

Optimization using SVD

1 JEE Maths - 65 C-1

This is Problem-30

1. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

Solution: The given equation can be written as

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \quad (1)$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \quad (2)$$

$$\Rightarrow \mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \quad (3)$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \quad (4)$$

where

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{m}_2 = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \quad (5)$$

Assume

$$\mathbf{M} = (\mathbf{m}_1 \quad \mathbf{m}_2) \quad (6)$$

$$\mathbf{B} - \mathbf{A} = (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \quad (7)$$

$$= \mathbf{M}\boldsymbol{\lambda} + \mathbf{k} \quad (8)$$

$$\text{where } \boldsymbol{\lambda} \triangleq \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{k} = (\mathbf{x}_2 - \mathbf{x}_1) = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \quad (9)$$

We can formulate an unconstrained optimization problem as below:

$$\min_{\lambda} \quad \|\mathbf{B} - \mathbf{A}\|^2 \quad (10)$$

Substituting (8) in (10)

$$(10) \implies \min_{\lambda} \quad \|\mathbf{M}\lambda + \mathbf{k}\|^2 \quad (11)$$

Using Singular Value Decomposition, \mathbf{M} can be written as

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^\top \quad (12)$$

Computing SVD using python code, yields

$$\mathbf{U} = \begin{pmatrix} -0.903 & 0.424 & 0.072 \\ -0.42 & -0.834 & -0.358 \\ -0.092 & -0.353 & 0.931 \end{pmatrix} \quad (13)$$

$$\Sigma = \begin{pmatrix} 5.858 & 0 \\ 0 & 2.384 \\ 0 & 0 \end{pmatrix} \quad (14)$$

$$\mathbf{V} = \begin{pmatrix} -0.539 & -0.842 \\ -0.842 & 0.539 \end{pmatrix} \quad (15)$$

Then, the solution for (11) is given by

$$\lambda_{min} = \sum_{i=1}^r \frac{-\mathbf{u}_i^\top \mathbf{k}}{\sigma_i} \mathbf{v}_i \quad (16)$$

where r is rank of matrix \mathbf{M} .

Computing using python code,

$$\lambda_{min} = \begin{pmatrix} -1.4 \\ 0.969 \end{pmatrix} \quad (17)$$

$$\mathbf{A} = \begin{pmatrix} 3.8 \\ 3.2 \\ 1.4 \end{pmatrix} \quad (18)$$

$$\mathbf{B} = \begin{pmatrix} 3.85 \\ 2.97 \\ 2 \end{pmatrix} \quad (19)$$

$$\|\mathbf{B} - \mathbf{A}\| = 0.6445 \text{ units} \quad (20)$$

The relevant figure is shown in 1.

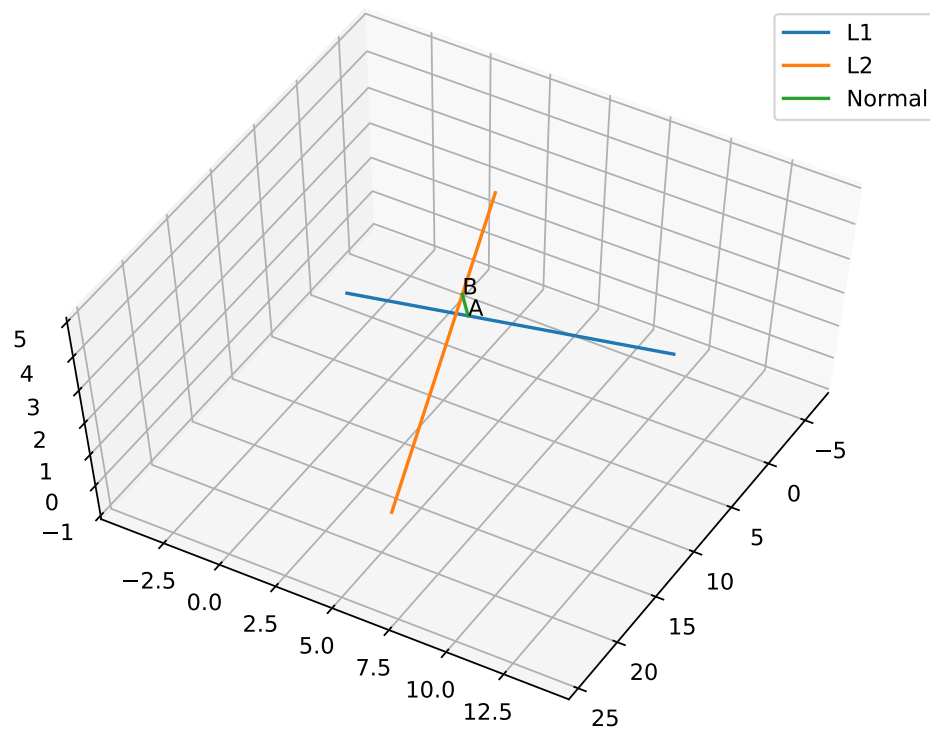


Figure 1