Conic Sections - Circle

1 10th Maths - Chapter 10

This is Problem-1 from Exercise 10.2

1. From a point $\mathbf{Q}$, the length of the tangent to a circle is $24cm$ and the distance of $\mathbf{Q}$ from the centre is $25cm$. Find the radius of the circle. Draw the circle and the tangents.

   Solution: Let $\mathbf{Q}$ be $\mathbf{Q}\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Let $\mathbf{O}$ be the centre of the circle. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the two points on the circle such that R_1Q and R_2Q are tangents to the circle from the point $\mathbf{Q}$. Given that,

$$OQ = 25, R_1Q = R_2Q = 24 \tag{1}$$

$$\therefore \mathbf{O} = \begin{pmatrix} 25 \\ 0 \end{pmatrix} \tag{2}$$

$$r = OR_1 = \sqrt{OQ^2 - R_1Q^2} \tag{3}$$

$$= \sqrt{25^2 - 24^2} \tag{4}$$

$$= 7cm \tag{5}$$

We have to find points $\mathbf{R}_1$ and $\mathbf{R}_2$.
We know that the equation to the circle is given as

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top\mathbf{u} + f = 0 \tag{6}$$

where

$$\mathbf{u} = -\mathbf{O} = -\begin{pmatrix} 25 \\ 0 \end{pmatrix} \text{ and} \tag{7}$$

$$f = \|\mathbf{O}\|^2 - r^2 = 576 \tag{8}$$

$$\Sigma = (\mathbf{Q} + \mathbf{u})(\mathbf{Q} + \mathbf{u})^\top - \left(\|\mathbf{Q}\|^2 + 2\mathbf{u}^\top\mathbf{Q} + f\right)\mathbf{I} \tag{9}$$

$$
= \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 25 \\ 0 \end{pmatrix}\right)\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 25 \\ 0 \end{pmatrix}\right)^\top \\
- \left(0 - 2\begin{pmatrix} 25 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 576\right)\mathbf{I} \tag{10}
$$

$$= \begin{pmatrix} -25 \\ 0 \end{pmatrix}\begin{pmatrix} -25 & 0 \end{pmatrix} - (576)\mathbf{I} \tag{11}$$

$$= \begin{pmatrix} 625 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 576 & 0 \\ 0 & 576 \end{pmatrix} \tag{12}$$

$$= \begin{pmatrix} 49 & 0 \\ 0 & -576 \end{pmatrix} \tag{13}$$

From (13), we can deduce Eigen pairs as follow:

$$\lambda_1 = 49, \lambda_2 = -576 \tag{14}$$

$$\mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{p_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{15}$$

Then

$$\mathbf{n_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{|\lambda_1|} \\ \sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 7 \\ 24 \end{pmatrix} \tag{16}$$

$$\mathbf{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{|\lambda_1|} \\ -\sqrt{|\lambda_2|} \end{pmatrix} = \begin{pmatrix} 7 \\ -24 \end{pmatrix} \tag{17}$$

The points of contact of a tangent on a circle from an external point is

given by

$$\mathbf{q_{ij}} = \left(\pm r \frac{\mathbf{n_j}}{\|\mathbf{n_j}\|} - \mathbf{u} \right), \quad i, j = 1, 2 \tag{18}$$

$$\mathbf{q_{i1}} = \left(\pm r \frac{\mathbf{n_1}}{\|\mathbf{n_1}\|} - \mathbf{u} \right) \tag{19}$$

$$= \left(\pm \frac{7}{25} \begin{pmatrix} 7 \\ 24 \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix} \right) \tag{20}$$

$$= \left(\pm \begin{pmatrix} \frac{49}{25} \\ \frac{168}{25} \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix} \right) \tag{21}$$

$$= \begin{pmatrix} \frac{674}{25} \\ \frac{168}{25} \end{pmatrix}, \begin{pmatrix} \frac{576}{25} \\ -\frac{168}{25} \end{pmatrix} \tag{22}$$

$$\mathbf{q_{i2}} = \left(\pm r \frac{\mathbf{n_2}}{\|\mathbf{n_2}\|} - \mathbf{u} \right) \tag{23}$$

$$= \left(\pm \frac{7}{25} \begin{pmatrix} 7 \\ -24 \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix} \right) \tag{24}$$

$$= \left(\pm \begin{pmatrix} \frac{49}{25} \\ \frac{-168}{25} \end{pmatrix} + \begin{pmatrix} 25 \\ 0 \end{pmatrix} \right) \tag{25}$$

$$= \begin{pmatrix} \frac{674}{25} \\ \frac{-168}{25} \end{pmatrix}, \begin{pmatrix} \frac{576}{25} \\ \frac{168}{25} \end{pmatrix} \tag{26}$$

$$\therefore \mathbf{R}_1 = \mathbf{q_{22}} = \begin{pmatrix} \frac{576}{25} \\ \frac{168}{25} \end{pmatrix} \tag{27}$$

$$\mathbf{R}_2 = \mathbf{q_{12}} = \begin{pmatrix} \frac{576}{25} \\ -\frac{168}{25} \end{pmatrix} \tag{28}$$
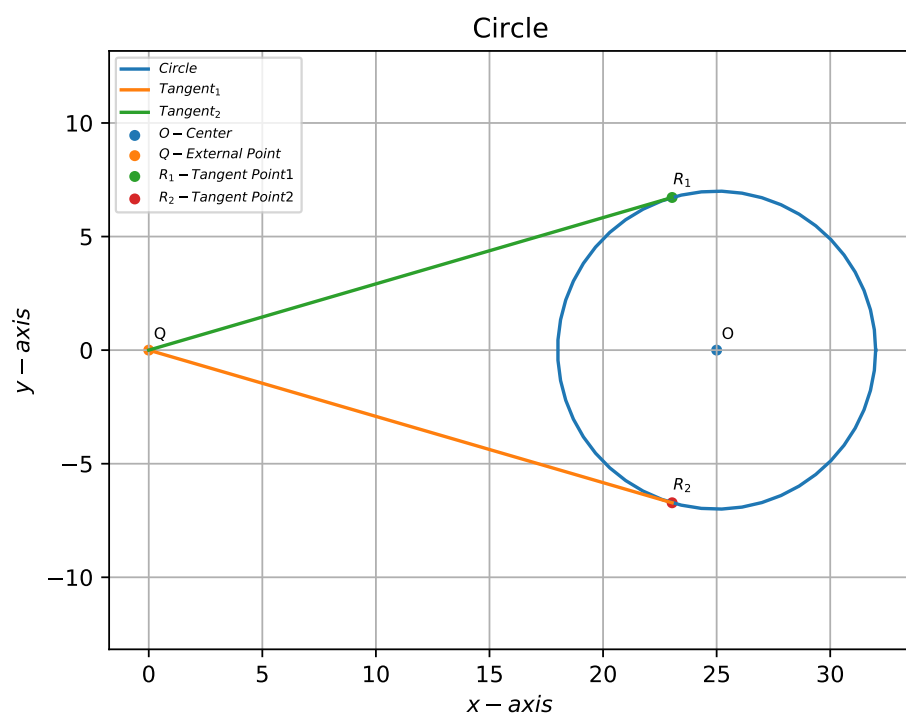
The figure is as shown in 1

Figure 1