

Geometric Programming

1 12th Maths - Chapter 6

This is Problem-26 from Exercise 6.5

1. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1} \left(\frac{1}{3} \right)$.

Solution:

2. Let r, h, l be the radius, height and slant height of the right circular cone respectively. Let S be the given surface area and V be the volume of the cone. We have

$$l^2 = r^2 + h^2 \quad (1)$$

$$S = \pi r l + \pi r^2 \quad (2)$$

$$\implies l = \frac{S - \pi r^2}{\pi r} \quad (3)$$

$$V = \frac{1}{3} \pi r^2 h \quad (4)$$

The given problem can be formulated as

$$\max_{r,h} V = \frac{1}{3} \pi r^2 h \quad (5)$$

$$\text{with constraints } S > 0 \implies r, h > 0 \quad (6)$$

$$(4) \implies V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2} \quad (7)$$

$$V^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \quad (8)$$

$$= \frac{1}{9}\pi^2 r^4 \left(\left(\frac{S - \pi r^2}{\pi r} \right)^2 - r^2 \right) \quad (9)$$

$$= \frac{1}{9}\pi^2 r^4 \left(\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2} \right) \quad (10)$$

$$= \frac{1}{9}r^2 \left((S - \pi r^2)^2 - \pi^2 r^4 \right) \quad (11)$$

$$= \frac{1}{9}r^2 (S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4) \quad (12)$$

$$= \frac{1}{9} (S^2 r^2 - 2\pi S r^4) \quad (13)$$

Differentiating wrt r ,

$$2V \frac{dV}{dr} = \frac{S^2}{9} 2r - \frac{2\pi S}{9} 4r^3 \quad (14)$$

$$= \frac{2rS}{9} (S - 4\pi r^2) \quad (15)$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\implies \frac{2rS}{9} (S - 4\pi r^2) = 0 \quad (16)$$

$$\implies r = 0 \text{ or } S - 4\pi r^2 = 0 \quad (17)$$

Since r can't be equal to 0,

$$\implies S - 4\pi r^2 = 0 \quad (18)$$

$$\implies S = 4\pi r^2 \quad (19)$$

$$\implies r^2 = \frac{S}{4\pi} \quad (20)$$

$$\implies r^2 = \frac{\pi r l + \pi r^2}{4\pi} \quad (21)$$

$$\implies 4\pi r^2 = \pi r l + \pi r^2 \quad (22)$$

$$\implies 3\pi r^2 = \pi r l \quad (23)$$

$$\implies l = 3r \quad (24)$$

Let θ be the semi-vertical angle in Figure 1. Then,

$$\sin \theta = \frac{OA}{CA} = \frac{r}{l} \quad (25)$$

$$\sin \theta = \frac{r}{3r} \quad (26)$$

$$\Rightarrow \theta = \sin^{-1} \frac{1}{3} \quad (27)$$

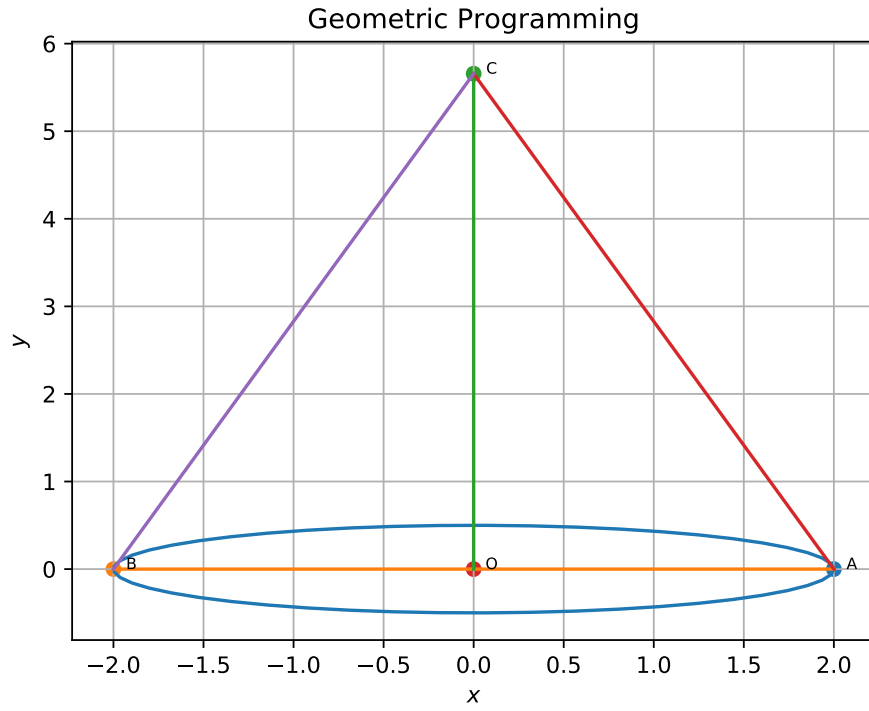


Figure 1