Project 3
Quark-photon vertex

ECT* Doctoral Training Program, May 2–18, 2022
Hadron Physics with Functional Methods

Introduction

A well-established and time-honored way to probe the internal structure of hadrons is to measure their **electromagnetic form factors**. For example, the leading diagram in electron-nucleon scattering is a one-photon exchange, where a virtual photon couples to the nucleon. In general, the coupling of a photon to an onshell spin-1/2 fermion with mass m is described by the electromagnetic current matrix element

$$J^\mu(k,Q) = i\bar{u}(k_+)\left[F_1(Q^2)\,\gamma^\mu - \frac{F_2(Q^2)}{2m}\,\sigma^{\mu\nu}Q^\nu\right]u(k_-)\,. \tag{1}$$

Here, Q^μ is the photon four-momentum, $k_\pm^\mu$ are the outgoing and incoming momenta of the fermion, $\sigma^{\mu\nu} = -\frac{i}{2}[\gamma^\mu,\gamma^\nu]$, and $u(k_\pm)$ are the Dirac spinors satisfying the Dirac equation:

$$\not{p}\,u(p) = imu(p)\,, \qquad \bar{u}(p)\,\not{p} = im\bar{u}(p) \qquad \text{with} \quad p^2 = -m^2\,. \tag{2}$$

Because the fermion is onshell, we have $k_\pm^2 = -m^2$. If we define the average momentum k by

$$k_\pm = k \pm \frac{Q}{2} \quad \Rightarrow \quad k_\pm^2 = k^2 + \frac{Q^2}{4} \pm k\cdot Q = -m^2\,, \tag{3}$$

then it follows that $k^2 = -m^2 - Q^2/4$ and $k\cdot Q = 0$, so that the process is fully characterized by the squared momentum transfer $Q^2 \geq 0$. The electromagnetic structure of the fermion is thus described by its Lorentz-invariant form factors, the **Dirac** form factor $F_1(Q^2)$ and the **Pauli** form factor $F_2(Q^2)$. For vanishing photon momentum transfer $Q^2 = 0$, the Pauli form factor $F_2(0)$ defines the **anomalous magnetic moment** of the fermion.

When a photon couples to a hadron, then microscopically it must always couple to a quark. The elementary quantity is therefore the **quark-photon vertex**, which describes the electromagnetic coupling of quarks to photons and thus encodes their electromagnetic properties. An elementary quark is, however, not *onshell* and does not have a well-defined mass. This means we must relax the constraints above, i.e., we neither contract with onshell spinors on the left and right nor do we impose $k_\pm^2 = -m^2$. As a result, the general quark-photon vertex depends on three Lorentz invariants k^2, Q^2 and $k\cdot Q$, and it features a much richer Lorentz-Dirac tensor structure: Instead of γ^μ and $\sigma^{\mu\nu}Q^\nu$ that enter in Eq. (1) it depends on 12 tensors, and instead of two form factors $F_1(Q^2)$ and $F_2(Q^2)$ it will thus depend on 12 dressing functions $F_i(k^2, k\cdot Q, Q^2)$ that describe the electromagnetic properties of the quark. The goal of this project is to determine these functions by solving the quark-photon vertex BSE [1].

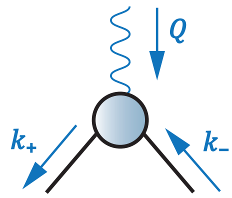

Figure 1: Momentum routing in the quark-photon vertex

Quark-photon vertex

The kinematics in the quark-photon vertex $\Gamma^\mu(k,Q)$ are shown in Fig. 1: The vertex depends on two independent momenta, the incoming quark momentum k_- and outgoing momentum k_+ (these are four-vectors), or equivalently the photon momentum Q and relative quark momentum k which are related by

$$k_\pm = k \pm \frac{Q}{2} \quad \Leftrightarrow \quad Q = k_+ - k_- \,, \quad k = \frac{k_+ + k_-}{2} \quad \Rightarrow \quad k_\pm^2 = k^2 + \frac{Q^2}{4} \pm k\cdot Q \,. \tag{4}$$

The tree-level vertex has the simple form $e\,Z_2\,i\gamma^\mu$, where Z_2 is the quark renormalization constant, e is the electric charge (in natural units) and $\alpha_{QED} = e^2/(4\pi) \approx 1/137$ the electromagnetic coupling.

The full quark-photon vertex depends on 12 linearly independent Lorentz-Dirac tensors. There is no common convention for them in the literature, but it is convenient to work with the following decomposition [2]:

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{4} g_j(k^2,\omega,Q^2)\, iG_j^\mu(k,Q) + \sum_{j=1}^{8} f_j(k^2,\omega,Q^2)\, iT_j^\mu(k,Q)\,. \tag{5}$$

Here, the $g_j(k^2,\omega,Q^2)$ and $f_j(k^2,\omega,Q^2)$ are the 12 dressing functions which depend on the three possible Lorentz invariants k^2, $\omega = k\cdot Q$ and Q^2, and the G_j^μ and T_j^μ are the corresponding tensors:

$$\begin{aligned}
G_1^\mu &= \gamma^\mu\,, & T_1^\mu &= t_{QQ}^{\mu\nu}\,\gamma^\nu\,, & T_5^\mu &= t_{QQ}^{\mu\nu}\,ik^\nu\,, \\
G_2^\mu &= k^\mu \not{k}\,, & T_2^\mu &= \omega\, t_{QQ}^{\mu\nu}\,\tfrac{i}{2}[\gamma^\nu,\not{k}]\,, & T_6^\mu &= t_{QQ}^{\mu\nu}\,k^\nu \not{k}\,, \\
G_3^\mu &= ik^\mu\,, & T_3^\mu &= \tfrac{i}{2}[\gamma^\mu,\not{Q}]\,, & T_7^\mu &= \omega\, t_{Qk}^{\mu\nu}\,\gamma^\nu\,, \\
G_4^\mu &= \omega\,\tfrac{i}{2}[\gamma^\mu,\not{k}]\,, & T_4^\mu &= \tfrac{1}{6}[\gamma^\mu,\not{k},\not{Q}]\,, & T_8^\mu &= t_{Qk}^{\mu\nu}\,\tfrac{i}{2}[\gamma^\nu,\not{k}]\,.
\end{aligned} \tag{6}$$

The quantities $t_{ab}^{\mu\nu}$ are defined by

$$t_{ab}^{\mu\nu} = a\cdot b\,\delta^{\mu\nu} - b^\mu a^\nu\,, \tag{7}$$

which is convenient because it entails that

$$a^\mu\, t_{ab}^{\mu\nu} = a\cdot b\, a^\nu - a\cdot b\, a^\nu = 0\,, \qquad t_{ab}^{\mu\nu}\, b^\nu = a\cdot b\, b^\mu - a\cdot b\, b^\mu = 0\,. \tag{8}$$

The commutators are given by

$$[A,B] = AB - BA\,, \qquad [A,B,C] = [A,B]\,C + [B,C]\,A + [C,A]\,B\,. \tag{9}$$

The decomposition (5) is advantageous for several reasons:

■ The quark-photon vertex must satisfy electromagnetic gauge invariance in the form of the **Ward-Takahashi identity** (WTI)

$$Q^\mu \Gamma^\mu(k,Q) = S(k_+)^{-1} - S(k_-)^{-1}\,, \tag{10}$$

which relates its longitudinal part with the quark propagator. From Eqs. (8–9) you can see that the tensors T_j^μ are transverse to the photon momentum Q^μ, i.e., $Q^\mu T_j^\mu = 0$, so they drop out from the WTI. Therefore, the WTI only affects the dressing functions g_j which are completely determined by the quark propagator. Inserting the decomposition (2) from Project 1 for the quark propagator,

$$S(p)^{-1} = A(p^2)\left(i\not{p} + M(p^2)\right) \quad \Leftrightarrow \quad S(p) = \frac{1}{A(p^2)}\,\frac{-i\not{p} + M(p^2)}{p^2 + M(p^2)^2} = -i\not{p}\,\sigma_v(p^2) + \sigma_s(p^2)\,. \tag{11}$$

one can easily show that this entails

$$g_1 = \Sigma_A\,, \qquad g_2 = 2\Delta_A\,, \qquad g_3 = -2\Delta_B\,, \qquad g_4 = 0\,, \tag{12}$$

where

$$\Sigma_A = \frac{A(k_+^2) + A(k_-^2)}{2}\,, \qquad \Delta_A = \frac{A(k_+^2) - A(k_-^2)}{k_+^2 - k_-^2} \tag{13}$$

and likewise for Δ_B, where $B(p^2) = A(p^2)\,M(p^2)$ and $k_+^2 - k_-^2 = 2k\cdot Q = 2\omega$. Therefore, once we know the two dressing functions $A(p^2)$ and $M(p^2)$ of the quark propagator, we already know a great deal about the quark-photon vertex! Note in particular that for a tree-level propagator ($A=1$, $M=m$) Eq. (12) reduces to $g_1 = 1$ and all other $g_j = 0$, so the vertex simplifies to

$$\Gamma^\mu(k,Q) = i\gamma^\mu + [\text{transverse part}]. \tag{14}$$

■ The vertex has a charge-conjugation symmetry and must satisfy

$$\bar{\Gamma}^\mu(k,Q) := -C\,\Gamma^\mu(-k,-Q)^T\,C^T \stackrel{!}{=} \Gamma^\mu(k,-Q)\,, \tag{15}$$

where $C = \gamma^4\gamma^2$ is the charge-conjugation matrix and the superscript T denotes a Dirac matrix transpose. One can show that each individual basis element G_j^μ, T_j^μ satisfies the same relation, i.e., we already chose the basis to satisfy Eq. (15). As a consequence, the dressing functions g_j and f_j must be even in the variable ω and can only depend on ω^2. This is easily verified for the g_j in Eq. (12): If we perform a Taylor expansion with respect to the variable $k_\pm^2$ from Eq. (4), we find

$$A(k_\pm^2) = A\left(k^2 + \frac{Q^2}{4} \pm \omega\right) = A\left(k^2 + \frac{Q^2}{4}\right) \pm \omega A'\left(k^2 + \frac{Q^2}{4}\right) + \ldots, \tag{16}$$

$$\Rightarrow \quad \Sigma_A = A\left(k^2 + \frac{Q^2}{4}\right) + \mathcal{O}(\omega^2)\,, \qquad \Delta_A = A'\left(k^2 + \frac{Q^2}{4}\right) + \mathcal{O}(\omega^2)\,, \tag{17}$$

where the primes denote the derivatives with respect to the arguments.

■ The tensor basis (6) is free of kinematic constraints, which entails that all dressing functions $g_j(k^2,\omega,Q^2)$ and $f_j(k^2,\omega,Q^2)$ become constant in either of the kinematic limits $Q^\mu = 0$ or $k^\mu = 0$. Again, this is easily verified for the g_j since we have

$$Q^2 = 0 \;\rightarrow\; \Sigma_A = A(k^2),\;\; \Delta_A = A'(k^2), \quad k^2 = 0 \;\rightarrow\; \Sigma_A = A\left(\frac{Q^2}{4}\right),\;\; \Delta_A = A'\left(\frac{Q^2}{4}\right). \tag{18}$$

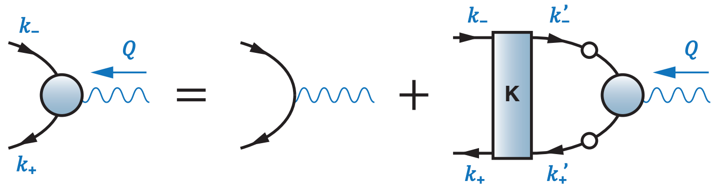

Figure 2: Inhomogeneous Bethe-Salpeter equation for the quark-photon vertex

All in all, even without knowing anything about the dynamics, in this way we have derived some very useful constraints on the dressing functions from symmetry arguments alone.

To calculate the quark-photon vertex dynamically, we must solve its **Bethe-Salpeter equation** (BSE) shown in Fig. 2. This is again an exact, nonperturbative equation in QCD. However, it requires knowledge of the quark-antiquark kernel K, which contains the sum of all possible $q\bar{q}$ irreducible gluon interactions between the quarks that are mediated by the strong coupling α_{QCD}. (In principle it also contains the electromagnetic interactions through photon exchanges, but those are much smaller since they involve α_{QED}.)

In practice we will use an ansatz for the kernel. However, this ansatz cannot be completely arbitrary because the kernel is closely linked with the quark propagator. In particular, electromagnetic gauge invariance can only hold if the kernel of the BSE is consistent with that of the quark DSE. The **rainbow-ladder truncation** satisfies that constraint: Here we approximate the full kernel by a gluon exchange between the quark and antiquark, with the same effective interaction that goes into the quark DSE (see Project 1) and absorbs the dressing functions of the gluon propagator and quark-gluon vertex:

$$g(k^2) = Z_2^2\,\frac{16\pi}{3}\,\frac{\alpha(k^2)}{k^2}\,, \qquad \alpha(k^2) = \pi\eta^7 x^2\, e^{-\eta^2 x} + \frac{2\pi\gamma_m\left(1-e^{-k^2/\Lambda_t^2}\right)}{\ln\left[e^2-1+\left(1+k^2/\Lambda_{\text{QCD}}^2\right)^2\right]}\,, \qquad x = \frac{k^2}{\Lambda^2}\,. \tag{19}$$

This is the Maris-Tandy model [1, 3], where the second term with the parameters $\Lambda_t = 1$ GeV, $\Lambda_{\text{QCD}} = 0.234$ GeV and $\gamma_m = 12/25$ ensures the correct perturbative behavior at large momenta, and the first term with the parameters $\Lambda = 0.72$ GeV and $\eta = 1.8$ dominates the small-momentum behavior and is important for the dynamical generation of a quark mass.

The resulting BSE reads explicitly:

$$\Gamma^\mu(k,Q) = Z_2\, i\gamma^\mu + \int_{k'} g(l^2)\, T_l^{\alpha\beta}\,\gamma^\alpha\, S(k'_+)\,\Gamma^\mu(k',Q)\,S(k'_-)\,\gamma^\beta\,. \tag{20}$$

Here, k' is the relative momentum of the vertex inside the momentum loop, $k'_\pm = k' \pm Q/2$ are the quark momenta, $l = k-k'$ is the gluon momentum, and $T_l^{\mu\nu} = \delta^{\mu\nu} - l^\mu l^\nu/l^2$ is the transverse projector arising from the gluon propagator. Keep in mind that $\Gamma^\mu(k,Q)$ is a 4×4 Dirac matrix with another Lorentz index $\mu = 1\dots 4$. The integral measure $\int_{k'}$ is given by

$$\int_{k'} = \int \frac{d^4k'}{(2\pi)^4} = \frac{1}{(2\pi)^4}\,\frac{1}{2}\int_0^{L^2} dk'^2\, k'^2 \int_{-1}^{1} dz'\,\sqrt{1-z'^2}\int_{-1}^{1} dy \int_0^{2\pi} d\phi\,, \tag{21}$$

where L is the cutoff in the system (a typical value is $L = 10^3$ GeV).

Solving the BSE

In principle, Eqs. (19–20) are all we need to solve the vertex BSE numerically. Suppose we write the decomposition (5) as

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{12} F_j(k^2,\omega,Q^2)\, i t_j^\mu(k,Q)\,, \tag{22}$$

with 12 dressing functions $F_j \in \{g_j, f_j\}$ and corresponding tensors $t_j^\mu \in \{G_j^\mu, T_j^\mu\}$, and plug this into the BSE. To project out the F_j on the l.h.s., we contract the equation with the charge-conjugate basis elements $\bar{t}_i^\mu$, with charge conjugation defined as in Eq. (15). The t_i^μ are not orthonormal, so we have

$$H_{ij}(k^2,\omega,Q^2) = \frac{1}{4}\,\text{Tr}\left\{\bar{t}_i^\mu(k,Q)\, t_j^\mu(k,Q)\right\} \neq \delta_{ij}\,. \tag{23}$$

As a result, the BSE turns into

$$H_{ij}(k^2,\omega,Q^2)\, F_j(k^2,\omega,Q^2) = F_i^0(k^2,\omega,Q^2) + \int\limits_{k'} K_{ij}\, F_j(k'^2,\omega',Q^2) \tag{24}$$

with

$$\begin{aligned} F_j^0(k^2,\omega,Q^2) &= Z_2\,\frac{1}{4}\,\text{Tr}\,\left\{\bar{t}_i^\mu(k,Q)\,\gamma^\mu\right\}\,, \\ K_{ij} &= g(l^2)\,T_l^{\alpha\beta}\,\frac{1}{4}\,\text{Tr}\,\left\{\bar{t}_i^\mu(k,Q)\,\gamma^\alpha\, S(k'_+)\, t_j^\mu(k',Q)\, S(k'_-)\,\gamma^\beta\right\}\,. \end{aligned} \tag{25}$$

The matrices H and K are known, so this is an inhomogeneous linear integral equation for the F_i. Note that it involves a matrix inversion for H.

There is still a more efficient way to handle the problem. The underlying idea [4] is to construct an **orthonormal basis** where $H_{ij} = \delta_{ij}$, so that we can bypass the matrix inversion. In addition, we will see that the resulting equation forms two orthogonal subspaces which can be handled independently, and the traces become very simple so we can work them out directly. To start with, we write down the four-vectors in a given coordinate frame:

$$Q^\mu = \sqrt{Q^2}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad k^\mu = \sqrt{k^2}\begin{bmatrix} 0 \\ 0 \\ \sqrt{1-z^2} \\ z \end{bmatrix}, \qquad k'^\mu = \sqrt{k'^2}\begin{bmatrix} 0 \\ \sqrt{1-z'^2}\,\sqrt{1-y^2} \\ \sqrt{1-z'^2}\, y \\ z' \end{bmatrix}. \tag{26}$$

How exactly we distribute the components inside the four-vectors is irrelevant because all that matters in the end are the six possible Lorentz invariants Q^2, k^2, k'^2, $k\cdot Q$, $k'\cdot Q$ and $k\cdot k'$, which determine the system completely and correspond to the six entries in the vectors, where the three angular variables are related to z, z', $y \in [-1,1]$. However, in this way we can define alternative vectors

$$d^\mu = \hat{Q}^\mu = \frac{Q^\mu}{\sqrt{Q^2}}, \qquad r^\mu = \widehat{k_\perp}^\mu = \frac{k_\perp^\mu}{\sqrt{k_\perp^2}}\,, \qquad k_\perp^\mu = k^\mu - \frac{k\cdot Q}{Q^2}\,Q^\mu\,. \tag{27}$$

If we perform these operations on the four-vectors in Eq. (26), we arrive at

$$d^\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad r^\mu = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad r'^\mu = \begin{bmatrix} 0 \\ \sqrt{1-y^2} \\ y \\ 0 \end{bmatrix}. \tag{28}$$

In this way, their Lorentz invariants become trivial,

$$d^2 = r^2 = r'^2 = 1\,, \qquad d\cdot r = d\cdot r' = 0\,, \qquad r\cdot r' = y\,, \tag{29}$$

and we have

$$\not{d} = \gamma^4\,, \qquad \not{r} = \gamma^3\,, \qquad \not{r}' = \sqrt{1-y^2}\,\gamma^2 + y\,\gamma^3\,. \tag{30}$$

Moreover, we can define transversely projected $\gamma-$matrices $\gamma_\perp^\mu$ by

$$\gamma_\perp^\mu = \gamma^\mu - d^\mu\,\not{d} - r^\mu\,\not{r}\,, \qquad (\gamma_\perp^\mu)' = \gamma^\mu - d^\mu\,\not{d} - r'^\mu\,\not{r}'\,, \tag{31}$$

which entails for example

$$\gamma_\perp^1 = \gamma^1\,, \qquad \gamma_\perp^2 = \gamma^2\,, \qquad \gamma_\perp^3 = \gamma_\perp^4 = 0\,. \tag{32}$$

In this way we can express the quark-photon vertex in the following basis:

$$\Gamma^\mu(k,Q) = \sum_{j=1}^{12} a_j(k^2,z,Q^2)\,i\tau_j^\mu(k,Q) \tag{33}$$

with

$$\begin{array}{llllll}
\tau_1^\mu & = \frac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,, & \tau_5^\mu & = r^\mu\,\mathbb{1}\,, & \tau_9^\mu & = d^\mu\,\mathbb{1}\,, \\
\tau_2^\mu & = \frac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,\not{d}\,, & \tau_6^\mu & = r^\mu\,\not{d}\,, & \tau_{10}^\mu & = d^\mu\,\not{d}\,, \\
\tau_3^\mu & = \frac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,\not{r}\,, & \tau_7^\mu & = r^\mu\,\not{r}\,, & \tau_{11}^\mu & = d^\mu\,\not{r}\,, \\
\tau_4^\mu & = \frac{1}{\sqrt{2}}\,\gamma_\perp^\mu\,\not{r}\,\not{d}\,, & \tau_8^\mu & = r^\mu\,\not{r}\,\not{d}\,, & \tau_{12}^\mu & = d^\mu\,\not{r}\,\not{d}\,,
\end{array} \tag{34}$$

which looks a lot simpler than the previous one. In particular, the τ_i^μ do not depend on any kinematic variable in the system since they are constructed only from unit vectors. The conjugate basis elements according to Eq. (15) become

$$\begin{array}{ll}
\bar{\tau}_j^\mu = \tau_j^\mu & \ldots\; j = 1,5,6,7,9,10,11 \\
\bar{\tau}_j^\mu = -\tau_j^\mu & \ldots\; j = 2,3,4,8,12
\end{array} \tag{35}$$

Then, using the relations

$$\not{d}^2 = \not{r}^2 = 1\,, \qquad \not{r}\,\not{d} = -\not{d}\,\not{r}\,, \qquad \gamma_\perp^\mu\,\not{d} = -\not{d}\,\gamma_\perp^\mu\,, \qquad \gamma_\perp^\mu\,\not{r} = -\not{r}\,\gamma_\perp^\mu\,, \qquad \gamma_\perp^\mu\,\gamma_\perp^\mu = 2 \tag{36}$$

you can verify that we have indeed arrived at an orthonormal basis with the orthonormality relation

$$\frac{1}{4}\,\mathrm{Tr}\left\{\bar{\tau}_i^\mu\,\tau_j^\mu\right\} = \delta_{ij}\,. \tag{37}$$

Note that the '1' and '2' appearing in (36) are Dirac matrices, i.e. they are understood to be multiplied by a 4×4 unit matrix in Dirac space, hence the $1/4$ in the orthogonality relation.

The two decompositions (6) and (34) are completely equivalent, except that (34) no longer satisfies the symmetries of the original basis, which means that the a_j are no longer even functions in $\omega = \sqrt{k^2}\sqrt{Q^2}z$. In any case, one can work out the linear relations between the two sets of dressing functions; abbreviating $s = \sqrt{1-z^2}$, then for the g_j these are

$$g_1 = a_{10} - \frac{z}{s}\,a_{11}\,, \qquad g_2 = \frac{1}{k^2\,zs}\,a_{11}\,, \qquad g_3 = -\frac{i}{kz}\,a_9\,, \qquad g_4 = \frac{i}{k^2\,Q\,zs}\,a_{12}\,. \tag{38}$$

One can show that these are consistent with the WTI: If we evaluate Eq. (10) using the decomposition (34), then only the longitudinal tensors $\tau^\mu_{9\dots 12}$ survive, which determines the dressing functions $a_{9\dots 12}$. Plugging them into the expressions above, the resulting g_i are identical to those in Eq. (12). The basis transformations for the f_j, on the other hand, are given by

$$
\begin{aligned}
f_1 &= \frac{1}{Q^2 s^2}\left(\frac{a_1}{\sqrt{2}} + zs\,(a_6+a_{11}) - z^2\,a_7 - s^2\,a_{10}\right), & f_5 &= -\frac{i}{k\,Q^2\,s}\left(a_5 - \frac{s}{z}\,a_9\right), \\
f_2 &= -\frac{i}{k^2\,Q^3\,s^2}\left(\frac{a_2}{\sqrt{2}} - a_8 + \frac{s}{z}\left(\frac{a_3}{\sqrt{2}} + a_{12}\right)\right), & f_6 &= -\frac{1}{k^2\,Q^2\,s^2}\left(\frac{a_1}{\sqrt{2}} - a_7 + \frac{s}{z}\,a_{11}\right), \\
f_3 &= \frac{i}{\sqrt{2}\,Q}\left(-a_2 + \frac{z}{s}\,a_3\right), & f_7 &= -\frac{1}{k^2\,Q^2\,s^2}\left(\frac{a_1}{\sqrt{2}} - a_7 + \frac{s}{z}\,a_6\right), \\
f_4 &= \frac{1}{\sqrt{2}\,k\,Q\,s}\,a_4\,, & f_8 &= \frac{i}{k^2\,Q\,s^2}\left(\frac{a_2}{\sqrt{2}} - a_8\right).
\end{aligned}
\tag{39}
$$

Before applying all this to the BSE (20), let us employ another convenient trick. Let us define

$$\Psi^\mu(k,Q) = S(k_+)\,\Gamma^\mu(k,Q)\,S(k_-) \tag{40}$$

as the quark-photon vertex with external quark propagators attached. Since this quantity has the same structural form as the vertex itself (4×4 Dirac matrix, one Lorentz index μ, two momenta k and Q), it must have the same basis decomposition (33) except with different dressing functions; let us call them b_j:

$$\Psi^\mu(k,Q) = \sum_{j=1}^{12} b_j(k^2,z,Q^2)\,i\tau^\mu_j(k,Q)\,. \tag{41}$$

In this way, we can split the BSE into a two-step process:

$$
\begin{aligned}
\Gamma^\mu(k,Q) &= Z_2\,i\gamma^\mu + \int\limits_{k'} g(l^2)\,T_l^{\alpha\beta}\,\gamma^\alpha\,\Psi^\mu(k',Q)\,\gamma^\beta\,, \\
\Psi^\mu(k,Q) &= S(k_+)\,\Gamma^\mu(k,Q)\,S(k_-)\,,
\end{aligned}
\tag{42}
$$

which avoids the need for evaluating the quark propagators inside the momentum loop and therefore saves some CPU time.

If we now insert the vertex decomposition (33) and employ the orthonormality relation (37), we arrive at purely scalar equations where all Lorentz and Dirac indices have disappeared,

$$
\begin{aligned}
a_i(k^2,z,Q^2) &= Z_2\,a_i^0 + \sum_{j=1}^{12}\int\limits_{k'} g(l^2)\,K_{ij}(k^2,k'^2,z,z',y,Q^2)\,b_j(k'^2,z',Q^2)\,, \\
b_i(k^2,z,Q^2) &= \sum_{j=1}^{12} G_{ij}(k^2,z,Q^2)\,a_j(k^2,z,Q^2)\,,
\end{aligned}
\tag{43}
$$

where $a_i^0 = \frac{1}{4}\,\mathrm{Tr}\,\{\bar\tau^\mu_i\,\gamma^\mu\}$ and the kernel matrix and propagator matrix are given by

$$
\begin{aligned}
K_{ij}(k^2,k'^2,z,z',y,Q^2) &= T_l^{\alpha\beta}\,\frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau^\mu_i(k,Q)\,\gamma^\alpha\,\tau^\mu_j(k',Q)\,\gamma^\beta\right\}, \\
G_{ij}(k^2,z,Q^2) &= \frac{1}{4}\,\mathrm{Tr}\left\{\bar\tau^\mu_i(k,Q)\,S(k_+)\,\tau^\mu_j(k,Q)\,S(k_-)\right\}.
\end{aligned}
\tag{44}
$$

Now you can see the *power* of using orthonormal bases: Had we used the original basis (6), the kernel matrix K_{ij} would be a complicated 12×12 matrix which depends on six variables. However, due to the simplicity of the τ_i^μ it simplifies dramatically:

- Because $d\cdot r = d\cdot r' = d\cdot\gamma_\perp = d\cdot\gamma'_\perp = 0$, the equations for the tensors $\tau^\mu_{1\dots 8}$ decouple completely from those for the $\tau^\mu_{9\dots 12}$. That is, both K_{ij} and G_{ij} fall apart into two non-interacting blocks,

$$K = \left(\begin{array}{c|c} 8\times 8 & 0 \\ \hline 0 & 4\times 4 \end{array}\right), \qquad G = \left(\begin{array}{c|c} 8\times 8 & 0 \\ \hline 0 & 4\times 4 \end{array}\right), \tag{45}$$

  so that the equations for $a_{1\dots 8}$ and those for $a_{9\dots 12}$ can be solved independently. The underlying reason is the WTI in Eq. (10), which says that the longitudinal part of the vertex defined by the $a_{9\dots 12}$ is completely determined by the quark propagator and decoupled from the dynamics.

- Because of $r\cdot\gamma_\perp = 0$, for the propagator matrix also the blocks for $a_{1\dots 4}$ and $a_{5\dots 8}$ decouple. In addition, because $r^2 = d^2 = 1$, the two lower blocks are identical:

$$G = \left(\begin{array}{c|c|c} G^{(1)}_{4\times 4} & 0 & 0 \\ \hline 0 & G^{(2)}_{4\times 4} & 0 \\ \hline 0 & 0 & G^{(2)}_{4\times 4} \end{array}\right). \tag{46}$$

  This does not happen for the kernel because

$$r\cdot r' \neq 0, \qquad r\cdot\gamma'_\perp \neq 0\,, \qquad r'\cdot\gamma_\perp \neq 0\,, \tag{47}$$

  so the equations for $a_{1\dots 4}$ and $a_{5\dots 8}$ do not decouple after all. However, they are also not overly complicated because the $\tau_i^\mu(k,Q)$ do not depend on any kinematic variable, and the only kinematic variable that enters in the $\tau_i^\mu(k',Q)$ is y. Thus, the trace on the r.h.s. of the kernel in Eq. (44) can only depend on y and nothing else.

- From charge conjugation definition (15) one can show that the matrix G must be symmetric.

- In addition, the squared photon momentum Q^2 is an external variable, i.e., the equations for the $a_i(k^2,z,Q^2)$ are solved for each Q^2 separately. The kernel K also does not depend on Q^2 (which is a consequence of employing a rainbow-ladder interaction, because the gluon momentum $l = k - k'$ does not depend on Q) and Q^2 only enters in the propagator matrix G.

With this the ingredients of the BSE (43) are completely specified. For the integral $\int_{k'}$ in Eq. (21), the innermost integration over ϕ is trivial. Moreover, the amplitudes a_j and b_j in Eq. (43) do not depend on the variable y, so one can integrate over y right away to obtain $K'_{ij} = \int dy\, g(l^2)\, K_{ij}$. The strategy to solve the BSE is then as follows:

- Compute the kernel K'_{ij} by integrating over y.
- Loop over Q^2, since we want to solve the BSE for each Q^2.
- Solve the transverse ($a_{1\dots 8}$) and longitudinal ($a_{9\dots 12}$) equations separately for each Q^2. In each case, start with some initial guess for the a_i, calculate the b_i by applying the propagator matrix, and determine again the a_i by applying the kernel and integrating over the momentum. Proceed until converged.
- Convert the a_i into the g_j and f_j from Eq. (5) by applying the formulas (38–39).

Explicit traces

While it is straightforward to compute the traces (44) numerically in the code, an alternative is to work them out explicitly in advance. In principle this is not necessary, but especially with the orthonormal basis the resulting expressions are quite manageable. The inhomogeneous term in front of the BSE reads

$$a_i^0 = \frac{1}{4}\,\mathrm{Tr}\,\{\bar{\tau}_i^\mu\,\gamma^\mu\} = \begin{cases} \sqrt{2} & i=1 \\ 1 & i=7,10 \\ 0 & \text{else.} \end{cases} \tag{48}$$

From Eq. (26) the squared gluon momentum is

$$l^2 = (k-k')^2 = k^2 + k'^2 - 2k\cdot k' = k^2 + k'^2 - 2kk'\,(zz' + y\sqrt{1-z^2}\sqrt{1-z'^2})\,. \tag{49}$$

To make the following expressions more compact, we define

$$\begin{aligned} u &= k\sqrt{1-z^2}\,, \\ u' &= k'\sqrt{1-z'^2}\,, \end{aligned} \qquad V = \frac{kz-k'z'}{l^2}\,, \qquad w = \frac{u^2}{l^2}\,, \qquad w' = \frac{u'^2}{l^2}\,, \qquad X = \frac{uu'}{l^2}\,. \tag{50}$$

The entries of the kernel matrix then become

$$\begin{aligned}
K_{11} &= -\tfrac{1+y^2}{2} - y\,(1-y^2)X\,, & K_{16} &= \sqrt{2}\,(1-y^2)\,u'V\,, \\
K_{22} &= -\tfrac{1+y^2}{2}\,(1-2l^2V^2) + y\,(1-y^2)X\,, & K_{61} &= -\sqrt{2}\,(1-y^2)\,uV\,, \\[1ex]
K_{33} &= y(1-2l^2V^2) - (1-y^2)X\,, & K_{17} &= -\tfrac{1-y^2}{\sqrt{2}}\,(1+2w'-2yX)\,, \\
K_{44} &= y + (1-y^2)X\,, & K_{71} &= -\tfrac{1-y^2}{\sqrt{2}}\,(1+2w-2yX)\,, \\[1ex]
K_{55} &= 3y\,, & K_{23} &= \big(2yu - (1+y^2)\,u'\big)\,V\,, \\
K_{66} &= -y\,(1+2l^2V^2)\,, & K_{32} &= -\,\big(2yu' - (1+y^2)\,u\big)\,V\,, \\[1ex]
K_{77} &= -y^2(3-2l^2V^2) + 2y\,(1-y^2)X\,, & K_{67} &= 2y\,(u'-yu)\,V\,, \\
K_{88} &= y^2 - 2y\,(1-y^2)X\,, & K_{76} &= -2y\,(u-yu')\,V\,,
\end{aligned} \tag{51}$$

together with

$$\begin{aligned}
K_{28} &= K_{71} + \sqrt{2}\,(1-y^2)\,, \\
K_{82} &= K_{17} + \sqrt{2}\,(1-y^2)\,, \\
K_{38} &= -K_{61}\,, \\
K_{83} &= -K_{16}\,,
\end{aligned} \qquad \begin{bmatrix} K_{99} \\ K_{10,10} \\ K_{10,11} \\ K_{11,10} \\ K_{11,11} \\ K_{12,12} \end{bmatrix} = \frac{1}{y} \begin{bmatrix} K_{55} \\ K_{66} \\ K_{67} \\ K_{76} \\ K_{77} \\ K_{88} \end{bmatrix} \tag{52}$$

and all other elements zero.

Concerning the propagator matrix, from the general decomposition of the quark propagator

$$S(k) = \frac{1}{A(k^2)}\,\frac{-i\not{k} + M(k^2)}{k^2 + M(k^2)^2} = \sigma_v(k^2)\,\big(-i\not{k} + M(k^2)\big) \tag{53}$$

and the appearance of $S(k_+)$ and $S(k_-)$ in Eq. (44) it is clear that there will be a common factor

$\sigma_v(k_+^2)\,\sigma_v(k_-^2)$ in front, where the rest can only depend on the mass function, i.e.

$$G_{ij}(k^2,z,Q^2) = \sigma_v(k_+^2)\,\sigma_v(k_-^2)\,\widetilde{G}_{ij}(k^2,z,Q^2)\,. \tag{54}$$

In analogy to Eq. (13), we define

$$\Sigma_M = \frac{M(k_+^2)+M(k_-^2)}{2}\,, \qquad \Delta_M = \frac{M(k_+^2)-M(k_-^2)}{k_+^2-k_-^2}\,, \qquad \overline{M}^2 = M(k_+^2)\,M(k_-^2)\,. \tag{55}$$

For the first 4×4 block, the explicit calculation gives

$$\begin{aligned}
&& \widetilde{G}_{12} &= iQ\,(\Sigma_M - 2k^2z^2\Delta_M)\,,\\
\widetilde{G}_{11} &= \overline{M}^2 + k^2 - \tfrac{Q^2}{4}\,, & \widetilde{G}_{13} &= -2ik^2\,Q\,z\sqrt{1-z^2}\,\Delta_M\,,\\
\widetilde{G}_{22} &= \overline{M}^2 - (1-2z^2)\,k^2 - \tfrac{Q^2}{4}\,, & \widetilde{G}_{14} &= -k\,Q\sqrt{1-z^2}\,,\\
\widetilde{G}_{33} &= \overline{M}^2 + (1-2z^2)\,k^2 + \tfrac{Q^2}{4}\,, & \widetilde{G}_{23} &= 2k^2z\sqrt{1-z^2}\,,\\
\widetilde{G}_{44} &= \overline{M}^2 - k^2 + \tfrac{Q^2}{4}\,, & \widetilde{G}_{24} &= 2ik\sqrt{1-z^2}\,\Sigma_M,\,,\\
&& \widetilde{G}_{34} &= ikz\,(Q^2\Delta_M - 2\Sigma_M)\,,
\end{aligned} \tag{56}$$

where the remaining entries are determined from the fact that G is symmetric, i.e., $G_{21}=G_{12}$ etc. The remaining two blocks can be reconstructed from the first block:

$$\begin{bmatrix} G_{55}\\ G_{56}\\ G_{66}\\ G_{77}\\ G_{78}\\ G_{88}\end{bmatrix} = \begin{bmatrix} G_{99}\\ G_{9,10}\\ G_{10,10}\\ G_{11,11}\\ G_{11,12}\\ G_{12,12}\end{bmatrix} = \begin{bmatrix} G_{44}\\ G_{34}\\ G_{33}\\ G_{22}\\ G_{12}\\ G_{11}\end{bmatrix}, \qquad \begin{bmatrix} G_{57}\\ G_{58}\\ G_{67}\\ G_{68}\end{bmatrix} = \begin{bmatrix} G_{9,11}\\ G_{9,12}\\ G_{10,11}\\ G_{10,12}\end{bmatrix} = -\begin{bmatrix} G_{24}\\ G_{14}\\ G_{23}\\ G_{13}\end{bmatrix}. \tag{57}$$

Improving the accuracy

Although the orthonormal basis is simple to implement, it has a disadvantage: While the g_i and f_i only show a weak dependence in the angular variable z (remember $\omega = \sqrt{k^2}\sqrt{Q^2}\,z$), the a_i and b_i pick up a strong z dependence by Eqs. (38–39) through factors $1/\sqrt{1-z^2}$ or $1/(1-z^2)$, which diverge at $z\to\pm 1$, and this makes a polynomial expansion in z fairly difficult. For this reason, the numerical results would be actually more accurate if we solved the BSE directly for the g_i and f_i, however at the price that we would need a lot more CPU time to do so.

There is, however, a simple way to implement the best of both worlds. Suppose we take the basis (6) and divide out all factors of k^2 and Q^2, such that the remaining basis elements $t_i'^{\mu}$ only depend on z (cf. Eq. (26)). This is equivalent to redefining the g_i and f_i in terms of functions a_i':

$$\begin{aligned}
g_1 &= a_9'\,, & g_2 &= \frac{a_{10}'}{k^2}\,, & g_3 &= \frac{a_{11}'}{k}\,, & g_4 &= \frac{a_{12}'}{Q\,k^2}\,,\\
f_1 &= \frac{a_1'}{Q^2}\,, & f_2 &= \frac{a_2'}{Q^3\,k^2}\,, & f_3 &= \frac{a_3'}{Q}\,, & f_4 &= \frac{a_4'}{Q\,k}\,,\\
f_5 &= \frac{a_5'}{Q^2\,k}\,, & f_6 &= \frac{a_6'}{Q^2\,k^2}\,, & f_7 &= \frac{a_7'}{Q^2\,k^2}\,, & f_8 &= \frac{a_8'}{Q\,k^2}\,.
\end{aligned} \tag{58}$$

Then, the dressing functions a_i in Eq. (33) are related to the a_i' by a 12×12 transformation matrix $U(z)$:

$$a_i = \sum_{j=1}^{12} U_{ij}\, a_j'\,, \qquad U_{ij} = \frac{1}{4}\,\mathrm{Tr}\left\{\bar{\tau}_i^\mu\, t'^\mu_j\right\}\,. \tag{59}$$

Applied to the BSE (43), changing the basis to go from $a_i \to a_i'$ merely amounts to replacing

$$a_i^0 \;\to U(z)^{-1}_{ij}\, a_j^0\,, \qquad K \;\to\; U(z)^{-1}\, K\, U(z')\,, \qquad G \;\to\; U(z)^{-1}\, G\, U(z)\,, \tag{60}$$

where K is the kernel in Eqs. (51–52) and G the propagator matrix in Eqs. (56–57). $U(z)$ only depends on the variable z and is given by

$$\begin{aligned}
U_{11} = U_{19} &= \sqrt{2}\,, \\
U_{17} &= \sqrt{2}\,z^2\,, \\
U_{22} = U_{28} = U_{2,12} &= \sqrt{2}\,iz^2\,, \\
U_{23} &= \sqrt{2}\,i\,, \\
U_{32} = U_{38} = U_{3,12} &= \sqrt{2}\,isz\,, \\
U_{44} &= \sqrt{2}s\,, \\
U_{55} = U_{5,11} &= is\,, \\
U_{66} = -U_{67} = U_{6,10} = U_{11,10} &= sz\,,
\end{aligned}
\qquad
\begin{aligned}
U_{71} = U_{79} = U_{10,9} &= 1\,, \\
U_{76} = U_{7,10} &= s^2\,, \\
U_{77} = U_{10,10} &= z^2\,, \\
U_{82} = U_{8,12} &= iz^2\,, \\
U_{83} = U_{88} &= i\,, \\
U_{9,11} &= iz\,, \\
U_{12,12} &= -isz\,,
\end{aligned} \tag{61}$$

with $s=\sqrt{1-z^2}$. Note that the new K has $U(z')$ on the right, i.e. one must replace $z\to z'$ and $s\to s'=\sqrt{1-z'^2}$. (G does not depend on z, here it is $U(z)$ on the right.)

The inverse of the matrix can be calculated numerically, but since it is also very simple it is faster to enter by hand. Writing $W=U^{-1}$, it reads

$$\begin{aligned}
W_{11} = -W_{61} = -W_{71} &= \tfrac{1}{\sqrt{2}\,s^2}\,, \\
W_{16} = W_{1,11} = -W_{9,11} &= \tfrac{z}{s}\,, \\
W_{17} &= -\tfrac{z^2}{s^2}\,, \\
W_{9,10} = -W_{1,10} &= 1\,, \\
W_{82} = -W_{22} &= \tfrac{i}{\sqrt{2}\,s^2}\,, \\
W_{23} &= -\tfrac{i}{\sqrt{2}\,sz}\,, \\
W_{28} = -W_{88} &= \tfrac{i}{s^2}\,,
\end{aligned}
\qquad
\begin{aligned}
W_{12,12} = -W_{2,12} &= \tfrac{i}{sz}\,, \\
W_{10,11} = -W_{6,11} = -W_{76} &= \tfrac{1}{sz}\,, \\
W_{32} &= -\tfrac{i}{\sqrt{2}}\,, \\
W_{33} &= \tfrac{iz}{\sqrt{2}\,s}\,, \\
W_{44} &= \tfrac{1}{\sqrt{2}\,s}\,, \\
W_{55} &= -\tfrac{i}{s}\,, \\
W_{59} = -W_{11,9} &= \tfrac{i}{z}\,, \\
W_{67} = W_{77} &= \tfrac{1}{s^2}\,.
\end{aligned} \tag{62}$$

Note that U is complex, so the new K and G will be complex as well. A simple check after implementing U and U^{-1} to make sure that everything is correct is to test if $UU^{-1}=\mathbb{1}$.

After this the BSE solution works in the same way as before, i.e., nothing else needs to be changed with one exception: the new K and G no longer decouple into 8×8 and 4×4 blocks, so one needs to solve the BSE as a 12×12 system. In the end, once the system has converged in terms of the a_i', we must reinstate the Q^2 and k^2 factors to obtain the actual dressing functions g_i and f_i according to Eq. (58) (whereas the basis transformations (38–39) are no longer needed).

Tasks:

- Solve the quark-photon vertex BSE from Eqs. (43–44). For the quark propagator dressing functions, use the parametrization

$$\begin{aligned} A(x) &= 0.95 + \frac{0.3}{\ln(x+2)} + \frac{0.1}{1+x} + 0.29\, e^{-0.1x} - 0.18\, e^{-3x}\,, \\ M(x) &= \frac{0.06}{1+x} + 0.44\, e^{-0.66x} + \frac{0.009}{\ln(x+2)^{\gamma_m}} \end{aligned} \tag{63}$$

  with $x = p^2/(0.7\,\text{GeV})^2$ and $\gamma_m = 12/25$, which provides a fair representation of the rainbow-ladder results for light u/d quarks and $p^2 > 0$. For the quark renormalization constant, use $Z_2 = 0.97$.

- Check how well the WTI (12) is satisfied. To this end, is advantageous to introduce a Pauli-Villars cutoff in the kernel of the quark-photon vertex BSE, i.e., multiply $\alpha(k^2)$ in Eq. (19) with a factor

$$\frac{1}{1 + k^2/\Lambda_{\text{PV}}^2}\,, \qquad \Lambda_{\text{PV}} = 200\,\text{GeV}\,. \tag{64}$$

- Plot the 12 dressing functions $g_j(k^2, z=0, Q^2)$ and $f_j(k^2, z=0, Q^2)$ (or, if you use a Chebyshev expansion in z, their zeroth Chebyshev moments).

- **(Extra)** Implement the quark propagator from Project 1 that is obtained from its quark DSE. From Eq. (44), you need the quark propagator for arguments

$$k_\pm^2 = k^2 + \frac{Q^2}{4} \pm \sqrt{k^2}\sqrt{Q^2}\, z \in \mathbb{R}_+\,. \tag{65}$$

  You could solve the quark DSE directly at those arguments (slow), or you could implement the calculated propagator functions using splines (faster). Check again how well the WTI is satisfied.

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