Quadratric Programming

1 12^{th} Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve $x^2 = 4y$ and passing through the point $(1, 2)$.

   Solution: The given equation of the curve can be written as

   $$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

   where

   $$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

   $$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

   $$f = 0 \tag{4}$$

   We are given that

   $$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5}$$

   This can be formulated as optimization problem as below:

   $$\min_{\mathbf{x}} \quad \mathrm{f}(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \tag{6}$$

   $$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{7}$$

   First we show that, (7) is not convex. Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfy $g(\mathbf{x}) = 0$. Then,

   $$\mathbf{x_1}^\top\mathbf{V}\mathbf{x_1} + 2\mathbf{u}^\top\mathbf{x_1} + f = 0 \tag{8}$$

   $$\mathbf{x_2}^\top\mathbf{V}\mathbf{x_2} + 2\mathbf{u}^\top\mathbf{x_2} + f = 0 \tag{9}$$

Then, for any $0 \leq \lambda \leq 1$, substituting

$$\mathbf{x}_\lambda \leftarrow \lambda \mathbf{x_1} + (1-\lambda)\,\mathbf{x_2} \tag{10}$$

into (7), we get

$$g\left(\mathbf{x}_\lambda\right) = \lambda\left(\lambda-1\right)\left(\mathbf{x_1}-\mathbf{x_2}\right)^\top \mathbf{V}\left(\mathbf{x_1}-\mathbf{x_2}\right) + f \neq 0 \tag{11}$$

Hence, the optimization problem is nonconvex. The constraints throw an error when *cvxpy* is used.

We will use Lagrange multipliers method to find the optimum value. Define

$$H\left(\mathbf{x},\lambda\right) = f\left(\mathbf{x}\right) - \lambda g\left(\mathbf{x}\right) \tag{12}$$

and we find that

$$\nabla f\left(\mathbf{x}\right) = 2\left(\mathbf{x}-\mathbf{h}\right) \tag{13}$$
$$\nabla g\left(\mathbf{x}\right) = 2\left(\mathbf{V}\mathbf{x}+\mathbf{u}\right) \tag{14}$$

We have to find $\lambda \in \mathbb{R}$ such that

$$\nabla H\left(\mathbf{x},\lambda\right) = 0 \tag{15}$$
$$\implies 2\left(\mathbf{x}-\mathbf{h}\right) - 2\lambda\left(\mathbf{V}\mathbf{x}+\mathbf{u}\right) = 0 \tag{16}$$
$$\implies \mathbf{x}-\mathbf{h} = \lambda\left(\mathbf{V}\mathbf{x}+\mathbf{u}\right) \tag{17}$$
$$\implies \left(\mathbf{I}-\lambda\mathbf{V}\right)\mathbf{x} = \lambda\mathbf{u}+\mathbf{h} \tag{18}$$
$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \lambda\begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{19}$$
$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ -2\lambda+2 \end{pmatrix} \tag{20}$$

We have 2 cases to considers here.

(a) When $\lambda \neq 1$. Writing augmented matrix,

$$\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1 & -2\lambda+2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{1-\lambda}} \begin{pmatrix} 1 & 0 & \frac{1}{1-\lambda} \\ 0 & 1 & -2\lambda+2 \end{pmatrix} \tag{21}$$

Then, we get

$$\mathbf{x}_m = \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} \tag{22}$$

Substituring this value in (7)

$$\begin{pmatrix} \frac{1}{1-\lambda} & -2\lambda + 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} + 2 \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \end{pmatrix} = 0 \tag{23}$$

$$\implies \begin{pmatrix} \frac{1}{1-\lambda} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} - 4\left(-2\lambda + 2\right) = 0 \tag{24}$$

$$\implies 8\left(\lambda^3 - 3\lambda^2 + 3\lambda - 1\right) + 1 = 0 \tag{25}$$

$$\implies \left(\lambda^3 - 3\lambda^2 + 3\lambda - 1\right) = -\frac{1}{8} \tag{26}$$

$$\implies (\lambda - 1)^3 = -\frac{1}{8} \tag{27}$$

$$\implies \lambda - 1 = -\frac{1}{2} \tag{28}$$

$$\implies \lambda = \frac{1}{2} \tag{29}$$

Substituting the value of λ in (22)

$$\mathbf{x}_m = \mathbf{q} = \begin{pmatrix} \frac{1}{1-\frac{1}{2}} \\ -2\frac{1}{2} + 2 \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{31}$$

(b) When $\lambda = 1$.

$$(20) \implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{32}$$

This is an invalid solution.

Given the point of contact $\mathbf{q}$, the equation to the normal is given by

$$\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)^\top \mathbf{R}\left(\mathbf{x}-\mathbf{q}\right) = 0 \tag{33}$$

$$\implies \left(\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}+\begin{pmatrix}0\\-2\end{pmatrix}\right)^\top \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\1\end{pmatrix}\right) = 0 \tag{34}$$

$$\implies \begin{pmatrix}2 & -2\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\1\end{pmatrix}\right) = 0 \tag{35}$$

$$\implies \begin{pmatrix}2 & -2\end{pmatrix}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\left(\mathbf{x}-\begin{pmatrix}2\\1\end{pmatrix}\right) = 0 \tag{36}$$

$$\implies \begin{pmatrix}1 & 1\end{pmatrix}\mathbf{x} = 3 \tag{37}$$

The relevant figure is shown in 1

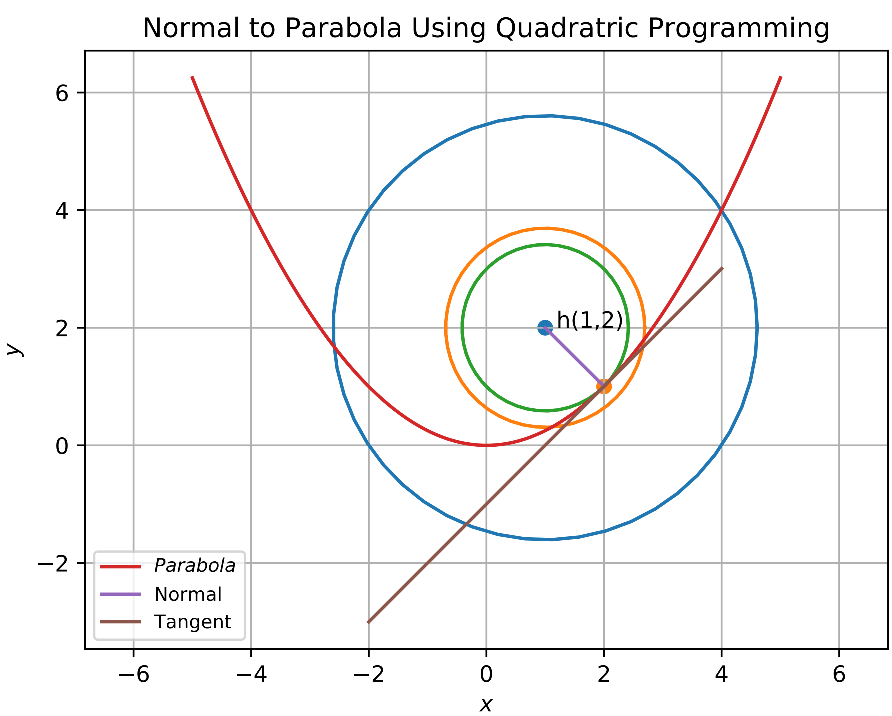

Figure 1