

Convex Optimization

1 11th Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

Solution: The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^\top \mathbf{x} + 8 = 0 \quad (1)$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (2)$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (3)$$

Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4)$$

Here, \mathbf{A} is a point on the given line. We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \quad (5)$$

$$(4) \implies \mathbf{x} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (6)$$

Let \mathbf{O} be the origin. The perpendicular distance will be the minimum distance from \mathbf{O} to the line. Let \mathbf{P} be the foot of the perpendicular.

This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \quad (7)$$

$$\Rightarrow \min_{\lambda} \left\| \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\|^2 \quad (8)$$

$$\Rightarrow \min_{\lambda} \left\| \begin{pmatrix} \lambda - 8 \\ \frac{\lambda}{\sqrt{3}} \end{pmatrix} \right\|^2 \quad (9)$$

$$\Rightarrow f(\lambda) = (\lambda - 8)^2 + \left(\frac{\lambda}{\sqrt{3}} \right)^2 \quad (10)$$

$$= \lambda^2 - 16\lambda + 64 + \frac{\lambda^2}{3} \quad (11)$$

$$= \frac{4}{3}\lambda^2 - 16\lambda + 64 \quad (12)$$

\because the coefficient of $\lambda^2 > 0$, equation (12) is a convex function.

$$f'(\lambda) = \frac{8}{3}\lambda - 16 \quad (13)$$

$$f''(\lambda) = \frac{8}{3} \quad (14)$$

$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \quad (15)$$

$$f'(\lambda_{min}) = \frac{8}{3}\lambda_{min} - 16 = 0 \quad (16)$$

$$\therefore \lambda_{min} = \frac{16 \times 3}{8} = 6 \quad (17)$$

Substituting this value in equation (6)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (18)$$

$$= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} \quad (19)$$

$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \quad (20)$$

$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \quad (21)$$

$$= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \quad (22)$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \quad (23)$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left(\frac{2\sqrt{3}}{-2} \right) \quad (24)$$

$$= \tan^{-1} (-\sqrt{3}) \quad (25)$$

$$= 120^\circ \quad (26)$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \quad (27)$$

The relevant figure is as shown in 1 and 2

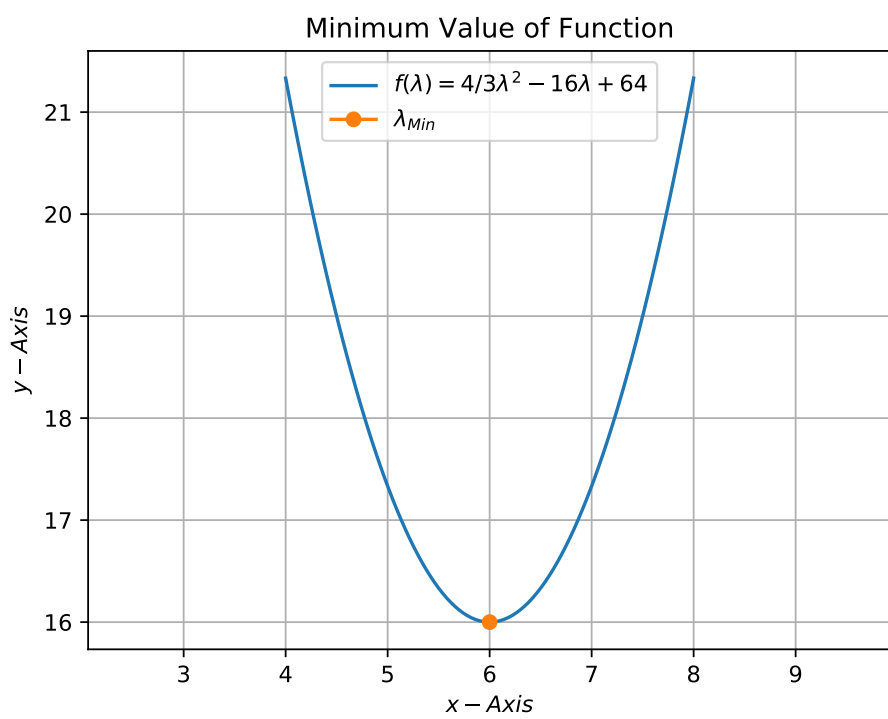


Figure 1

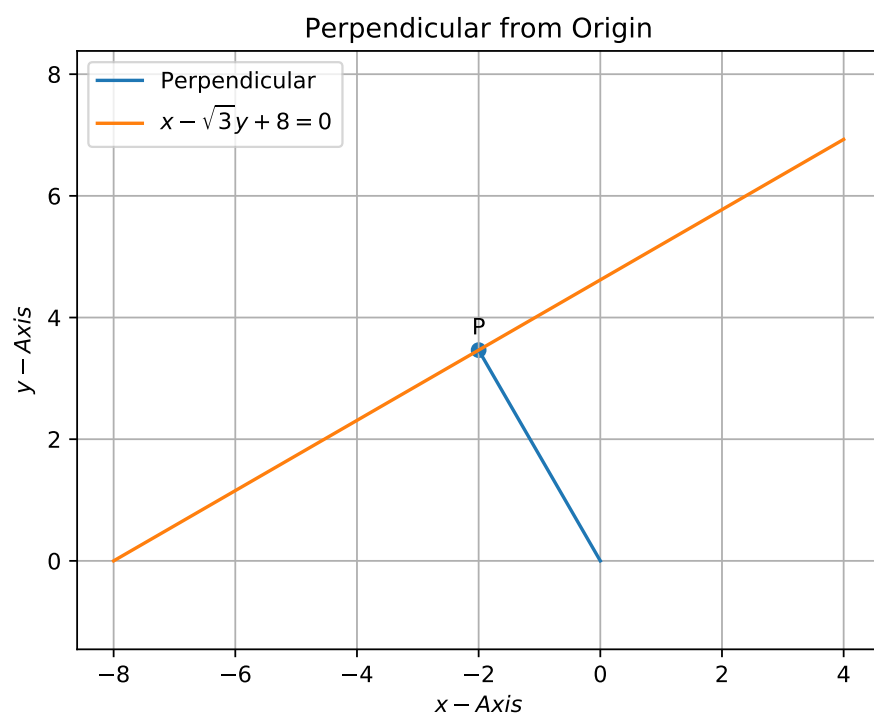


Figure 2