Conic Sections - Ellipse

1 11^{th} Maths - Chapter 11

This is Problem-1 from Exercise 11.3

1. Find the coordinates of the focii, the vertices, the length of major and minor axes, the eccentricity and the length of the latus rectum of an ellipse whose equation is given by $\frac{x^2}{36} + \frac{y^2}{16} = 1$.
Solution: The given equation of the ellipse can be rearranged as

$$4x^2 + 9y^2 - 144 = 0 \tag{1}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2}$$

Comparing coefficients of both equations (1) and (2)

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \tag{3}$$

$$\mathbf{u} = 0 \tag{4}$$

$$f = -144 \tag{5}$$

From equation (3), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 4 \tag{6}$$

$$\lambda_2 = 9 \tag{7}$$

The Eigen vector $\mathbf{p_1}$ corresponding to Eigen value λ_1 is computed as shown below

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \tag{8}$$

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 4 - \lambda_1 & 0 \\ 0 & 9 - \lambda_1 \end{pmatrix} \tag{9}$$

Substituting value of λ_1 from (6) in (9)

$$(9) \implies \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

x_1 is free variable and $x_2 = 0$.

$$\therefore \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

The Eigen vector $\mathbf{p_2}$ corresponding to Eigen value λ_2 is computed as shown below

$$\mathbf{V} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \tag{13}$$

$$\mathbf{V} - \lambda_2 \mathbf{I} = \begin{pmatrix} 4 - \lambda_2 & 0 \\ 0 & 9 - \lambda_1 \end{pmatrix} \tag{14}$$

Substituting value of λ_2 from (7) in (14)

$$(14) \implies \begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} -5 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{16}$$

x_2 is free variable and $x_1 = 0$.

$$\therefore \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{17}$$

(a) The eccentricity of the ellipse is given as

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{18}$$

$$= \sqrt{1 - \frac{4}{9}} \tag{19}$$

$$= \frac{\sqrt{5}}{3} \tag{20}$$

(b) Finding the coordinates of Focii:

$$\mathbf{n} = \sqrt{\lambda_2}\mathbf{p_1} \tag{21}$$

$$= \sqrt(9)\begin{pmatrix}1\\0\end{pmatrix} \tag{22}$$

$$= \begin{pmatrix}3\\0\end{pmatrix} \tag{23}$$

$$c = \frac{e\mathbf{u}^\top\mathbf{n} \pm \sqrt{e^2\left(\mathbf{u}^\top\mathbf{n}\right)^2 - \lambda_2\left(e^2-1\right)\left(\|\mathbf{u}\|^2 - \lambda_2 f\right)}}{\lambda_2 e\left(e^2-1\right)} \tag{24}$$

Substituting values of $e, \mathbf{u}, \mathbf{n}, \lambda_2$ and f in (24)

$$= \frac{0 \pm \sqrt{0 - 9\left(\frac{5}{9} - 1\right)\left(0 + 9\left(144\right)\right)}}{9\frac{\sqrt{5}}{3}\left(\frac{5}{9} - 1\right)} \tag{25}$$

$$= \frac{\pm 72}{-4\frac{\sqrt{5}}{3}} \tag{26}$$

$$= \frac{\pm 54}{\sqrt{5}} \tag{27}$$

The focus $\mathbf{F}$ of ellipse is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{28}$$

$$= \frac{\frac{\pm 54}{\sqrt{5}}\left(\frac{5}{9}\right)\begin{pmatrix}3\\0\end{pmatrix}}{9} \tag{29}$$

$$= \pm\begin{pmatrix}2\sqrt{5}\\0\end{pmatrix} \tag{30}$$

(c) The length of the major axis $2a$ is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_1}\right|} \tag{31}$$

$$\tag{32}$$

where

$$f_0 = \mathbf{u}^\top \mathbf{V}^{-1}\mathbf{u} - f \tag{33}$$

$$f_0 = 144 \because \mathbf{u} = 0 \tag{34}$$

$$(31) \implies 2\sqrt{\left|\frac{144}{4}\right|} \tag{35}$$

$$= 12 \tag{36}$$

(d) The length of the minor axis is given by

$$2\sqrt{\left|\frac{f_0}{\lambda_2}\right|} \tag{37}$$

$$= 2\sqrt{\left|\frac{144}{9}\right|} \tag{38}$$

$$= 8 \tag{39}$$

(e) The vertices of the ellipse are given by

$$\pm \begin{pmatrix} \frac{2a}{2} \\ 0 \end{pmatrix} \tag{40}$$

$$= \pm \begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{41}$$

(f) The length of the latus rectum is given as

$$2\frac{\sqrt{|f_0 \lambda_1|}}{\lambda_2} \tag{42}$$

$$= 2\frac{\sqrt{|144\,(4)|}}{9} \tag{43}$$

$$= \frac{16}{3} \tag{44}$$

The relevant diagram is shown in Figure 1

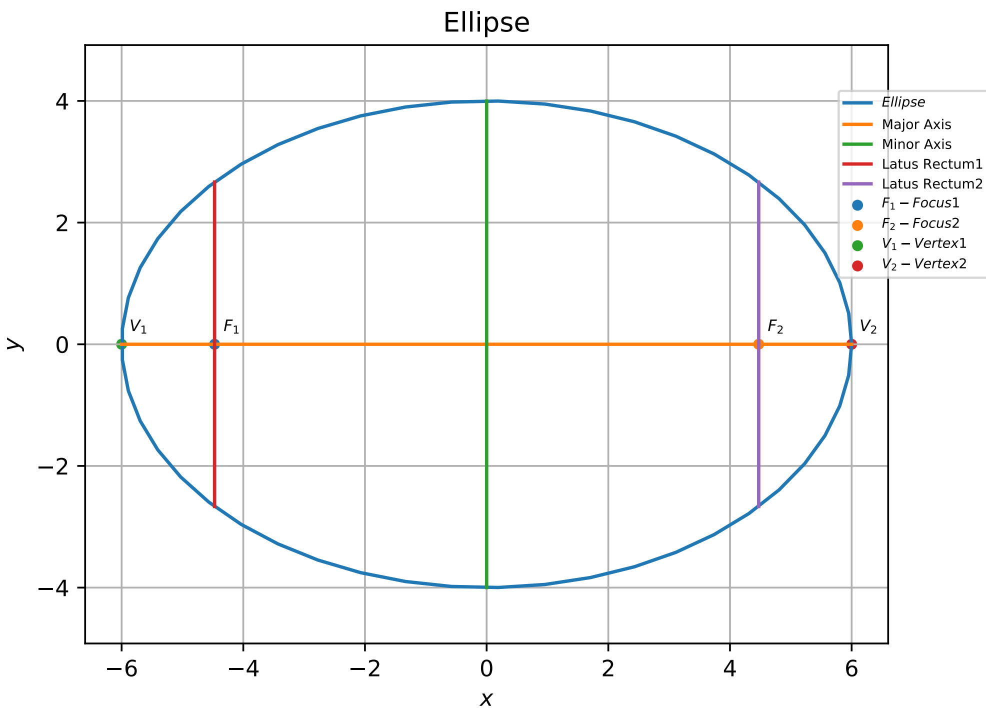
Ellipse
Ellipse
Major Axis
Minor Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$
V_1
F_1
F_2
V_2
4
2
0
−2
−4
−6
−4
−2
0
2
4
6
x
y

Figure 1