

Convex Optimization

1 11th Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

Solution: The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^T \mathbf{x} + 8 = 0 \quad (1)$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (2)$$

$$\Rightarrow \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (3)$$

Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \quad (4)$$

Here, \mathbf{A} is a point on the given line.

Let \mathbf{O} be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from \mathbf{O} to the line. Let \mathbf{P} be the foot of the perpendicular. This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 \quad (5)$$

$$\text{s.t. } \mathbf{n}^T \mathbf{x} = c \quad (6)$$

(a) Using parametric equation:

Substituting (4) in (5)

$$\implies \min_{\lambda} \|\mathbf{A} + \lambda \mathbf{m} - \mathbf{O}\|^2 \quad (7)$$

$$\implies \min_{\lambda} \|\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})\|^2 \quad (8)$$

$$\implies f(\lambda) = [\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})]^\top [\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})] \quad (9)$$

$$= [\lambda \mathbf{m}^\top + (\mathbf{A} - \mathbf{O})^\top] [\lambda \mathbf{m} + (\mathbf{A} - \mathbf{O})] \quad (10)$$

$$= \lambda^2 \|\mathbf{m}\|^2 + \lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \lambda \mathbf{m}^\top (\mathbf{A} - \mathbf{O}) + \|\mathbf{A} - \mathbf{O}\|^2 \quad (11)$$

$$= \lambda^2 \|\mathbf{m}\|^2 + 2\lambda (\mathbf{A} - \mathbf{O})^\top \mathbf{m} + \|\mathbf{A} - \mathbf{O}\|^2 \quad (12)$$

\therefore the coefficient of $\lambda^2 > 0$, equation (12) is a convex function.

$$f'(\lambda) = 2\lambda \|\mathbf{m}\|^2 + 2(\mathbf{A} - \mathbf{O})^\top \mathbf{m} \quad (13)$$

$$f''(\lambda) = 2\|\mathbf{m}\|^2 \quad (14)$$

$$\therefore f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \quad (15)$$

$$f'(\lambda_{min}) = 2\lambda_{min} \|\mathbf{m}\|^2 + 2(\mathbf{A} - \mathbf{O})^\top \mathbf{m} = 0 \quad (16)$$

$$\lambda_{min} = -\frac{(\mathbf{A} - \mathbf{O})^\top \mathbf{m}}{\|\mathbf{m}\|^2} \quad (17)$$

We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \quad (18)$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (19)$$

Substituting the values of \mathbf{A} , \mathbf{O} and \mathbf{m} in equation (17)

$$\lambda_{min} = -\frac{\left(\begin{pmatrix} -8 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)^\top \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}\right\|^2} \quad (20)$$

$$= \frac{(8 \ 0) \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}}{\frac{4}{3}} \quad (21)$$

$$= 6 \quad (22)$$

Substituting this value in equation (4)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \quad (23)$$

$$= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} \quad (24)$$

$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \quad (25)$$

$$OP = \|\mathbf{P} - \mathbf{O}\|^2 \quad (26)$$

$$= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \quad (27)$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \quad (28)$$

(b) Solving using cvxpy, with

$$\mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \quad (29)$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (30)$$

$$c = -8 \quad (31)$$

$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{O}\|^2 = 4, \mathbf{x}_{min} = \begin{pmatrix} -2 \\ 3.46 \end{pmatrix} \quad (32)$$

The angle θ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left(\frac{2\sqrt{3}}{-2} \right) \quad (33)$$

$$= \tan^{-1} \left(-\sqrt{3} \right) \quad (34)$$

$$= 120^\circ \quad (35)$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \quad (36)$$

The relevant figures are shown in 1 and 2

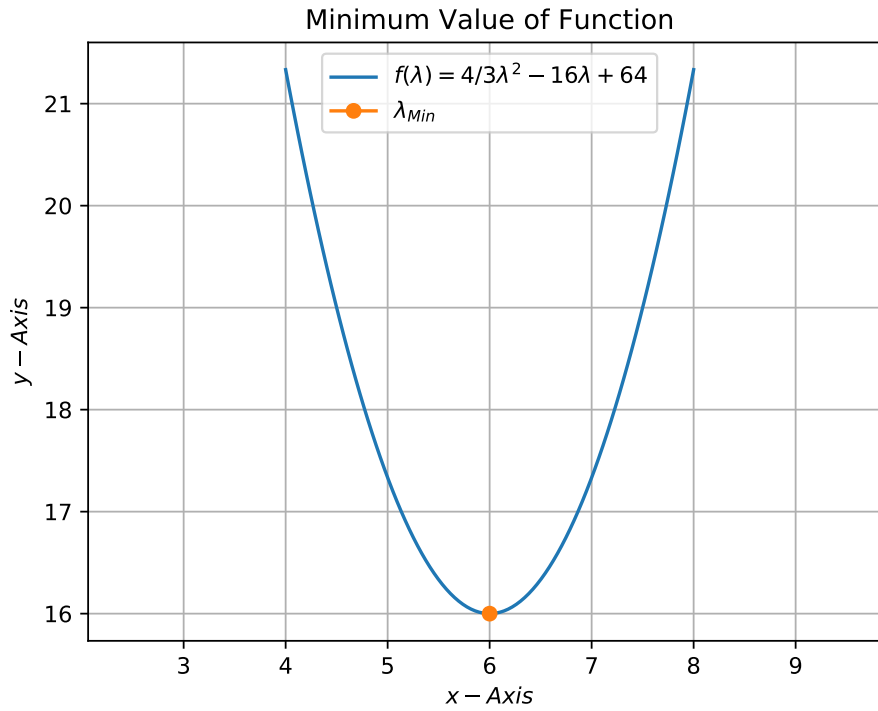


Figure 1

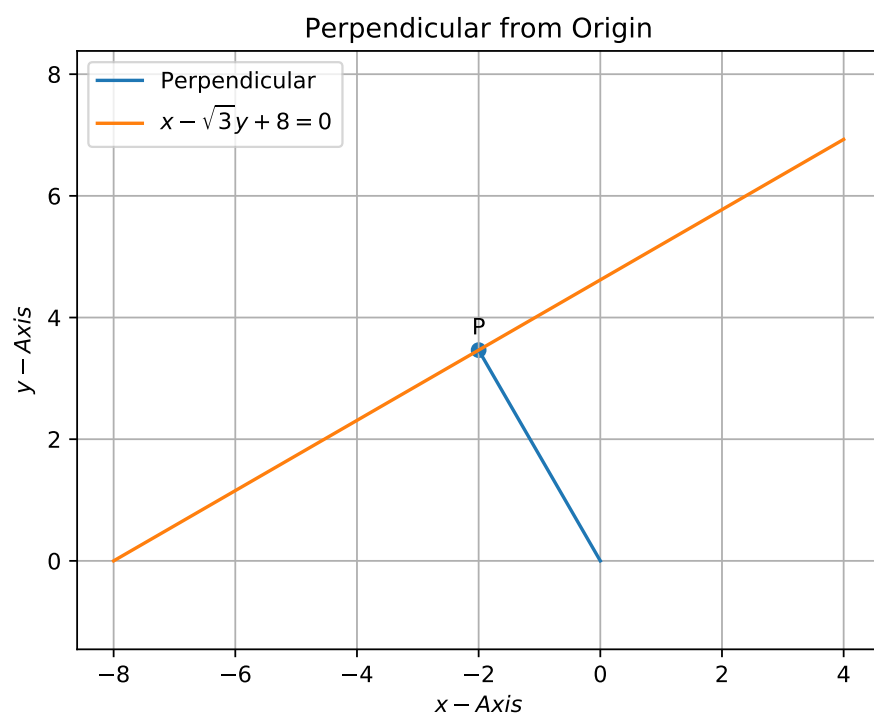


Figure 2