Quadratric Programming

1 12^{th} Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve $x^2 = 4y$ and passing through the point $(1, 2)$.

   Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{1}$$

   where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

$$f = 0 \tag{4}$$

   We are given that

$$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5}$$

   This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad \mathrm{f}(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \tag{6}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{7}$$

   It is already proved that the optimization problem is nonconvex. The constraints throw an error when *cvxpy* is used.

We will use Gradient Descent method to find the optimum value. Define

$$
\begin{aligned}
x_{n+1} &= x_n - \alpha \nabla g\left(x_n\right) && (8)\\
\text{with loop condition as } \left(\mathbf{x}-\mathbf{h}\right)^\top \nabla g\left(x_n\right) &\quad != 0 && (9)
\end{aligned}
$$

Choosing

(a) $\alpha = 0.001$

(b) precision = 0.001

(c) n = 10000

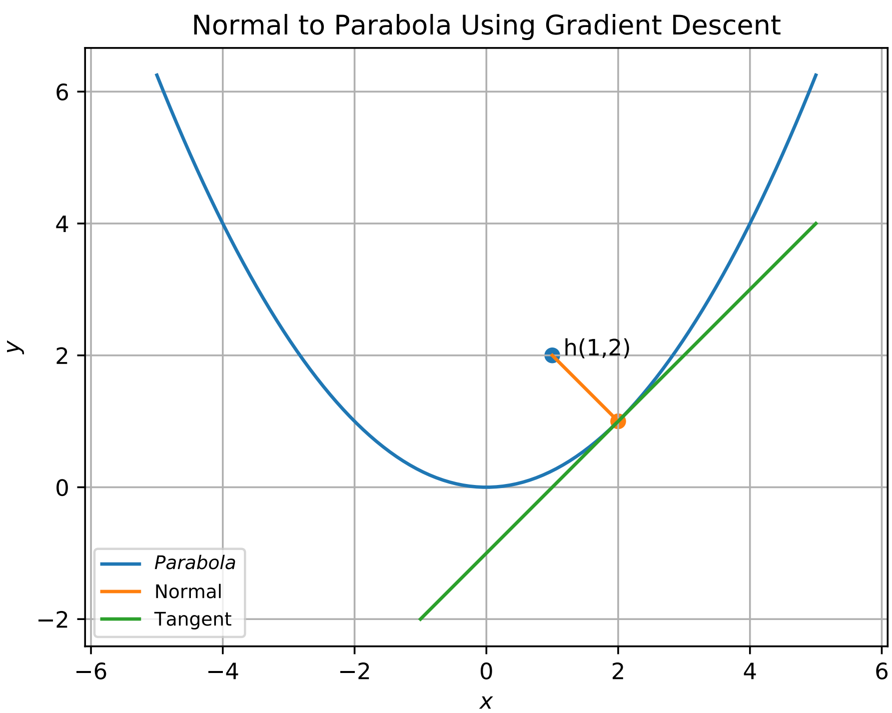
(d) $x_0 = 4$

$$
\mathbf{x}_{min} = \mathbf{q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{10}
$$

Given the point of contact $\mathbf{q}$, the equation to the normal is given by

$$
\begin{aligned}
&\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top \mathbf{R}\left(\mathbf{x}-\mathbf{q}\right) = 0 && (11)\\
&\implies \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 && (12)\\
&\implies \begin{pmatrix} 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 && (13)\\
&\implies \begin{pmatrix} 2 & 2 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 && (14)\\
&\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 && (15)
\end{aligned}
$$

The relevant figure is shown in 1



Normal to Parabola Using Gradient Descent
h(1,2)
Parabola
Normal
Tangent
6
4
2
0
−2
−6
−4
−2
0
2
4
6
x
y

Figure 1