

# Quadratic Programming

## 1 12<sup>th</sup> Maths - Chapter 6

This is Problem-23 from Exercise 6.6

1. Find the equation of the normal to the curve  $x^2 = 4y$  and passing through the point  $(1, 2)$ .

**Solution:** The given equation of the curve can be written as

$$g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (1)$$

where

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (2)$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \quad (3)$$

$$f = 0 \quad (4)$$

We are given that

$$\mathbf{h} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (5)$$

This can be formulated as optimization problem as below:

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) = \|\mathbf{x} - \mathbf{h}\|^2 \quad (6)$$

$$\text{s.t.} \quad g(\mathbf{x}) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (7)$$

First we show that, (7) is not convex. Suppose  $\mathbf{x}_1$  and  $\mathbf{x}_2$  satisfy  $g(\mathbf{x}) = 0$ . Then,

$$\mathbf{x}_1^T \mathbf{V} \mathbf{x}_1 + 2\mathbf{u}^T \mathbf{x}_1 + f = 0 \quad (8)$$

$$\mathbf{x}_2^T \mathbf{V} \mathbf{x}_2 + 2\mathbf{u}^T \mathbf{x}_2 + f = 0 \quad (9)$$

Then, for any  $0 \leq \lambda \leq 1$ , substituting

$$\mathbf{x}_\lambda \leftarrow \lambda \mathbf{x}_1 + (1 - \lambda) \mathbf{x}_2 \quad (10)$$

into (7), we get

$$g(\mathbf{x}_\lambda) = \lambda(\lambda - 1)(\mathbf{x}_1 - \mathbf{x}_2)^\top \mathbf{V}(\mathbf{x}_1 - \mathbf{x}_2) + f \neq 0 \quad (11)$$

Hence, the optimization problem is nonconvex. The constraints throw an error when *cvxpy* is used.

We will use Lagrange multipliers method to find the optimum value. Define

$$H(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda g(\mathbf{x}) \quad (12)$$

and we find that

$$\nabla f(\mathbf{x}) = 2(\mathbf{x} - \mathbf{h}) \quad (13)$$

$$\nabla g(\mathbf{x}) = 2(\mathbf{V}\mathbf{x} + \mathbf{u}) \quad (14)$$

We have to find  $\lambda \in \mathbb{R}$  such that

$$\nabla H(\mathbf{x}, \lambda) = 0 \quad (15)$$

$$\implies 2(\mathbf{x} - \mathbf{h}) - 2\lambda(\mathbf{V}\mathbf{x} + \mathbf{u}) = 0 \quad (16)$$

$$\implies \mathbf{x} - \mathbf{h} = \lambda(\mathbf{V}\mathbf{x} + \mathbf{u}) \quad (17)$$

$$\implies (\mathbf{I} - \lambda\mathbf{V})\mathbf{x} = \lambda\mathbf{u} + \mathbf{h} \quad (18)$$

$$\implies \begin{pmatrix} 1 - \lambda & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \lambda \begin{pmatrix} 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad (19)$$

$$\implies \begin{pmatrix} 1 - \lambda & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ -2\lambda + 2 \end{pmatrix} \quad (20)$$

We have 2 cases to consider here.

(a) When  $\lambda \neq 1$ . Writing augmented matrix,

$$\left( \begin{array}{ccc|c} 1 - \lambda & 0 & 1 & 1 \\ 0 & 1 & -2\lambda + 2 & -2\lambda + 2 \end{array} \right) \xrightarrow{R_1 \leftarrow \frac{R_1}{1 - \lambda}} \left( \begin{array}{ccc|c} 1 & 0 & \frac{1}{1 - \lambda} & \frac{1}{1 - \lambda} \\ 0 & 1 & -2\lambda + 2 & -2\lambda + 2 \end{array} \right) \quad (21)$$

Then, we get

$$\mathbf{x}_m = \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} \quad (22)$$

Substituting this value in (7)

$$\begin{aligned} & \begin{pmatrix} \frac{1}{1-\lambda} & -2\lambda + 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} \\ & \quad + 2 \begin{pmatrix} 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \end{pmatrix} = 0 \\ \implies & \begin{pmatrix} \frac{1}{1-\lambda} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{1-\lambda} \\ -2\lambda + 2 \end{pmatrix} - 4(-2\lambda + 2) = 0 \\ \implies & 8(\lambda^3 - 3\lambda^2 + 3\lambda - 1) + 1 = 0 \\ \implies & (\lambda^3 - 3\lambda^2 + 3\lambda - 1) = -\frac{1}{8} \\ \implies & (\lambda - 1)^3 = -\frac{1}{8} \\ \implies & \lambda - 1 = -\frac{1}{2} \\ \implies & \lambda = \frac{1}{2} \quad (23) \end{aligned}$$

Substituting the value of  $\lambda$  in (22)

$$\mathbf{x}_m = \mathbf{q} = \begin{pmatrix} \frac{1}{1-\frac{1}{2}} \\ -2\left(\frac{1}{2}\right) + 2 \end{pmatrix} \quad (24)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (25)$$

(b) When  $\lambda = 1$ .

$$(20) \implies \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (26)$$

This is an invalid solution.

Given the point of contact  $\mathbf{q}$ , the equation to the normal is given by

$$(\mathbf{V}\mathbf{q} + \mathbf{u})^\top \mathbf{R}(\mathbf{x} - \mathbf{q}) = 0 \quad (27)$$

$$\Rightarrow \left( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right)^\top \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (28)$$

$$\Rightarrow (2 \quad -2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left( \mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (29)$$

$$\Rightarrow (2 \quad 2) \left( \mathbf{x} - \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = 0 \quad (30)$$

$$\Rightarrow (1 \quad 1) \mathbf{x} = 3 \quad (31)$$

The relevant figure is shown in 1

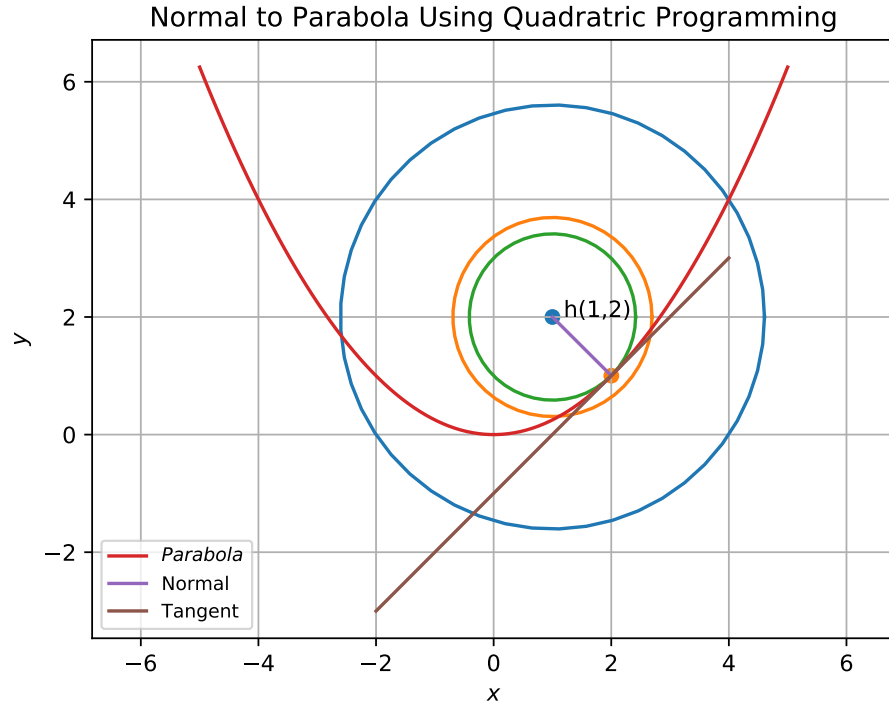


Figure 1