

# 3D Lines - Gradient Descent Method

## 1 JEE Maths - 65 C-1

This is Problem-30

1. Find the shortest distance between the lines  $\frac{x-1}{2} = \frac{y+1}{3} = z$  and  $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

**Solution:** The given equation can be written as

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \quad (1)$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \quad (2)$$

$$\Rightarrow \mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (3)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (4)$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \quad (5)$$

Assume

$$\mathbf{M} = (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (6)$$

$$\mathbf{B} - \mathbf{A} = (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \quad (7)$$

$$= \mathbf{M}\boldsymbol{\lambda} + \mathbf{k} \quad (8)$$

$$\text{where } \boldsymbol{\lambda} \triangleq \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{k} = (\mathbf{x}_2 - \mathbf{x}_1) \quad (9)$$

$$(10)$$

We can formulate an unconstrained optimization problem as below:

$$\min_{\lambda} \quad \|\mathbf{B} - \mathbf{A}\|^2 \quad (11)$$

Substituting (8) in (11)

$$(11) \implies \min_{\lambda} \quad \|\mathbf{M}\lambda + \mathbf{k}\|^2 \quad (12)$$

$$\implies f(\lambda) = (\mathbf{M}\lambda + \mathbf{k})^\top (\mathbf{M}\lambda + \mathbf{k}) \quad (13)$$

$$= (\lambda^\top \mathbf{M}^\top + \mathbf{k}^\top) (\mathbf{M}\lambda + \mathbf{k}) \quad (14)$$

$$= \lambda^\top \mathbf{M}^\top \mathbf{M} \lambda + 2\mathbf{k}^\top \mathbf{M} \lambda + \|\mathbf{k}\|^2 \quad (15)$$

Equation (15) is a quadratic vector equation and it is already proved that the function is convex.

We will use Gradient Descent method to find the optimum value. Define

$$\lambda_{n+1} = \lambda_n - \alpha \nabla f(\lambda_n) \quad (16)$$

Choosing

(a)  $\alpha = 0.001$

(b) precision = 0.0000001

(c) n = 10000

(d)  $\lambda_0 = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$$\lambda_{min} = \begin{pmatrix} -1.4 \\ 0.969 \end{pmatrix} \quad (17)$$

$$\mathbf{A} = \begin{pmatrix} 3.8 \\ 3.2 \\ 1.4 \end{pmatrix} \quad (18)$$

$$\mathbf{B} = \begin{pmatrix} 3.85 \\ 2.97 \\ 2 \end{pmatrix} \quad (19)$$

$$\|\mathbf{B} - \mathbf{A}\| = 0.6445 \text{ units} \quad (20)$$

The relevant figure is shown in 1.

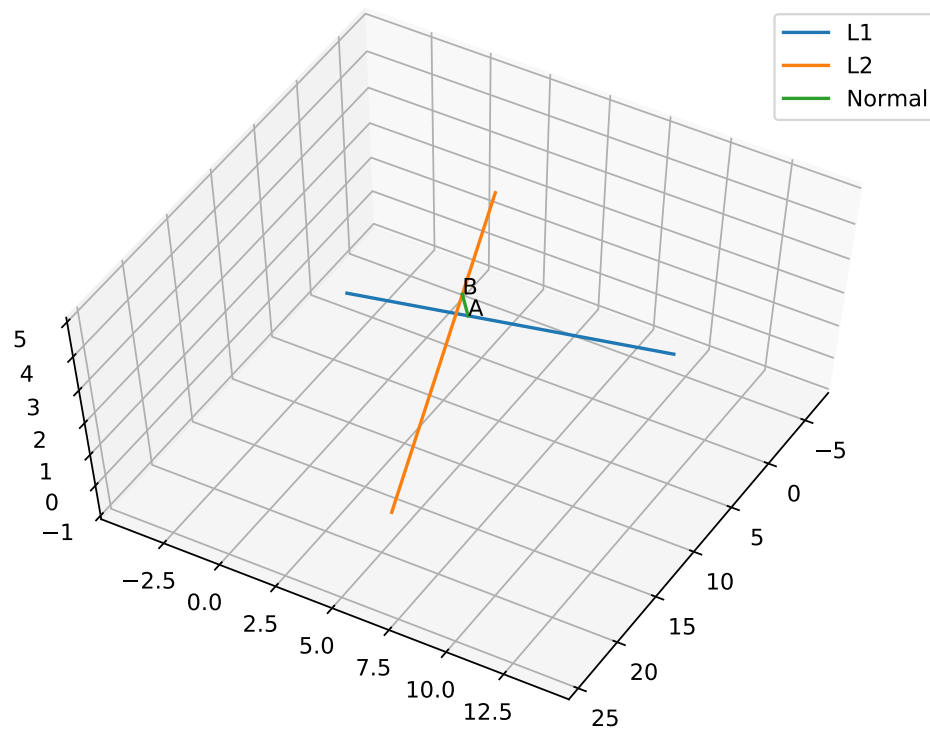


Figure 1