Gradient Descent

1 11^{th} Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   Solution: Equation for a line can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{1}$$

   We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{2}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{4}$$

$$\text{yielding } f(\lambda) = \frac{4}{3}\lambda^2 - 16\lambda + 64 \tag{5}$$

$$f'(\lambda) = \frac{8}{3}\lambda - 16 \tag{6}$$

   Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha f'(\lambda_n) \tag{7}$$

$$\lambda_{n+1} = \lambda_n \left(1 - \frac{8}{3}\alpha\right) + 16\alpha \tag{8}$$

Taking the one-sided Z-transform on both sides of (8),

$$z\Lambda(z) = \left(1 - \frac{8}{3}\alpha\right)\Lambda(z) + \frac{16\alpha}{1 - z^{-1}} \tag{9}$$

$$\Lambda(z) = \frac{16\alpha z^{-1}}{\left(1 - z^{-1}\right)\left(1 - \left(1 - \frac{8}{8}\alpha\right) z^{-1}\right)} \tag{10}$$

$$= 6\left(\frac{1}{1 - z^{-1}} - \frac{1}{1 - \left(1 - \frac{8}{3}\alpha\right) z^{-1}}\right) \tag{11}$$

$$= 6\sum_{k=0}^{\infty}\left(1 - \left(1 - \frac{8}{3}\alpha\right)^{k}\right) z^{-k} \tag{12}$$

From (12), the ROC is

$$|z| > \max\left\{1, \left|1 - \frac{8}{3}\alpha\right|\right\} \tag{13}$$

$$\implies 0 < \left|1 - \frac{8}{3}\alpha\right| < 1 \tag{14}$$

$$\implies 0 < \alpha < \frac{6}{8} \tag{15}$$

Thus, if α satisfies (15), then from (12),

$$\lim_{n\to\infty} \lambda_n = 6 \tag{16}$$

Choosing

(a) $\alpha = 0.001$
(b) precision = 0.0000001
(c) n = 10000000
(d) λ_0 = -5

$$\lambda_{min} = 6 \tag{17}$$

Substituting the values of $\mathbf{A}$, $\mathbf{m}$ and λ_{min} in equation (1)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{18}$$

$$= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \tag{20}$$

$$OP = \left\| \mathbf{P} - \mathbf{O} \right\|^2 \tag{21}$$

$$= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \tag{22}$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{23}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left(\frac{2\sqrt{3}}{-2} \right) \tag{24}$$

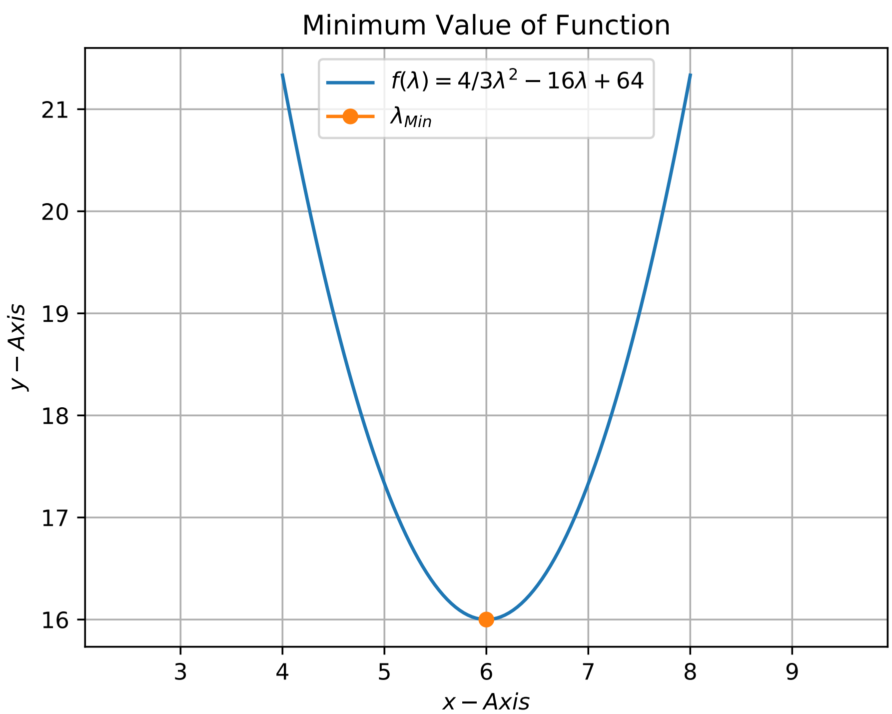
$$= \tan^{-1} \left(-\sqrt{3} \right) \tag{25}$$

$$= 120^\circ \tag{26}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{27}$$

The relevant figure is as shown in 1 and 2



Minimum Value of Function
$f(\lambda) = 4/3\lambda^2 - 16\lambda + 64$
λ_{Min}
21
20
19
18
17
16
3
4
5
6
7
8
9
$x - Axis$
$y - Axis$

Figure 1

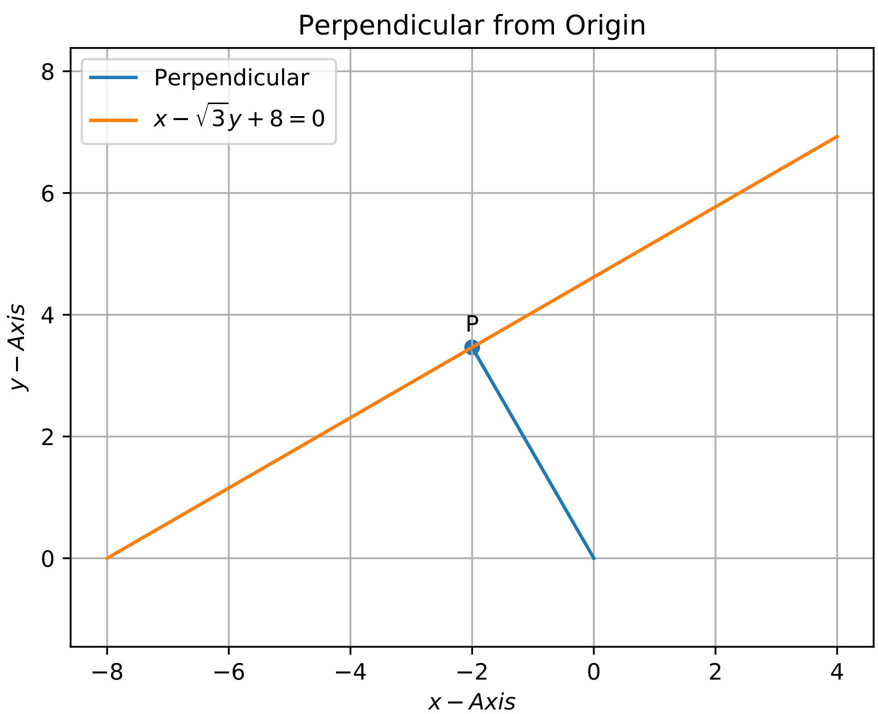

Perpendicular from Origin
Perpendicular
$x - \sqrt{3}y + 8 = 0$
P
8
6
4
2
0
−8
−6
−4
−2
0
2
4
$y - Axis$
$x - Axis$

Figure 2