Tangents and Normals

1 12^{th} Maths - Chapter 6

This is Problem-2 from Exercise 6.3

1. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
   Solution:

   (a) The given equation of the curve can be rearranged as

   $$xy - x - 2y + 1 = 0 \tag{1}$$

   $$\implies \mathbf{x}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{2}$$

   The above equation can be equated to the generic equation of conic sections

   $$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3}$$

   Comparing coefficients of both equations (2) and (3)

   $$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{4}$$

   $$\mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{5}$$

   $$f = 1 \tag{6}$$

   Given the point of contact $\mathbf{q}$, the normal vector of the tangent to (3) is

   $$\kappa \mathbf{n} = \mathbf{V}\mathbf{q} + \mathbf{u}, \kappa \in \mathbb{R} \tag{7}$$

For the given point of contact with $\mathbf{q}_1 = 10$,

$$\mathbf{q}_2 = \frac{10-1}{10-2} = \frac{9}{8} \tag{8}$$

$$\therefore \mathbf{q} = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{9}$$

$$(7) \implies \kappa\mathbf{n} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{10}$$

$$= \left(\begin{pmatrix} \frac{9}{16} \\ 5 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \right) \tag{11}$$

$$\therefore \mathbf{n} = \alpha \begin{pmatrix} 1 \\ 64 \end{pmatrix} \tag{12}$$

$$\mathbf{m} = \alpha \begin{pmatrix} 1 \\ \frac{-1}{64} \end{pmatrix} \tag{13}$$

(b) Now, we have to determine the nature of the conic. The matrix($\mathbf{A}$) of the quadratic equation is represented as

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^\top & f \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & -1 \\ -\frac{1}{2} & -1 & 1 \end{pmatrix} \tag{15}$$

$$\left|A\right| = \frac{1}{4} \tag{16}$$

$$\left|A_{33}\right| = -\frac{1}{4} \tag{17}$$

$\because \left|A\right| \neq 0$ and $\left|A_{33}\right| < 0$, the conic is a hyperbola. Moreover, the Eigen vectors for $\mathbf{V}$, which are given as below, indicate that the axes of hyperbola are rotated by 45°.

$$\begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{18}$$

To summarize, the conic is a 45° rotated hyperbola.

The relevant diagram is shown in Figure 1

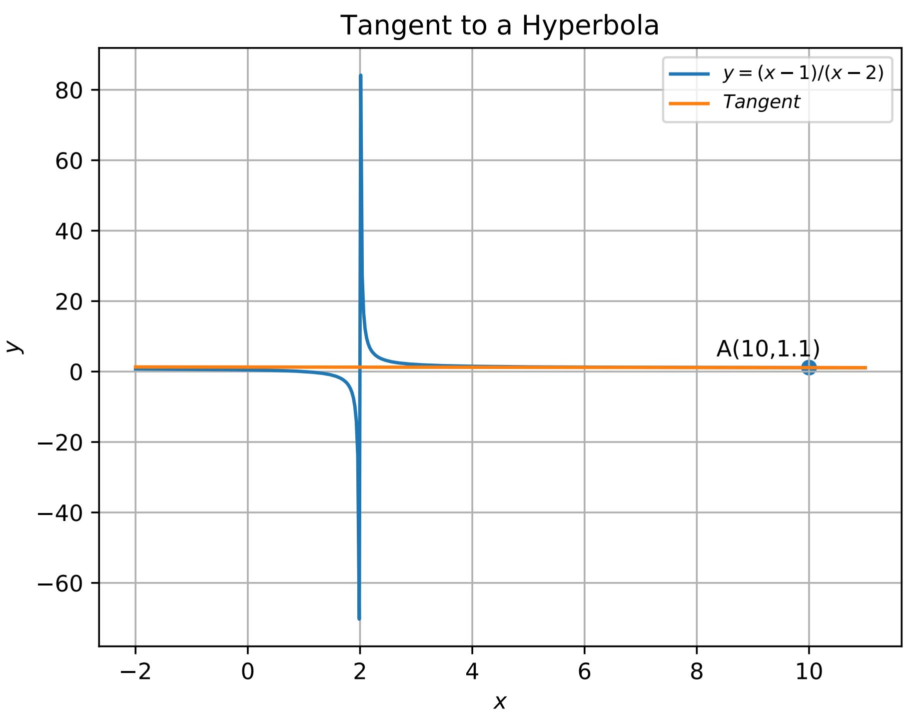
Tangent to a Hyperbola
$y = (x-1)/(x-2)$
Tangent
A(10,1.1)
80
60
40
20
0
−20
−40
−60
y
−2
0
2
4
6
8
10
x

Figure 1