Conic Sections - Circle

1 10^{th} Maths - Chapter 10

This is Problem-1 from Exercise 10.2

1. From a point $\mathbf{Q}$, the length of the tangent to a circle is $24cm$ and the distance of $\mathbf{Q}$ from the centre is $25cm$. Find the radius of the circle. Draw the circle and the tangents.

   Solution: Let $\mathbf{Q}$ be $\mathbf{Q}\begin{pmatrix}0\\0\end{pmatrix}$. Let $\mathbf{c}$ be the centre of the circle. Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be the two points on the circle such that $\mathbf{R_1Q}$ and $\mathbf{R_2Q}$ are tangents to the circle from the point $\mathbf{Q}$. Given that,

$$\|\mathbf{cQ}\| = 25, \|\mathbf{R_1Q}\| = \|\mathbf{R_2Q}\| = 24 \tag{1}$$

$$\therefore \mathbf{c} = \mathbf{c}\begin{pmatrix}25\\0\end{pmatrix} \tag{2}$$

$$r = \|\mathbf{cR_1}\| = \sqrt{\|\mathbf{cQ}\|^2 - \left\|\mathbf{R_1Q}^2\right\|} \tag{3}$$

$$= \sqrt{25^2 - 24^2} \tag{4}$$

$$= 7 \tag{5}$$

   We have to find points $\mathbf{R}_1$ and $\mathbf{R}_2$.
   We know that the equation to the circle is given as

$$\|\mathbf{x}\|^2 + 2\mathbf{x}^\top\mathbf{u} + f = 0 \tag{6}$$

   where

$$\mathbf{u} = -\mathbf{c} = -\begin{pmatrix}25\\0\end{pmatrix} \text{ and} \tag{7}$$

$$f = \|\mathbf{c}\|^2 - r^2 = 576 \tag{8}$$

$$\mathbf{\Sigma} = (\mathbf{Q}+\mathbf{u})(\mathbf{Q}+\mathbf{u})^\top - \left(\|\mathbf{Q}\|^2 + 2\mathbf{u}^\top\mathbf{Q} + f\right)\mathbf{I} \tag{9}$$

$$= \left(\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}25\\0\end{pmatrix}\right)\left(\begin{pmatrix}0\\0\end{pmatrix} - \begin{pmatrix}25\\0\end{pmatrix}\right)^\top - \left(0 - 2\begin{pmatrix}25 & 0\end{pmatrix}\begin{pmatrix}0\\0\end{pmatrix} + 576\right)\mathbf{I} \tag{10}$$

$$= \begin{pmatrix}-25\\0\end{pmatrix}\begin{pmatrix}-25 & 0\end{pmatrix} - (576)\,\mathbf{I} \tag{11}$$

$$= \begin{pmatrix}625 & 0\\0 & 0\end{pmatrix} - \begin{pmatrix}576 & 0\\0 & 576\end{pmatrix} \tag{12}$$

$$= \begin{pmatrix}49 & 0\\0 & -576\end{pmatrix} \tag{13}$$

From (13), we can deduce Eigen pairs as follow:

$$\lambda_1 = 49, \lambda_2 = -576 \tag{14}$$

$$\mathbf{p_1} = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{p_2} = \begin{pmatrix}0\\1\end{pmatrix} \tag{15}$$

Then

$$\mathbf{n_1} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{|\lambda_1|}\\\sqrt{|\lambda_2|}\end{pmatrix} = \begin{pmatrix}7\\24\end{pmatrix} \tag{16}$$

$$\mathbf{n_2} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{|\lambda_1|}\\-\sqrt{|\lambda_2|}\end{pmatrix} = \begin{pmatrix}7\\-24\end{pmatrix} \tag{17}$$

The points of contact of a tangent on a circle from an external point is given by

$$\mathbf{q_{ij}} = \left(\pm r\frac{\mathbf{n_j}}{\|\mathbf{n_j}\|} - \mathbf{u}\right), \quad i,j = 1,2 \tag{18}$$

$$\mathbf{q_{i1}} = \left(\pm r\frac{\mathbf{n_1}}{\|\mathbf{n_1}\|} - \mathbf{u}\right) \tag{19}$$

$$= \left(\pm\frac{7}{25}\begin{pmatrix}7\\24\end{pmatrix} + \begin{pmatrix}25\\0\end{pmatrix}\right) \tag{20}$$

$$= \left(\pm\begin{pmatrix}\frac{49}{25}\\\frac{168}{25}\end{pmatrix} + \begin{pmatrix}25\\0\end{pmatrix}\right) \tag{21}$$

$$= \begin{pmatrix}\frac{674}{25}\\\frac{168}{25}\end{pmatrix}, \begin{pmatrix}\frac{576}{25}\\-\frac{168}{25}\end{pmatrix} \tag{22}$$

$$\mathbf{q_{i2}} = \left(\pm r\frac{\mathbf{n_2}}{\|\mathbf{n_2}\|} - \mathbf{u}\right) \tag{23}$$

$$= \left(\pm\frac{7}{25}\begin{pmatrix}7\\-24\end{pmatrix} + \begin{pmatrix}25\\0\end{pmatrix}\right) \tag{24}$$

$$= \left(\pm\begin{pmatrix}\frac{49}{25}\\\frac{-168}{25}\end{pmatrix} + \begin{pmatrix}25\\0\end{pmatrix}\right) \tag{25}$$

$$= \begin{pmatrix}\frac{674}{25}\\\frac{-168}{25}\end{pmatrix}, \begin{pmatrix}\frac{576}{25}\\\frac{168}{25}\end{pmatrix} \tag{26}$$

$$\therefore \mathbf{R}_1 = \mathbf{q22} = \begin{pmatrix}\frac{576}{25}\\\frac{168}{25}\end{pmatrix}, \mathbf{R}_2 = \mathbf{q12} = \begin{pmatrix}\frac{576}{25}\\-\frac{168}{25}\end{pmatrix} \tag{27}$$

The figure is as shown in 1

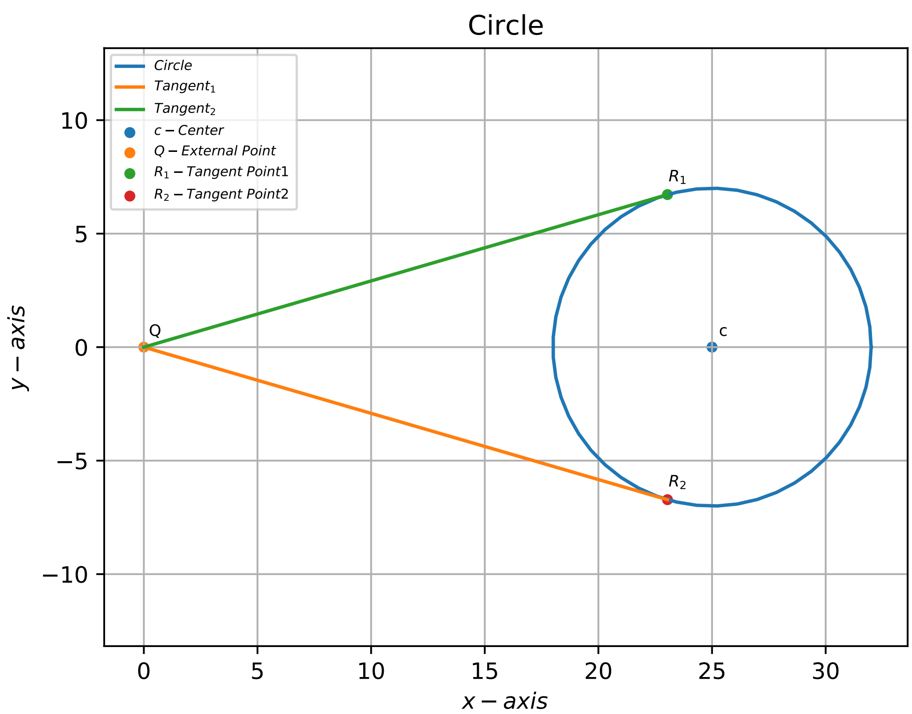

Figure 1