Optimization using SVD

1 JEE Maths - 65 C-1

This is Problem-30

1. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

   Solution: The given equation can be written as

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \tag{1}$$
$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \tag{2}$$
$$\implies \mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \tag{3}$$
$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \tag{4}$$

   where

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

   Assume

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \tag{6}$$

$$\mathbf{B} - \mathbf{A} = (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \tag{7}$$
$$= \mathbf{M}\boldsymbol{\lambda} + \mathbf{k} \tag{8}$$
$$\text{where } \boldsymbol{\lambda} \triangleq \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{k} = (\mathbf{x}_2 - \mathbf{x}_1) = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \tag{9}$$

We can formulate an unconstrained optimization problem as below:

$$\min_{\boldsymbol{\lambda}} \quad \|\mathbf{B} - \mathbf{A}\|^2 \tag{10}$$

Substituting (8) in (10)

$$(10) \implies \min_{\boldsymbol{\lambda}} \quad \|\mathbf{M}\boldsymbol{\lambda} + \mathbf{k}\|^2 \tag{11}$$

Using Singular Value Decomposition, $\mathbf{M}$ cn be written as

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^\top \tag{12}$$

Computation of SVD as follow

$$\mathbf{M}\mathbf{M}^\top = \begin{pmatrix} 29 & 11 & 2 \\ 11 & 10 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{13}$$

To compute eigen values for the matrix, we find

$$\left|\mathbf{M}\mathbf{M}^\top - \alpha\mathbf{I}\right| = \begin{vmatrix} 29-\alpha & 11 & 2 \\ 11 & 10-\alpha & 3 \\ 2 & 3 & 1-\alpha \end{vmatrix} = 0 \tag{14}$$

$$\implies (29-\alpha)\left(\alpha^2 - 11\alpha + 1\right) - 11\left(5 - 11\alpha\right) + 2\left(13 + 2\alpha\right) = 0 \tag{15}$$

$$\implies \alpha^3 - 40\alpha^2 + 195\alpha = 0 \tag{16}$$

$$\implies \alpha\left(\alpha^2 - 40\alpha + 195\right) = 0 \tag{17}$$

$$\implies \alpha = 20 \pm \sqrt{205}, 0 \tag{18}$$

$$
\mathbf{\Sigma} = \begin{bmatrix} \sqrt{20-\sqrt{205}} & 0 \\ 0 & \sqrt{\sqrt{205}+20} \end{bmatrix} \tag{19}
$$

$$
\mathbf{U}_1 = \begin{bmatrix}
\frac{-\frac{2\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\cdots}{\sqrt{\left(\frac{6}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\frac{\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}\right)^{2}+\left(-\frac{2\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\cdots\right.}} \\
\frac{-\frac{3\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}-\cdots}{\sqrt{\left(\frac{6}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\frac{\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}\right)^{2}+\left(-\frac{2\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\cdots\right.}} \\
\frac{-\frac{\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}-\cdots}{\sqrt{\left(\frac{6}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\frac{\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}\right)^{2}+\left(-\frac{2\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}}\sqrt{20-\sqrt{205}}}+\cdots\right.}}
\end{bmatrix} \tag{20}
$$

$$
\mathbf{U}_2 = \begin{bmatrix}
\frac{\frac{2\sqrt{205}}{13\sqrt{\sqrt{205}+20}\sqrt{\left(-\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}+1}}+\cdots}{\sqrt{\left(-\frac{6}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}+\frac{\sqrt{205}}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}\right)^{2}+\left(-\frac{5}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}+\cdots\right.}} \\
\frac{-\frac{5}{13\sqrt{\sqrt{205}+20}\sqrt{\left(-\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}+1}}+\cdots}{\sqrt{\left(-\frac{6}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}+\frac{\sqrt{205}}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}\right)^{2}+\left(-\frac{5}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}+\cdots\right.}} \\
\frac{-\frac{6}{13\sqrt{\sqrt{205}+20}\sqrt{\left(-\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^{2}+1}}+\cdots}{\sqrt{\left(-\frac{6}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}+\frac{\sqrt{205}}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}\right)^{2}+\left(-\frac{5}{13\sqrt{\sqrt{205}+20}\sqrt{\left(\frac{6}{13}-\frac{\sqrt{205}}{13}\right)^{2}+1}}+\cdots\right.}}
\end{bmatrix} \tag{21}
$$

$$\mathbf{U}_3 = \left[\begin{array}{l} -\dfrac{\cdots}{-169\sqrt{205}\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2\left(-\frac{3\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}\sqrt{20-\sqrt{205}}}-\frac{5}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}\sqrt{20-\sqrt{205}}}\right)^2-169\sqrt{205}\left(-\frac{\cdots}{13\sqrt{1+\left(\frac{6}{13}\cdots\right.}}\right.\cdots} \\ \\ -\dfrac{\cdots}{-169\sqrt{205}\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2\left(-\frac{3\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}\sqrt{20-\sqrt{205}}}-\frac{5}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}\sqrt{20-\sqrt{205}}}\right)^2-169\sqrt{205}\left(-\frac{\cdots}{13\sqrt{1+\left(\frac{6}{13}+\cdots\right.}}\right.\cdots} \\ \\ -\dfrac{\cdots}{-169\sqrt{205}\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2\left(-\frac{3\sqrt{205}}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}\sqrt{20-\sqrt{205}}}-\frac{5}{13\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}\sqrt{20-\sqrt{205}}}\right)^2-169\sqrt{205}\left(-\frac{\cdots}{13\sqrt{1+\left(\frac{6}{13}\cdots\right.}}\right.\cdots} \end{array}\right. \tag{22}$$

$$\mathbf{V} = \begin{bmatrix} \frac{-\frac{\sqrt{205}}{13}-\frac{6}{13}}{\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}} & \frac{-\frac{6}{13}+\frac{\sqrt{205}}{13}}{\sqrt{\left(-\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2+1}} \\ \frac{1}{\sqrt{1+\left(\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2}} & \frac{1}{\sqrt{\left(-\frac{6}{13}+\frac{\sqrt{205}}{13}\right)^2+1}} \end{bmatrix} \tag{23}$$

Then, the solution for (11) is given by

$$\boldsymbol{\lambda}_{min} = \Sigma_{i=1}^{r} \frac{-\mathbf{u}_i^\top \mathbf{k}}{\sigma_i} \mathbf{v}_i \tag{24}$$

where r is rank of matrix $\mathbf{M}$.

Computing using python code,

$$\boldsymbol{\lambda}_{min} = \begin{pmatrix} -1.4 \\ 0.969 \end{pmatrix} \tag{25}$$

$$\mathbf{A} = \begin{pmatrix} 3.8 \\ 3.2 \\ 1.4 \end{pmatrix} \tag{26}$$

$$\mathbf{B} = \begin{pmatrix} 3.85 \\ 2.97 \\ 2 \end{pmatrix} \tag{27}$$

$$\|\mathbf{B} - \mathbf{A}\| = 0.6445 \text{ units} \tag{28}$$

The relevant figure is shown in 1.

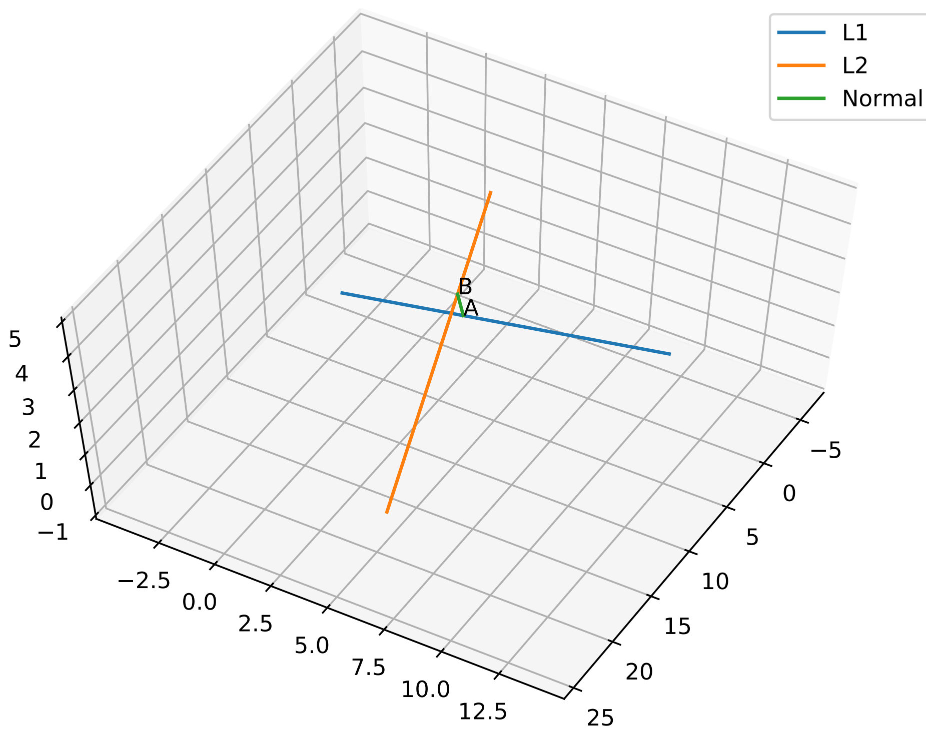

Figure 1