3D Lines

1 JEE Maths - 65 C-1

This is Problem-30

1. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

   Solution: The given equation can be written as

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \tag{1}$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \tag{2}$$

$$\implies \mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \tag{3}$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \tag{4}$$

where

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

Assume

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \tag{6}$$

$$\mathbf{B} - \mathbf{A} = (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \tag{7}$$

$$= \mathbf{M}\boldsymbol{\lambda} + \mathbf{k} \tag{8}$$

$$\text{where } \boldsymbol{\lambda} \triangleq \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{k} = (\mathbf{x}_2 - \mathbf{x}_1) \tag{9}$$

$$\tag{10}$$

We can formulate an unconstrained optimization problem as below:

$$\min_{\boldsymbol{\lambda}} \quad \|\mathbf{B}-\mathbf{A}\|^2 \tag{11}$$

Substituting (8) in (11)

$$
\begin{aligned}
(11) \implies \min_{\boldsymbol{\lambda}} \quad & \|\mathbf{M}\boldsymbol{\lambda}+\mathbf{k}\|^2 && (12)\\
\implies \mathrm{f}\left(\boldsymbol{\lambda}\right) &= \left(\mathbf{M}\boldsymbol{\lambda}+\mathbf{k}\right)^\top \left(\mathbf{M}\boldsymbol{\lambda}+\mathbf{k}\right) && (13)\\
&= \left(\boldsymbol{\lambda}^\top\mathbf{M}^\top+\mathbf{k}^\top\right)\left(\mathbf{M}\boldsymbol{\lambda}+\mathbf{k}\right) && (14)\\
&= \boldsymbol{\lambda}^\top\mathbf{M}^\top\mathbf{M}\boldsymbol{\lambda}+2\mathbf{M}\boldsymbol{\lambda}\mathbf{k}^\top+\|\mathbf{k}\|^2 && (15)
\end{aligned}
$$

Equation (15) is a quadratric vector equation. To check whether it is convex or not, we will compute the value of $\mathbf{M}^\top\mathbf{M}$.

$$\mathbf{M}^\top\mathbf{M} = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 3 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 14 & 13 \\ 13 & 26 \end{pmatrix} \tag{16}$$

The eigen values of $\mathbf{M}^\top\mathbf{M}$ are 34.73 and 5.27, which are greater than 0. Therefore $\mathbf{M}^\top\mathbf{M}$ is a positive definite matrix implying (15) is convex.

Setting parameters of equation (11) in cvxpy and solving, yields

$$\boldsymbol{\lambda}_{min} = \begin{pmatrix} -1.4 \\ 0.969 \end{pmatrix} \tag{17}$$

$$\mathbf{A} = \begin{pmatrix} 3.8 \\ 3.2 \\ 1.4 \end{pmatrix} \tag{18}$$

$$\mathbf{B} = \begin{pmatrix} 3.85 \\ 2.97 \\ 2 \end{pmatrix} \tag{19}$$

$$\|\mathbf{B}-\mathbf{A}\| = 0.6445 \text{ units} \tag{20}$$

The relevant figure is shown in 1.

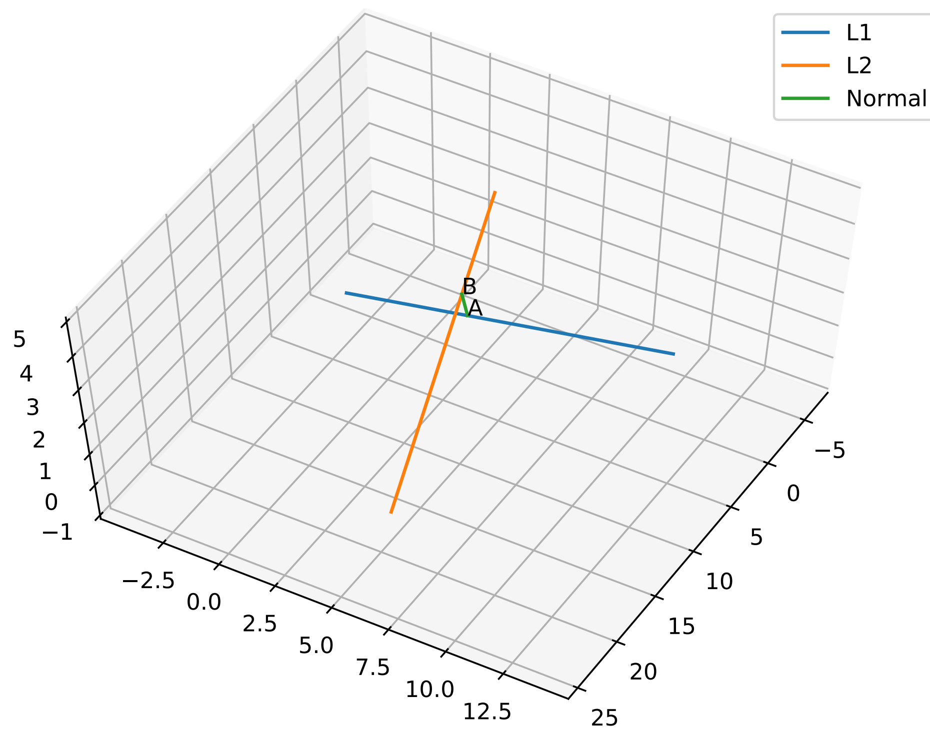

Figure 1