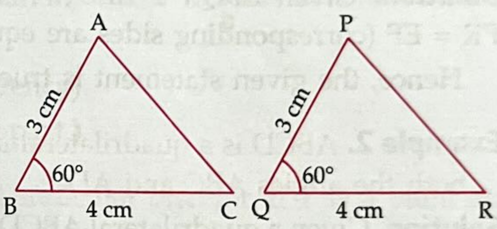
and $$AC = PR$$

$\therefore$ $$\Delta ABC \cong \Delta PQR$$

Note that SSS stands for **Side-Side-Side.**

Now, let us see whether two sides and the included angle of one triangle are equal to two sides and the included angle of another triangle is enough for the congruence of two triangles.

Draw two triangles ABC and PQR such that $BC = QR = 4$ cm, $\angle B = \angle Q = 60°$ and $AB = PQ = 3$ cm



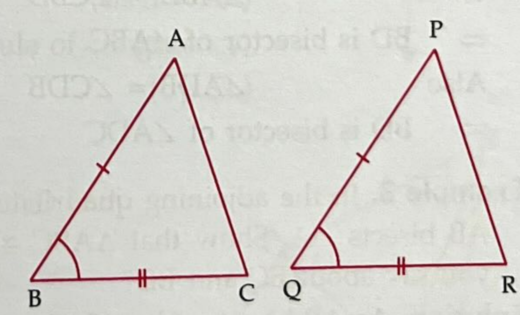
Make a trace-copy of ΔABC and place it over ΔPQR. We observe that the two triangles cover each other exactly and so they are congruent.

Repeat this activity with more pairs of triangles satisfying these conditions. We observe that equality of two sides and the included angle is enough for the congruence of two triangles. We record it as:

- **SAS congruence rule**

Two triangles are congruent if two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle.

In the adjoining figure,



$$AB = PQ,\ BC = QR \text{ and}$$

$$\angle B = \angle Q$$

$\therefore$ $$\Delta ABC \cong \Delta PQR$$

This is known as **Side-Angle-Side** criterion (or rule) of congruency.