Geometric Programming

1 12^{th} Maths - Chapter 6

This is Problem-26 from Exercise 6.5

1. Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

   Solution:

2. Let r, h, l be the radius, height and slant height of the right circular cone respectively. Let S be the given surface area and V be the volume of the cone. We have

$$l^2 = r^2 + h^2 \tag{1}$$

$$S = \pi rl + \pi r^2 \tag{2}$$

$$\implies l = \frac{S - \pi r^2}{\pi r} \tag{3}$$

$$V = \frac{1}{3}\pi r^2 h \tag{4}$$

   The given problem can be formulated as

$$\max_{r,h} V = \frac{1}{3}\pi r^2 h \tag{5}$$

$$\text{with constraints } S <= 75.42857 \tag{6}$$

(a) Theoritical proof:

$$(4) \implies V = \frac{1}{3}\pi r^2\sqrt{l^2 - r^2} \tag{7}$$

$$V^2 = \frac{1}{9}\pi^2 r^4 \left(l^2 - r^2\right) \tag{8}$$

$$= \frac{1}{9}\pi^2 r^4 \left(\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right) \tag{9}$$

$$= \frac{1}{9}\pi^2 r^4 \left(\frac{\left(S - \pi r^2\right)^2 - \pi^2 r^4}{\pi^2 r^2}\right) \tag{10}$$

$$= \frac{1}{9}r^2 \left(\left(S - \pi r^2\right)^2 - \pi^2 r^4\right) \tag{11}$$

$$= \frac{1}{9}r^2 \left(S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4\right) \tag{12}$$

$$= \frac{1}{9}\left(S^2 r^2 - 2\pi S r^4\right) \tag{13}$$

Differentiating wrt r,

$$2V\frac{dV}{dr} = \frac{S^2}{9}2r - \frac{2\pi S}{9}4r^3 \tag{14}$$

$$= \frac{2rS}{9}\left(S - 4\pi r^2\right) \tag{15}$$

For maximum volume, $\frac{dV}{dr} = 0$

$$\implies \frac{2rS}{9}\left(S - 4\pi r^2\right) = 0 \tag{16}$$

$$\implies r = 0 \text{ or } S - 4\pi r^2 = 0 \tag{17}$$

Since r can't be equal to 0,

$$\implies S - 4\pi r^2 = 0 \tag{18}$$
$$\implies S = 4\pi r^2 \tag{19}$$
$$\implies r^2 = \frac{S}{4\pi} \tag{20}$$
$$\implies r^2 = \frac{\pi r l + \pi r^2}{4\pi} \tag{21}$$
$$\implies 4\pi r^2 = \pi r l + \pi r^2 \tag{22}$$
$$\implies 3\pi r^2 = \pi r l \tag{23}$$
$$\implies l = 3r \tag{24}$$

For V to be maximum, $\frac{d^2V}{dr^2} < 0$

$$(14) \implies \frac{dV}{dr} = \frac{S}{3}\left[\frac{S - 4\pi r^2}{\sqrt{S^2 - 2\pi S r^2}}\right] \tag{25}$$
$$\frac{d^2V}{dr^2} = \frac{S}{3}\left[\frac{\sqrt{S^2 - 2\pi S r^2}\,(-8\pi r) + \frac{\left(S-4\pi r^2\right)(4\pi S r)}{2\sqrt{S^2-2\pi S r^2}}}{S^2 - 2\pi S r^2}\right] \tag{26}$$
$$= \frac{S}{3}\left[\frac{(S^2 - 2\pi S r^2)(-8\pi r) + (2\pi S r)(S - 4\pi r^2)}{(S^2 - 2\pi S r^2)^{\frac{3}{2}}}\right] \tag{27}$$
$$= \frac{S}{3}\left[\frac{-8\pi r S^2 + 16\pi^2 S r^3 + 2\pi S^2 r - 8\pi^2 S r^3}{(S^2 - 2\pi S r^2)^{\frac{3}{2}}}\right] \tag{28}$$
$$= \frac{S}{3}\left[\frac{8\pi^2 S r^3 - 6\pi r S^2}{(S^2 - 2\pi S r^2)^{\frac{3}{2}}}\right] \tag{29}$$

For maximum volume, substituting the value of S from (19) into

(29)

$$\implies \frac{d^2V}{dr^2} = \frac{4\pi r^2}{3}\left[\frac{(8\pi^2 r^3)\,4\pi r^2 - 6\pi r\,(16\pi^2 r^4)}{(16\pi^2 r^4 - 2\pi r^2\,(4\pi r^2))^{\frac{3}{2}}}\right] \tag{30}$$

$$= \frac{4\pi r^2}{3}\left[\frac{32\pi^3 r^5 - 96\pi^3 r^5}{(16\pi^2 r^4 - 8\pi^2 r^4)^{\frac{3}{2}}}\right] \tag{31}$$

$$= \frac{4\pi r^2}{3}\left[\frac{-64\pi^3 r^5}{(8\pi^2 r^4)^{\frac{3}{2}}}\right] \tag{32}$$

$$= \frac{-8\sqrt{2}\pi r}{3} \tag{33}$$

$$\therefore \frac{d^2V}{dr^2} < 0 \tag{34}$$

Let θ be the semi-vertical angle in Figure 1. Then,

$$\sin\theta = \frac{OA}{CA} = \frac{r}{l} \tag{35}$$

$$\sin\theta = \frac{r}{3r} \tag{36}$$

$$\implies \theta = sin^{-1}\frac{1}{3} \tag{37}$$

(b) Using Disciplined Geometric Programming (DGP) of cvxpy: Refer to equations (5) and (6) for formulation of optimization problem. Solving this problem, yields following results:

$$r = 2.45 \text{ units} \tag{38}$$

$$l = 7.35 \text{ units} \tag{39}$$

$$h = 6.93 \text{ units} \tag{40}$$

$$\text{Optimal } V \approx 43.557 \text{ cu.units} \tag{41}$$

It can be seen from solution that $l = 3r$ and semi-vertical angle is given as $\sin^{-1}\left(\frac{1}{3}\right)$. This is similar to what we proved theoritically.

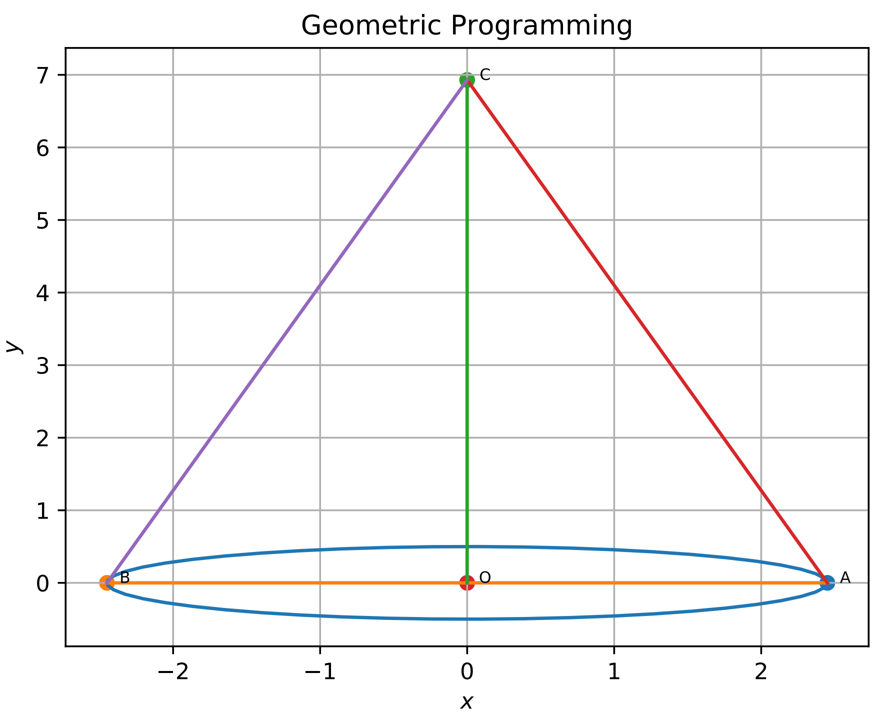

Figure 1