Geometric Programming

1 12^{th} Maths - Chapter 6

This is Problem-26 from Exercise 6.5

1. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

   Solution:

   (a) Theoritical proof: Let r, h, l be the radius, height and slant height of the right circular cone respectively. Let S be the given surface area and V be the volume of the cone. We have

$$l^2 = r^2 + h^2 \tag{1}$$

$$S = \pi r l + \pi r^2 \tag{2}$$

$$\implies l = \frac{S - \pi r^2}{\pi r} \tag{3}$$

$$V = \frac{1}{3}\pi r^2 h \tag{4}$$

   The given problem can be formulated as

$$\max_{r,h} V = \frac{1}{3}\pi r^2 h \tag{5}$$

$$\text{with constraints } S <= 75.42857 \tag{6}$$

$$(4) \implies V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2} \tag{7}$$

$$V^2 = \frac{1}{9}\pi^2 r^4\left(l^2 - r^2\right) \tag{8}$$

$$= \frac{1}{9}\pi^2 r^4\left(\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right) \tag{9}$$

$$= \frac{1}{9}\pi^2 r^4\left(\frac{\left(S - \pi r^2\right)^2 - \pi^2 r^4}{\pi^2 r^2}\right) \tag{10}$$

$$= \frac{1}{9}r^2\left(\left(S - \pi r^2\right)^2 - \pi^2 r^4\right) \tag{11}$$

$$= \frac{1}{9}r^2\left(S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4\right) \tag{12}$$

$$= \frac{1}{9}\left(S^2 r^2 - 2\pi S r^4\right) \tag{13}$$

Differentiating wrt r,

$$2V\frac{dV}{dr} = \frac{S^2}{9}2r - \frac{2\pi S}{9}4r^3 \tag{14}$$

$$= \frac{2rS}{9}\left(S - 4\pi r^2\right) \tag{15}$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\implies \frac{2rS}{9}\left(S - 4\pi r^2\right) = 0 \tag{16}$$

$$\implies r = 0 \text{ or } S - 4\pi r^2 = 0 \tag{17}$$

Since r can't be equal to 0,

$$\implies S - 4\pi r^2 = 0 \tag{18}$$

$$\implies S = 4\pi r^2 \tag{19}$$

$$\implies r^2 = \frac{S}{4\pi} \tag{20}$$

$$\implies r^2 = \frac{\pi r l + \pi r^2}{4\pi} \tag{21}$$

$$\implies 4\pi r^2 = \pi r l + \pi r^2 \tag{22}$$

$$\implies 3\pi r^2 = \pi r l \tag{23}$$

$$\implies l = 3r \tag{24}$$

Let θ be the semi-vertical angle in Figure 1. Then,

$$\sin\theta = \frac{OA}{CA} = \frac{r}{l} \tag{25}$$

$$\sin\theta = \frac{r}{3r} \tag{26}$$

$$\implies \theta = sin^{-1}\frac{1}{3} \tag{27}$$

(b) Using Disciplined Geometric Programming (DGP) of cvxpy: The objective function and constraints are defined as follow:

$$\max_{r,h} V = \frac{1}{3}\pi r^2 h \tag{28}$$

$$\text{such that } S <= 75.42857 \tag{29}$$

Solving this problem, yields following results

$$r = 2.45 \text{ units} \tag{30}$$

$$l = 7.35 \text{ units} \tag{31}$$

$$h = 6.93 \text{ units} \tag{32}$$

$$\text{Optimal } V = 43.557 \text{ cu.units} \tag{33}$$

It can be seen from solution that $l = 3r$ and semi-vertical angle is given as $\sin^{-1}\left(\frac{1}{3}\right)$. This is similar to what we proved theoritically.

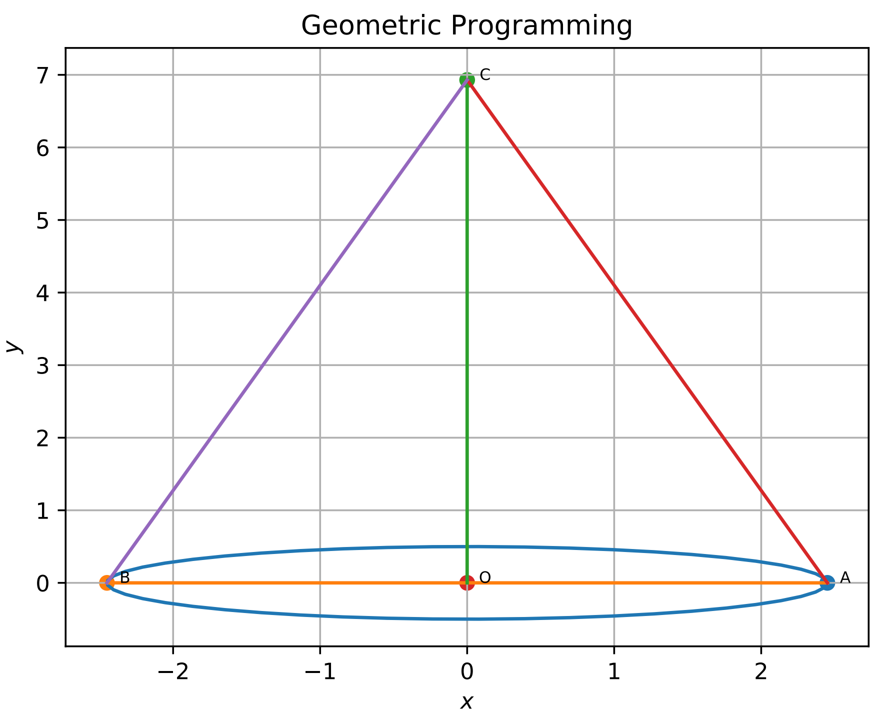

Figure 1