Conic Sections - Hyperbola

1 11^{th} Maths - Chapter 11

This is Problem-1 from Exercise 11.4

1. Find the coordinates of the focii, the vertices, the eccentricity and the length of the latus rectum of a hyperbola whose equation is given by $\frac{x^2}{16} - \frac{y^2}{9} = 1$.
Solution: From the given equation of hyperbola, we can get values of

$$a = 4 \tag{1}$$
$$b = 3 \tag{2}$$

The given equation of the hyperbola can be rearranged as

$$9x^2 - 16y^2 - 144 = 0 \tag{3}$$
$$\tag{4}$$

The above equation can be equated to the generic equation of conic sections

$$g(\mathbf{x}) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{5}$$

Comparing coefficients of both equations (3) and (5)

$$\mathbf{V} = \begin{pmatrix} 9 & 0 \\ 0 & -16 \end{pmatrix} \tag{6}$$
$$\mathbf{u} = 0 \tag{7}$$
$$f = -144 \tag{8}$$

From equation (6), since $\mathbf{V}$ is already diagonalized, the Eigen values λ_1 and λ_2 are given as

$$\lambda_1 = 9 \tag{9}$$
$$\lambda_2 = -16 \tag{10}$$

(a) The eccentricity of the hyperbola is given as

$$e = \sqrt{1 - \frac{\lambda_1}{\lambda_2}} \tag{11}$$

$$= \sqrt{1 - \frac{9}{-16}} \tag{12}$$

$$= \frac{5}{4} \tag{13}$$

(b) For the standard hyperbola, the coordinates of Focii are given as:

$$\mathbf{F} = \pm \frac{\left(\frac{1}{e\sqrt{1-e^2}}\right)\left(e^2\right)\sqrt{\frac{\lambda_2}{f_0}}}{\frac{\lambda_2}{f_0}}\mathbf{e1} \tag{14}$$

where

$$f_0 = -f \tag{15}$$

$$(14) \implies = \pm \frac{\left(\frac{1}{\frac{5}{4}\sqrt{1-\frac{25}{16}}}\right)\left(\frac{25}{16}\right)\sqrt{\frac{-16}{144}}\mathbf{e_1}}{\frac{-16}{144}} \tag{16}$$

$$= \pm \begin{pmatrix} 5 \\ 0 \end{pmatrix} \tag{17}$$

(c) The vertices of the hyperbola are given by

$$\pm \begin{pmatrix} a \\ 0 \end{pmatrix} \tag{18}$$

$$= \pm \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{19}$$

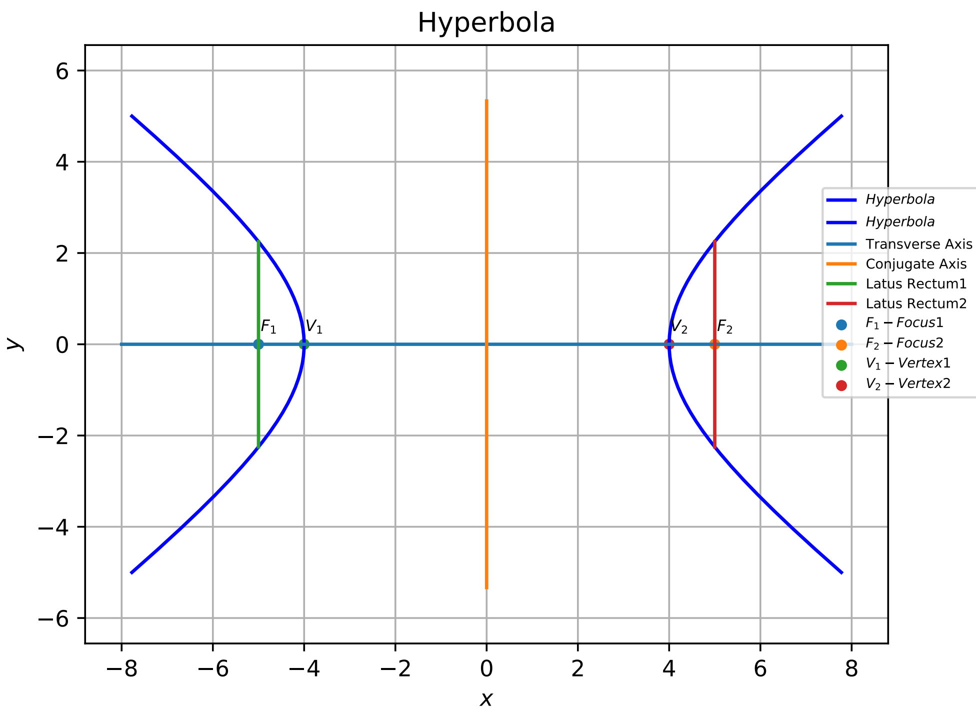
(d) The length of the latus rectum is given as

$$2\frac{\sqrt{|f_0 \lambda_1|}}{\lambda_2} \tag{20}$$

$$= 2\frac{\sqrt{|144\,(9)|}}{-16} \tag{21}$$

$$= \frac{9}{2} \tag{22}$$

as length can't be negative.
The relevant diagram is shown in Figure 1



Hyperbola
6
4
2
0
−2
−4
−6
−8
−6
−4
−2
0
2
4
6
8
x
y
F_1
V_1
V_2
F_2
Hyperbola
Hyperbola
Transverse Axis
Conjugate Axis
Latus Rectum1
Latus Rectum2
$F_1 - Focus1$
$F_2 - Focus2$
$V_1 - Vertex1$
$V_2 - Vertex2$

Figure 1