Convex Optimization

1 11^{th} Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   Solution: The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^{\top} \mathbf{x} + 8 = 0 \tag{1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3}$$

   Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

   Here, $\mathbf{A}$ is a point on the given line. We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{5}$$

$$(4) \implies \mathbf{x} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{6}$$

   Let $\mathbf{O}$ be the origin. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of the perpendicular.

This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \tag{7}$$

$$\implies \min_{\lambda} \left\| \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|^2 \tag{8}$$

$$\implies \min_{\lambda} \left\| \begin{pmatrix} \lambda - 8 \\ \frac{\lambda}{\sqrt{3}} \end{pmatrix} \right\|^2 \tag{9}$$

$$\implies f(\lambda) = (\lambda - 8)^2 + \left(\frac{\lambda}{\sqrt{3}}\right)^2 \tag{10}$$

$$= \lambda^2 - 16\lambda + 64 + \frac{\lambda^2}{3} \tag{11}$$

$$= \frac{4}{3}\lambda^2 - 16\lambda + 64 \tag{12}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (12) is a convex function.

$$f'(\lambda) = \frac{8}{3}\lambda - 16 \tag{13}$$

(a) Computing λ_{min} using Derivate method:

$$f''(\lambda) = \frac{8}{3} \tag{14}$$

$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \tag{15}$$

$$f'(\lambda_{min}) = \frac{8}{3}\lambda_{min} - 16 = 0 \tag{16}$$

$$\therefore \lambda_{min} = \frac{16 \times 3}{8} = 6 \tag{17}$$

(b) Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha \nabla f(\lambda_n) \tag{18}$$

Choosing

i. $\alpha = 0.001$
ii. precision = 0.0000001
iii. n = 10000000

iv. λ_0 = -5

$$\lambda_{min} = 6 \tag{19}$$

Both methods yielded same value of λ_{min}. Substituring this value in equation (6)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} \tag{21}$$

$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \tag{22}$$

$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \tag{23}$$

$$= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \tag{24}$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{25}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left(\frac{2\sqrt{3}}{-2} \right) \tag{26}$$

$$= \tan^{-1} \left(-\sqrt{3} \right) \tag{27}$$

$$= 120^\circ \tag{28}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{29}$$

The relevant figure is as shown in 1

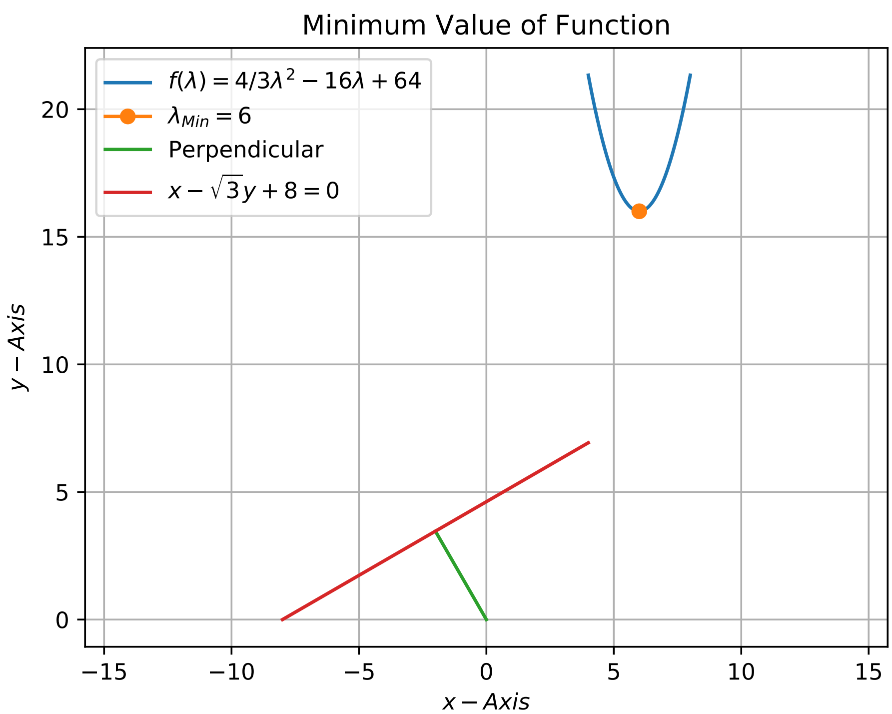
Minimum Value of Function
$f(\lambda) = 4/3\lambda^2 - 16\lambda + 64$
$\lambda_{Min} = 6$
Perpendicular
$x - \sqrt{3}y + 8 = 0$
$y - Axis$
$x - Axis$
20
15
10
5
0
−15
−10
−5
0
5
10
15

Figure 1