Conic Sections - Parbola

1 11^{th} Maths - Chapter 11

This is Problem-1 from Exercise 11.2

1. Find the coordinates of the focus, equation of the directrix, axis and length of the latus rectum of a parabola whose equation is given by $y^2 = 12x$.
   Solution: The given equation of the parabola can be rearranged as

   $$y^2 - 12x = 0 \tag{1}$$

   The above equation can be equated to the generic equation of conic sections

   $$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2}$$

   Comparing coefficients of both equations (1) and (2)

   $$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{3}$$

   $$\mathbf{u} = -\begin{pmatrix} 6 \\ 0 \end{pmatrix} \tag{4}$$

   $$f = 0 \tag{5}$$

   (a) From equation (3), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

   $$\lambda_1 = 0 \tag{6}$$

   $$\lambda_2 = 1 \tag{7}$$

The Eigen vector $\mathbf{p_1}$ corresponding to Eigen value λ_1 is computed as shown below

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

$$\mathbf{V} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 - \lambda_1 & 0 \\ 0 & 1 - \lambda_1 \end{pmatrix} \tag{9}$$

Substituting value of λ_1 from (6) in (9)

$$(9) \implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{10}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{11}$$

x_1 is free variable and $x_2 = 0$.

$$\therefore \mathbf{p_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

$$\mathbf{n} = \sqrt{\lambda_2} \mathbf{p_1} \tag{13}$$

$$= \sqrt{(}1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{15}$$

$$c = \frac{\|\mathbf{u}^2\| - \lambda_2 f}{2\mathbf{u}^\top \mathbf{n}} \tag{16}$$

Substituting values of $\mathbf{u}, \mathbf{n}, \lambda_2$ and f in (16)

$$c = \frac{6^2 - 1\,(0)}{-2 \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = -3 \tag{17}$$

$$\tag{18}$$

The focus $\mathbf{F}$ of parabola is expressed as

$$\mathbf{F} = \frac{ce^2\mathbf{n} - \mathbf{u}}{\lambda_2} \tag{19}$$

$$= \frac{-3\,(1)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \end{pmatrix}}{1} \tag{20}$$

$$= \begin{pmatrix} 3 \\ 0 \end{pmatrix} \tag{21}$$

(b) Equation of directrix is given as

$$\mathbf{n}^\top \mathbf{x} = c \tag{22}$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = -3 \tag{23}$$

(c) The equation for the axis of parabola passing through $\mathbf{F}$ and orthogonal to the directrix is given as

$$\mathbf{m}^\top (\mathbf{x} - \mathbf{F}) = 0 \tag{24}$$

where $\mathbf{m}$ is the normal vector to the axis and also the slope of the directrix.

$$\because \mathbf{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{25}$$

$$(24) \implies \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) \tag{26}$$

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \mathbf{x} = 0 \tag{27}$$

(d) The latus rectum of a parabola is given by

$$l = \frac{\eta}{\lambda_2} \tag{28}$$

$$= \frac{2\mathbf{u}^\top \mathbf{p_1}}{\lambda_2} \tag{29}$$

$$= \frac{2 \begin{pmatrix} 6 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{1} \tag{30}$$

$$= 12 \text{ units} \tag{31}$$

The relevant diagram is shown in Figure 1

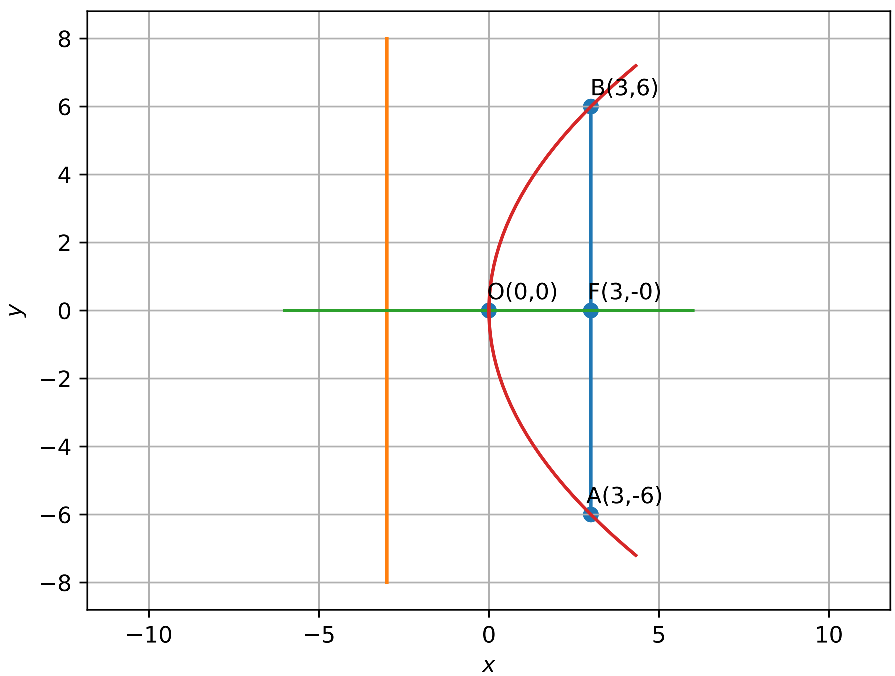

Figure 1