Circles

1 11^{th} Maths - Chapter 10

This is Problem-3 from Exercise 10.4

1. Find the centre of a circle passing though the points $(6,-6),(3,-7)$ and $(3,3)$.
   Solution: The equation of the crcle is given by

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top\mathbf{u} + f = 0 \tag{1}$$

   where

$$\mathbf{u} = -\mathbf{c} \text{ and} \tag{2}$$
$$f = \|\mathbf{c}\|^2 - r^2 \tag{3}$$

   Given points are

$$\mathbf{x_1}\begin{pmatrix}6\\-6\end{pmatrix}, \mathbf{x_2}\begin{pmatrix}3\\-7\end{pmatrix}, \mathbf{x_3}\begin{pmatrix}3\\3\end{pmatrix} \tag{4}$$

   Substituting points from (4) into (1)

$$\left(6^2+(-6)^2\right) + 2\begin{pmatrix}6 & -6\end{pmatrix}\mathbf{u} + f = 0 \implies 2\begin{pmatrix}6 & -6\end{pmatrix}\mathbf{u} + f = -72 \tag{5}$$
$$\left(3^2+(-7)^2\right) + 2\begin{pmatrix}3 & -7\end{pmatrix}\mathbf{u} + f = 0 \implies 2\begin{pmatrix}3 & -7\end{pmatrix}\mathbf{u} + f = -58 \tag{6}$$
$$\left(3^2+3^2\right) + 2\begin{pmatrix}3 & 3\end{pmatrix}\mathbf{u} + f = 0 \implies 2\begin{pmatrix}3 & 3\end{pmatrix}\mathbf{u} + f = -18 \tag{7}$$

   Representing the above system of equations in matrix form

$$\begin{pmatrix}6 & -14 & 1\\ 12 & -12 & 1\\ 6 & 6 & 1\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ f\end{pmatrix} = \begin{pmatrix}-58\\ -72\\ -18\end{pmatrix} \tag{8}$$

The augmented matrix is expressed as

$$\left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 12 & -12 & 1 & -72 \\ 6 & 6 & 1 & -18 \end{array}\right) \tag{9}$$

Performing sequence of row operations to transform into an Echelon form

$$\xleftrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 6 & 6 & 1 & -18 \end{array}\right) \tag{10}$$

$$\xleftrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 0 & 20 & 0 & 40 \end{array}\right) \tag{11}$$

$$\xleftrightarrow{R_3 \to R_3 - \frac{20}{16}R_2} \left(\begin{array}{ccc|c} 6 & -14 & 1 & -58 \\ 0 & 16 & -1 & 44 \\ 0 & 0 & \frac{20}{16} & -15 \end{array}\right) \tag{12}$$

$$\text{So, } \frac{20}{16}f = -15 \tag{13}$$

$$f = -12 \tag{14}$$

$$16u_2 - f = 44 \tag{15}$$

$$16u_2 + 12 = 44 \tag{16}$$

$$u_2 = 2 \tag{17}$$

$$6u_1 - 14u_2 + f = -58 \tag{18}$$

$$6u_1 - 28 - 12 = -58 \tag{19}$$

$$6u_1 = -18 \tag{20}$$

$$u_1 = -3 \tag{21}$$

Since $\mathbf{u} = -\mathbf{c}$,

$$\mathbf{c} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \tag{22}$$

$$(3) \implies r^2 = \left(3^2 + (-2)^2\right) + 12 \tag{23}$$

$$r = 5 \tag{24}$$

Therefore, the equation of the circle is

$$(x-3)^2 + (y+2)^2 = 25 \tag{25}$$

The relevant diagram is shown in Figure 1

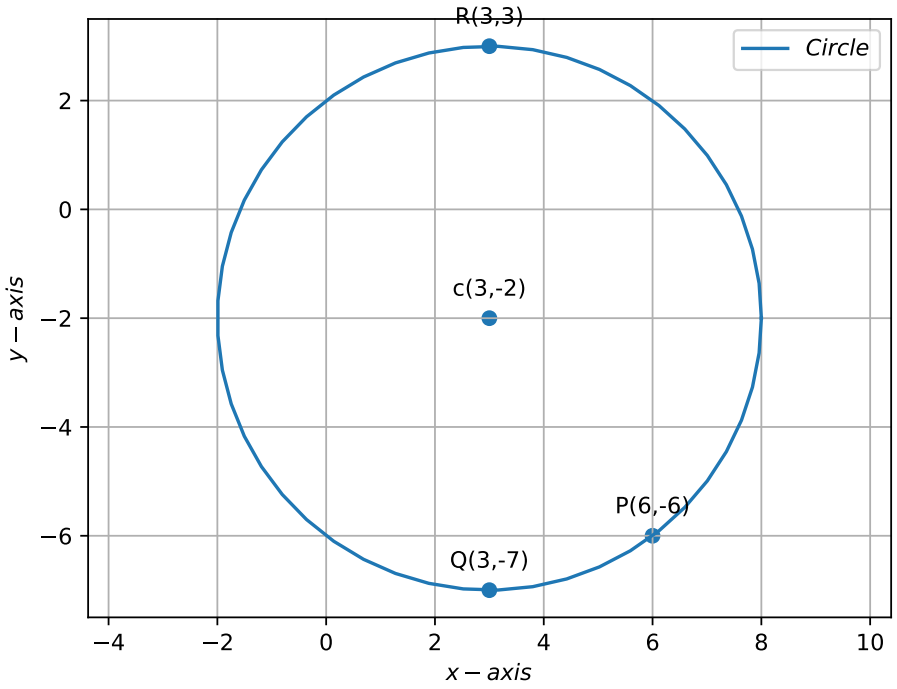

Figure 1