Properties of Quadrilaterals

1 10^{th} Maths - Chapter 7

This is Problem-8 from Exercise 7.4

1. ABCD is a rectangle formed by the points $\mathbf{A}(-1,-1), \mathbf{B}(-1,4), \mathbf{C}(5,4)$ and $\mathbf{D}(5,-1)$. $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ and $\mathbf{S}$ are the mid-points of $\mathbf{AB}, \mathbf{BC}, \mathbf{CD}$ and $\mathbf{DA}$ respectively. Is the quadrilateral PQRS a square? a rectangle? or a rhombus? Justify your answer.
   Solution: Refer figure 1

$$\mathbf{P} = \frac{1}{2}(\mathbf{A}+\mathbf{B}) = \frac{1}{2}\left(\begin{pmatrix}-1\\-1\end{pmatrix}+\begin{pmatrix}-1\\4\end{pmatrix}\right) = \begin{pmatrix}-1\\1\frac{1}{2}\end{pmatrix} \tag{1}$$

$$\mathbf{Q} = \frac{1}{2}(\mathbf{B}+\mathbf{C}) = \frac{1}{2}\left(\begin{pmatrix}-1\\4\end{pmatrix}+\begin{pmatrix}5\\4\end{pmatrix}\right) = \begin{pmatrix}2\\4\end{pmatrix} \tag{2}$$

$$\mathbf{R} = \frac{1}{2}(\mathbf{C}+\mathbf{D}) = \frac{1}{2}\left(\begin{pmatrix}5\\4\end{pmatrix}+\begin{pmatrix}5\\-1\end{pmatrix}\right) = \begin{pmatrix}5\\1\frac{1}{2}\end{pmatrix} \tag{3}$$

$$\mathbf{S} = \frac{1}{2}(\mathbf{D}+\mathbf{A}) = \frac{1}{2}\left(\begin{pmatrix}5\\-1\end{pmatrix}+\begin{pmatrix}-1\\-1\end{pmatrix}\right) = \begin{pmatrix}2\\-1\end{pmatrix} \tag{4}$$

$$\tag{5}$$

   We know that PQRS is a parallelogram. To know, if it is a rectangle, we need to ascertain whether any of the two adjacent sides are perpendicular. That means $(\mathbf{Q}-\mathbf{P})^{\top}(\mathbf{R}-\mathbf{Q})$ should be equal to zero.

$$\mathbf{Q}-\mathbf{P} = \begin{pmatrix}2\\4\end{pmatrix}-\begin{pmatrix}-1\\1\frac{1}{2}\end{pmatrix} = \begin{pmatrix}3\\2\frac{1}{2}\end{pmatrix} \tag{6}$$

$$\mathbf{R}-\mathbf{Q} = \begin{pmatrix}5\\1\frac{1}{2}\end{pmatrix}-\begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}3\\-2\frac{1}{2}\end{pmatrix} \tag{7}$$

$$(\mathbf{Q}-\mathbf{P})^{\top}(\mathbf{R}-\mathbf{Q}) = \begin{pmatrix}3 & 2\frac{1}{2}\end{pmatrix}\begin{pmatrix}3\\-2\frac{1}{2}\end{pmatrix} \neq 0 \tag{8}$$

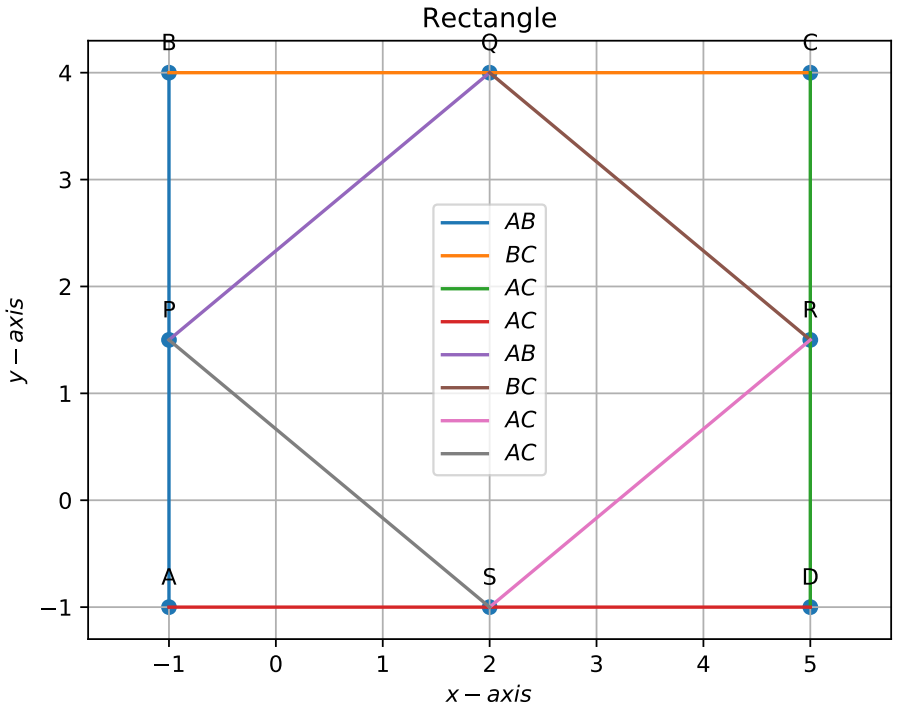

Figure 1

Therefore PQRS is not a rectangle. Let us check if it is a rhombus. For a rhombus, the diagonals bisect perpendicularly. That means $(\mathbf{R}-\mathbf{P})^{\top}(\mathbf{S}-\mathbf{Q})$ should be equal to zero.

$$\mathbf{R}-\mathbf{P}=\begin{pmatrix}5\\1\frac{1}{2}\end{pmatrix}-\begin{pmatrix}-1\\1\frac{1}{2}\end{pmatrix}=\begin{pmatrix}6\\0\end{pmatrix} \tag{9}$$

$$\mathbf{S}-\mathbf{Q}=\begin{pmatrix}2\\-1\end{pmatrix}-\begin{pmatrix}2\\4\end{pmatrix}=\begin{pmatrix}0\\-5\end{pmatrix} \tag{10}$$

$$(\mathbf{R}-\mathbf{P})^{\top}(\mathbf{S}-\mathbf{Q})=\begin{pmatrix}6 & 0\end{pmatrix}\begin{pmatrix}0\\-5\end{pmatrix}=0 \tag{11}$$

Therefore PQRS is a rhombus.