Convex Optimization

1 11^{th} Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   Solution: The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^\top \mathbf{x} + 8 = 0 \tag{1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3}$$

   Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

   Here, $\mathbf{A}$ is a point on the given line.

   Let $\mathbf{O}$ be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of the perpendicular. This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \tag{5}$$

$$\implies \min_{\lambda} \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{O}\|^2 \tag{6}$$

$$\implies \min_{\lambda} \|\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \tag{7}$$

$$\implies f(\lambda) = \left[\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right]^\top \left[\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right] \tag{8}$$

$$= \left[\lambda\mathbf{m}^\top + (\mathbf{A} - \mathbf{O})^\top\right] \left[\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right] \tag{9}$$

$$= \lambda^2 \|\mathbf{m}\|^2 + \lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \lambda\mathbf{m}^\top (\mathbf{A} - \mathbf{O}) + \|\mathbf{A} - \mathbf{O}\|^2 \tag{10}$$

$$= \lambda^2 \|\mathbf{m}\|^2 + 2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + + \|\mathbf{A} - \mathbf{O}\|^2 \tag{11}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (11) is a convex function.

$$f'(\lambda) = 2\lambda \|\mathbf{m}\|^2 + 2 (\mathbf{A} - \mathbf{O})^\top \mathbf{m} \tag{12}$$

$$f''(\lambda) = 2 \|\mathbf{m}\|^2 \tag{13}$$

$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \tag{14}$$

$$f'(\lambda_{min}) = 2\lambda_{min} \|\mathbf{m}\|^2 + 2 (\mathbf{A} - \mathbf{O})^\top \mathbf{m} = 0 \tag{15}$$

$$\lambda_{min} = -\frac{(\mathbf{A} - \mathbf{O})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \tag{16}$$

We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{17}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{18}$$

Substituting the values of $\mathbf{A}$, $\mathbf{O}$ and $\mathbf{m}$ in equation (16)

$$\lambda_{min} = -\frac{\left(\begin{pmatrix}-8\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right)^\top \begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix}}{\left\|\begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix}\right\|^2} \tag{19}$$

$$= \frac{\begin{pmatrix}8 & 0\end{pmatrix}\begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix}}{\frac{4}{3}} \tag{20}$$

$$= 6 \tag{21}$$

Substituring this value in equation (4)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix}-8\\0\end{pmatrix} + 6\begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix} \tag{22}$$

$$= \begin{pmatrix}-8\\0\end{pmatrix} + \begin{pmatrix}6\\\frac{6}{\sqrt{3}}\end{pmatrix} \tag{23}$$

$$= \begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix} \tag{24}$$

$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \tag{25}$$

$$= \left\|\begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right\| \tag{26}$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{27}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) \tag{28}$$

$$= \tan^{-1}\left(-\sqrt{3}\right) \tag{29}$$

$$= 120° \tag{30}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix}\cos 120°\\\sin 120°\end{pmatrix}^\top \mathbf{x} = 4 \tag{31}$$

The relevant figure is as shown in 1 and 2

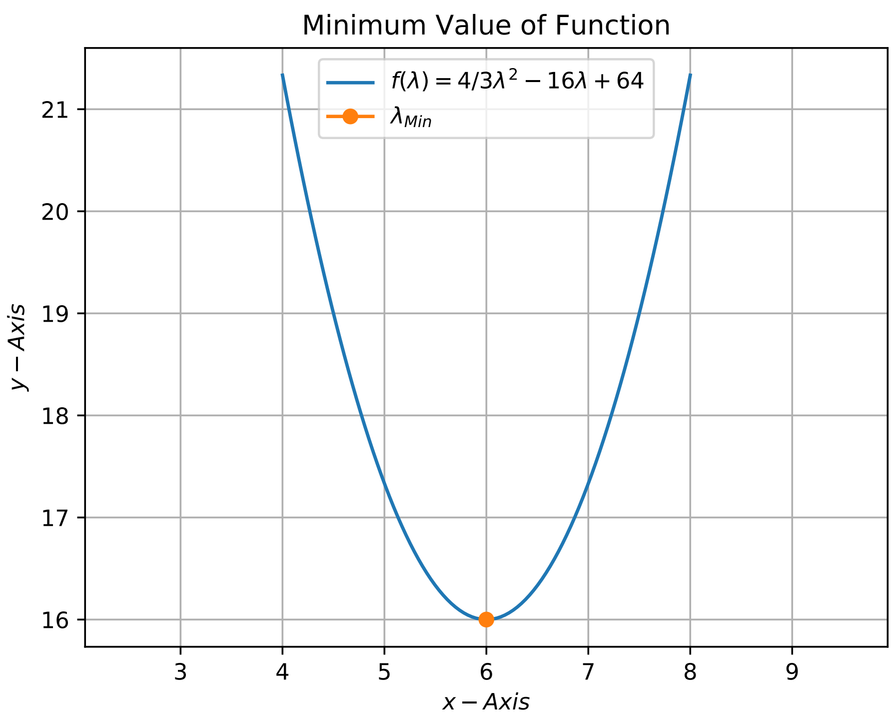
Minimum Value of Function
$f(\lambda) = 4/3\lambda^2 - 16\lambda + 64$
λ_{Min}
21
20
19
18
17
16
3
4
5
6
7
8
9
$y - Axis$
$x - Axis$

Figure 1

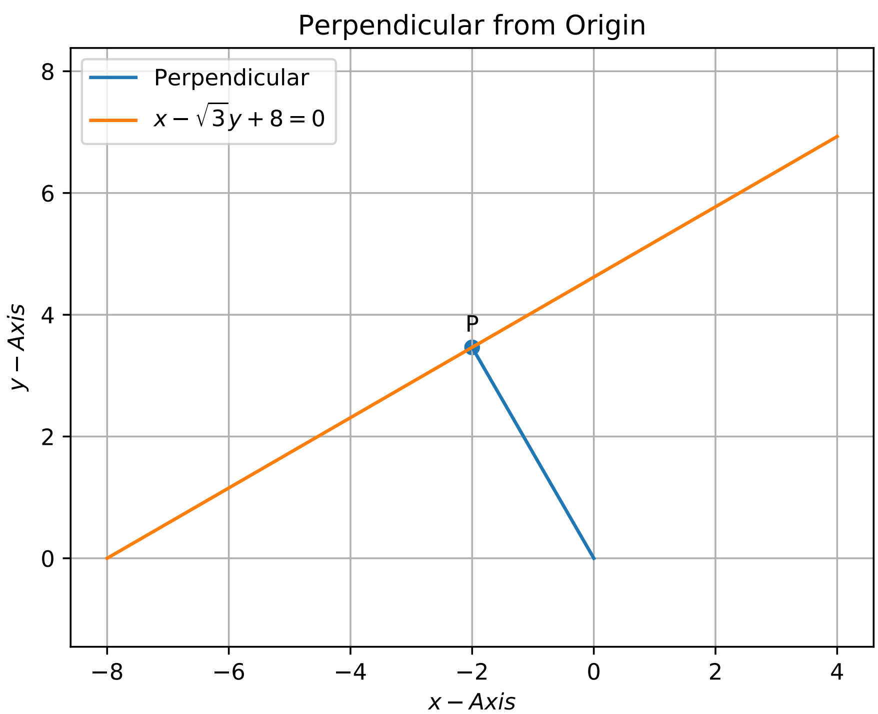
Perpendicular from Origin
Perpendicular
$x-\sqrt{3}y+8=0$
P
8
6
4
2
0
−8
−6
−4
−2
0
2
4
$x-Axis$
$y-Axis$

Figure 2