23/3/21

2.20 to 4.30

Unit - III :-

Continous probability Dis^n and Bivariate data

Continous prob Dis^n
- N.D
- E.D
- Unif Dis^n

} Mid - I

Bivariate data
- Mdf
- property Sum & probt

Exponential Distribution / Negative Exponential

Reliability of item

Def:- If 'x' is a c.r.v defined in the interval $[0, \infty)$ with the parameter $(\theta > 0)$, and probability density function is defined as

$$E(x; \theta > 0) = f(x) = \begin{cases} \theta e^{-\theta x} & \theta > 0 \\ & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Cumulative distribution function [CdF]

$$P(X \le x) = F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{0}^{x} f(x)\, dx$$

$$= \int_{0}^{x} \theta\, e^{-\theta x}\, dx$$

$$= \left. \theta \frac{e^{-\theta x}}{-\theta} \right]_{0}^{x}$$

$$= \left. \frac{\cancel{\theta}}{-\cancel{\theta}}\, e^{-\theta x} \right]_{0}^{x}$$

$$= -e^{-\theta x} + 1$$

$$\boxed{P(X \le x) = F(x) = 1 - e^{-\theta x}}$$

Derive Moment Generating function and compute Exp and Variance ?

Proof:- $M_x(t) = E[e^{tx}]$

$= \int_0^\infty e^{tx} \cdot f(x)\, dx$

$= \int_0^\infty e^{tx} \cdot \theta e^{-\theta x}\, dx$

$= \theta \int_0^\infty e^{tx} \cdot e^{-\theta x}\, dx$

$= \theta \int_0^\infty e^{tx - \theta x}\, dx$

$= \theta \int_0^\infty e^{-x(\theta - t)}\, dx$

$= \theta \left[\frac{e^{-x(\theta - t)}}{-(\theta - t)} \right]_0^\infty$

$= \theta \left[0 + \frac{1}{\theta - t} \right] = \frac{\theta}{\theta - t}$

$$M_x(t) = \frac{\theta}{\theta - t}$$

$$= \frac{\cancel{\theta}}{\cancel{\theta}(1 - t/\theta)}$$

$$\boxed{M_x(t) = (1 - t/\theta)^{-1}}$$

$|t| < \theta$

Expectation & Variance:-

$$\mu_r' = E(x^r) = \frac{d^r}{dt^r} M_x(t) \Big]_{t=0}$$

Put $(r = 1)$

$$\mu_1' = E(x) = \text{Mean} = \frac{d}{dt} M_x(t) \Big]_{t=0}$$

$$= \frac{d}{dt}[1 - t/\theta)^{-1}$$

$$= (-1)(1 - t/\theta)^{-2} \cdot (-1/\theta)$$

$$\mu_1' = E(x) = (+1)(1-t/\theta)^{-2}(1/\theta)$$

$$= \frac{1}{\theta}(1-t/\theta)^{-2}\Big]_{t=0}$$

$$= 1/\theta\,(1-0)^{-2} = 1/\theta$$

$$\boxed{\mu_1' = E(x) = M = 1/\theta} \Rightarrow \boxed{\mu_1' = E(x) = \frac{1}{\text{Parameter}}}$$

$\therefore$ Mean is reciprocal of Parameter (Vice Versa) put $r=2$

$$\mu_2' = E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big]_{t=0}$$

$$= \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]$$

$$= \frac{d}{dt}\left[\frac{1}{\theta}\left(1-t/\theta\right)^{-2}\right]$$

$$= \frac{1}{\theta}(-2)\left(1-t/\theta\right)^{-3}\left(-1/\theta\right)$$

$$= \frac{2}{\theta^2}\left(1-t/\theta\right)^{-3}\Big]_{t=0}$$

$$\mu_2' = 2/\theta^2 (1-0)^{-3}$$

$$\boxed{\mu_2' = E(x^2) = 2/\theta^2}$$

$$Var(x) = \mu_2' - (\mu_1')^2$$

$$= 2/\theta^2 - (1/\theta)^2$$

$$\boxed{Var(x) = 1/\theta^2}$$

Cumulants:-

$$k_1 = E(x) = \mu_1' = \text{coefficient } \frac{t^1}{1!} = 1/\theta$$

$$k_2 = Var(x) = \mu_2 = \text{coeff } \frac{t^2}{2!} = 1/\theta^2$$

$$k_3 = \mu_3 = \text{coefficient } \frac{t^3}{3!} = 2/\theta^3$$

$$k_4 = \text{coefficient } \frac{t^4}{4!} = 6/\theta^4$$

$$\mu_4 = k_4 + 3k_2^2$$

$$= 6/\theta^4 + 3(1/\theta^4) = 9/\theta^4$$

$\therefore$ Exponential Dis^n, Mean & Variance are

$$\boxed{\begin{aligned} E(x) &= \text{mean} = 1/\theta \\ Var(x) &= \mu_2 = 1/\theta^2 \end{aligned}}$$

Cumulative Generating function [C.G.F]

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots$$

$$= -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} \cdots\right)$$

$$K_x(t) = \log_e M_x(t)$$

$$= \log_e (1 - t/\theta)^{-1}$$

$$= (-1) \log_e (1 - t/\theta)$$

$$= (-1) * -\left(t/\theta + \frac{1}{2}(t/\theta)^2 + 1/3\,(t/\theta)^3 + 1/4\,(t/\theta)^4 + \cdots\right]$$

$$= \frac{1}{1}(t/\theta)^1 + \frac{1}{2}(t/\theta)^2 + 1/3\,(t/\theta)^3 + 1/4\,(t/\theta)^4 + \cdots$$

$$= \frac{t^1}{1}(1/\theta) + \frac{t^2}{2}(1/\theta^2) + \frac{t^3}{3}(1/\theta^3) + \frac{t^4}{4}(1/\theta^4) + \cdots$$

Coefficient of Kurtosis:-

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9/\theta^4}{(1/\theta^2)^2} = \frac{9/\cancel{\theta^4}}{1/\cancel{\theta^4}} = 9 > 3$$

$$\gamma_2 = \beta_2 - 3 = 9 - 3 = 6 > 0$$

Exponential probability curve is Lepto kurtic

$(\beta_2 > 3, \ \gamma_2 > 0)$

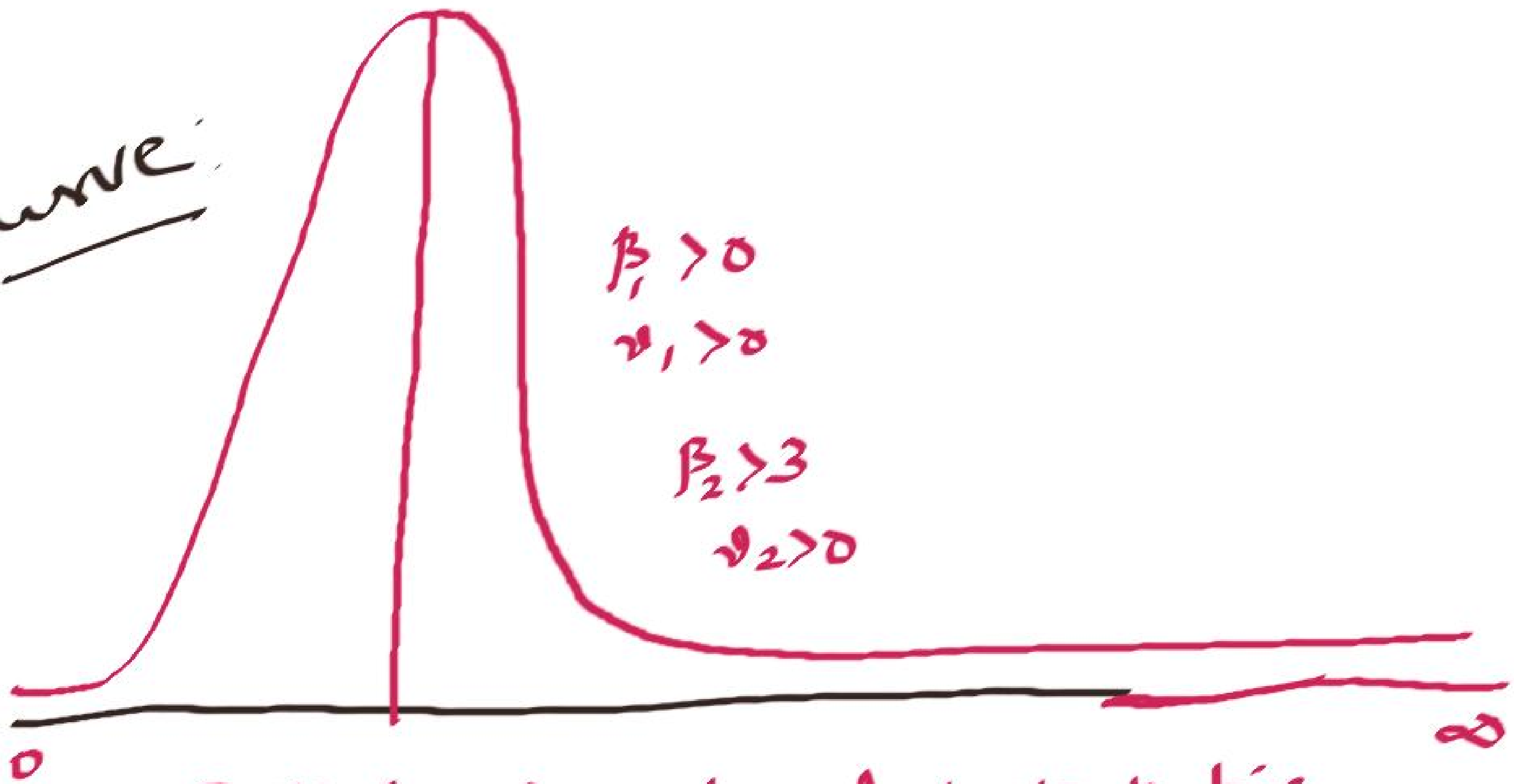
Coefficient of Skewness:-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2/\theta^3)^2}{(1/\theta^2)^3} = 4/\theta^6 \times \frac{\theta^6}{1} = 4$$

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{4} = 2$$

$\therefore$ Exponential r.v probability curve is positively skewed

Exponential curve:



Positively skewed and Leptokutic

(1) Average service time of the customer is 10 mins, find the prob at least two customers are served at given point of time ?

Sol:- $E(x) = 1/\theta = 10 \Rightarrow \theta = 1/10$

$$f(x) = \frac{1}{10} e^{-x/10}$$

$$P(x \geq 2) = \int_2^\infty f(x)\,dx$$

$$= \int_2^\infty \frac{1}{10} e^{-x/10}\,dx$$

$$= \cancel{\tfrac{1}{10}} \left.\frac{e^{-x/10}}{-\cancel{1/10}}\right]_2^\infty$$

$$= 0 + e^{-2/10}$$

$$= e^{-1/5} = e^{-0.2} = 0.818$$

Uniform Dis^n (or) Rectangular Dis^n

Def:- If 'x' is a Uniform random variable defined in the interval (a, b) $(a<b)$ $(a, b>0)$ then probability density function is defined as

$$U(a,b) = f(x) = \begin{cases} K & a<x<b \\ & a<b \\ 0 & \text{otherwise} \end{cases}$$

$$\int_a^b f(x)\,dx = 1$$

$$\int_a^b K\,dx = 1$$

$$K\int_a^b dx = 1$$

$\Rightarrow \quad k\,[x]_a^b = 1$

$\Rightarrow \quad k[b-a] = 1$

$\Rightarrow \quad k = \dfrac{1}{b-a}$

$$\therefore \boxed{f(x) = \begin{cases} \dfrac{1}{b-a} & a<x<b \\ & a<b \\ 0 & \text{otherwise} \end{cases}}$$

<u>Note:-</u>

$x \in (-a, a)$

$(a>0)$

$f(x) = \dfrac{1}{a-(-a)}$

$$\boxed{\begin{aligned} f(x) &= 1/2a \\ &= 0 \text{ otherwise} \end{aligned}}$$

$U(-a,a) = f(x) = \dfrac{1}{2a}$

Cumulative distribution function (a, b) :-

$$P(X \le x) = F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{a}^{x} f(x)\,dx$$

$$= \int_{a}^{x} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \int_{a}^{x} dx$$

$$= \frac{1}{b-a} \left[x\right]_{a}^{x}$$

$$= \frac{1}{b-a} [x-a]$$

$$\therefore F(x) = \begin{cases} \dfrac{x-a}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$

$$\left[\begin{aligned} F(a) &= \frac{a-a}{b-a} = 0 \\ F(b) &= \frac{b-a}{b-a} = 1 \end{aligned}\right]$$

Moment Generating function (MGF)

$$M_x(t) = E[e^{tx}] = \int_a^b e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{t(b-a)} \left[e^{bt} - e^{at}\right]$$

$$\therefore M_x(t) = E[e^{tx}] = \frac{e^{bt} - e^{at}}{t(b-a)}$$

Moments (non-central about origin)

$$\mu_r' = E(x^r) = \int_a^b x^r \cdot f(x)\, dx$$

$$= \frac{1}{b-a} \int_a^b x^r\, dx$$

$$= \frac{1}{b-a} \left. \frac{x^{r+1}}{r+1} \right]_a^b$$

$$\boxed{\mu_r' = E(x^r) = \frac{1}{b-a}\left[\frac{b^{r+1} - a^{r+1}}{r+1}\right]} \rightarrow (1)$$

case (1) :- put $r=1$ in (1)

$$\mu_1' = E(x) = \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{\cancel{b-a}} \frac{\cancel{(b-a)}(b+a)}{2}$$

$$\boxed{\mu_1' = \frac{b+a}{2}}$$

$$\left[E(x) = \mu_1' = \frac{a+b}{2}\right]$$

Put $r = 2$ in ①

$$\mu_2' = \frac{1}{b-a}\left[\frac{b^3 - a^3}{3}\right]$$

$$= \frac{1}{\cancel{b-a}}\left[\frac{(\cancel{b-a})(b^2 + ab + a^2)}{3}\right]$$

$$\boxed{\mu_2' = \frac{b^2 + ab + a^2}{3}}$$

$$\mathrm{Var}(x) = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{1}{4}(a^2 + b^2 + 2ab)$$

$$= \frac{4b^2 + 4ab + 4a^2 - 3a^2 - 3b^2 - 6ab}{12}$$

$$= \frac{b^2 + a^2 - 2ab}{12}$$

$$\boxed{\mathrm{Var}(x) = (b-a)^2/12}$$

Put $r=3$ in eq ①

$$\mu_3' = \frac{1}{b-a}\left[\frac{b^4-a^4}{4}\right]$$

$$= \frac{1}{b-a}\left[\frac{(b^2)^2-(a^2)^2}{4}\right]$$

$$= \frac{(b^2-a^2)(b^2+a^2)}{4(b-a)}$$

$$= \frac{(b+a)(b^2+a^2)}{4}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$= \frac{(b+a)(b^2+a^2)}{4} - \cancel{3}\cdot\left(\frac{b^2+ab+a^2}{\cancel{3}}\right)\left(\frac{a+b}{2}\right)$$

$$+ 2\left(\frac{b+a}{2}\right)^3$$

$$= \frac{a+b}{4}\left[b^2+a^2-2b^2-2ab-2a^2+ +b^3+a^3+3a^2b+3ab^2\right]$$

$\mu_3 = 0$ Symmmetry. $\boxed{\checkmark}$

30/03/2021
2.20 to 4.30PM

Normal Distribution / Gaussian Dis^n :-

Def:- If 'x' is continous r.v defined $(-\infty, +\infty)$ with mean (μ) and variance (σ^2), then the probability density funtion is defined as

$$N(x;\mu,\sigma^2) = f(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} & \begin{array}{l} -\infty < x < +\infty \\ -\infty < \mu < +\infty \\ 0 < \sigma < \infty \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

Standared Normal r.v (Z)

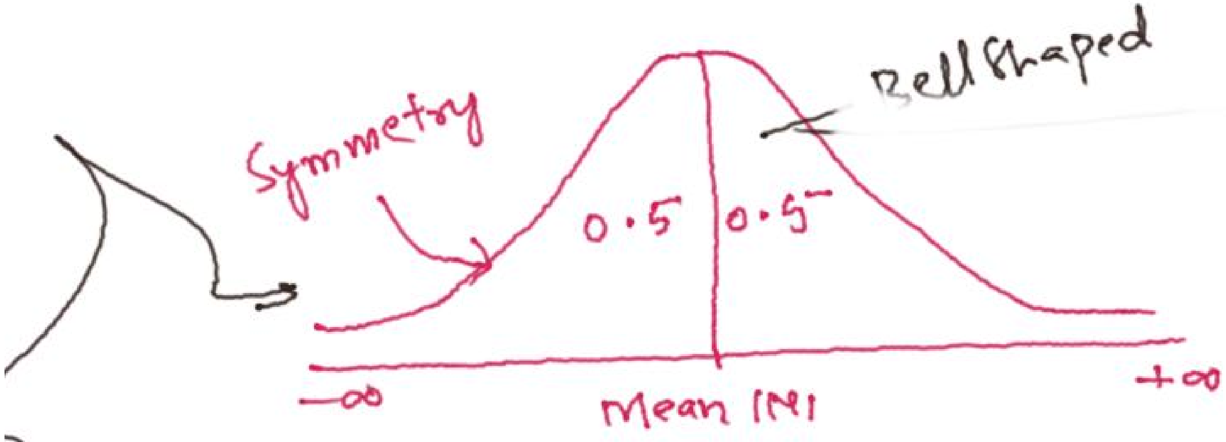
Def:- If 'Z' is a normal r.v with mean '0' and s.d is '1' then the r.v is Standared Normal r.v. and its p.d.f is defined as

$$f(z) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \\ 0 \quad \text{otherwise} \end{cases}$$

Mathematically it is defined as

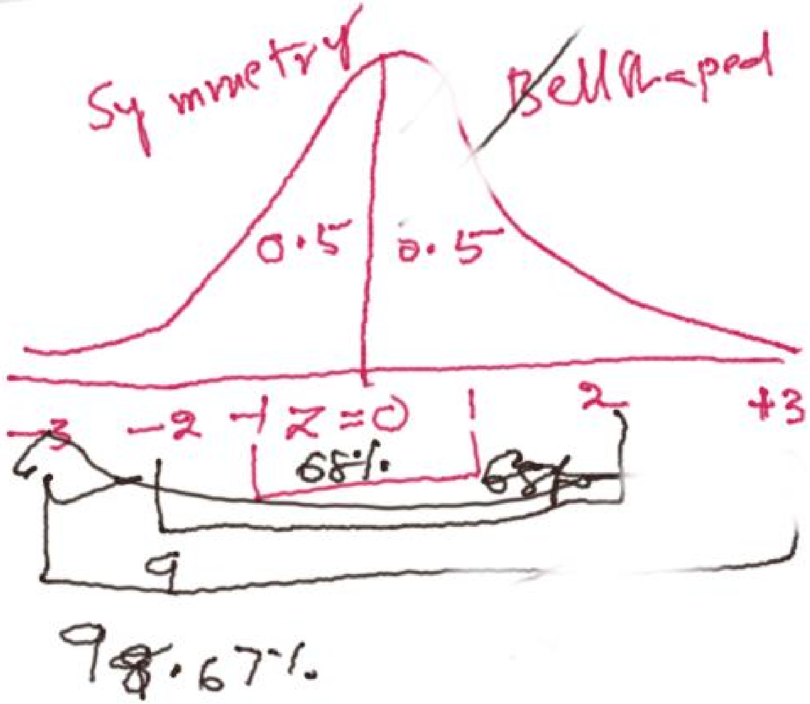
$$\boxed{Z = \frac{x - E(x)}{s.d(x)}} \quad (-3 \leq Z \leq +3)$$

Charateristics of Normal Curve:-



Normal curve

Standard Normal curve



(1) It is a Bell shaped curve. and symmetry about Mean

(2) Edges of the curve never touches the horizontal axis

(3) $M = Md = Mo = \mu$

(4) Prob is Max at Mean (μ): $f(x) = \frac{1}{\sigma\sqrt{2\pi}}$

(5) $P(-1 < z < +1) = 68\%$

$P(-2 < z < 2) = 95\%$

$P(-3 < z < 3) = 99.86\%$

Derive M.G.F of Normal Dis^n:- and find E(x), Var(x):

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{+\infty} e^{tx} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = w \Rightarrow x = w\sigma + \mu$

$dx = \sigma\, dw$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(w\sigma + \mu)} \; e^{-\frac{1}{2}w^2} \; \cancel{\sigma} \, dw$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tw\sigma} \, e^{t\mu} \, e^{-\frac{1}{2}w^2} \, dw$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w^2 - 2tw\sigma)} \, dw$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(w - t\sigma)^2 - t^2\sigma^2\right]} \, dw$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w - t\sigma)^2} dw$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \times 2 \left[\int_{0}^{\infty} e^{-\frac{1}{2}(w - t\sigma)^2} dw \right]$$

Put $w - t\sigma = y \Rightarrow dw = dy$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \times 2 \int_{0}^{\infty} e^{-\frac{1}{2}y^2} dy$$

put $\frac{1}{2}y^2$

put $\frac{1}{2}y^2 = s \Rightarrow y^2 = 2s$

$$\Rightarrow \cancel{2}\, y\, dy = \cancel{2}\, ds$$

$$dy = \frac{ds}{\sqrt{2s}}$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}}\, \cancel{2} \int_0^\infty e^{-s}\, s^{-1/2} \frac{ds}{\cancel{\sqrt{2}}}$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{\pi}} \int_0^\infty e^{-s}\, s^{1/2 - 1}\, ds$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\sqrt{\pi}}\, \Gamma\tfrac{1}{2}$$

$$= \frac{e^{t\mu + \frac{t^2\sigma^2}{2}}}{\cancel{\sqrt{\pi}}}\, \cancel{(\sqrt{\pi})}$$

$$\boxed{M_X(t) = e^{t\mu + \frac{t^2\sigma^2}{2}}}$$

Derive Exp and Variance:

$$\mu_r' = E(x^r) = \frac{d^r}{dt^r} M_x(t) \Big]_{t=0}$$

put $(r=1)$

$$\mu_1' = E(x') = \frac{d}{dt} M_x(t) \Big]_{t=0}$$

$$= \frac{d}{dt}\left(e^{t\mu + \frac{t^2\sigma^2}{2}}\right)$$

$$= \frac{d}{dt}\left[e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}}\right)$$

$$= e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} \cancel{2}t\frac{\sigma^2}{\cancel{2}} + e^{\frac{t^2\sigma^2}{2}} e^{t\mu} \cdot \mu \Big]_{t=0}$$

$$= 1 \cdot 1 \cdot 0 + 1 \cdot 1 \cdot \mu$$

$$\boxed{\mu_1' = E(x) = \mu}$$

$$Var(x) = \mu_2' - (\mu_1')^2$$

Put ($r=2$)

$$\mu_2' = E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big]_{t=0}$$

$$\mu_2' = E(x^2) = \frac{d^2}{dt^2} M_x(t)$$

$$= \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]$$

$$= \frac{d}{dt}\left(e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} t\sigma^2 + e^{\frac{t^2\sigma^2}{2}} e^{t\mu}\mu\right)$$

$$= \Big[e^{t\mu}\cdot\mu\, e^{\frac{t^2\sigma^2}{2}} \cancel{t\sigma^2} + e^{t\mu} e^{\frac{t^2\sigma^2}{2}} \cancel{t\sigma^2 \cdot t\sigma^2}$$

$$+ e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} \sigma^2 + e^{\frac{t^2\sigma^2}{2}} \cdot \cancel{t\sigma^2}\, e^{t\mu}\mu + e^{\frac{t^2\sigma^2}{2}} e^{t\mu}\cdot\mu^2\Big]_{t=0}$$

$$= \underline{\sigma^2 + \mu^2}$$

$$\mu_2' = E(x^2) = \mu^2 + \sigma^2$$

$$\therefore \quad V(x) = \mu_2 = \mu_2' - (\mu_1')^2$$

$$= \mu^2 + \sigma^2 - (\mu)^2$$

$$\boxed{Var(x) = \sigma^2}$$

Cumulative Generating function :—

$$K_x(t) = \log_e M_x(t) = \log_e\left(e^{t\mu + \frac{t^2\sigma^2}{2}}\right)$$

$$= \frac{t\mu}{1!} + \frac{t^2\sigma^2}{2}$$

$$K_1 = \text{coeff } \frac{t^1}{1!} = E(x) = \mu_1' = \mu$$

$$K_2 = \text{coeff } \frac{t^2}{2!} = V(x) = \sigma^2$$

$$K_3 = \mu_3 = \text{coeff } \frac{t^3}{3!} = 0$$

$$K_4 = \text{coeff } \frac{t^4}{4!} = 0$$

$$\mu_4 = K_4 + 3K_2^2$$

$$= 0 + 3(\sigma^2)^2$$

$$= 3\sigma^4$$

$$\therefore \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0$$

$$\gamma_1 = 0$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3$$

$$\gamma_2 = \beta_2 - 3 = 0$$

$\therefore$ Normal curve is symmetry ($\mu_3 = 0$) and

Mesokurtic ($\beta_2 = 3$)

— x —

Problems:-

If 'x' is a normal r.v with $E(x) = 40$; $\sigma_x = 5$ find

(i) $P(X < 50)$ (ii) $P(X < 30)$ (iii) $P(X > 45)$ (iv) $P(X > 38)$

Sol:- (i) $P(X < 50)$

Standard Normal

$$Z = \frac{x - E(x)}{s.d(x)}$$

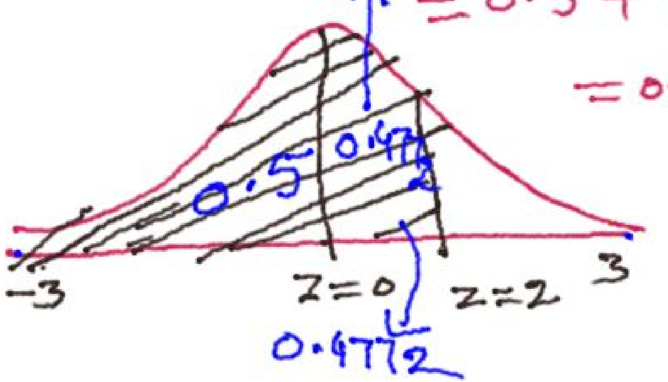
$$= \frac{50 - 40}{5} = 2$$

$P(X < 50) = P(Z < 2)$

$= P(-3 < Z < 0) + P(0 < Z < 2)$

$= 0.5 + 0.4772$

$= 0.9772$



(ii) $P(x < 30)$

$$Z = \frac{x - E(x)}{sd(x)}$$

$$Z = \frac{30 - 40}{5} = -2$$

$$\Rightarrow P(x < 30) = P(Z < -2)$$

$$= P(0 < Z < -3) - P(0 < Z < -2)$$

$$= 0.5 - 0.4772$$

$$= 0.0228$$

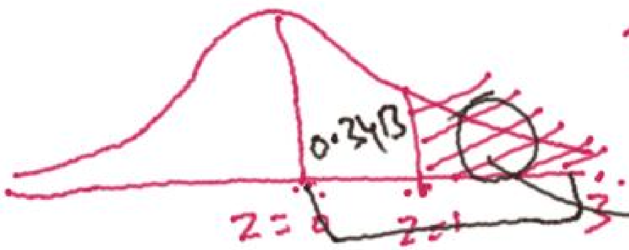
(iii) $P(x > 45)$

$$Z = \frac{x - E(x)}{s.d(x)} = \frac{45 - 40}{5} = 1$$

$$P(x > 45) = P(z > 1)$$

$$= P(0 < z < 3) - P(0 < z < 1)$$

$$= 0.5 - 0.3413$$



$$= 0.1587 \;//$$

(iv) $P(X > 38)$

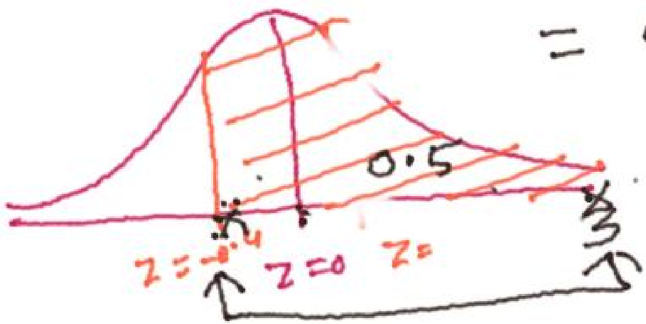
$$Z = \frac{x - E(x)}{s.d(x)} = \frac{38 - 40}{5} = \frac{-2}{5} = -0.4$$

$$P(X > 38) = P(Z > -0.4)$$

$$= P(0 < Z < -0.4) + P(0 < Z < 3)$$

$$= 0.1554 + 0.5$$

$$= 0.6554$$



(Prob 2). If x is a normal r.v $E(X) = 50$; $\sigma_x = 8$

SV

31/3/21
1.10 – 3.30

find (i) $P(X < 40)$ (ii) $P(42 < X < 58)$ (iii) $P(58 < X < 66)$

(iv) $P(|X - 50| < 8)$ (v) $P(X > 66)$

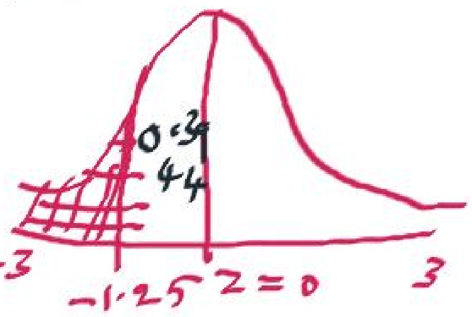
Sol:– (i) $P(X < 40)$

$$Z = \frac{x - E(x)}{sd(x)} = \frac{40 - 50}{8} = -\frac{10}{8} = -1.25$$

$$P(X < 40) = P(Z < -1.25)$$

$$= P(0 < Z < -3) - P(0 < Z < -1.25)$$

$$= 0.5 - 0.3944 = 0.1056$$



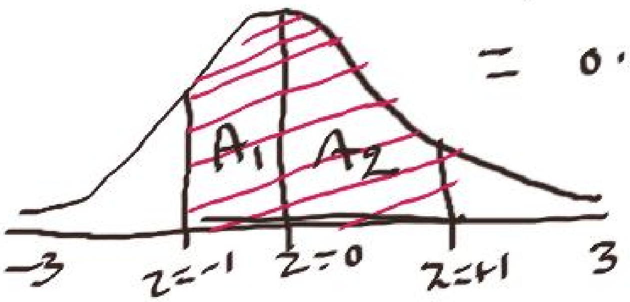
(ii) $P(\overset{x_1}{42} < X < \overset{x_2}{58})$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{42-50}{8} = -\frac{8}{8} = -1$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{58-50}{8} = 8/8 = 1$$

$$\Rightarrow P(42 < X < 58) = P(-1 < Z < +1)$$

$$= P(0 < Z < -1) + P(0 < Z < +1)$$



$$= 0.3413 + 0.3413$$

$$= \underline{\underline{0.6826}}$$

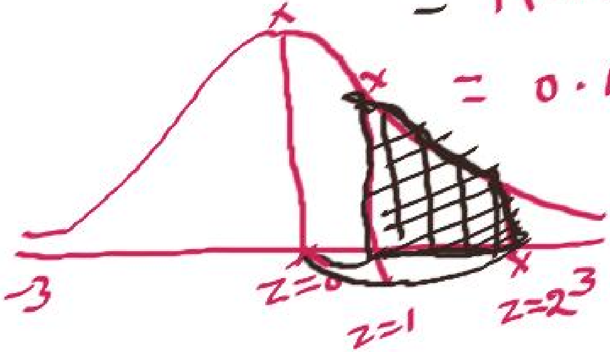
(iii) $P(58 < x < 66)$
$\quad\quad\;\; x_1 \quad\quad\quad x_2$

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{58 - 50}{8} = 8/8 = 1$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{66 - 50}{8} = \frac{16}{8} = 2$$

$$P(58 < x < 66) = P(1 < z < 2)$$

$$= P(0 < z < 2) - P(0 < z < 1)$$



$$= 0.4772 - 0.3413$$

$$= 0.1359$$

(iv) $P(|x-50|<8) = P(-8 < x-50 < 8)$

$= P(-8+50 < x-\cancel{50}+\cancel{50} < 8+50)$

$= P(42 < x < 58)$

$= 0.6826$ // (solution from prb (ii))

(v) $P(x > 66)$

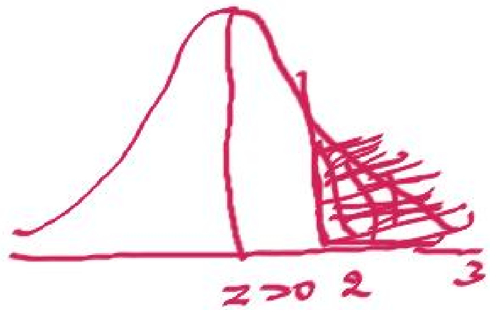
$$Z = \frac{x - E(x)}{\sigma} = \frac{66-50}{8} = 2$$

$P(x>66) = P(Z>2)$

$= P(0<Z<3) - P(0<Z<2)$

$= 0.5 - 0.4772$

$= 0.0228$ //



Prob (iii) A Group has 850 students, who average mark is 58 and s.d is 7.5 find the numbers of students

(i) $P(x < 60)$

(ii) $P(x > 75)$

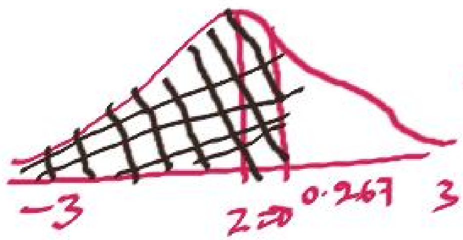
(iii) $P(65 < x < 85)$

(iv) $P(x < 70)$

(i) $P(x < 60)$

$$z = \frac{x - E(x)}{s.d(x)} = \frac{60 - 58}{7.5} = \frac{2}{7.5} = 0.267$$

$$P(x < 60) = P(Z < 0.267)$$
$$= P(0 < Z < -3) + P(0 < z < 0.267)$$
$$= 0.5 + 0.1064 = 0.6064$$



$\therefore$ No of students < 60 are $=$
$$= 0.6064 \times 850$$
$$= 515$$

(ii) $P(x > 75)$

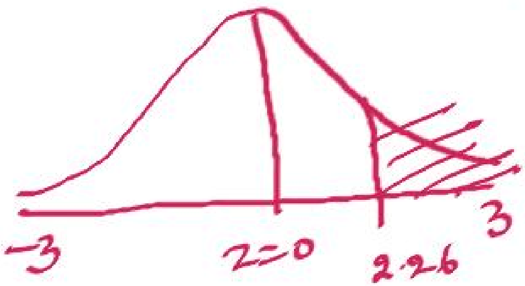
$$Z = \frac{x - E(x)}{sd(x)} = \frac{75 - 58}{7.5} = 2.26$$

$$P(x > 75) = P(Z > 2.26)$$

$$= P(0 < Z < 3) - P(0 < Z < 2.26)$$

$$= 0.5 - 0.4881$$

$$= 0.0119$$



No of students greater than 75 $\Big\} = 0.0119 \times 850$

$$= 10$$

(iv) $P(65 < x < 75)$
$\quad\quad x_1 \quad\quad x_2$

$$Z_1 = \frac{x_1 - E(x)}{\sigma_x} = \frac{65-58}{7.5} = \frac{7}{7.5} = 0.933$$

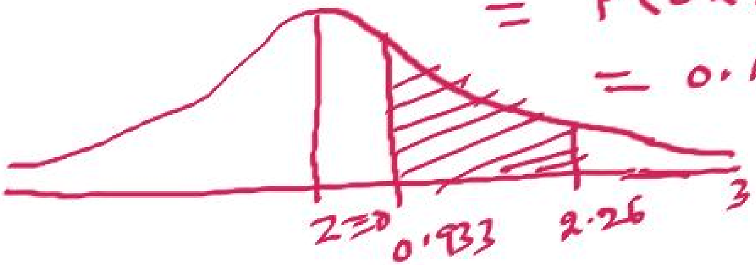
$$Z_2 = \frac{x_2 - E(x)}{\sigma_x} = \frac{75-58}{7.5} = \frac{17}{7.5} = 2.26$$

$$P(65 < x < 75) = P(0.933 < Z < 2.26)$$

$$= P(0 < Z < 2.26) - P(0 < Z < 0.933)$$

$$= 0.4881 - 0.3238$$

$$= 0.1643$$



$\therefore$ No of Students $= 0.1643 \times 850$

$= 140$ //

(v) $P(X < 70)$

$$Z = \frac{x - E(x)}{s.d(x)} = \frac{70 - 58}{7.5} = \frac{12}{7.5} \doteq 1.6$$

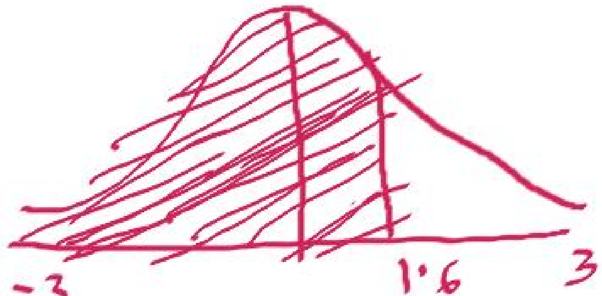
$$P(X < 70) = P(Z < 1.6) = P(0 < Z < -3) + P(0 < Z < 1.6)$$

$$= 0.5 + 0.4452$$

$$= 0.9452$$

$\therefore$ No of students $= 0.9452 \times 850$

$$= 803$$



Bivariate data:-

Two dimensional r.v's:. Connecting two outcomes at a time with one real value, provided those two outcomes must be drawn from same sample space. And its data or distribution is known as Bivariate data / Bivariate distribution.

Here we need to compute statistical averages such as [Exp, Var, cov, cond Ex, cond prob fuct etc]?

X \ Y	y_1	y_2	y_3	y_4	...
x_1	p_{11}	p_{12}	p_{13}	p_{14}	
x_2	p_{21}				
x_3	p_{31}				
x_4	p_{41}				
⋮	⋮				

$p_{11} \rightarrow$ Joint probabilities

$f(x, y)$ $\Downarrow$ Joint prob density function [JPdf] $\Downarrow$ Continous r.v

$p(x, y)$ $\Downarrow$ Joint prob Mass function [JPmf] $\Downarrow$ Discrete r.V

Case(i) Given two dimensional r.v's are continous r.v (x, y) :

If the r.v's are continous (x, y) and its corresponding prob function is known as joint probability density function (JPdf) represented as $f(x, y)$ and its Jointcumulative distribution function represend as $F(x, y) \Rightarrow$

$$\left[\begin{array}{l} JCdF - F(x, y) \\ Jpdf - f(x, y) \end{array}\right]$$

Main point is joint probability are given $(f(x, y))$ how to find statistical averages of individual r.v's ?

Once JPdf $(f(x,y))$ is given first we must find individual r.v probability function

Marginal probability density function

Imp

$$M.d.f(x) = \boxed{f(x) = \int_y f(x,y)\,dy}$$

$$M.d.f(y) = \boxed{f(y) = \int_x f(x,y)\,dx}$$

② If x and y are continous independent r.v's iff

$$f(x, y) = f(x) \cdot f(y)$$

Joint pdf = product (md f(x) · md f(y))

(3) Relation b/w JcdF & Jpdf is

$$\boxed{\begin{gathered} \frac{\partial^2}{\partial x \partial y} F(x, y) = f(x, y) \\ F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x, y)\, dy\, dx \end{gathered}}$$

④ Condition Expectation is $E(x/y) = \frac{E(xy)}{E(y)}$

$$\Rightarrow E(x \cdot y) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} x \cdot y \; f(x,y) \, dy \, dx$$

— x —

Case (ii) If given two dimensional r.v's are discrete (x, y)

If the r.v's are discrete (x, y) then the corresponding probability function is known as joint probability mass function (J.P.m.f)

Denoted by "$p(x,y)$"

Given probability function is $p(x,y)$ then the respective marginal function are

$$\boxed{\begin{aligned} &\text{Marginal Mass function}(x) \quad p(x) = \sum_y p(x,y) \\ &\text{Marginal Mass function}(y) \quad p(y) = \sum_x p(x,y) \end{aligned}}$$

Recall:-

$E(x) = \int x \, f(x) \, dx \quad \to \text{mdf}(x)$

$= \sum x \cdot p(x)$

$V(x) = \int x^2 f(x) \, dx - \left(\int x \, f(x) \, dx\right)^2$

$Cov(x,y) = E(xy) - E(x)E(y)$

Problems:-

If x and y are two dimentional continous r.v's with Joint pdf is $f(x,y) = k\,x\cdot y$, $0 < x < 2$, $0 < y < 2$

find
(i) k
(ii) mdf(x), mdf(y)
(3) $E(x)$, $E(y)$, $V(y)$
(4) $E(x|y)$
(5) $cov(x,y)$
(6) Check x and y are independent or not?

Sol:- Given $f(x,y) = kxy$ $\quad 0<x<2$, $0<y<2$

(i) $\int_0^2 \int_0^2 f(x,y)\, dy\, dx = 1$

$k \int_{x=0}^2 \int_{y=0}^2 xy\, dy\, dx = 1$

$k \int_{x=0}^2 x \left(\int_{y=0}^2 y\, dy \right) dx = 1$

$k \int_{x=0}^2 x \left. \frac{y^2}{2} \right]_0^2 dx = 1$

$\frac{k}{2} \int_{x=0}^2 x\,(4-0)\, dx = 1$

$2k \int_0^2 x\, dx = 1$

$k\, [x^2]_0^2 = 1$

$4k = 1$

$(k = 1/4)$ +ve real no

Mdf's

$$f(x) = \int_{y=0}^{2} f(x,y)\, dy$$

$$= \frac{1}{4}\int_{y=0}^{2} xy\, dy$$

$$= \frac{x}{4}\int_{y=0}^{2} y\, dy$$

$$= \frac{x}{8}\, y^2\Big]_0^2$$

$$= \frac{4x}{8} = \frac{x}{2}$$

$$\therefore \boxed{f(x) = \begin{cases} x/2 & 0<x<2 \\ 0 & \text{otherwise} \end{cases}}$$

$$f(y) = \int_{x=0}^{2} f(x,y)\, dx$$

$$= \frac{1}{4}\, y \int_{x=0}^{2} x\, dx$$

$$= \frac{1}{4}\, y\, \frac{x^2}{2}\Big]_0^2$$

$$\boxed{f(y) = \begin{cases} y/2 & 0<y<2 \\ 0 & \text{otherwise} \end{cases}}$$

③ find $E(x)$, $E(y)$ & $V(y)$

$$E(x) = \int_x x \cdot f(x)\, dx$$

$$= \int_{x=0}^{2} x \cdot x/2 \, dx$$

$$= \frac{1}{2} \int_{x=0}^{2} x^2 \, dx$$

$$= 1/2 \; x^3/3 \Big]_0^2$$

$$E(x) = \frac{1}{2}\left[8/3\right] = 4/3 \;//$$

$$E(y) = \int_y y \cdot f(y)\, dy$$

$$= \int_0^2 y \cdot y/2 \, dy$$

$$= 1/2 \; y^3/3 \Big]_0^2$$

$$= 1/2 \left[8/3\right]$$

$$\boxed{E(y) = 4/3}$$

$$E(y^2) = \int_0^2 y^2 f(y)\, dy$$

$$= \int_0^2 y^2 \cdot y/_2\, dy$$

$$= 1/_2 \left. y^4/_4 \right]_0^2$$

$$= \frac{1}{2}\left[\frac{16}{4}\right]^2$$

$$\boxed{E(y^2) = 2}$$

$$Var(y) = E(y^2) - (E(y))^2$$

$$= 2 - (4/_3)^2$$

$$\boxed{Var(y) = 2/_9}$$

(iii) $E(x/y) = \dfrac{E(xy)}{E(y)}$

$$= \frac{16/9}{4/3} = 4/3$$

$$E(xy) = \int_x \int_y x \cdot y \; f(x, y)\, dy\, dx$$

$$= \frac{1}{4} \int_0^2 \int_0^2 (x \cdot y)\,(x \cdot y)\, dy\, dx$$

$$= \frac{1}{4} \int_{x=0}^{2} x^2 \left(\int_0^2 y^2\, dy \right) dx$$

$$= 2/3 \int_{x=0}^{2} x^2\, dx$$

$$= 2/3 \times 8/3 = 16/9$$

(VI) $\operatorname{COV}(x, y) = E(xy) - E(x)\,E(y)$

$= 16/9 - 4/3 \times 4/3$

$= 16/9 - 16/9 = 0$

(VII) Check whether x and y independent or not

$f(x, y) = f(x) \cdot f(y)$

$= x/2 \cdot y/2$

$xy/4 = xy/4$

$\therefore f(x, y) = xy/4$

So given r.v's are independent r.v

(Prob 2)

6/4/21

2.20 to 4.30

If x, y two dimensional continous r.v's and $f(x,y) =$

$$f(x,y) = K \cdot x \cdot y \, e^{-(x^2+y^2)} \quad x > 0, \, y > 0$$

(i) K (ii) mdf(x), mdf(y) (iii) $E(x)$; $E(y)$ (iv) $E(x \cdot y)$

(v) check x and y are indep or not?

Sol:- Since $\int_0^\infty \int_0^\infty f(x,y) \, dx \, dy = 1$

$$\int_{x=0}^\infty \int_{y=0}^\infty K \cdot x \cdot y \, e^{-(x^2+y^2)} \, dy \, dx = 1$$

$$K \int_{x=0}^\infty x e^{-x^2} \left(\int_{y=0}^\infty y \cdot e^{-y^2} \, dy \right) dx = 1$$

put $y^2 = t$

$y = t^{1/2}$

$dy = \frac{1}{2} t^{-1/2} dt$

$$K\int_{x=0}^{\infty} x \cdot e^{-x^2}\left(\int_{y=0}^{\infty} y\, e^{-y^2}\,dy\right)dx = 1$$

$$K\int_{x=0}^{\infty} x\, e^{-x^2}\left(\int_{t=0}^{\infty} \cancel{t^{1/2}}\, e^{-t}\ \frac{1}{2}\cancel{t^{-1/2}}\,dt\right)dx = 1$$

$$\frac{K}{2}\int_{x=0}^{\infty} x\ e^{-x^2}\left(\int_{t=0}^{\infty} e^{-t}\,dt\right)dx = 1$$

$$\frac{K}{2}\int_{x=0}^{\infty} x\, e^{-x^2}\ \left[\frac{e^{-t}}{-1}\right]_0^{\infty} dx = 1$$

$$\frac{K}{2}\int_{x=0}^{\infty} x\ e^{-x^2}\,dx = 1$$

$$x^2 = s \Rightarrow x = s^{1/2}$$

$$\Rightarrow dx = \tfrac{1}{2}\, s^{-1/2}\,ds$$

$$\frac{K}{2}\int_{s=0}^{\infty} s^{1/2}\, e^{-s}\, \frac{1}{2}\, s^{-1/2}\, ds = 1$$

$$\frac{K}{4}\int_{s=0}^{\infty} e^{-s}\, ds = 1$$

$$\frac{K}{4}\left.\frac{e^{-s}}{-1}\right]_{s=0}^{\infty} = 1$$

$$\frac{K}{4}(1) = 1 \Rightarrow \boxed{K = 4}$$

(ii) Marginal density function $f(x) = \int_y f(x,y)\,dy$

$$= 4\int_{y=0}^{\infty} x\,y\,e^{-x^2}e^{-y^2}\,dy$$

$$= 4xe^{-x^2}\left[\int_{y=0}^{\infty} y\,e^{-y^2}\,dy\right]$$

$$= 4xe^{-x^2}\int_{t=0}^{\infty} \cancel{\sqrt{t}}\,e^{-t}\,\frac{\cancel{t^{-1/2}}}{2}\,dt$$

$$= 2xe^{-x^2}\int_{t=0}^{\infty} e^{-t}\,dt$$

$$= 2x e^{-x^2} \int_{t=0}^{\infty} e^{-t}\, dt$$

$$= 1$$

$$\text{marginal} = f(x) = \begin{cases} 2x\, e^{-x^2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{mdf}(y) = f(y) = \begin{cases} 2y\, e^{-y^2} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(x) = \int_{x=0}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{x=0}^{\infty} x \cdot (2x\, e^{-x^2}\, dx)$$

$$E(x) = \int_{x=0}^{\infty} 2x^2 e^{-x^2}\, dx$$

$$= 2 \int_{x=0}^{\infty} x^2 e^{-x^2}\, dx$$

$$= \cancel{2} \int_{s=0}^{\infty} s\, e^{-s}\, \frac{1}{\cancel{2}}\, s^{-1/2}\, ds$$

$$= \int_{s=0}^{\infty} s^{1/2} e^{-s}\, ds$$

$$= \int_{s=0}^{\infty} e^{-s}\, s^{3/2-1}\, ds$$

$$= \Gamma(3/2)$$

$$= \frac{1}{2}\Gamma(1/2)$$

$$\boxed{E(x) = \frac{\sqrt{\pi}}{2}}$$

$$\boxed{E(Y) = \frac{\sqrt{\pi}}{2}}$$

$$E(x\cdot y) = \int_{x=0}^{\infty}\int_{y=0}^{\infty} x\cdot y \; f(x,y)\, dx\, dy$$

$$= 4\int_{x=0}^{\infty}\int_{y=0}^{\infty} x\cdot y \;\; xy \;\; e^{-(x^2+y^2)}\, dx\, dy$$

$$= 4\int_{x=0}^{\infty} x^2 e^{-x^2}\left(\int_{y=0}^{\infty} y^2 e^{-y^2}\, dy\right) dx$$

$$= 2\int_{x=0}^{\infty} x^2 e^{-x^2}\left(\int_{y=0}^{\infty} 2y^2 e^{-y^2}\, dy\right) dx$$

$$= \left(2\int_{x=0}^{\infty} x^2 e^{-x^2}\, dx\right) \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2}$$

$$\boxed{E(x\cdot y) = \pi/4}$$

$\underline{V}$ Check x & y are i.d or not

$$f(x,y) = f(x) \cdot f(y)$$

$$\text{Jpdf} = \text{md } f(x) \cdot \text{md } f(y)$$

$$= 2x e^{-x^2} \cdot 2y \cdot e^{-y^2}$$

$$= 4xy \, e^{-x^2-y^2}$$

$$f(x,y) = 4xy \, e^{-(x^2+y^2)}$$

$\therefore$ Given r.v' are independent r.v's

(Prob 3) If x, y are two dimensional discrete r.v's and its joint Probability function is

$Y \backslash X$	1	2	3	Marginal Mass function Y
1	$1/12$	$1/6$	0	$3/12$
2	0	$1/9$	$1/5$	$14/45$
3	$1/18$	$1/4$	$2/15$	$79/180$
Marginal function	$5/36$	$17/36$	$5/15$	1

$(y=2, x=3)$

find (i) Marginal Mass function of x, Marginal Mass function y

(ii) $P(X>1 / Y=3)$ (iii) $P(Y>1 / X=3)$ (iv) $E(x)$, $E(y)$

Marginal Probability of 'x'

x	1	2	3
p(x)	5/36	19/36	5/15

Marginal probabilities of 'y'

y	1	2	3
p(y)	3/12	14/45	$\boxed{79/180}$

(ii) $P(X>1/Y=3) =$

$$= \frac{P(X>1 \cap Y=3)}{P(Y=3)}$$

$$= \frac{P(x=2, y=3 + x=3, y=3)]}{P(Y=3)}$$

$$= \frac{1/4 + 2/15}{79/180} = \frac{69}{79}$$

$$P(Y>1 / x=3) = \frac{P(Y>1 \cap X=3)}{P(X=3)}$$

$$= \frac{P(Y=2 \cap X=3 + Y=3 \cap X=3)}{P(X=3)}$$

$$= \frac{1/5 + 2/15}{5/15} = \frac{5}{5} = 1$$

$$E(X) = \sum_{x=1}^{3} x \cdot p(x)$$

$$= 1 \cdot P(1) + 2 \cdot P(2) + 3 \cdot P(3)$$

$$= 1 \cdot 5/36 + 2 \cdot 19/36 + 3 \cdot 5/15 = 79/36$$

$$E(y) = \sum_{y=1}^{3} y \cdot p(y)$$

$$= 1 \cdot p(1) + 2 \cdot p(2) + 3 \cdot p(3)$$

$$= 1 \cdot 3/12 + 2 \cdot \frac{14}{45} + 3 \cdot \frac{79}{180}$$

$$= \frac{197}{90}$$

Prob (4) - If x, y are two dimensional d.r.v's

$$p(x,y) = \frac{x_1 + x_2}{21} \qquad x = 1, 2, 3 \qquad y = 1, 2, 3$$

find (i) m.m.f (x) : m.m.f (y)

(ii) $P(X > 0.5 / y = 2)$, $P(Y > 0.5 / x = 2)$

Sol:-

x_2 \ x_1	

Sol:-

$$p(x,y) = \frac{x+y}{21}$$

$y \downarrow$ \ $x \rightarrow$	1	2	3	m.m.f(y)
1	2/21	3/21	4/21	9/21
2	3/21	4/21	5/21	12/21
m.m.f(x)	5/21	7/21	9/21	1

Marginal Mass function (x):

$x = 1$	2	3
$p(x)$: $5/21$	$7/21$	$9/21$

Marginal Mass funtion (y)

$y = 1$	$y = 2$
$9/21$	$12/21$

$$P(X > 0.5 \mid Y = 2) =$$

$$= \frac{P(X = 1 \cap Y = 2) + P(X = 2 \cap Y = 2)}{P(Y = 2)}$$

$$= \frac{3/21 + 4/21}{12/21} = \frac{7}{12}$$