Normal to a Parabola

1 12^{th} Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve $x^2 = 4y$ and passing through the point $(1, 2)$.

   Solution: The given equation of the curve can be written as

   $$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

   where

   $$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

   $$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

   $$f = 0 \tag{4}$$

   We are given that

   $$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5}$$

   A point $\mathbf{h}$ lies on a normal to the conic in (1) , if

   $$\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\right)^2\left(\mathbf{n}^\top\mathbf{V}\mathbf{n}\right) - 2\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)\left(\mathbf{m}^\top(\mathbf{V}\mathbf{h}+\mathbf{u})\mathbf{n}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)\right) + \mathrm{g}\left(\mathbf{h}\right)\left(\mathbf{m}^\top\mathbf{V}\mathbf{n}\right)^2 = 0 \tag{6}$$

   where $\mathbf{m}$ is directional vector of the tangent(or normal vector of the normal) and $\mathbf{n}$ is the normal vector of the tangent (or directional vector of the normal). Assume

$$\mathbf{m} = \begin{pmatrix} 1 \\ m \end{pmatrix} \tag{7}$$

$$\mathbf{n} = \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} \tag{8}$$

Then

$$\mathbf{Vh} + \mathbf{u} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{9}$$

$$\mathbf{m}^\top \mathbf{Vn} = \begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = 1 \tag{10}$$

$$\mathbf{n}^\top \mathbf{Vn} = \begin{pmatrix} 1 & -\frac{1}{m} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{m} \end{pmatrix} = 1 \tag{11}$$

$$\mathrm{g}\left(\mathbf{h}\right) = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 8 = -7 \tag{13}$$

$$\begin{aligned} (6) \implies & \left(\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)^2 (1) \\ & - 2\left(1\right) \left(\begin{pmatrix} 1 & m \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{m} \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right) + \left(-7\right)\left(1\right)^2 = 0 \end{aligned} \tag{14}$$

$$\implies \left(1 - 2m\right)^2 - 2\left(1 - 2m\right)\left(1 + \frac{2}{m}\right) - 7 = 0 \tag{15}$$

$$\implies 1 - 4m + 4m^2 - 2\left(1 - 4 + \frac{2}{m} - 2m\right) - 7 = 0 \tag{16}$$

$$\implies 4m^2 - \frac{4}{m} = 0 \tag{17}$$

$$\implies 4m^3 = 4 \tag{18}$$

$$m = 1 \tag{19}$$

The equation of the normal is given by

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{h}) = 0 \tag{20}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right) = 0 \tag{21}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} (\mathbf{x}) = 3 \tag{22}$$

The relevant figure is shown in 1

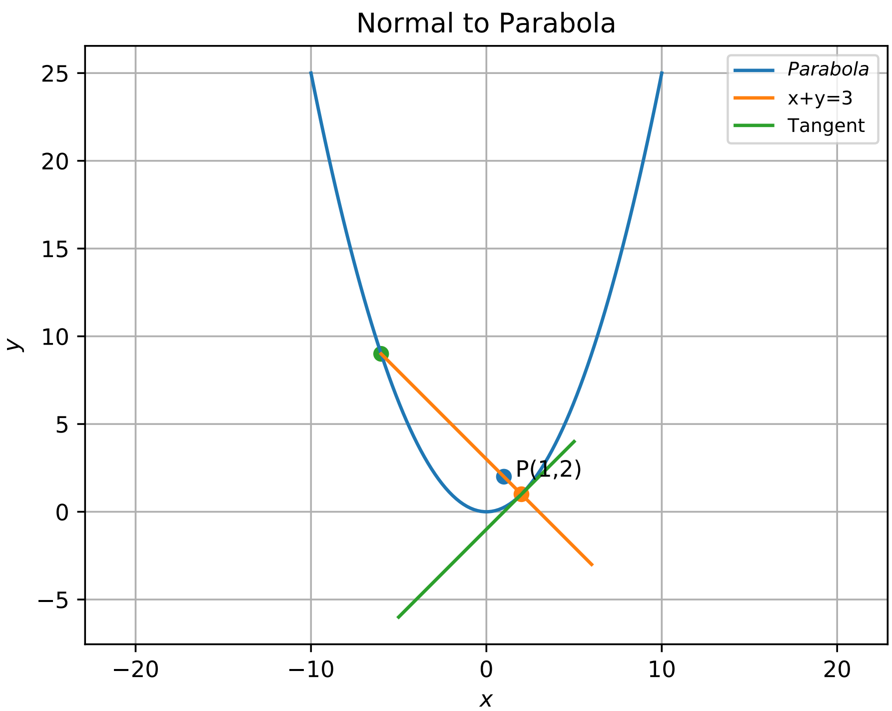

Figure 1