Geometric Programming

1 12^{th} Maths - Chapter 6

This is Problem-26 from Exercise 6.5

1. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $sin^{-1}\left(\frac{1}{3}\right)$.

   Solution:

2. Let r, h, l be the radius, height and slant height of the right circular cone respectively. Let S be the given surface area and V be the volume of the cone. We have

$$
\begin{aligned}
l^2 &= r^2 + h^2 && (1)\\
S &= \pi r l + \pi r^2 && (2)\\
\implies l &= \frac{S - \pi r^2}{\pi r} && (3)\\
V &= \frac{1}{3}\pi r^2 h && (4)\\
&= \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2} && (5)\\
V^2 &= \frac{1}{9}\pi^2 r^4 \left(l^2 - r^2\right) && (6)\\
&= \frac{1}{9}\pi^2 r^4 \left(\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right) && (7)\\
&= \frac{1}{9}\pi^2 r^4 \left(\frac{\left(S - \pi r^2\right)^2 - \pi^2 r^4}{\pi^2 r^2}\right) && (8)\\
&= \frac{1}{9} r^2 \left(\left(S - \pi r^2\right)^2 - \pi^2 r^4\right) && (9)
\end{aligned}
$$

$$= \frac{1}{9}r^2\left(S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4\right) \tag{10}$$
$$= \frac{1}{9}\left(S^2 r^2 - 2\pi S r^4\right) \tag{11}$$

Differentiating wrt r,

$$2V\frac{dV}{dr} = \frac{S^2}{9}2r - \frac{2\pi S}{9}4r^3 \tag{12}$$
$$= \frac{2rS}{9}\left(S - 4\pi r^2\right) \tag{13}$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\implies \frac{2rS}{9}\left(S - 4\pi r^2\right) = 0 \tag{14}$$
$$\implies r = 0 \text{ or } S - 4\pi r^2 = 0 \tag{15}$$

Since r can't equal to 0,

$$\implies S - 4\pi r^2 = 0 \tag{16}$$
$$\implies S = 4\pi r^2 \tag{17}$$
$$\implies r^2 = \frac{S}{4\pi} \tag{18}$$
$$\implies r^2 = \frac{\pi r l + \pi r^2}{4\pi} \tag{19}$$
$$\implies 4\pi r^2 = \pi r l + \pi r^2 \tag{20}$$
$$\implies 3\pi r^2 = \pi r l \tag{21}$$
$$\implies l = 3r \tag{22}$$

Let θ be the semi-vertical angle. Then,

$$sin\theta = \frac{r}{l} \tag{23}$$
$$sin\theta = \frac{r}{3r} \tag{24}$$
$$\implies \theta = sin^{-1}\frac{1}{3} \tag{25}$$

The relevant diagram is shown in Figure 1

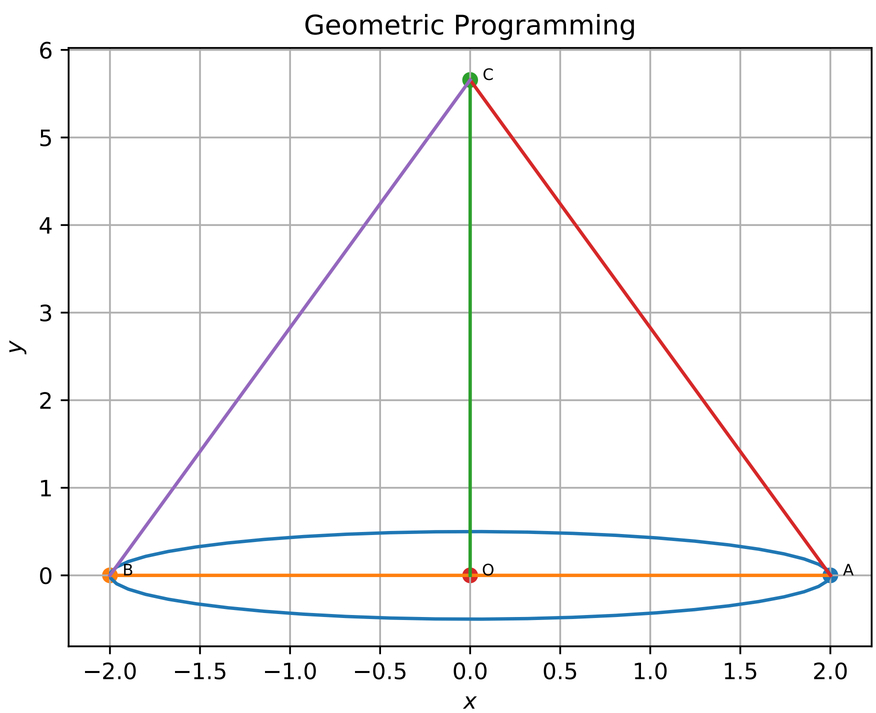
Geometric Programming
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Figure 1