Convex Optimization

1 12^{th} Maths - Chapter 6

This is Problem-1(i) from Exercise 6.5

1. Detrmine whether the function $f(x) = (2x-1)^2 + 3$ is convex or not.
   Solution: A single variable function f is said to be convex if

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2), \tag{1}$$

for $\quad 0 < \lambda < 1$ and $x_1, x_2 \in \mathbb{R}$.

For a generic quadratic function $ax^2 + bc + c$, let us determine the sufficient condition for it to be convex. Let

$$f(x) = ax^2 + bx + c \tag{2}$$

Substituting LHS of inequality from (1) in (2)

$$f[\lambda x_1 + (1-\lambda)x_2] = f[x_2 + \lambda(x_1 - x_2)] \tag{3}$$

$$\begin{aligned} &= a[x_2 + \lambda(x_1 - x_2)]^2 \\ &\quad + b[x_2 + \lambda(x_1 - x_2)] + c \end{aligned} \tag{4}$$

$$\begin{aligned} &= ax_2^2 + a\lambda^2 x_1^2 + a\lambda^2 x_2^2 - 2a\lambda^2 x_1 x_2 \\ &\quad + 2a\lambda x_1 x_2 - 2a\lambda x_2^2 + bx_2 + b\lambda x_1 - b\lambda x_2 + c \end{aligned} \tag{5}$$

Substituting RHS of inequality from (1) in (2)

$$\begin{aligned} \lambda f(x_1) + (1-\lambda) f(x_2) &= a\lambda x_1^2 + b\lambda x_1 + \lambda c \\ &\quad + (1-\lambda)\left(ax_2^2 + bx_2 + c\right) \end{aligned} \tag{6}$$

$$= a\lambda x_1^2 + b\lambda x_1 + ax_2^2 + bx_2 + c - a\lambda x_2^2 - b\lambda x_2 \tag{7}$$

Combining (5) and (7) with ineqiality and simplifying

$$\begin{aligned}a\lambda^2x_1^2 + a\lambda^2x_2^2 - 2a\lambda^2x_1x_2 + 2a\lambda x_1x_2 - 2a\lambda x_2^2 \\ \leq a\lambda x_1^2 - a\lambda x_2^2\end{aligned} \tag{8}$$

$$a\lambda^2x_1^2 + a\lambda^2x_2^2 - 2a\lambda^2x_1x_2 + 2a\lambda x_1x_2 - a\lambda x_2^2 - a\lambda x_1^2 \leq 0 \tag{9}$$

$$= x_1^2\left(a\lambda^2 - a\lambda\right) + x_2^2\left(a\lambda^2 - a\lambda\right) - 2x_1x_2\left(a\lambda^2 - a\lambda\right) \leq 0 \tag{10}$$

$$= \left(a\lambda^2 - a\lambda\right)(x_1 - x_2)^2 \leq 0 \tag{11}$$

$$= a\lambda\left(1 - \lambda\right)(x_1 - x_2)^2 \geq 0 \tag{12}$$

For the ineqality in (12) to be true,

$$a \geq 0 \because \lambda, 1 - \lambda \geq 0, (x_1 - x_2)^2 \geq 0 \tag{13}$$

However, $a \neq 0$, since it is a quadratice function. Hence $a > 0$, for $f(x)$ to be convex.

The given function is

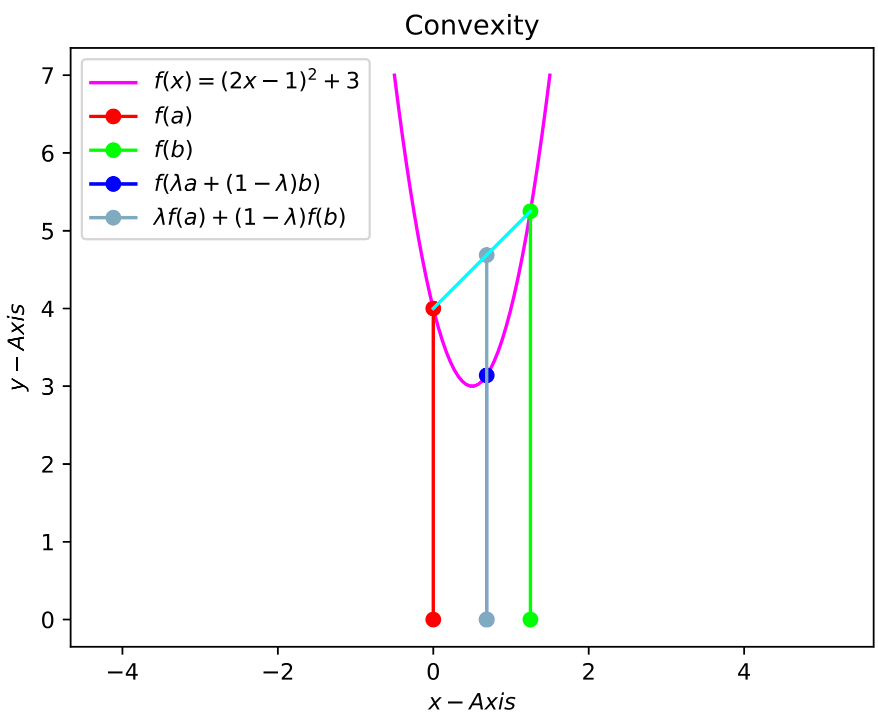
$$f(x) = (2x - 1)^2 + 3 \tag{14}$$

$$= 4x^2 + 4x + 4 \tag{15}$$

$$\therefore a = 4, > 0 \tag{16}$$

Hence, the function in equation (14) is convex.

The figure is as shown in Fig1



Convexity
$f(x) = (2x-1)^2 + 3$
$f(a)$
$f(b)$
$f(\lambda a + (1-\lambda)b)$
$\lambda f(a) + (1-\lambda)f(b)$
$y - Axis$
$x - Axis$
7
6
5
4
3
2
1
0
−4
−2
0
2
4

Figure 1