Quadratric Programming

1 12^{th} Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve $x^2 = 4y$ and passing through the point $(1, 2)$.

   Solution: The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{1}$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{3}$$

$$f = 0 \tag{4}$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{5}$$

This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad \mathrm{f}(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \tag{6}$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{7}$$

First we show that, (7) is not convex. Suppose $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfy $g(\mathbf{x}) = 0$. Then,

$$\mathbf{x_1}^\top\mathbf{V}\mathbf{x_1} + 2\mathbf{u}^\top\mathbf{x_1} + f = 0 \tag{8}$$

$$\mathbf{x_2}^\top\mathbf{V}\mathbf{x_2} + 2\mathbf{u}^\top\mathbf{x_2} + f = 0 \tag{9}$$

Then, for any $0 \leq \lambda \leq 1$, substituting

$$\mathbf{x}_\lambda \leftarrow \lambda \mathbf{x_1} + (1-\lambda)\,\mathbf{x_2} \tag{10}$$

into (7), we get

$$g\left(\mathbf{x}_\lambda\right) = \lambda\,(\lambda-1)\,(\mathbf{x_1}-\mathbf{x_2})^\top \mathbf{V}\,(\mathbf{x_1}-\mathbf{x_2}) + f \neq 0 \tag{11}$$

Hence, the optimization problem is nonconvex. The constraints throw an error when *cvxpy* is used.

We will use Lagrange multipliers method to find the optimum value. Define

$$H\left(\mathbf{x},\lambda\right) = f\left(\mathbf{x}\right) - \lambda g\left(\mathbf{x}\right) \tag{12}$$

and we find that

$$\nabla f\left(\mathbf{x}\right) = 2\left(\mathbf{x}-\mathbf{h}\right) \tag{13}$$

$$\nabla g\left(\mathbf{x}\right) = 2\left(\mathbf{V}\mathbf{x}+\mathbf{u}\right) \tag{14}$$

We have to find $\lambda \in \mathbb{R}$ such that

$$\nabla H\left(\mathbf{x},\lambda\right) = 0 \tag{15}$$

$$\implies 2\left(\mathbf{x}-\mathbf{h}\right) - 2\lambda\left(\mathbf{V}\mathbf{x}+\mathbf{u}\right) = 0 \tag{16}$$

$$\implies \mathbf{x}-\mathbf{h} = \lambda\left(\mathbf{V}\mathbf{x}+\mathbf{u}\right) \tag{17}$$

$$\implies \left(\mathbf{I}-\lambda\mathbf{V}\right)\mathbf{x} = \lambda\mathbf{u}+\mathbf{h} \tag{18}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \lambda\begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{19}$$

$$\implies \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1 \end{pmatrix}\mathbf{x} = \begin{pmatrix} 1 \\ -2\lambda+2 \end{pmatrix} \tag{20}$$

We have 2 cases to considers here.

(a) When $\lambda \neq 1$. Writing augmented matrix,

$$\begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1 & -2\lambda+2 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{R_1}{1-\lambda}} \begin{pmatrix} 1 & 0 & \frac{1}{1-\lambda} \\ 0 & 1 & -2\lambda+2 \end{pmatrix} \tag{21}$$

Then, we get

$$\mathbf{x}_m = \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} \tag{22}$$

Substituring this value in (7)

$$\begin{aligned}
\begin{pmatrix} \frac{1}{1-\lambda} & -2\lambda + 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} \\
+ 2 \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \end{pmatrix} = 0 \\
\implies \begin{pmatrix} \frac{1}{1-\lambda} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} - 4 \left(-2\lambda + 2\right) = 0 \\
\implies 8 \left(\lambda^3 - 3\lambda^2 + 3\lambda - 1\right) + 1 = 0 \\
\implies \left(\lambda^3 - 3\lambda^2 + 3\lambda - 1\right) = -\frac{1}{8} \\
\implies \left(\lambda - 1\right)^3 = -\frac{1}{8} \\
\implies \lambda - 1 = -\frac{1}{2} \\
\implies \lambda = \frac{1}{2} \quad (23)
\end{aligned}$$

Substituting the value of λ in (22)

$$\mathbf{x}_m = \mathbf{q} = \begin{pmatrix} \frac{1}{1-\frac{1}{2}} \\ -2\left(\frac{1}{2}\right) + 2 \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{25}$$

(b) When $\lambda = 1$.

$$(20) \implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{26}$$

This is an invalid solution.

Given the point of contact $\mathbf{q}$, the equation to the normal is given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \tag{27}$$

$$\implies \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix}\right)^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \tag{28}$$

$$\implies \begin{pmatrix} 2 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \tag{29}$$

$$\implies \begin{pmatrix} 2 & 2 \end{pmatrix}\left(\mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = 0 \tag{30}$$

$$\implies \begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 3 \tag{31}$$

The relevant figure is shown in 1

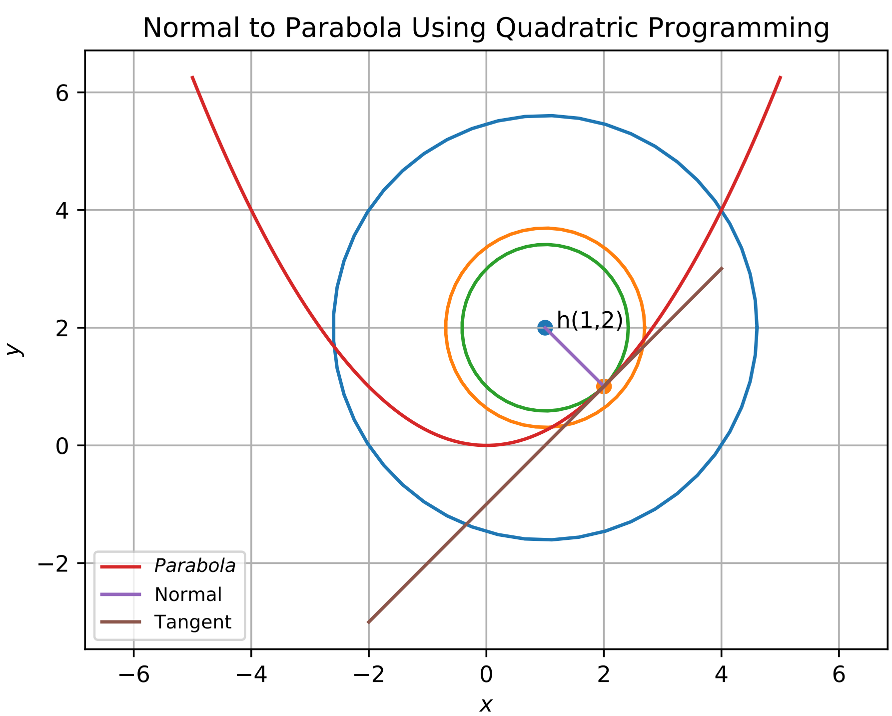

Figure 1