Lagrange Multipliers

1 11^{th} Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   Solution:

   The equation of the given line is

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^{\top} \mathbf{x} + 8 = 0 \tag{1}$$

   Let $\mathbf{O}$ be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of the perpendicular. This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} f(\mathbf{x}) = \|\mathbf{x} - \mathbf{O}\|^2 \tag{2}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{n}^T \mathbf{x} - c = 0 \tag{3}$$

   where

$$\mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{4}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{5}$$

$$\text{and } c = -8 \tag{6}$$

Define

$$H(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \tag{7}$$

and we find that

$$\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{O}) \tag{8}$$
$$\nabla g(\mathbf{x}) = \mathbf{n} \tag{9}$$

We have to find $\lambda \in \mathbb{R}$ such that

$$\nabla H(\mathbf{x}, \lambda) = 0 \tag{10}$$
$$\implies 2(\mathbf{x} - \mathbf{O}) - \lambda \mathbf{n} = 0 \tag{11}$$
$$\implies \mathbf{x} = \frac{\lambda}{2}\mathbf{n} + \mathbf{O} \tag{12}$$

Substituting (12) in (1)

$$\mathbf{n}^\top \left(\frac{\lambda}{2}\mathbf{n} + \mathbf{O}\right) - c = 0 \tag{13}$$
$$\implies \lambda = \frac{2\left(c - \mathbf{n}^\top \mathbf{O}\right)}{\|\mathbf{n}\|^2} \tag{14}$$

Substituting the value of λ in (11),

$$\mathbf{x}_{min} = \mathbf{P} = \mathbf{O} + \frac{\mathbf{n}\left(c - \mathbf{n}^\top \mathbf{O}\right)}{\|\mathbf{n}\|^2} \tag{15}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}\left(-8 - \begin{pmatrix} 1 & -\sqrt{3} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)}{4} \tag{16}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{17}$$
$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \tag{18}$$
$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \tag{19}$$
$$= \left\|\begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\| \tag{20}$$
$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{21}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) \tag{22}$$

$$= \tan^{-1}\left(-\sqrt{3}\right) \tag{23}$$

$$= 120^\circ \tag{24}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{25}$$

The relevant figure is shown in 1

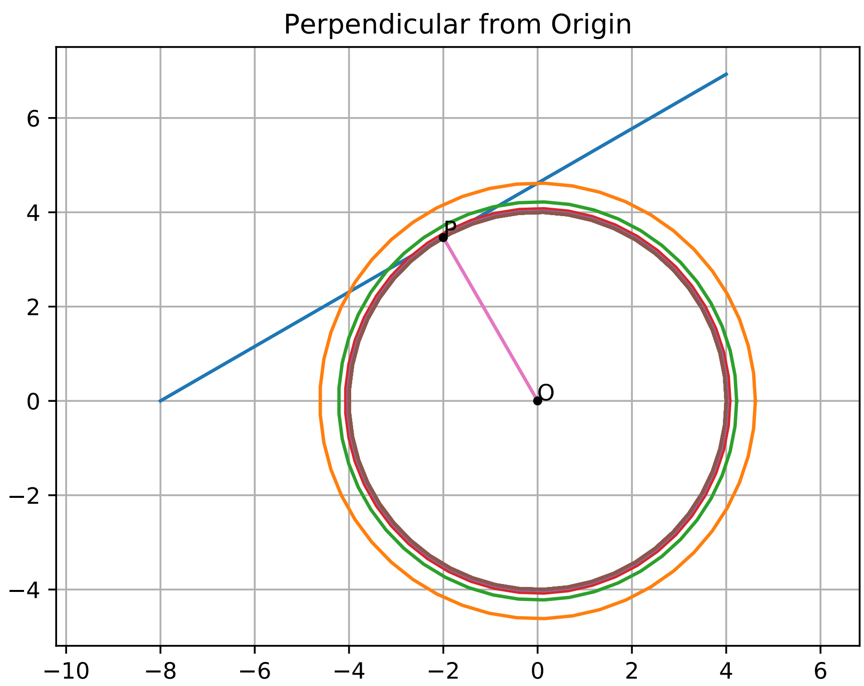

Figure 1