CS105 Discrete Structures: Propositions, Predicates
Exercise Problem Set 1

Part 1

1. (a) Construct the truth table of the compound proposition $(p \wedge \neg q) \rightarrow (p \vee q)$.

   (b) How can this English sentence be translated into a logical expression?
   "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

   (c) Verify that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent using the truth tables of both the expressions.

   (d) Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent using De Morgan's laws. Verify the same again using truth tables.

2. Give the converse and contrapositive for the following propositions:

   (a) If it rains today, then my hostel room will leak.

   (b) If $|x| = x$, then $x \geq 0$.

   (c) If n is greater than 3, then n^2 is greater than 9.

   Which of the above statements/propositions are true? Are their converses also true?

3. True or False

   (a) A proposition is equivalent to its contrapositive.

   (b) A proposition is equivalent to its converse.

   (c) A proposition is equivalent to the converse of its converse.

4. For each of the following propositions, write their negation, such that the negated proposition begins with the quantifier: "there exists a natural number n" or "for all natural numbers n". Also convince yourself the negation is false if the original proposition is true, and vice-versa.

   (a) For all natural numbers n, n is a multiple of 2 or n is a multiple of 3.

   (b) For all natural numbers n, if n is prime, then it is odd.

   (c) There exists a natural number n which is greater than 100.

   (d) There exists a natural number n such that $n^2 = 7$.

5. For each of the following propositions, write its negation. Is the negation true?

   (a) If it rains today, then my hostel room will leak.

   (b) There exists $n \in \mathbb{N}$ such that $n \geq 5$ and $n^2 < 25$.

   (c) For all $n \in \mathbb{N}$, n is a prime or n^2 is a prime, but n^3 is not a prime.

   (d) All computer science students like coffee.

Part 2

6. Prove or disprove the following:
   (a) For any real number x, if x^3 is irrational, then so is x.
   (b) For any real number x, if x is irrational, then so is x^3.
   (c) There exists a nonnegative integer $n^2 > 10^{1000}$.

7. Prove that for all $a, b \in \mathbb{Z}$, we have $a^2 - 4b \neq 2$.

8. Assume that the symbols $>$, $<$ and $=$ are given their natural interpretations as the "greater than", "less than" and "equal to" binary relations, and that the domain of discourse is over reals $\mathbb{R}$.
   (a) Convert the following to English.
      i. $(\forall x(x^2 > x)) \wedge (\exists x(x^2 = 2))$
      ii. $\forall x \forall y \exists z((x < y) \rightarrow ((x < z) \wedge (z < y)))$
      iii. $\forall x \forall y \forall z(((x = y) \wedge (y = z)) \rightarrow (x = z))$
   (b) Also, for each of the above statements, write their negation as propositions with quantifiers.

CS105 2024 Discrete Structures
Basics, Strong Induction
Exercise Problem Set 2

Part 1

1. Prove or disprove: For any $n \in \mathbb{N}$ if n^3 is odd then $n^2 + n$ is even.

2. Prove (by induction) or disprove: For every positive integer n,

   (a) $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}$.

   (b) if $h > -1$, then $1 + nh \leq (1+h)^n$.

   (c) 12 divides $n^4 - n^2$.

3. Prove that if x is irrational and $x \geq 0$, then $\sqrt{x}$ is irrational. What if the condition $x \geq 0$ is removed?

4. Consider the proposition: for all natural numbers $n \geq 5, n^2 < 2^n$.

   (a) Give a proof by induction.

   (b) Also give a proof by contradiction, using Well-Ordering-Principle.

5. Given $n \in \mathbb{N}$, consider a sequence of numbers where the n^{th} term of the sequence is defined by a "recurrence". That is,

   $$u_0 = 1, u_1 = 3, u_n = 2u_{n-1} - u_{n-2}, \text{ for } n \geq 2$$

   Prove using Strong Induction that for all $n \geq 0$, $u_n = 2n + 1$. Also highlight why you need the Strong Induction in your proof.

Part 2

6. Prove or disprove: there is no rational solution to the equation $x^5 + x^4 + x^3 + x^2 + 1 = 0$.

7. In a cricket tournament, every two teams played against each other exactly once. After all games were over, each team wrote down the names of the other teams they defeated, and the names of those teams defeated by some team they defeated. Give 2 proofs, one using induction and one using contradiction, that at least one team listed the names of every other team! (**Note:** Assume no ties.)

8. There are n identical cars placed at various points on a circular track. Each car has some amount of fuel in its tank, and the total amount of fuel among all n cars is exactly enough for one car to complete a full lap around the track. Suppose that whenever a car passes by another car, it can collect all the fuel from that car. Show, using mathematical induction, that there exists at least one car which can complete a full lap by collecting fuel from the other cars as it moves along the track.

9. Consider the following game:

- There are two piles of matches.
- Two players take turns removing any positive (i.e., non-zero) number of matches they want from one of the two piles.
- The player who removes the last match wins.

Show (by strong induction!) that, if the two piles contain the same number of matches initially, then the second player can always win the game.

10. Prove that the following division algorithm (that you may have learnt in high school) is correct: For any $m, n \in \mathbb{N}$, $m \neq 0$, there exists a unique quotient q and remainder r ($q, r \in \mathbb{N}$), such that

$$n = q \cdot m + r, \quad 0 \leq r < m$$

(hint: try induction/strong induction on n)

CS105 2025 Discrete Structures
More on Proofs and Induction
Exercise Problem Set 3

Part 1

1. For each of the following pair of propositions, are they logically equivalent? If yes, prove it, else give a counter-example to show that they aren't.
   (a) $(p \rightarrow q) \vee (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$
   (b) $\neg(p \rightarrow q) \rightarrow p$ and $\top$
   (c) $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$

   In the above, $\top$ is the symbol used for $True$, the proposition that always has value true! Similarly $\perp$ is used for $False$.

2. For what integers n is the proposition $3^n < n!$ true? Prove your answer by induction. Recall that $n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n$.

3. Use strong induction to prove that $\sqrt{2}$ is irrational.

4. Suppose that $a_{m,n}$ is defined recursively for $(m, n) \in \mathbb{N} \times \mathbb{N}$ by

$$a_{0,0} = 0$$

   and

$$a_{m,n} = \begin{cases} a_{m-1,n} + 1, & \text{if } n = 0 \text{ and } m > 0, \\ a_{m,n-1} + n, & \text{if } n > 0. \end{cases}$$

   Show that

$$a_{m,n} = m + \frac{n(n+1)}{2}$$

   for all $(m, n) \in \mathbb{N} \times \mathbb{N}$, that is, for all pairs of nonnegative integers.

Part 2

5. Prove using (Strong) Induction: Every integer greater than 7 is the sum of a nonnegative integer multiple of 3 and a nonnegative integer multiple of 5.

6. Use the Well Ordering-Principle to show the following:
   (a) Define g as the GCD (greatest common divisor) of natural numbers a and b if $g|a$ and $g|b$, and for all natural numbers d, if $d|a$ and $d|b$ then $d|g$.

   Show that any two positive integers a, b have a unique greatest common divisor (hint: consider the set of numbers of the form $ax + by$).

(b) The equation $4a^3 + 2b^3 = c^3$ does not have any solutions over $\mathbb{N} - \{0\}$. What about the equation $a^4 + b^4 + c^4 = d^4$ over $\mathbb{Z}$?

7. For any $n \in \mathbb{N}$, $n \geq 2$, prove that

$$\sqrt{2\sqrt{3\sqrt{4\ldots\sqrt{(n-1)\sqrt{n}}}}} < 3$$

CS105 (Discrete Structures)
Exercise Problem Set 3

August 22, 2025

Instructions:

- Attempt *all* questions.
- *Some* of the answers will be discussed during the help sessions, but again you are expected to have attempted *all* the questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!
- In the following, “disprove” means you have to give a counterexample.

Part 1

1. Give examples of infinite sets A, B and functions f, g, h all from A to B such that
   - f is an injection but not a surjection,
   - g is a surjection but not an injection, and
   - h is an injection and surjection (and therefore a bijection).

   Can you also give examples of finite sets (and functions between them) such that the above statements hold?

2. Consider non-empty finite sets A, B. Which of the following statements are true about them? Give a proof or a counter-example
   (a) There always exists a bijection from A to $A \cup B$.
   (b) There can never exist a bijection from A to $A \cup B$.
   (c) If there is a bijection from A to B then there always exists a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$ (i.e., between their power-sets, set of subsets).

   Which of the above statements hold when A and B are countably infinite?

3. Prove or disprove the following (with complete justifications): Let A, B, C be non-empty sets.
   (a) If there is an injection from A to B then there is a surjection from B to A.
   (b) If there is an injection from A to B then there is a surjection from A to B.
   (c) If there is an bijection from A to B and an injection from B to C, then there is an injection from A to C.

(d) If $A \subseteq B$ but $A \neq B$, then there must exist an injection from A to B but there can exist no surjection from A to B.

4. Show that the function $f(x) = |x|$ from set of real numbers to set of nonnegative real numbers has no inverse, but if the domain is restricted to the set of nonnegative real numbers, the resulting function has an inverse.

5. Prove (formally) that there exists a bijection from the set of all integers $\mathbb{Z}$ to the set of all natural numbers $\mathbb{N}$.

Part 2

6. Let A be any infinite set. Prove carefully that there is a surjection from A to $\mathbb{N}$. We said in class that this implies that the natural numbers are the "smallest" infinite set! Do you agree? What about the set of even numbers or set of all primes? Discuss.

7. Construct an explicit bijection f from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Of course you need to show that your answer is correct, i.e., f is a bijection.

8. *(Schröder-Bernstein Theorem) For any two sets A, B, show that if there exist injective functions $f : A \to B$ and $g : A \to B$ between the sets A, B, then there exists a bijection between A and B. *Note: If you cannot prove it yourself, read the proof!*

CS105 (DIC on Discrete Structures)
Exercise Problem Sheet 5

Instructions:

- Attempt *all* questions.
- If you have any doubts or you find any typos in the questions, post them on piazza at once!

Part 1

1. True or False. You must provide a short justification for your answer.
   (a) Any subset of a countable set is countable.
   (b) An intersection of an uncountable set and a countable set must be uncountable.
   (c) The set of all irrational numbers (i.e., reals that are not rational) is countable.
   (d) The set $\{a^b \mid a, b \in \mathbb{Q}^+\}$ is countable, where $\mathbb{Q}^+$ stands for positive rationals.
2. Give an example for each of the following, if such an example exists. Else prove why it cannot exist.
   (a) A relation that is irreflexive, symmetric and not transitive.
   (b) Relations R_1 and R_2 on set S such that both are symmetric but $R_1 \cap R_2$ is not symmetric.
3. Suppose R_1 and R_2 are two equivalence relations on set S.
   (a) Is $R_1 \cap R_2$ an equivalence relation?
   (b) Is $R_1 \cup R_2$ an equivalence relation?

   For each of the above, if your answer is "yes", you must prove it, and if your answer is "no", you must provide a counterexample.
4. Prove or disprove:
   (a) There is a surjection but no injection from $\mathbb{Q} \cap [0, 1]$ to $\mathbb{N} \times \mathbb{N}$.
   (b) There is a bijection between $\mathbb{Z} \times \mathbb{N}$ and $\mathbb{Q} \times \mathbb{N} \times (\mathbb{N} \cup \{\sqrt{2}\})$.
5. Construct a bijection
   (a) from $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$
   (b) from the open interval $(0, 1)$ on the real line to the closed interval $[0, 1]$.

   Prove that the function you constructed is indeed a bijection.

Part 2

6. In class we showed that there is no bijection from $\mathbb{N}$ to the set of subsets of $\mathbb{N}$. Prove that for *any* non-empty set S, there is no bijection from S to the set of all subsets of S.

7. Prove that there exists a bijection from $\mathbb{R}$ to set of all subsets of $\mathbb{N}$. Can you construct it explicitly? Also, can you conclude whether $\mathbb{R}$ is countable or uncountable from this?

8. Which of the following sets are countable? Justify with a formal proof.

   (a) Set of all functions from $\mathbb{N}$ to $\mathbb{N}$.

   (b) Set of all non-increasing functions from $\mathbb{N}$ to $\mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{N}$ is said to be non-increasing if for all $x, y \in \mathbb{N}$, if $x \leq y$ then $f(x) \geq f(y)$.

9. (*) Prove that there does not exist a C-program which will always determine whether an arbitrary input-free C-program will halt.

CS105 (DIC on Discrete Structures)
Problem set 6

- Attempt *all* questions.
- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.

Part 1

1. Consider the set of all bit-strings, i.e., finite sequences over $\{0, 1\}$. EATVA.g., $0101, 000, 11111110$ are all bit-strings.

   For any two bit-strings u, v, let us define a relation by uRv iff u and v contain the same number of 1's.

   Is R an equivalence relation? Why or why not?

2. Consider the set $S = \{1, 2, 3, 4\}$ with the subseteq relation $\subseteq$. We know that $(\mathcal{P}(S), \subseteq)$ is a poset.
   (a) Draw its Hasse diagram.
   (b) What is the length of the longest chain (chain with maximum possible number of elements) in this poset?
   (c) What is the size of the largest anti-chain in this poset?
   (d) Give two (different) topological sorts of the poset.

3. Consider the poset $(\mathbb{Z}^+, |)$, i.e., positive integers with divisbility ordering.
   (a) Give an example of a chain of length 5 in this poset.
   (b) Give an example of an anti-chain of length 5 in this poset.
   (c) Does this poset have:
      i. a minimal element (an element in the poset such that there is no element smaller than it in the poset)
      ii. a maximal element (an element in the poset such that there is no element larger than it in the poset)

iii. a minimum or least element (i.e., it is smaller than all elements in the poset).
iv. a maximum or greatest element (i.e., it is larger than all elements in the poset).
v. an infinite chain (a chain with infinitely many elements)
vi. an infinite anti-chain (an anti-chain with infinitely many elements)

For each of the above, if you claim there exists one, give an example, otherwise explain why there can't be any.

Part 2

4. A maximal chain is a chain that is not a subset of a larger chain. Prove or disprove: every maximal chain in a finite poset $(S, \preceq)$ contains a minimal element of S.

5. Consider a necklace made of 3 beads, each of which can be either red, white or blue. Let S be the set of all such necklaces. Define the following relation R on S as: $N_1 \; R \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating it (and *not* allowing to flip the necklace).

   (a) Show that R is an equivalence relation.

   (b) What are the equivalence classes of R?

   (c) Is the number of elements in each equivalence class the same? Is there a relationship between the number of elements in an equivalence class of R and the total number of elements in S?

   (d) If in the definition of the relation, we allow flipping of the necklace as well: that is, $N_1 \; R' \; N_2$ iff necklace N_2 can be obtained from necklace N_1 by rotating or flipping it. Is R' an equivalence relation? Why or why not?

CS105 (DIC on Discrete Structures)
Problem set 7

- Attempt *all* questions.
- Apart from things proved in lecture, you cannot assume anything as "obvious". Either quote previously proved results or provide clear justification for each statement.
- Note that questions on lattices (Qns 1, 2, 5 below) are not in syllabus for the midsem exam.

Part 1

1. Give an example of a poset with five elements that is a lattice and an example of another poset with five elements which is not a lattice.
2. Prove or disprove: Every totally ordered set is a lattice.
3. Give an example of a poset that has 3 maximal elements and 2 minimal elements. Can it have a maximum element?
4. Prove or disprove: the intersection between a chain and an antichain can have at most one element.

Part 2

5. Let L be a lattice. For any two elements $x, y \in L$ we use $x \vee y$ to denote the least upper bound of $\{x, y\}$ and $x \wedge y$ to denote the greatest lower bound of $\{x, y\}$ (note that both these elements exist by the definition of a lattice).

   (a) Show the following properties for all $x, y, z \in L$,

      i. (commutative laws) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$

      ii. (associative laws) $((x \vee y) \vee z) = (x \vee (y \vee z))$ and $((x \wedge y) \wedge z) = (x \wedge (y \wedge z))$.

iii. (absorption laws) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$

iv. (idempotency laws) $x \vee x = x$ and $x \wedge x = x$.

(b) Use the above to prove that every finite nonempty subset of a lattice must have a greatest lower bound and a least upper bound.

6. For all $t > 0$, prove formally that any poset with n elements must have either a chain of length greater than t or an antichain with at least $\frac{n}{t}$ elements.

7. *Consider a permutation of the numbers from 1 to n arranged as a sequence from left to right on a line. Using Mirsky's theorem done in class, prove that there exists a $\sqrt{n}$-length subsequence of these numbers that is completely increasing or completely decreasing as you move from right to left.

   For example, the sequence 2, 3, 4, 7, 9, 5, 6, 1, 8 has an increasing subsequence of length 3, for example: 2, 3, 4, and a decreasing subsequence of length 3, for example: 9, 6, 1. (Hint: Use the previous question!)