Tangents and Normals

1 12^{th} Maths - Chapter 6

This is Problem-2 from Exercise 6.3

1. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.
Solution: The given equation of the curve can be rearranged as

$$xy - x - 2y + 1 = 0 \tag{1}$$

$$\implies \mathbf{x}^\top \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} -1 & -2 \end{pmatrix} \mathbf{x} + 1 = 0 \tag{2}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{3}$$

Comparing coefficients of both equations (2) and (3)

$$\mathbf{V} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \tag{4}$$

$$\mathbf{u} = -\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \tag{5}$$

$$f = 1 \tag{6}$$

Given the point of contact $\mathbf{q}$, the equation of a tangent to (3) is

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{x} + \mathbf{u}^\top \mathbf{q} + f = 0 \tag{7}$$

For the given point of contact with $\mathbf{q}_x = 10$,

$$\mathbf{q}_y = \frac{10-1}{10-2} = \frac{9}{8} \tag{8}$$

$$\therefore \mathbf{q} = \begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} \tag{9}$$

$$(7) \implies \left(\begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\right)^\top \mathbf{x} - \begin{pmatrix} \frac{1}{2} & 1 \end{pmatrix}\begin{pmatrix} 10 \\ \frac{9}{8} \end{pmatrix} + 1 = 0 \tag{10}$$

$$\implies \left(\begin{pmatrix} \frac{9}{16} \\ 5 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}\right)^\top \mathbf{x} - \frac{41}{8} = 0 \tag{11}$$

$$\implies \begin{pmatrix} \frac{1}{16} & 4 \end{pmatrix}\mathbf{x} - \frac{41}{8} = 0 \tag{12}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ 64 \end{pmatrix} \tag{13}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{-1}{64} \end{pmatrix} \tag{14}$$

The relevant diagram is shown in Figure 1

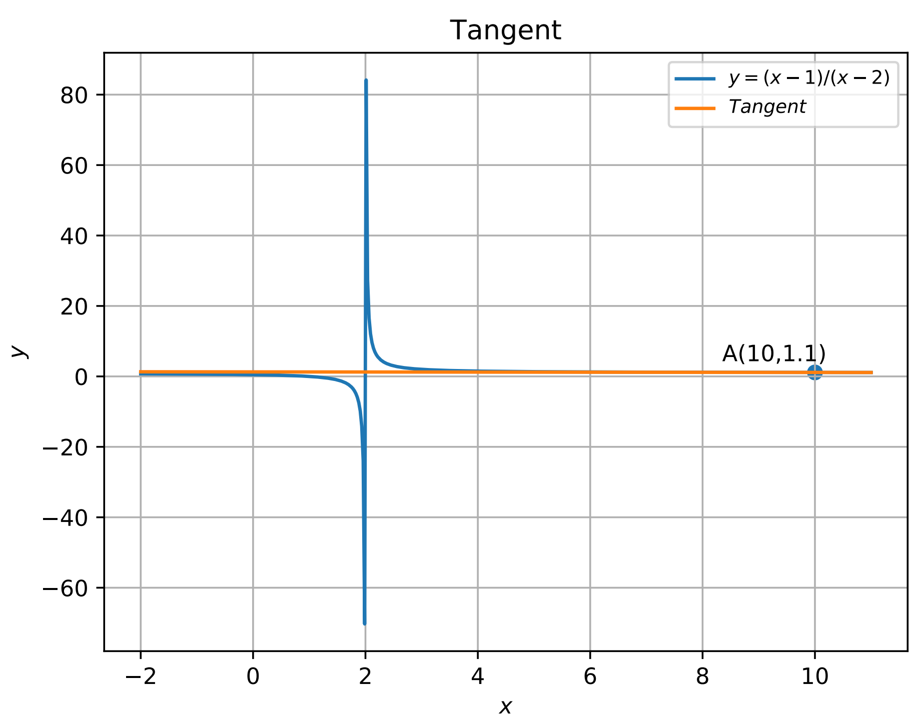

Figure 1