Quadratric Programming

1 12^{th} Maths - Chapter 6

This is Problem-22 from Exercise 6.6

1. Find the equation of the normal at the point $(1, 1)$ on the curve $2y + x^2 = 3$.

   Solution: The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^\top \mathbf{x} + 8 = 0 \tag{1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3}$$

   Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

   Here, $\mathbf{A}$ is a point on the given line.

   Let $\mathbf{O}$ be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of the perpendicular. This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \tag{5}$$

$$\text{s.t.} \quad \mathbf{n}^T \mathbf{x} = c \tag{6}$$

(a) Using parametric equation:
Substituting (4) in (5)

$$\implies \min_{\lambda} \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{O}\|^2 \tag{7}$$

$$\implies \min_{\lambda} \|\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \tag{8}$$

$$\implies f(\lambda) = \left[\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\right]^\top \left[\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\right] \tag{9}$$

$$= \left[\lambda \mathbf{m}^\top + (\mathbf{A} - \mathbf{O})^\top\right] \left[\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\right] \tag{10}$$

$$= \lambda^2 \|\mathbf{m}\|^2 + \lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \lambda \mathbf{m}^\top (\mathbf{A} - \mathbf{O}) + \|\mathbf{A} - \mathbf{O}\|^2 \tag{11}$$

$$= \lambda^2 \|\mathbf{m}\|^2 + 2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \|\mathbf{A} - \mathbf{O}\|^2 \tag{12}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (12) is a convex function.

$$f'(\lambda) = 2\lambda \|\mathbf{m}\|^2 + 2(\mathbf{A} - \mathbf{O})^\top \mathbf{m} \tag{13}$$

$$f''(\lambda) = 2\|\mathbf{m}\|^2 \tag{14}$$

$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \tag{15}$$

$$f'(\lambda_{min}) = 2\lambda_{min} \|\mathbf{m}\|^2 + 2(\mathbf{A} - \mathbf{O})^\top \mathbf{m} = 0 \tag{16}$$

$$\lambda_{min} = -\frac{(\mathbf{A} - \mathbf{O})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \tag{17}$$

We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{18}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

Substituting the values of $\mathbf{A}$, $\mathbf{O}$ and $\mathbf{m}$ in equation (17)

$$\lambda_{min} = -\frac{\left(\begin{pmatrix}-8\\0\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right)^\top \begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix}}{\left\|\begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix}\right\|^2} \tag{20}$$

$$= \frac{\begin{pmatrix}8 & 0\end{pmatrix}\begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix}}{\frac{4}{3}} \tag{21}$$

$$= 6 \tag{22}$$

Substituring this value in equation (4)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix}-8\\0\end{pmatrix} + 6\begin{pmatrix}1\\\frac{1}{\sqrt{3}}\end{pmatrix} \tag{23}$$

$$= \begin{pmatrix}-8\\0\end{pmatrix} + \begin{pmatrix}6\\\frac{6}{\sqrt{3}}\end{pmatrix} \tag{24}$$

$$= \begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix} \tag{25}$$

$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \tag{26}$$

$$= \left\|\begin{pmatrix}-2\\2\sqrt{3}\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right\| \tag{27}$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{28}$$

(b) Solving using cvxpy, with

$$\mathbf{n} = \begin{pmatrix}1\\-\sqrt{3}\end{pmatrix} \tag{29}$$

$$\mathbf{O} = \begin{pmatrix}0\\0\end{pmatrix} \tag{30}$$

$$c = -8 \tag{31}$$

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 = 4, \mathbf{x}_{min} = \begin{pmatrix}-2\\3.46\end{pmatrix} \tag{32}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) \tag{33}$$

$$= \tan^{-1}\left(-\sqrt{3}\right) \tag{34}$$

$$= 120^\circ \tag{35}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{36}$$

The relevant figures are shown in 1 and 2

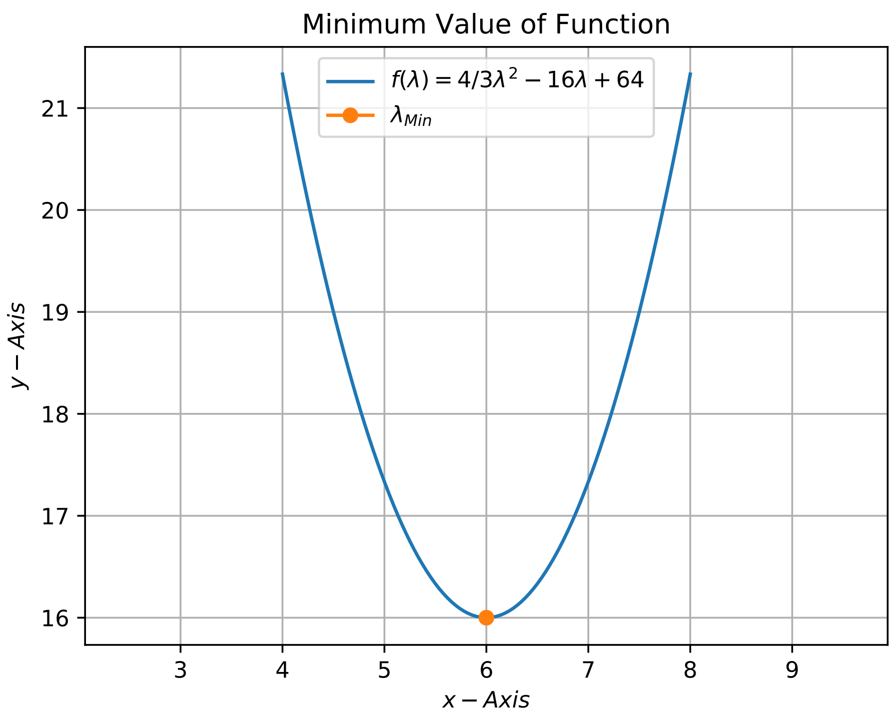
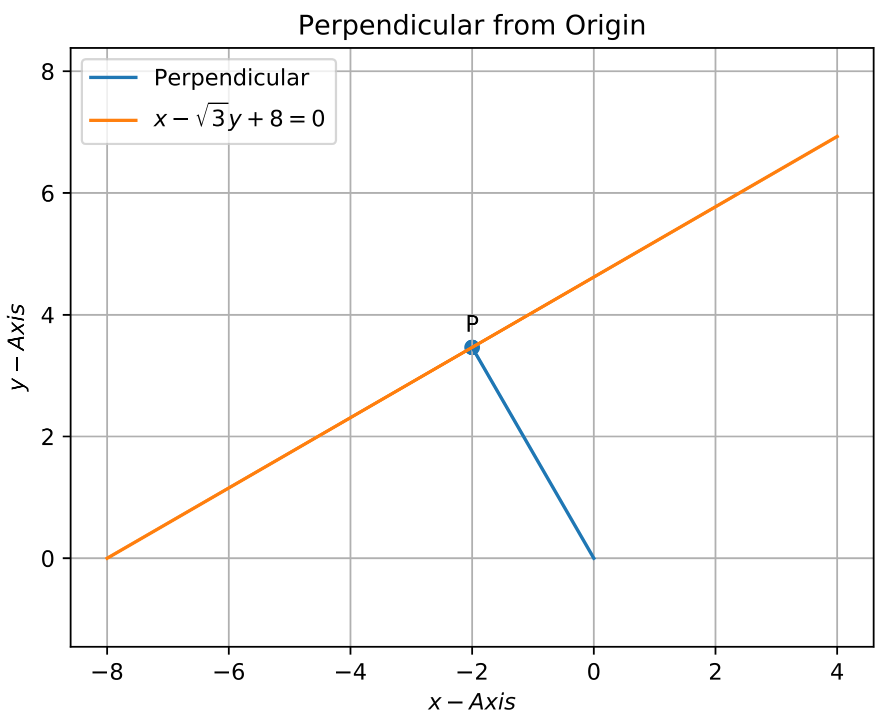


Figure 1



Perpendicular from Origin
Perpendicular
$x - \sqrt{3}y + 8 = 0$
P
8
6
4
2
0
−8
−6
−4
−2
0
2
4
$y - Axis$
$x - Axis$

Figure 2