Chords

1 12^{th} Maths - Chapter 8

This is Problem-13 from Exercise 8.1

1. Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line $y = 3$.

   Solution:

2. The given equation of the curve can be rearranged as

$$y^2 - 4x = 0 \tag{1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} -2 & 0 \end{pmatrix} \mathbf{x} + 0 = 0 \tag{2}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3}$$

Comparing coefficients of both equations (2) and (3)

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

$$\mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \tag{5}$$

$$f = 0 \tag{6}$$

For the given line $y = 3$, the parameters are

$$\mathbf{h} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{7}$$

To calculate the point of contact of line with the conic, we use

$$\mu^2 \mathbf{m}^\top \mathbf{V}\mathbf{m} + 2\mu \mathbf{m}^\top \left(\mathbf{V}\mathbf{h} + \mathbf{u}\right) + g\left(\mathbf{h}\right) = 0 \tag{8}$$

$$\begin{aligned} g\left(\mathbf{h}\right) = \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \\ + 2\begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + 0 \\ \implies g\left(\mathbf{h}\right) = \begin{pmatrix} 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + 2\left(0\right) \\ \implies g\left(\mathbf{h}\right) = 9 \end{aligned} \tag{9}$$

$$\begin{aligned} (8) \implies \mu^2 \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ + 2\mu \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right) + 9 = 0 \\ \implies \mu^2\left(0\right) + 2\mu \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 9 = 0 \\ \implies -4\mu + 9 = 0 \\ \implies \mu = \frac{9}{4} \end{aligned} \tag{10}$$

The point of contact is given as

$$\mathbf{a}_0 = \begin{pmatrix} \frac{9}{4} \\ \\ 3 \end{pmatrix} \tag{11}$$

The desired area of the region is given as

$$\int_0^3 \frac{y^2}{4}\, dx = \frac{1}{12}\left[y^3\right]_0^3 \tag{12}$$

$$= \frac{1}{12}\left(27 - 0\right) \tag{13}$$

$$= \frac{9}{4} \text{ sq.units} \tag{14}$$

The relevant diagram is shown in Figure 1

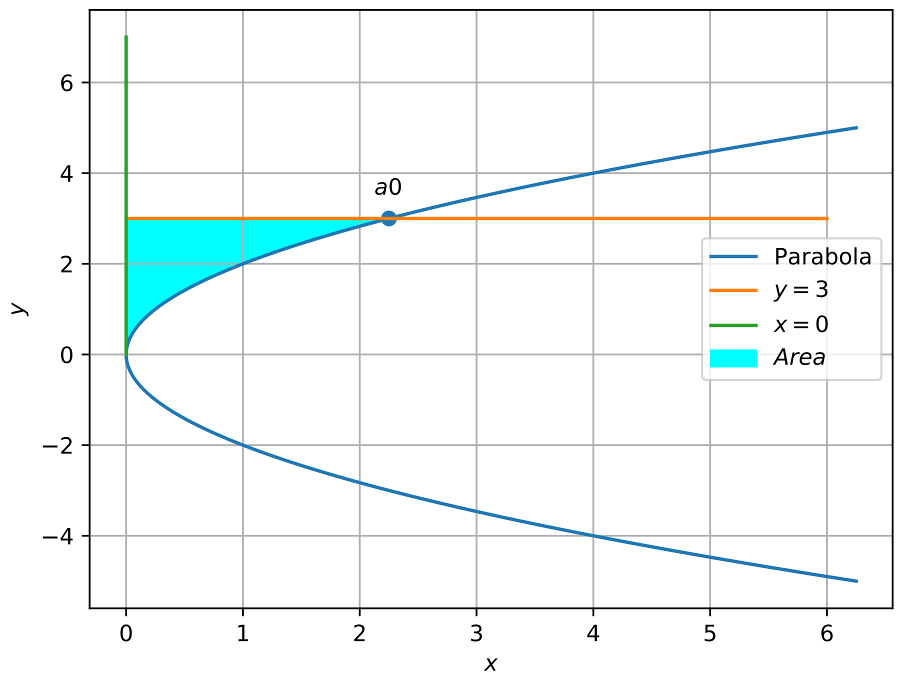

Figure 1