3D Lines

1 JEE Maths - 65 C-1

This is Problem-30

1. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

   Solution: The given equation can be written as

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \tag{1}$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \tag{2}$$

$$\implies \mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{3}$$

$$\mathbf{x} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \tag{4}$$

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

   We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1}, \ \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{6}$$

   intersect if

$$\mathbf{M}\lambda = \mathbf{x_2} - \mathbf{x_1} \tag{7}$$

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{8}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{9}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 2 & 5 \\ 3 & 1 \\ 1 & 0 \end{pmatrix}, \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \tag{10}$$

$$\implies \begin{pmatrix} 2 & 5 \\ 3 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \tag{11}$$

Let us check whether the equation (11) has a solution.

The augmented matrix is given by,

$$\left(\begin{array}{cc|c} 2 & 5 & -2 \\ 3 & 1 & 3 \\ 1 & 0 & 2 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 - \frac{3}{2}R_1} \tag{12}$$

$$\left(\begin{array}{cc|c} 2 & 5 & -2 \\ 0 & -\frac{13}{2} & 6 \\ 0 & -\frac{5}{2} & 3 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - \frac{5}{13}R_2} \tag{13}$$

$$\left(\begin{array}{cc|c} 2 & 5 & -2 \\ 0 & -\frac{13}{2} & 6 \\ 0 & 0 & \frac{9}{13} \end{array}\right) \tag{14}$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (6) are given by

$$\mathbf{M}^{\top}\mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^{\top}\left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{15}$$

$$\implies \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 1 \\ 1 & 0 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \tag{16}$$

$$\implies \begin{pmatrix} 14 & 13 \\ 13 & 26 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 7 \\ -7 \end{pmatrix} \tag{17}$$

The augmented matrix of the above equation (17) given by,

$$\left(\begin{array}{cc|c} 14 & 13 & 7 \\ 13 & 26 & -7 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{13}{14} R_1} \tag{18}$$

$$\left(\begin{array}{cc|c} 14 & 13 & 7 \\ 0 & \frac{195}{14} & \frac{-27}{2} \end{array}\right) \xleftrightarrow{R_1 \leftarrow (R_1) - \frac{14}{15} R_2} \tag{19}$$

$$\left(\begin{array}{cc|c} 14 & 0 & \frac{98}{5} \\ 0 & \frac{195}{14} & \frac{-27}{2} \end{array}\right) \xleftrightarrow[R_2 \leftarrow \frac{14}{195} R_2]{R_1 \leftarrow \frac{1}{14} (R_1)} \tag{20}$$

$$\left(\begin{array}{cc|c} 1 & 0 & \frac{7}{5} \\ 0 & 1 & \frac{-63}{65} \end{array}\right) \tag{21}$$

resulting

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{7}{5} \\ \frac{-63}{65} \end{pmatrix} \tag{22}$$

The closest points $\mathbf{A}$ on line l_1 and $\mathbf{B}$ on line l_2 are given by,

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{7}{5} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{19}{5} \\ \frac{16}{5} \\ \frac{7}{5} \end{pmatrix} \tag{23}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + \frac{63}{65} \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{50}{13} \\ \frac{193}{65} \\ 2 \end{pmatrix} \tag{24}$$

The minimum distance between the lines is given by

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \begin{pmatrix} \frac{3}{65} \\ \frac{-15}{65} \\ \frac{3}{5} \end{pmatrix} \right\| = 3\sqrt{\frac{3}{65}} = 0.6445 \text{ units} \tag{25}$$

The relevant figure is shown in 1.

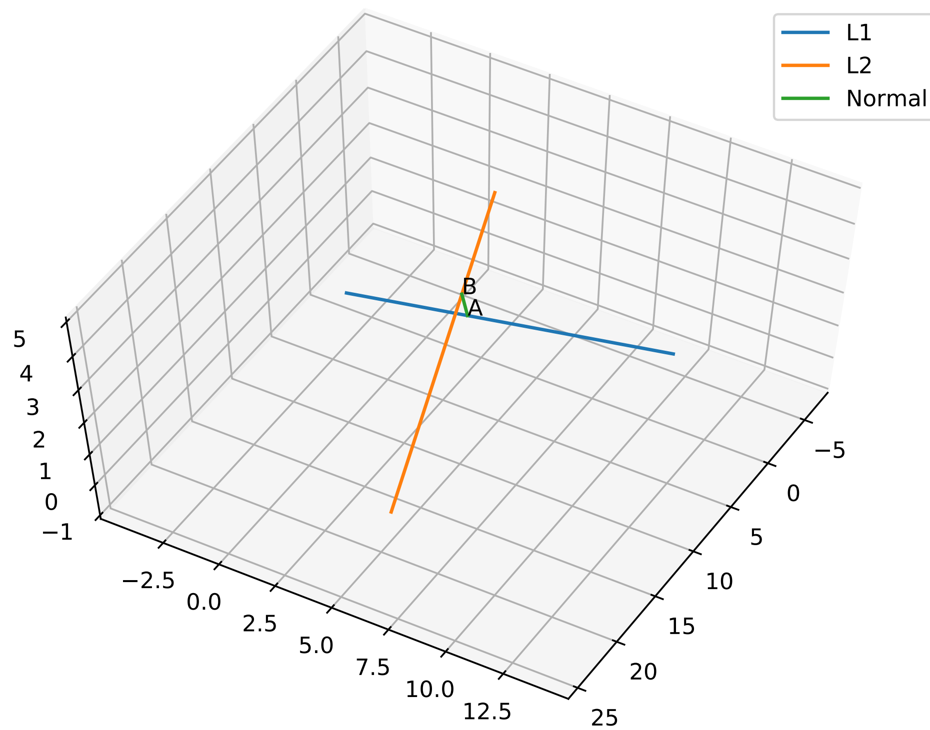

Figure 1