# Convex Optimization

## 1 $11^{th}$ Maths - Chapter 10

This is Problem-3.1 from Exercise 10.3

1. Reduce $x - \sqrt{3}y + 8 = 0$ into normal form. Find its perpendicular distance from the origin and angle between perpendicular and the positive x-axis.

   **Solution:** The given equation can be written as

$$\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}^\top \mathbf{x} + 8 = 0 \tag{1}$$

$$\implies \mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{2}$$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{3}$$

   Equation (1) can be represented in parametric form as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m} \tag{4}$$

   Here, $\mathbf{A}$ is a point on the given line.

   Let $\mathbf{O}$ be the point from where we have to find the perpendicular distance. The perpendicular distance will be the minimum distance from $\mathbf{O}$ to the line. Let $\mathbf{P}$ be the foot of the perpendicular. This problem can be formulated as an optimization problem as follow:

$$\min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{O}\right\|^2 \tag{5}$$

$$\text{s.t.} \quad \mathbf{n}^T\mathbf{x} = c \tag{6}$$

(a) Using parateric equation:

$$(5) \implies \min_{\lambda} \left\|\mathbf{A} + \lambda\mathbf{m} - \mathbf{O}\right\|^2 \tag{7}$$

$$\implies \min_{\lambda} \left\|\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right\|^2 \tag{8}$$

$$\implies f(\lambda) = \left[\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right]^{\top} \left[\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right] \tag{9}$$

$$= \left[\lambda\mathbf{m}^{\top} + (\mathbf{A} - \mathbf{O})^{\top}\right] \left[\lambda\mathbf{m} + (\mathbf{A} - \mathbf{O})\right] \tag{10}$$

$$= \lambda^2 \left\|\mathbf{m}\right\|^2 + \lambda (\mathbf{A} - \mathbf{O})^{\top} \mathbf{m} + \lambda\mathbf{m}^{\top} (\mathbf{A} - \mathbf{O}) + \left\|\mathbf{A} - \mathbf{O}\right\|^2 \tag{11}$$

$$= \lambda^2 \left\|\mathbf{m}\right\|^2 + 2\lambda (\mathbf{A} - \mathbf{O})^{\top} \mathbf{m} + + \left\|\mathbf{A} - \mathbf{O}\right\|^2 \tag{12}$$

$\because$ the coefficient of $\lambda^2 > 0$, equation (12) is a convex function.

$$f'(\lambda) = 2\lambda \left\|\mathbf{m}\right\|^2 + 2 (\mathbf{A} - \mathbf{O})^{\top} \mathbf{m} \tag{13}$$

$$f''(\lambda) = 2 \left\|\mathbf{m}\right\|^2 \tag{14}$$

$$\because f''(\lambda) > 0, f'(\lambda_{min}) = 0, \text{ for } \lambda_{min} \tag{15}$$

$$f'(\lambda_{min}) = 2\lambda_{min} \left\|\mathbf{m}\right\|^2 + 2 (\mathbf{A} - \mathbf{O})^{\top} \mathbf{m} = 0 \tag{16}$$

$$\lambda_{min} = -\frac{(\mathbf{A} - \mathbf{O})^{\top} \mathbf{m}}{\left\|\mathbf{m}\right\|^2} \tag{17}$$

We choose

$$\mathbf{A} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} \tag{18}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{19}$$

Substituting the values of **A**, **O** and **m** in equation (17)

$$\lambda_{min} = -\frac{\left(\begin{pmatrix} -8 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)^{\top} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}}{\left\|\begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}\right\|^2} \tag{20}$$

$$= \frac{\begin{pmatrix} 8 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix}}{\frac{4}{3}} \tag{21}$$

$$= 6 \tag{22}$$

Substituring this value in equation (4)

$$\mathbf{x}_{min} = \mathbf{P} = \begin{pmatrix} -8 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 1 \\ \frac{1}{\sqrt{3}} \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} -8 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ \frac{6}{\sqrt{3}} \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} \tag{25}$$

$$OP = \left\|\mathbf{P} - \mathbf{O}\right\|^2 \tag{26}$$

$$= \left\| \begin{pmatrix} -2 \\ 2\sqrt{3} \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\| \tag{27}$$

$$= \sqrt{2^2 + 12} = \sqrt{16} = 4 \tag{28}$$

(b) Solving using cvxpy, with

$$\mathbf{n} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} \tag{29}$$

$$\mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{30}$$

$$c = -8 \tag{31}$$

$$\min_{\mathbf{x}} \left\|\mathbf{x} - \mathbf{O}\right\|^2 = 4, \mathbf{x} = \begin{pmatrix} -2 \\ 3.46 \end{pmatrix} \tag{32}$$

The angle $\theta$ made by this perpendicular with x-axis is given by

$$\theta = \tan^{-1} \left( \frac{2\sqrt{3}}{-2} \right) \tag{33}$$

$$= \tan^{-1} \left( -\sqrt{3} \right) \tag{34}$$

$$= 120^\circ \tag{35}$$

The normal form of equation for straight line is given by

$$\begin{pmatrix} \cos 120^\circ \\ \sin 120^\circ \end{pmatrix}^\top \mathbf{x} = 4 \tag{36}$$

The relevant figures are shown in 1 and 2

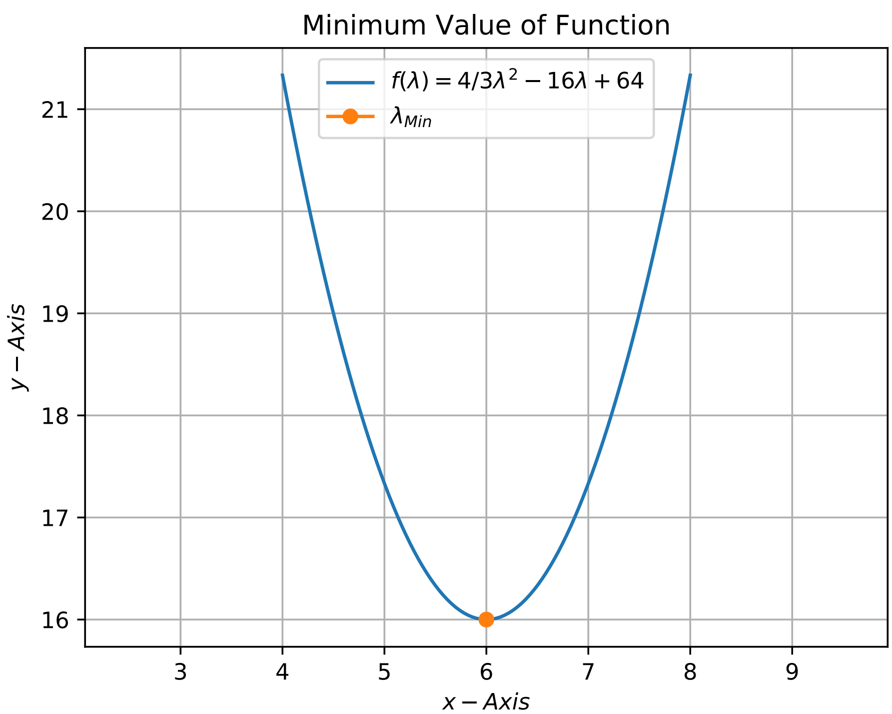
Minimum Value of Function
$f(\lambda) = 4/3\lambda^2 - 16\lambda + 64$
$\lambda_{Min}$
21
20
19
18
17
16
3
4
5
6
7
8
9
$x - Axis$
$y - Axis$

Figure 1

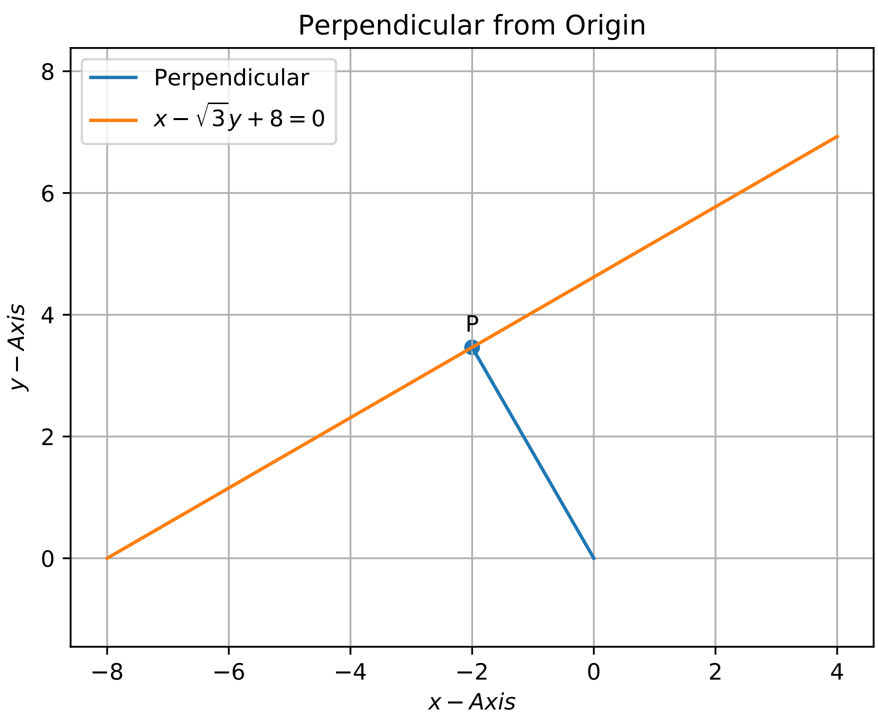
Perpendicular from Origin
Perpendicular
$x - \sqrt{3}y + 8 = 0$
P
8
6
4
2
0
−8
−6
−4
−2
0
2
4
$x - Axis$
$y - Axis$

Figure 2