Gradient Descent

1 11^{th} Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   Solution: Equation for a line can be represented in parametric form as

   $$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1}$$

   We choose

   $$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{2}$$

   $$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

   $$\text{yielding } f(\lambda) = \frac{4}{3}\lambda^2 - 16\lambda + 64 \tag{4}$$

   $$f'(\lambda) = \frac{8}{3}\lambda - 16 \tag{5}$$

   Computing λ_{min} using Gradient Descent method:

   $$\lambda_{n+1} = \lambda_n - \alpha \nabla f(\lambda_n) \tag{6}$$

   Choosing

   (a) $\alpha = 0.001$

   (b) precision $= 0.0000001$

(c) n = 10000000

(d) λ_0 = -5

$$\lambda_{min} = 6 \tag{7}$$

Substituting the values of $\mathbf{A}$, $\mathbf{m}$ and λ_{min} in equation (1)

$$\begin{aligned}
\mathbf{x}_{min} = \mathbf{P} &= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} && (8) \\
&= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} && (9) \\
&= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} && (10) \\
OP &= \left\|\mathbf{P} - \mathbf{O}\right\|^2 && (11) \\
&= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| && (12) \\
&= \sqrt{2^2 + 12} = \sqrt{16} = 4 && (13)
\end{aligned}$$

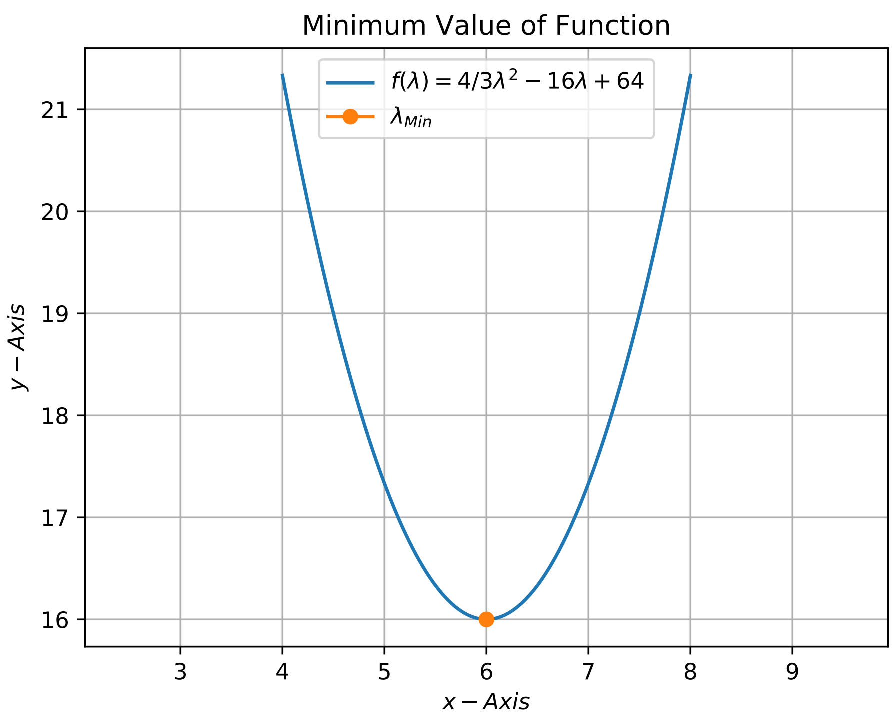
The angle θ made by this perpendicular with x-axis is given by

$$\begin{aligned}
\theta &= \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) && (14) \\
&= \tan^{-1}\left(-\sqrt{3}\right) && (15) \\
&= 120^\circ && (16)
\end{aligned}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{17}$$

The relevant figure is as shown in 1 and 2



Minimum Value of Function
$f(\lambda) = 4/3\lambda^2 - 16\lambda + 64$
λ_{Min}
21
20
19
18
17
16
$y - Axis$
3
4
5
6
7
8
9
$x - Axis$

Figure 1

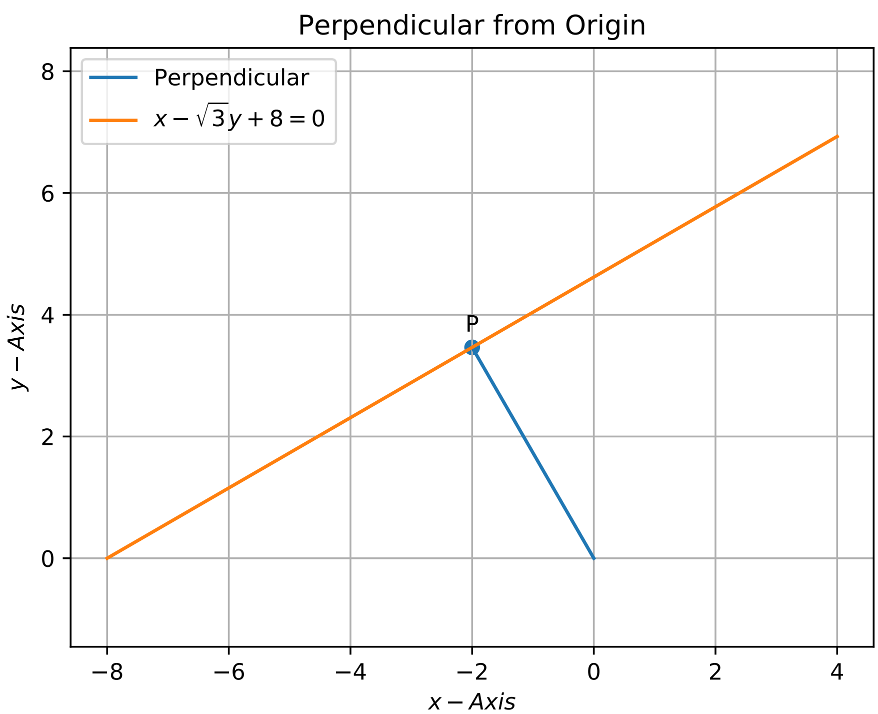

Perpendicular from Origin
Perpendicular
$x-\sqrt{3}y+8=0$
P
8
6
4
2
0
−8
−6
−4
−2
0
2
4
$x-Axis$
$y-Axis$

Figure 2