Chords

1 12^{th} Maths - Chapter 8

This is Problem-13 from Exercise 8.1

1. Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.

   Solution:

2. The given equation of the curve can be rearranged as

$$y^2 - 4x = 0 \tag{1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{2}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3}$$

Comparing coefficients of both equations (2) and (3)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{5}$$

$$f = 0 \tag{6}$$

For the given line $y = 3$, the parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}$$

To calculate the point of contact of line with the conic, we use

$$2\mu\mathbf{m}^\top\left(\mathbf{V}\mathbf{h}+\mathbf{u}\right)+g\left(\mathbf{h}\right)=0 \tag{8}$$

$$\begin{aligned}
g\left(\mathbf{h}\right)=\begin{pmatrix}0 & 3\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix}\\
+2\begin{pmatrix}-2 & 0\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix}+0\\
\implies g\left(\mathbf{h}\right)=\begin{pmatrix}0 & 3\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix}+2\left(0\right)\\
\implies g\left(\mathbf{h}\right)=9 \quad (9)
\end{aligned}$$

$$\begin{aligned}
(8)\implies 2\mu\begin{pmatrix}1 & 0\end{pmatrix}\left(\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}0\\3\end{pmatrix}+\begin{pmatrix}-2\\0\end{pmatrix}\right)+9=0\\
\implies 2\mu\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}-2\\3\end{pmatrix}+9=0\\
\implies -4\mu+9=0\\
\implies \mu=\frac{9}{4} \quad (10)
\end{aligned}$$

The point of contact is given as

$$\mathbf{a}_0=\begin{pmatrix}\frac{9}{4}\\ \\3\end{pmatrix} \tag{11}$$

The desired area of the region is given as

$$\int_0^3 \frac{y^2}{4}\,dx=\frac{9}{4}\text{ sq.units} \tag{12}$$

The relevant diagram is shown in Figure 1

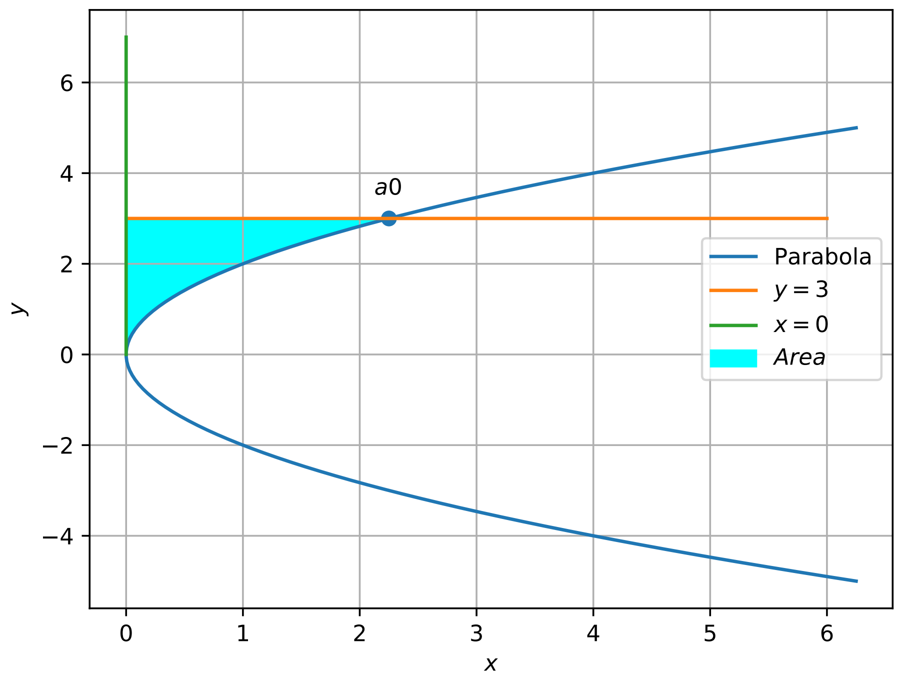

Figure 1