Gradient Descent

1 11^{th} Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   Solution: The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^{\top} \mathbf{x} + 8 = 0 \tag{1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3}$$

   Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

   Here, $\mathbf{A}$ is a point on the given line.

   Let $\mathbf{O}$ be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of the perpendicular. This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \left\| \mathbf{x} - \mathbf{O} \right\|^2 \tag{5}$$

$$\text{s.t.} \quad \mathbf{n}^T \mathbf{x} = c \tag{6}$$

We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{7}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{8}$$

$$\text{yielding } f\left(\lambda\right) = \frac{4}{3}\lambda^2 - 16\lambda + 64 \tag{9}$$

$$f'\left(\lambda\right) = \frac{8}{3}\lambda - 16 \tag{10}$$

Computing λ_{min} using Gradient Descent method:

$$\lambda_{n+1} = \lambda_n - \alpha \nabla f\left(\lambda_n\right) \tag{11}$$

Choosing

(a) $\alpha = 0.001$

(b) precision = 0.0000001

(c) n = 10000000

(d) λ_0 = -5

$$\lambda_{min} = 6 \tag{12}$$

Substituting the values of $\mathbf{A}$, $\mathbf{m}$ and λ_{min} in equation (4)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{13}$$

$$= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \tag{15}$$

$$OP = \left\|\mathbf{P} - \mathbf{O}\right\|^2 \tag{16}$$

$$= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \tag{17}$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{18}$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) \tag{19}$$

$$= \tan^{-1}\left(-\sqrt{3}\right) \tag{20}$$

$$= 120^\circ \tag{21}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{22}$$

The relevant figure is as shown in 1 and 2

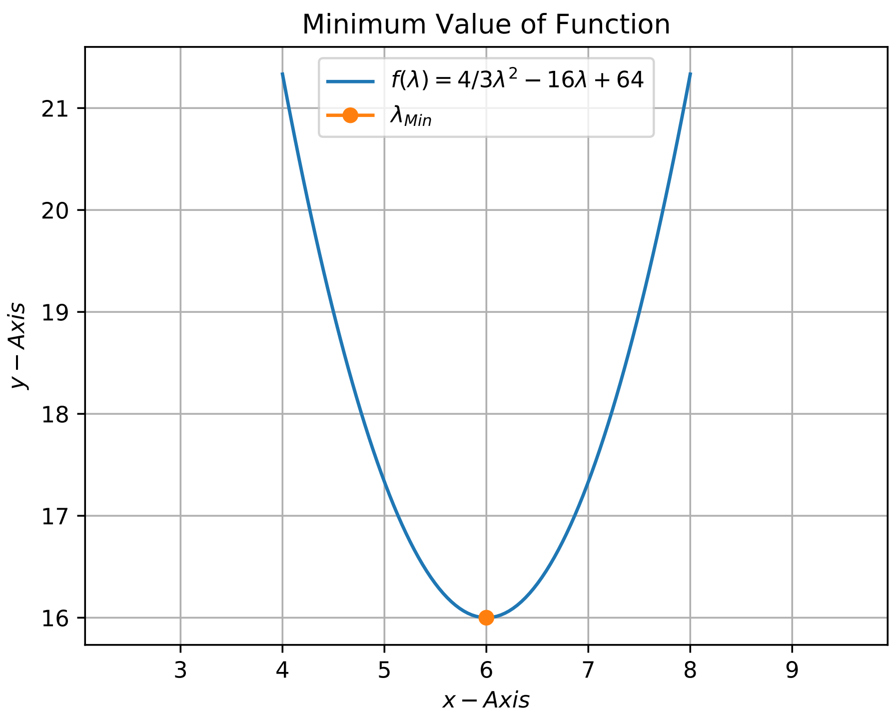
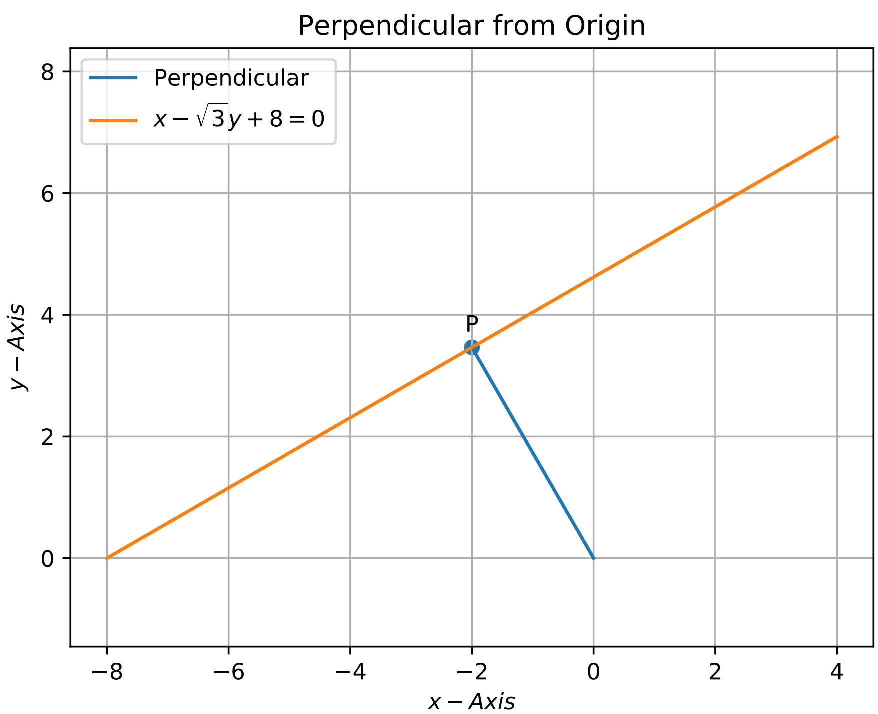


Figure 1



Perpendicular from Origin
Perpendicular
$x-\sqrt{3}y+8=0$
P
8
6
4
2
0
−8
−6
−4
−2
0
2
4
$x-Axis$
$y-Axis$

Figure 2