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Matrices

DEFINITION

A rectangular arrangement of elements in rows and columns, is called a matrix. Such a rectangular arrangement of numbers is enclosed by small () or big [] brackets. Generally a matrix is represented by a capital letter A, B, C..... etc. and its element are represented by small letters a, b, c, x, y etc.

Following are some examples of a matrix :

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{bmatrix} 1 & 5 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad C = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad D = [1, 5, 6], \quad E = [5]$$

ORDER OF MATRIX

A matrix which has m rows and n columns is called a matrix of order $m \times n$, and its represented by

$$A_{m \times n} \text{ or } A = [a_{ij}]_{m \times n}$$

It is obvious to note that a matrix of order $m \times n$ contains mn elements. Every row of such a matrix contains n elements and every column contains m elements.

TYPES OF MATRICES

Row matrix

If in a matrix, there is only one row, then it is called a Row Matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a row matrix if $m = 1$

Column Matrix

If in a matrix, there is only one column, then it is called a column matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a column matrix if $n = 1$.

Square matrix

If number of rows and number of columns in a matrix are equal, then it is called a square matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$.

Note : (a) If $m \neq n$ then matrix is called a rectangular matrix.

(b) The elements of a square matrix A for which $i = j$ i.e., $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are called principal diagonal elements and the line joining these elements is called the principal diagonal or leading diagonal of matrix A.

(c) Trace of a matrix : The sum of principal diagonal elements of a square matrix A is called the trace of matrix A which is denoted by trace A. $\text{Trace } A = a_{11} + a_{22} + \dots + a_{nn}$

Singleton matrix

If in a matrix there is only one element then it is called singleton matrix.

Thus $A = [a_{ij}]_{m \times n}$ is a singleton matrix if $m = n = 1$.

Null or zero matrix

If in a matrix all the elements are zero then it is called a zero matrix and it is generally denoted by O.

Thus $A = [a_{ij}]_{m \times n}$ is a zero matrix if $a_{ij} = 0$ for all i and j.

Diagonal matrix

If all elements except the principal diagonal in a square matrix are zero, it is called a diagonal matrix.

Thus a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

- Note : (a) No element of principal diagonal in diagonal matrix is zero.
 (b) Number of zero in a diagonal matrix is given by $n^2 - n$ where n is a order of the matrix.

Scalar Matrix

If all the elements of the diagonal in a diagonal matrix are equal, it is called a scalar matrix.

Thus a square matrix $A [a_{ij}]$ is a scalar matrix is

$$a_{ij} = \begin{cases} 0 & i \neq j \\ k & i = j \end{cases} \text{ where } k \text{ is a constant.}$$

Unit matrix

If all elements of principal diagonal in a diagonal matrix are 1, then it is called unit matrix. A unit matrix of order n is denoted by I_n .

Thus a square matrix

$$A = [a_{ij}] \text{ is a unit matrix if } a_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Note : Every unit matrix is a scalar matrix.

Triangular matrix

A square matrix $[a_{ij}]$ is said to be triangular if each element above or below the principal diagonal is zero. It is of two types -

- (a) Upper triangular matrix : A square matrix $[a_{ij}]$ is called the upper triangular matrix, if $a_{ij} = 0$ when $i > j$.
- (b) Lower triangular matrix : A square matrix $[a_{ij}]$ is called the lower triangular matrix, if $a_{ij} = 0$ when $i < j$

Note : Minimum number of zero in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

Equal matrix

Two matrices A and B are said to be equal if they are of same order and their corresponding elements are equal.

Singular matrix

Matrix A is said to be singular matrix if its determinant $|A| = 0$, otherwise non-singular matrix i.e.,

If $\det |A| = 0 \Rightarrow$ singular and $\det |A| \neq 0 \Rightarrow$ non-singular

ADDITION AND SUBTRACTION OF MATRICES

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of the same order then their sum $A + B$ is a matrix whose each element is the sum of corresponding elements.

i.e., $A + B = [a_{ij} + b_{ij}]_{m \times n}$

$A - B$ is defined as $A - B = [a_{ij} - b_{ij}]_{m \times n}$

Note : Matrix addition and subtraction can be possible only when matrices are of same order.

Properties of matrices addition

If A , B and C are matrices of same order, then-

- (i) $A + B = B + A$ (Commutative Law)
- (ii) $(A + B) + C = A + (B + C)$ (Associative law)
- (iii) $A + O = O + A = A$, where O is zero matrix which is additive identity of the matrix.
- (iv) $A + (-A) = 0 = (-A) + A$ where $(-A)$ is obtained by changing the sign of every element of A which is additive inverse of the matrix

$$(v) \left. \begin{array}{l} A + B = A + C \\ B + A = C + A \end{array} \right\} \Rightarrow B = C \text{ (Cancellation law)}$$

$$(vi) \text{ Trace } (A \pm B) = \text{trace } (A) \pm \text{trace } (B)$$

SCALAR MULTIPLICATION OF MATRICES

Let $A = [a_{ij}]_{m \times n}$ be a matrix and k be a number then the matrix which is obtained by multiplying every element of A by k is called scalar multiplication of A by k and it denoted by kA .

$$\text{Thus } A = [a_{ij}]_{m \times n} \Rightarrow kA = [ka_{ij}]_{m \times n}$$

Properties of scalar multiplication

If A, B are matrices of the same order and m, n are any numbers, then the following results can be easily established.

$$(i) m(A + B) = mA + mB \quad (ii) (m + n)A = mA + nA \quad (iii) m(nA) = (mn)A = n(mA)$$

MULTIPLICATION OF MATRICES

If A and B be any two matrices, then their product AB will be defined only when number of column in A is equal to the number of rows in B . If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ then their product $AB = C = [c_{ij}]$, will be matrix of order $m \times p$, where

$$(AB)_{ij} = C_{ij} = \sum_{r=1}^n a_{ir} b_{rj}$$

Properties of matrix multiplication

If A, B and C are three matrices such that their product is defined, then

- (i) $AB \neq BA$ (Generally not commutative)
- (ii) $(AB)C = A(BC)$ (Associative Law)
- (iii) $IA = A = AI$ I is identity matrix for matrix multiplication
- (iv) $A(B + C) = AB + AC$ (Distributive law)
- (v) If $AB = AC \Rightarrow B = C$ (cancellation Law is not applicable)
- (vi) If $AB = 0$ It does not mean that $A = 0$ or $B = 0$, again product of two non-zero matrix may be zero matrix.
- (vii) $\text{trace}(AB) = \text{trace}(BA)$

Note : (i) The multiplication of two diagonal matrices is again a diagonal matrix.

(ii) The multiplication of two triangular matrices is again a triangular matrix.

(iii) The multiplication of two scalar matrices is also a scalar matrix.

(iv) If A and B are two matrices of the same order, then

- (a) $(A + B)^2 = A^2 + B^2 + AB + BA$
- (b) $(A - B)^2 = A^2 + B^2 - AB - BA$
- (c) $(A - B)(A + B) = A^2 - B^2 + AB - BA$
- (d) $(A + B)(A - B) = A^2 - B^2 - AB + BA$
- (e) $A(-B) = (-A)B = -(AB)$

Positive Integral powers of a matrix

The positive integral powers of a matrix A are defined only when A is a square matrix.

Also then $A^2 = A \cdot A$ $A^3 = A \cdot A \cdot A = A^2 \cdot A$

Also for any positive integers m, n

- (i) $A^m A^n = A^{m+n}$
- (ii) $(A^m)^n = A^{mn} = (A^n)^m$
- (iii) $I^n = I, I^m = I$
- (iv) $A^\circ = I_n$ where A is a square matrices of order n.

TRANSPOSE OF MATRIX

If we interchange the rows to columns and columns to rows of a matrix A, then the matrix so obtained is called the transpose of A and it is denoted by

$$A^T \text{ or } A^t \text{ or } A'$$

From this definition it is obvious to note that

- (i) Order of A is $m \times n \Rightarrow$ order of A^T is $n \times m$
- (ii) $(A^T)_{ij} = (A)_{ji}, \forall i, j$

Properties of Transpose

If A, B are matrices of suitable order then

- (i) $(A^T)^T = A$
- (ii) $(A + B)^T = A^T + B^T$
- (iii) $(A - B)^T = A^T - B^T$
- (iv) $(kA)^T = kA^T$
- (v) $(AB)^T = B^T A^T$
- (vi) $(A_1 A_2 \dots A_n)^T = A_n^T \dots A_2^T A_1^T$
- (vii) $(A^n)^T = (A^T)^n, n \in N$

SYMMETRIC AND SKEW-SYMMETRIC MATRIX

(a) Symmetric matrix : A square matrix $A = [a_{ij}]$ is called symmetric matrix if $a_{ij} = a_{ji}$ for all $i = j$ or $A^T = A$.

Note : (i) Every unit matrix and square zero matrix are symmetric matrices.

(ii) Maximum number of different element in a symmetric matrix is $\frac{n(n+1)}{2}$

(b) Skew-symmetric matrix : A square matrix $A = [a_{ij}]$ is called skew-symmetric matrix if

$$a_{ij} = -a_{ji} \text{ for all } i, j \quad \text{or} \quad A^T = -A$$

Note : (i) All principal diagonal elements of a skew-symmetric matrix are always zero because for any diagonal element - $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$

(ii) Trace of a skew symmetric matrix is always 0

Properties of symmetric and skew-symmetric matrices

- (i) If A is a square matrix, then $A + A^T, AA^T, A^TA$ are symmetric matrices while $A - A^T$ is skew-symmetric matrices.
- (ii) If A, B are two symmetric matrices, then-
 - (a) $A \pm B, AB + BA$ are also symmetric matrices.
 - (b) $AB - BA$ is a skew-symmetric matrix.
 - (c) AB is a symmetric matrix when $AB = BA$
- (iii) If A, B are two skew-symmetric matrices, then-
 - (a) $A \pm B, AB - BA$ are skew-symmetric matrices.
 - (b) $AB + BA$ is a symmetric matrix.

- (iv) If A is a skew-symmetric matrix and C is a column matrix, then $C^T AC$ is a zero matrix.
(v) Every square matrix A can be uniquely be expressed as sum of a symmetric and skew symmetric matrix i.e.,

$$A = \left[\frac{1}{2}(A + A^T) \right] + \left[\frac{1}{2}(A - A^T) \right]$$

DETERMINANT OF A MATRIX

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix, then its determinant, denoted by $|A|$ or $\det(A)$ is

defined as $|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Properties of the determinant of a matrix

- (i) $|A|$ exist $\Leftrightarrow A$ is a square matrix
- (ii) $|AB| = |A| |B|$
- (iii) $|A^T| = |A|$
- (iv) $|kA| = k^n |A|$, if A is a square matrix of order n.
- (v) If A and B are square matrices of same order then $|AB| = |BA|$
- (vi) If A is skew symmetric matrix of odd order then $|A| = 0$
- (vii) If $A = \text{diag}(a_1, a_2, \dots, a_n)$ then $|A| = a_1 a_2 \dots a_n$
- (viii) $|A|^n = |A^n|$, $n \in \mathbb{N}$

ADJOINT OF A MATRIX

If every element of a square matrix A be replaced by its cofactor in $|A|$, then the transpose of the matrix so obtained is called the adjoint of A and it is denoted by $\text{adj } A$

Thus if $A = [a_{ij}]$ be a square matrix and C_{ij} be the cofactor of a_{ij} in $|A|$, then

$$\text{adj } A = [C_{ij}]^T$$

$$\Rightarrow (\text{adj } A)_{ij} = C_{ij}$$

$$\text{Hence if } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, \text{ then } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Properties of Adjoint Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

- (i) $A(\text{adj } A) = |A| I_n = (\text{adj } A) A$
(Thus $A(\text{adj } A)$ is always a scalar matrix)
- (ii) $|\text{adj } A| = |A|^{n-1}$
- (iii) $\text{adj } (\text{adj } A) = |A|^{n-2} A$

- | | |
|---|--|
| (iv) $ \text{adj}(\text{adj } A) = A ^{(n-1)^2}$ | (v) $\text{adj}(A^T) = (\text{adj } A)^T$ |
| (vi) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$ | (vii) $\text{adj}(A^m) = (\text{adj } A)^m, m \in \mathbb{N}$ |
| (viii) $\text{adj}(kA) = k^{n-1}(\text{adj } A), k \in \mathbb{R}$ | (ix) $\text{adj}(I_n) = I_n$ |
| (x) $\text{adj } 0 = 0$ | (xi) A is symmetric $\Rightarrow \text{adj } A$ is also symmetric. |
| (xii) A is diagonal $\Rightarrow \text{adj } A$ is also diagonal. | (xiii) A is triangular $\Rightarrow \text{adj } A$ is also triangular. |
| (xiv) A is singular $\Rightarrow \text{adj } A = 0$ | |

INVERSE MATRIX

If A and B are two matrices such that

$$AB = I = BA$$

then B is called the inverse of A and it is denoted by A^{-1} . Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

Further we may note from above property (i) of adjoint matrix that if $|A| \neq 0$, then

$$A \frac{\text{adj}(A)}{|A|} = I = \frac{(\text{adj } A)}{|A|} A \quad \Rightarrow \quad A^{-1} = \frac{1}{|A|} \text{adj } A$$

Thus A^{-1} exists $\Leftrightarrow |A| \neq 0$.

Note :

- (i) Matrix A is called invertible if A^{-1} exists.
- (ii) Inverse of a matrix is unique.

Properties of Inverse Matrix

- (i) $(A^{-1})^{-1} = A$
- (ii) $(A^T)^{-1} = (A^{-1})^T$
- (iii) $(AB)^{-1} = B^{-1}A^{-1}$
- (iv) $(A^n)^{-1} = (A^{-1})^n, n \in \mathbb{N}$
- (v) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$
- (vi) $|A^{-1}| = \frac{1}{|A|} = |A|^{-1}$
- (vii) $A = \text{diag}(a_1, a_2, \dots, a_n) \Rightarrow A^{-1} = \text{diag}(a_1^{-1}, a_2^{-1}, \dots, a_n^{-1})$
- (viii) A is symmetric $\Rightarrow A^{-1}$ is also symmetric.
- (ix) A is diagonal $|A| \neq 0 \Rightarrow A^{-1}$ is also diagonal.
- (x) A is scalar matrix $\Rightarrow A^{-1}$ is also scalar matrix.
- (xi) A is triangular $|A| \neq 0 \Rightarrow A^{-1}$ is also triangular.

SOME IMPORTANT CASES OF MATRICES

Orthogonal Matrix

A square matrix A is called orthogonal if

$$AA^T = I = A^TA \quad ; \quad \text{i.e., if } A^{-1} = A^T$$

Idempotent matrix

A square matrix A is called an idempotent matrix if $A^2 = A$

Involutory Matrix

A square matrix A is called an involutory matrix if $A^2 = I$ or $A^{-1} = A$

Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exist a $p \in N$ such that $A^p = 0$

Hermitian matrix

A square matrix A is skew-Hermitian matrix if $A^H = A$; i.e., $a_{ij} = -\bar{a}_{ji}$ " i, j

Skew hermitian matrix

A square matrix A is skew-hermitian is $A = -A^H$ i.e., $a_{ij} = -\bar{a}_{ji}$ " i, j

Period of a matrix

If for any matrix A $A^{k+1} = A$

then k is called period of matrix (where k is a least positive integer)

Differentiation of matrix

$$\text{If } A = \begin{bmatrix} f(x) & g(x) \\ h(x) & l(x) \end{bmatrix}$$

then $\frac{dA}{dx} = \begin{bmatrix} f'(x) & g'(x) \\ h'(x) & l'(x) \end{bmatrix}$ is a differentiation of matrix A

Submatrix

Let A be $m \times n$ matrix, then a matrix obtained by leaving some rows or columns or both of a is called a sub matrix of A

Rank of a matrix

A number r is said to be the rank of a $m \times n$ matrix A if

- (a) every square sub matrix of order $(r + 1)$ or more is singular and
- (b) there exists at least one square submatrix of order r which is non-singular.

Thus, the rank of matrix is the order of the highest order non-singular sub matrix.

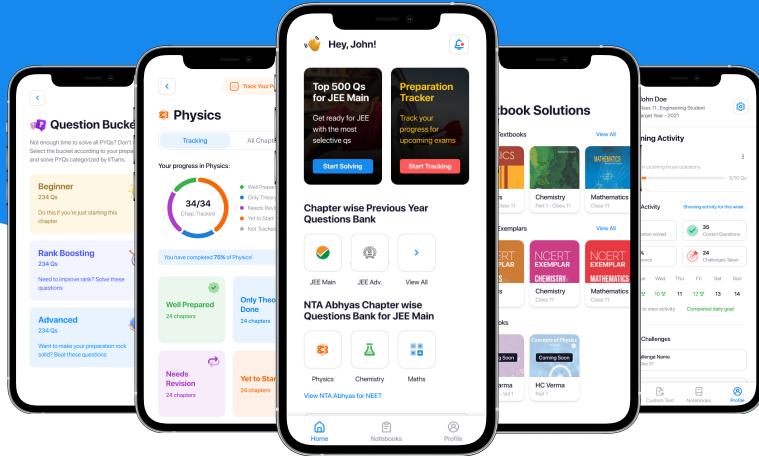
We have $|A| = 0$ therefore r (A) is less than 3, we observe that $\begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}$ is a non-singular square sub matrix of order 2 hence r (A) 2.

Note :

- (i) The rank of the null matrix is zero.
- (ii) The rank of matrix is same as the rank of its transpose i.e., $r(A) = r(A^T)$
- (iii) Elementary transformation of not alter the rank of matrix.



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