Optimization using SVD

1 JEE Maths - 65 C-1

This is Problem-30

1. Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{1}; z = 2$

   Solution: The given equation can be written as

$$\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-0}{1} \tag{1}$$

$$\frac{x+1}{5} = \frac{y-2}{1} = \frac{z-2}{0} \tag{2}$$

$$\implies \mathbf{A} = \mathbf{x}_1 + \lambda_1 \mathbf{m}_1 \tag{3}$$

$$\mathbf{B} = \mathbf{x}_2 + \lambda_2 \mathbf{m}_2 \tag{4}$$

   where

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \tag{5}$$

   Assume

$$\mathbf{M} = \begin{pmatrix} \mathbf{m}_1 & \mathbf{m}_2 \end{pmatrix} \tag{6}$$

$$\mathbf{B} - \mathbf{A} = (\lambda_2 \mathbf{m}_2 - \lambda_1 \mathbf{m}_1) + (\mathbf{x}_2 - \mathbf{x}_1) \tag{7}$$

$$= \mathbf{M}\boldsymbol{\lambda} + \mathbf{k} \tag{8}$$

$$\text{where } \boldsymbol{\lambda} \triangleq \begin{pmatrix} -\lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \mathbf{k} = (\mathbf{x}_2 - \mathbf{x}_1) = \begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} \tag{9}$$

We can formulate an unconstrained optimization problem as below:

$$\min_{\boldsymbol{\lambda}} \quad \|\mathbf{B}-\mathbf{A}\|^2 \tag{10}$$

Substituting (8) in (10)

$$(10) \implies \min_{\boldsymbol{\lambda}} \quad \|\mathbf{M}\boldsymbol{\lambda}+\mathbf{k}\|^2 \tag{11}$$

Using Singular Value Decomposition, $\mathbf{M}$ cn be written as

$$\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^\top \tag{12}$$

Computing SVD using python code, yields

$$\mathbf{U} = \begin{pmatrix} -0.903 & 0.424 & 0.072 \\ 0.42 & -0.834 & -0.358 \\ -0.092 & -0.353 & 0.931 \end{pmatrix} \tag{13}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} 5.858 & 0 \\ 0 & 2.383 \\ 0 & 0 \end{pmatrix} \tag{14}$$

$$\mathbf{V} = \begin{pmatrix} -0.539 & -0.842 \\ -0.842 & 0.539 \end{pmatrix} \tag{15}$$

Then, the solution for (11) is given by

$$\boldsymbol{\lambda}_{min} = \Sigma_{i=1}^{r} \frac{-\mathbf{u}_i^\top \mathbf{k}}{\sigma_i}\mathbf{v}_i \tag{16}$$

where r is rank of matrix $\mathbf{M}$.

Computing using python code,

$$\boldsymbol{\lambda}_{min} = \begin{pmatrix} -1.4 \\ 0.969 \end{pmatrix} \tag{17}$$

$$\mathbf{A} = \begin{pmatrix} 3.8 \\ 3.2 \\ 1.4 \end{pmatrix} \tag{18}$$

$$\mathbf{B} = \begin{pmatrix} 3.85 \\ 2.97 \\ 2 \end{pmatrix} \tag{19}$$

$$\|\mathbf{B}-\mathbf{A}\| = 0.6445 \text{ units} \tag{20}$$

The relevant figure is shown in 1.

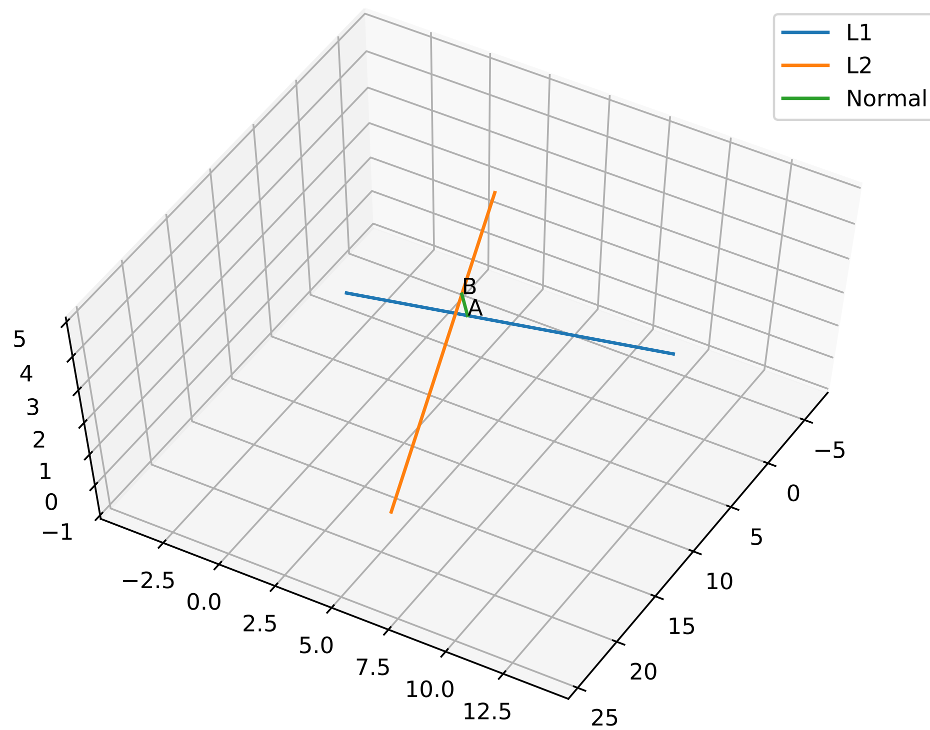

Figure 1