Overview

- Collections: sets, multi-sets, sequences.
- Relations: predicates, binary relations, n-ary relations.
- Functions: injective, surjective, bijective.
- Propositional logic.
- First-order logic.
- Validity.
- Normal forms.

Sets

Set: A collection of distinct elements. No ordering.

Empty set: $\emptyset$.

Universal set: U wrt some domain.

Basic Operations: membership: $a \in A$. union $A \cup B$, intersection $A \cap B$, absolute complement A' wrt U, relative complement $A - B$.

Sets Continued

Properties: $\emptyset$ identity for $\cup$, U identity for $\cap$, union and intersection are *idempotent*, *associative* and *commutative*, $\cup$ *distributes over* $\cap$ and vice-versa.

Subset: $A \subset B$ if every element which belongs to A also belongs to B. $\emptyset \subset A$ for all sets A.

Power set: The set of all subsets of A is $P(S)$. Given $S = \{a, b\}$, $P(S) = \{\{\}, \{a\}, \{b\}, \{a, b\}\}$.

Cartesian Product: Given two sets A and B, the cartesian product $A \times B = \{(a, b) | a \in A, b \in B\}$. For example, if $A = \{a, b\}$ and $B = \{0, 1, 2\}$, $A \times B = \{(a, 0), (a, 1), (a, 2), (b, 0), (b, 1), (b, 2)\}$.

Set Cardinality

- The number of elements $|S|$ in a set S is known as its cardinality.
- If $|A| = n$, then $|P(A)| = 2^n$.
- What is the cardinality of the set of all natural numbers $\mathcal{N}$, the cardinality of the set of all real numbers $\mathcal{R}$?

Multi-Sets

- Allows repeated elements: aka bag.
- For an element, it makes sense to ask how many occurrences there are of that element.

Sequences

- An *ordered* collection of elements (may contain repetitions).
- Can be put into 1:1 correspondence with $\mathcal{N}$.
- Index-set: $0, 1, 2, \ldots$.
- A *tuple* is a sequence of a specified length. A n-tuple is denoted as $(s_1, s_2, \ldots s_n)$. If $n = 2$, we have a *pair*.

Relations

A relation R over sets $S_1, S2, \ldots S_n$, is some subset of $S_1 \times S2 \times \ldots \times S_n$

- When $n = 1$, we have a unary relation, aka *predicate*.
- When $n = 2$, we have a binary relation. If $(s_1, s_2) \in R$, we say that $s_1 \; R \; s_2$.
- We say that R is true for $(s_1, s_2, \ldots s_n)$ iff $(s_1, s_2, \ldots s_n) \in R$.

Relation Properties

A binary relation $R \subset (A \times A)$ is:

- Reflexive: $a\,R\,a,\ \forall a \in A$.
- Irreflexive: There is no $a \in R$ such that $a\,R\,a$.
- Symmetric: $a\,R\,b \Rightarrow b\,R\,a$.
- Anti-symmetric: $a\,R\,b \wedge b\,R\,a \Rightarrow a = b$.
- Transitive: $a\,R\,b \wedge b\,R\,c \Rightarrow a\,R\,c$.
- Equivalence Relation: Reflexive, symmetric and transitive.
- Partial order: Reflexive, anti-symmetric and transitive.

Inverse Relation

Given a binary relation R, its inverse relation R^{-1} is defined such that if $(a, b) \in R$ iff $(b, a) \in R^{-1}$.

Functions

Given a set D called the domain and a set R called the range (or codomain), a relation $F \in (D \times R)$ is a function iff $\forall d \in D, (d, r_1) \in F \wedge (d, r_2) \in F \Rightarrow r_1 = r_2$. We say that $F : D \rightarrow R$; if $(d, r) \in F$, we say that $F(d) = r$.

- If F is defined for all $d \in D$, then we say that F is total. If F may or may not be defined for all $d \in D$, then we call it partial.
- The definition implies that for any value d in the domain, F maps it to at most one element in the range.

Function Properties

Given a function $F : A \to B$ it is:

- **surjective** If for all $b \in B$ there is a $a \in A$ such that $F(a) = b$. AKA onto.
- **injective** If $F(a) = F(b)$, then $a = b$. AKA one-to-one.
- **bijective** Surjective and injective. AKA one-to-one and onto or one-to-one correspondence.

The inverse of a bijective function F is also a function F^{-1}.

Propositional Logic Well-Formed Formulas

Constants: `true` or `false`.

Atoms: Variables p, q, etc. standing for either `true` or `false`.

Basic Operators: $\vee$ for *or*, $\wedge$ for *and*, $\neg$ for *not*.

Implication: $p \Rightarrow q$ equivalent to $\neg p \vee q$.

Equivalence: $p \Leftrightarrow q$ or $p \equiv q$ or p iff q equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$.

Operator precedence: (lowest) $\equiv$ and $\Rightarrow$, $\vee$, $\wedge$, $\neg$ (highest).

Propositional Operators Truth Table

p	q	$p \vee q$	$p \wedge q$	$\neg p$	$p \Rightarrow q$	$p \equiv q$
false	false	false	false	true	true	true
false	true	true	false	true	true	false
true	false	true	false	false	false	false
true	true	true	true	false	true	true

Tautologies

A WFF is satisfiable if there is some interpretation (assignment to `true` or `false`) for its atoms such that the WFF evaluates to `true`.

A WFF is a tautology if it is true under **all** interpretations.

Examples: $p \vee \neg p$, $p \equiv p$, $p \Rightarrow (p \vee q)$.

A WFF is a contradiction if it is false under **all** interpretations.

Examples: $p \wedge \neg p$, $p \equiv \neg p$.

First-Order Logic

Terms: used to denote objects from some non-empty domain. Represented using infinite set of n-ary function symbols $f_0^n, f_1^n, \ldots$ applied to n objects.

Predicates: used to represent relations. Represented using an infinite set of n-ary predicate symbols $p_0^n, p_1^n, \ldots$ applied to n objects.

Operators: Propositional operators.

First-Order Logic Continued

Variables: Standing for terms.

Quantifiers: $\forall x\, P$, $\exists x\, P$ where x is a variable and P is a WFF. Note that $\forall x\, P$ stands for $P(a_1) \wedge P(a_2) \ldots \wedge P(a_n)$ and $\exists x\, P$ stands for $P(a_1) \vee P(a_2) \ldots \vee P(a_n)$ where the domain consists of $a_1, a_2, \ldots a_n$.

Sentence: WFF without free variables.

Valid WFFs

A sentence is satisfiable if there is some domain and interpretation for its term and predicate symbols under which it is `true`.

A sentence is valid iff it is true under all domains and interpretations.

$$\neg\forall x\, p(x) \equiv \exists x\, \neg p(x)$$

$$\neg\exists x\, p(x) \equiv \forall x\, \neg p(x)$$

Normal Form

- Conjunctive normal form for propositional logic. Example: $(p_1 \vee \neg p_2) \wedge (\neg p_1 \vee p_3) \wedge (p_1 \vee p_2 \vee \neg p_3)$.
- Disjunctive normal form for propositional logic. Example: $(p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3)$.
- Clausal form for first-order logic. CNF with implicit universal quantification (existential quantifiers replaced by skolem functions). Example: The WFF

$$
\begin{aligned}
&[\forall X1 \forall Y1 \ \text{father}(X1, Y1) \Rightarrow \text{parent}(X1, Y1)] \wedge \\
&[\forall X2 \forall Y2 \ \text{mother}(X2, Y2) \Rightarrow \text{parent}(X2, Y2)] \wedge \\
&[\forall X3 \forall Y3 \ \text{parent}(X3, Y3) \Rightarrow \text{father}(X3, Y3) \vee \text{mother}(X3, Y3)]
\end{aligned}
$$

has clausal form:

$$
\begin{aligned}
&[\neg\text{father}(X1, Y1) \vee \text{parent}(X1, Y1)] \wedge \\
&[\neg\text{mother}(X2, Y2) \vee \text{parent}(X2, Y2)] \wedge \\
&[\neg\text{parent}(X3, Y3) \vee \text{father}(X3, Y3) \vee \text{mother}(X3, Y3)]
\end{aligned}
$$