

Gradient Descent

1 12th Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve $x^2 = 4y$ and passing through the point $(1, 2)$.

Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (3)$$

$$f = 0 \quad (4)$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5)$$

This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \quad (6)$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (7)$$

It is already proved that the optimization problem is nonconvex. The constraints throw an error when *cvxpy* is used.

We will use Gradient Descent method to find the optimum value.
Define

$$x_{n+1} = x_n - \alpha \nabla g(x_n) \quad (8)$$

$$\text{with loop condition as } (\mathbf{x} - \mathbf{h})^\top \nabla g(x_n) \neq 0 \quad (9)$$

Choosing

- (a) $\alpha = 0.001$
- (b) precision = 0.001
- (c) n = 10000
- (d) $x_0 = 4$

$$\mathbf{x}_{min} = \mathbf{q} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (10)$$

Given the point of contact \mathbf{q} , the equation to the normal is given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \quad (11)$$

$$\implies \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right)^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (12)$$

$$\implies (2 \quad -2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (13)$$

$$\implies (2 \quad 2) \left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (14)$$

$$\implies (1 \quad 1) \mathbf{x} = 3 \quad (15)$$

The relevant figure is shown in 1

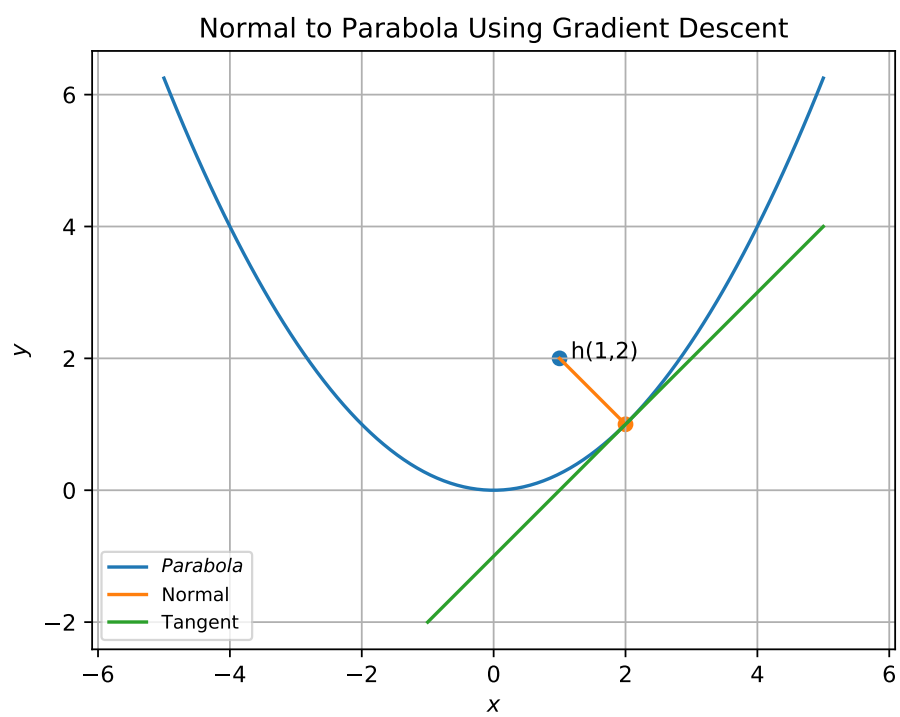


Figure 1