# DIGITAL FILTERS FOR NON-UNIFORMLY SAMPLED SIGNALS

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## ABSTRACT

We consider in this paper the problem of designing digital filters for non-uniformly sampled signals without any previous reconstruction of the equivalent uniformly sampled signal. In a first step, we propose a filtering method based on a numerical approximation of the convolution integral or differential equations, which is applicable to finite impulse response filters. The second method uses digitization of an equivalent analog filter and may be applied to infinite impulse response filters. Numerical examples show the validity of the theoretical analysis.

## 1. INTRODUCTION

In many physical systems, signals are only known by samples measured at non-periodic times. Such cases are found, for example, in laser velocimetry or Doppler analysis, in real-time data transmission via asynchronous telecommunication networks which need packet signal processing or, more generally, when signals are sampled in a deliberately non-uniformly manner using adaptive sampling techniques for data compression.

A main distinction concerns the possible values of the time intervals between successive samples. They can be integer multiples or not of a 'hidden sampling period' when a synchronous system can periodically consider the signal, without systematically sampling it.

For a long time, the processing of non-uniformly sampled signals (NUSS) had been limited by prior recovery of a uniformly sampled signal. Various extensions to Shannon's theorem have been demonstrated (see for example [1][6]). Real time recovery had been less studied. However, some methods can be applied [10][9][11].

As in the case of periodic sampling, filtering of these signals is a fundamental problem. In this paper, we study real-time linear filtering of NUSS and, more specifically, digitizing continuous-time systems.

After having described the signal model that we use, we present in section 2 a filtering method based on finite differences approximations. Section 3 deals with the digitizing of any analog filter. Numerical simulations show the validity of the proposed methods in section 4.

*Signal modeling and notations*

The more common model $\{x(t_k)\}$ of a NUSS [4] is given by a continuous-time random process $x(t)$ that is sampled by a point process $\Pi_{tk} = \{t_k\}$

$$x_k \equiv x(t_k)$$

In the case of 'hidden sampling period' signals, $x(t)$ can be modeled in a classical way (ARMA models). In the general case, continuous-time model (CARMA models [2]) must be used.

The point process $\Pi_{tk}$ to consider depends on applications. We will classically find Poisson processes (case of particle counting); processes whose time intervals have a constant probability density function (PDF) in an interval (case of some network data transmission); processes whose PDF are centered around a mean value (sampling jitter); processes whose time intervals are integer multiples of a hidden uniform sampling period (case of missing samples due to errors); Markov processes (packet data transmission on asynchronous telecommunication networks : HTTP, FTP... protocols).

The general case in which $\Pi_{t_k}$ is one or another will be referred to throughout this paper.

## 2. FINITE DIFFERENCES METHODS

Mathematically, a linear filter can be expressed as an integral convolution in the form

$$s(t) = \int_{-\infty}^{+\infty} h(\tau)\ e(t-\tau)\ \mathrm{d}\tau$$

where input signal values $e(t)$ are only known for some values of $t$.

Let $\delta_n = t_n - t_{n-1}$ be the time interval between two successive samples. For a causal system, the rectangular approximation of the previous integral is given by

$$\hat{s}_n = \sum_{k=0}^{+\infty} \left[ h(t_n - t_{n-k})\ e_{n-k}\ \delta_{n-k} \right]$$

This formula can easily be implemented in real time in the case of an $N$ points finite impulse response filter (FIR) or if the filter's impulse response is truncated.

Another estimate can be obtained with the trapezoidal method

$$\begin{aligned} \hat{s}_n &= \frac{1}{2} \sum_{k=0}^{N-1} \left[ h(t_n - t_{n-k})\ e_{n-k} \right. \\ &\quad \left. + h(t_n - t_{n-k-1})\ e_{n-k-1} \right]\ \delta_{n-k} \end{aligned}$$

This less traditional formula gives better results, particularly when $h(0) \neq 0$.

However, these approximations cannot be used for a real-time implementation of a infinite impulse response filter (IIR) for which a recursive structure is required.

If we denote $E(s)$, $H(s)$ et $S(s)$ the respective Laplace transforms of the signals $e(t)$, $h(t)$ and $s(t)$, the convolution integral becomes

$$S(s) = H(s)\ E(s)$$

and if $H(s)$ is a rational function in $s$

$$H(s) = \frac{\sum_{i=0}^{N} b_i\ s^i}{\sum_{i=0}^{N} a_i\ s^i} \qquad (a_N = 1) \tag{1}$$

the solution of the previous integral can be computed by solving the following differential equation

$$\begin{aligned} &\frac{\mathrm{d}^{(N)} s(t)}{\mathrm{d}t^{(N)}} + \cdots + a_1 \frac{\mathrm{d}s(t)}{\mathrm{d}t} + a_0 s(t) = \\ &b_N \frac{\mathrm{d}^{(N)} e(t)}{\mathrm{d}t^{(N)}} + \cdots + b_1 \frac{\mathrm{d}e(t)}{\mathrm{d}t} + b_0 e(t) \end{aligned} \tag{2}$$

The first idea is to get a simple finite difference approximation of the derivatives in the differential equations. For example, in the case of a second order transfer function, Euler approximation allows the estimation of the derivatives at sampling times $t_k$ with only the past values

$$\begin{aligned} \left.\frac{\mathrm{d}s(t)}{\mathrm{d}t}\right|_{t=t_k} &\approx \hat{s}'_k = \frac{s_k - s_{k-1}}{\delta_k} \\ \left.\frac{\mathrm{d}^2 s(t)}{\mathrm{d}t^2}\right|_{t=t_k} &\approx \hat{s}''_k = \frac{1}{\delta_k} \Big( \frac{s_k - s_{k-1}}{\delta_k} - \frac{s_{k-1} - s_{k-2}}{\delta_{k-1}} \Big) \end{aligned} \tag{3}$$

The output of a second order filter can now be computed in real-time with

$$\hat{s}_k = \alpha_k\ \hat{s}_{k-1} + \beta_k\ \hat{s}_{k-2} + \gamma_k\ e_k$$

where

$$\begin{aligned} \alpha_k &= \frac{\frac{a_1}{\delta_k} + \frac{1}{\delta_k^2} + \frac{1}{\delta_k\ \delta_{k-1}}}{a_0 + \frac{a_1}{\delta_k} + \frac{1}{\delta_k^2}} \\ \beta_k &= \frac{1}{\delta_k\ \delta_{k-1} \left[ a_0 + \frac{a_1}{\delta_k} + \frac{1}{\delta_k^2} \right]} \\ \gamma_k &= \frac{1}{a_0 + \frac{a_1}{\delta_k} + \frac{1}{\delta_k^2}} \end{aligned} \tag{4}$$

The estimate accuracy depends on the sampling times (i.e. the step size) and a high degree of accuracy [7] is thus only obtained for very small intervals.

The other classical methods of numerical integration of ordinary differential equations could be applied. But it is easy to extrapolate from this example that their complexity becomes very heavy, particularly because of the divisions that can be seen in 4. Then, another method had been used and is going to be presented now.

## 3. FROM CONTINUOUS TIME TO DISCRETE TIME WITH NUSS

Classically, numerical systems are discrete. The goal here is to obtain a corresponding digital filter from the transfer function $H(s)$ of a continuous time system.

In the uniform sampling case, some transforms $s = G(z)$ [8] are readily used to convert the analog $H(s)$ to the digital filter transfer function $H(z)$. On the other hand, in the case of NUSS, it is necessary to go back to the idea of numerical approximations of the derivatives or the integrals, as they can be found (in an implicit manner) in the Euler or Tustin methods.

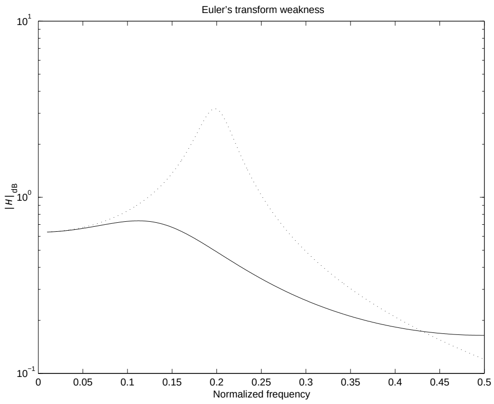
For example, for the Euler approximation (3), we obtain for a second order IIR filter

$$\begin{aligned} &\hat{s}_k \left[ a_0 + \frac{a_1}{\delta_k} + \frac{1}{\delta_k^2} \right] = \\ &\hat{s}_{k-1} \left[ \frac{a_1}{\delta_k} + \frac{1}{\delta_k^2} + \frac{1}{\delta_k\ \delta_{k-1}} \right] - \hat{s}_{k-2} \left[ \frac{1}{\delta_k\ \delta_{k-1}} \right] + e_k \end{aligned}$$

The filter output can be computed from the input values of the past sampling times with a high degree of computational efficiency.

Obviously, the well-known weaknesses of the Euler transform are also to be found with the NUSS (Figure 1).

Hence, we can hope to obtain a good degree of accuracy when the mean value of the sampling intervals $\delta_k$ is much greater than the inverse of the greatest frequency of the input signal.




$\cdots$ : real curve ; — : Euler approximation

FIGURE 1. Euler transform weakness

Now, if we take the Tustin transform

$$p \quad \xrightarrow{G} \quad \frac{2}{\delta}\frac{1 - z^{-1}}{1 + z^{-1}} \qquad \text{(periodic case)}$$

for a NUSS, the derivative of the signal $x(t)$ can be approximated by

$$\hat{x}'_k = -\,\hat{x}'_{k-1} + \frac{2}{\delta_k}\Big(x_k - x_{k-1}\Big)$$

So, we propose rewriting the formula (2) as

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}\mathbf{X}(t)}{\mathrm{d}t} = \mathbf{A}\ \mathbf{X}(t) + \mathbf{B}\ e(t) \\ \\ s(t) = \mathbf{C}\ \mathbf{X}(t) + \mathbf{D}\ e(t) \end{array}\right. \tag{5}$$

where

$\mathbf{X}(t)$ state vector $(1 \times N)$

$$\mathbf{A} = \begin{pmatrix} -b_{N-1} & -b_{N-2} & \cdots & -b_1 & -b_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$\mathbf{B} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^T$

$\mathbf{C} = \begin{pmatrix} a_{N-1} - a_N b_{N-1} & \cdots & a_0 - a_N b_0 \end{pmatrix}$

$\mathbf{D} = (a_N)$

Discretization of these equations using the approximates given by the Tustin transform can be expressed in the form

$$\left\{\begin{array}{l} \mathbf{X}_k = \mathbf{\Phi}_k\ \mathbf{X}_{k-1} + \mathbf{\Lambda}_k(e_k + e_{k-1}) \\ s_k = \mathbf{C}\ \mathbf{X}_k + \mathbf{D}\ e_k \end{array}\right. \tag{6}$$

where

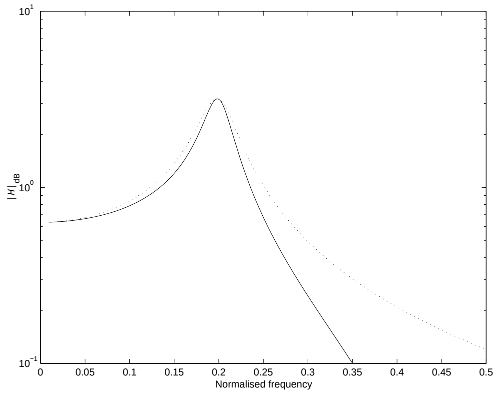
$$\left\{\begin{array}{l} \mathbf{\Phi}_k = \left(\mathbf{I} - \dfrac{\delta_k}{2}\mathbf{A}\right)^{-1}\left(\mathbf{I} + \dfrac{\delta_k}{2}\mathbf{A}\right) \\ \mathbf{\Lambda}_k = \dfrac{\delta_k}{2}\left(\mathbf{I} - \dfrac{\delta_k}{2}\mathbf{A}\right)^{-1}\mathbf{B} \end{array}\right.$$

Applying these formulae to filtering is now obvious: starting from the analog transfer function $H(s)$, we infer the corresponding differential equation (2) written in the form (5). Then, the application of the approximation formula (6) allows to obtain a recursive form.

It is possible to take in account the frequency distortion of the Tustin transform during the digitization by replacing poles and zeros of $H(s)$ by

$$\omega_c = \frac{2}{\delta}\tan\Big(\frac{\omega'_c\,\delta}{2}\Big)$$

The same example as before gives now much better results (Figure 2).




$\cdots$ : real curve ; — : Tustin approximation

FIGURE 2. Tustin's approximation without frequency distortion

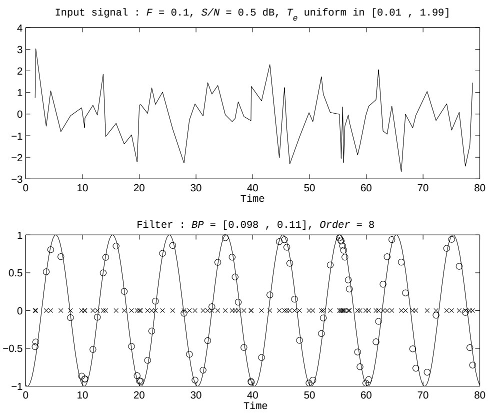
## 4. MAIN RESULTS

We must emphasize that, in order to validate NUSS processing, it is necessary to dispose of methods and tools capable of generating well characterized synthetic signals. As many methods in the literature [3] do not allow any control of the integration error, we have used the method presented in [5].

### *4.1. Noisy signal*

We show below (Figure 3) the results obtained with a noisy sinusoid input signal ($f_0 = 0,1$ Hz, $S/N = 0,5$ dB) sampled with a uniformly distributed sampling process in the boundaries $[0\,,\,2$ s$]$. This signal is applied to a selective recursive filter (8th order, central frequency equal to the sinus frequency).

As can be seen on figure 3, the Euler method gives unworkable results, as opposed to the proposed method.

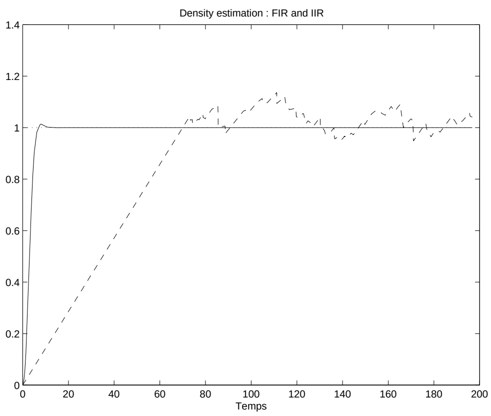



— : sinusoid without noise
Filter's output : x Euler approximate
o digitalization of the continuous filter

FIGURE 3. NUSS filtering of a noisy sinusoid

### *4.2. Particle counting*

The method that we propose has been used for the counting of particles (photons for example). The required information is carried out by the mean value of the time intervals between the arrivals of two photons. The sampling process has been modeled by a Poisson process, the density of which is $\lambda$. The density value has been estimated by a finite impulse response low-pass filter (FIR), which corresponds to the classical estimation method and then by an infinite impulse response filter (IIR), using the proposed method based on Tustin approximation.




... : real value ; - - FIR filter ; — IIR filter

FIGURE 4. Estimation of the time intervals mean value

Figure 4 shows the significantly better performances of the IIR filter as compared to the integrator: it accurately follows the changes of the physical quantity value, generally linked to the $\lambda$ coefficient of the Poisson law.

## 5. CONCLUSION

We have proposed two filtering methods for non uniformly sampled signals. The first one, based on an approximation of the convolution integral can be applied to finite impulse response filters. The other one uses the digitalization of the equivalent analog filter and can be applied to infinite impulse response filters.

The use of these results can help to design filters for systems in which data are intrinsically non periodic. One can think, for example, of spectral analysis methods by filter banks or process control algorithms where reference inputs and measurements are carried by asynchronous telecommunication channels.

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