Experimental Design on a Budget for Sparse Linear Models and Applications

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University of Wisconsin Madison

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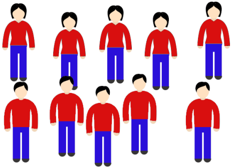
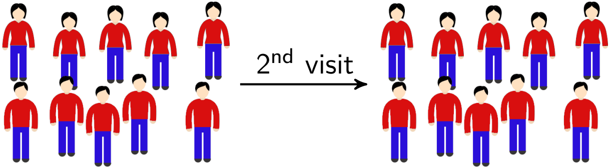
Longitudinal Study



Observe: (x, s_t)

Longitudinal Study



Observe: (x, s_t) (x, s_{t+1})

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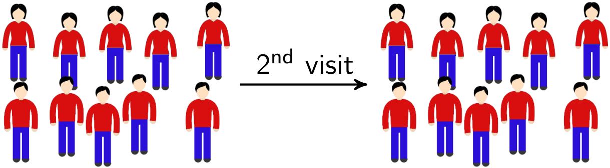

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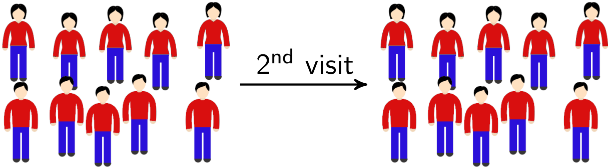
- $y = s_{t+1} - s_t$

Longitudinal Study



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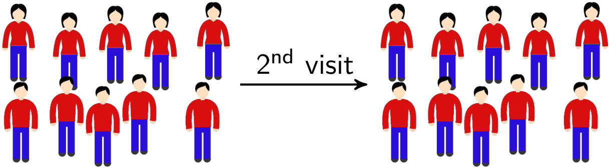

Observe: (x, s_t) (x, s_{t+1})

- $y = s_{t+1} - s_t$
- Assume $f : x \to y$. Specific examples of x, y?

Covariates/Response in Alzheimer's Disease Studies

Examples of Covariates $\mathbf{x}$:

Imaging

Covariates/Response in Alzheimer's Disease Studies

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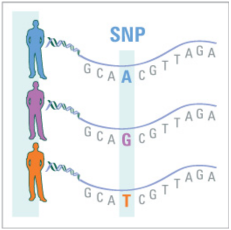

Genetics

Covariates/Response in Alzheimer's Disease Studies

Examples of Covariates $\mathbf{x}$:

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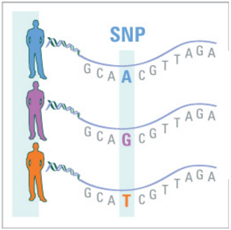

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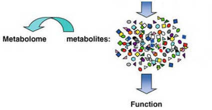

Metabolomics

Covariates/Response in Alzheimer's Disease Studies

Examples of Covariates $\mathbf{x}$:

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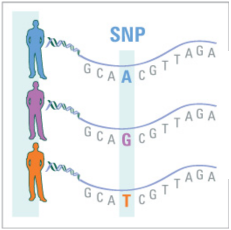

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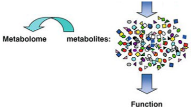
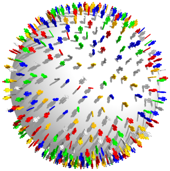


Metabolomics

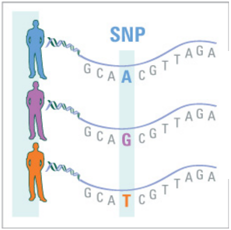


Demographics

Covariates/Response in Alzheimer's Disease Studies

Examples of Covariates **x** :

Imaging



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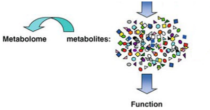

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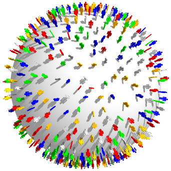


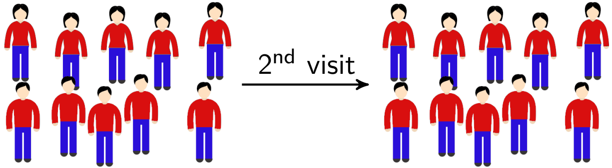
Demographics

Examples of Response **y** : Clinical Dementia Rating (CDR), Alzheimer's Disease Assessment Score (ADAS) etc..

Financial implications

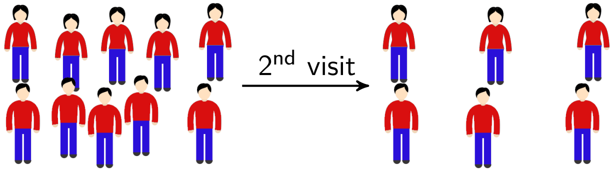
Financial implications

500 **patients with cost** $\$150$**/visit** $= \$150 \times 500 = \$75000$.



Financial implications

300 **patients with cost** $\$150$/**visit** = $\$150 \times 300 = \$45000$.



Savings=$\$30000$.

How to choose x_i's in a meaningful way?

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- Active Learning approaches

How to choose x_i's in a meaningful way?

NOT allowed to get y_i's on a cost to know basis.

- Active Learning approaches

How to choose x_i's in a meaningful way?

- Active Learning approaches
- Variational approaches

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Compute integrals, not scalable to high dimensions/large data setting.

How to choose x_i's in a meaningful way?

- Active Learning approaches
- Variational approaches
- Sensing Matrix

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Covariate data agnostic, not applicable here.

How to choose x_i's in a meaningful way?

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- Experimental Design

How to choose x_i's in a meaningful way?

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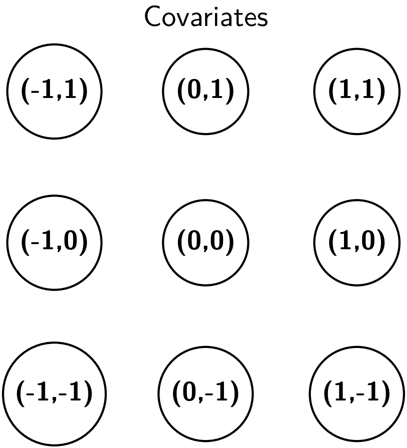
- Experimental Design ✓

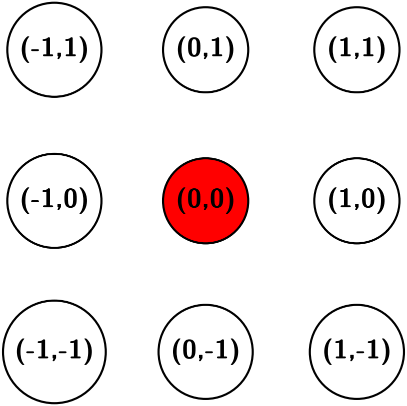
Well studied when f is linear. Often solved using Convex Optimization.

Example in 2−D: Choose 4 points



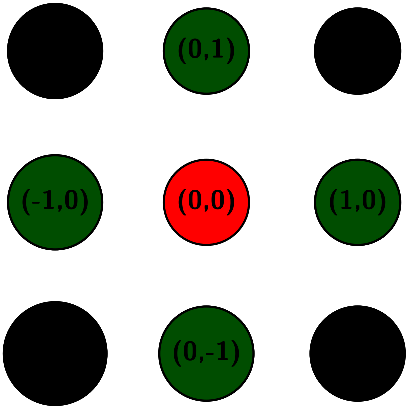
Example in 2−D: Choose 4 points

Origin is not useful.



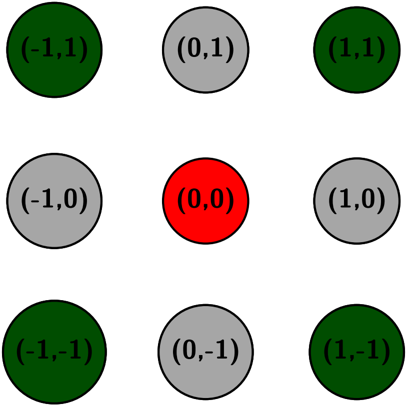
Example in 2−D: Choose 4 points

Not a great Choice!



Example in 2−D: Choose 4 points

Ah! Much better.



$D-$Optimal Design

$$\min_{\mathcal{S} \subseteq N} \log \det \left(\sum_{i \in \mathcal{S}} x_i {x_i}^T \right)^{-1} \quad \text{s.t.} \quad |\mathcal{S}| \leq B$$

$D-$Optimal Design

Information Matrix

$$\min_{\mathcal{S} \subseteq N} \log \det \left(\sum_{i \in \mathcal{S}} x_i {x_i}^T \right)^{-1} \quad \text{s.t.} \quad |\mathcal{S}| \leq B$$

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Information Matrix

−Shannon Differential Information Content

$D-$Optimal Design

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Full set of subjects

–Shannon Differential Information Content

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Information Matrix

Full set of subjects

–Shannon Differential Information Content

- $A-$Optimal: Replace $\log\det$ by trace and so on. $\log\det$ is just a placeholder.

MINLO Formulation

$$\min_{\mu_i \in \{0,1\}} \log\det\left(\sum_{i \in \mathcal{S}} \mu_i x_i {x_i}^T\right)^{-1} \quad \text{s.t.} \quad 1^T \mu \leq B$$

Convex Formulation

$$\min_{\mu_i \in [0,1]} \log\det\left(\sum_{i\in\mathcal{S}} \mu_i x_i x_i^T\right)^{-1} \quad \text{s.t.} \quad 1^T\mu \le B$$

Relax

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Costs

Relax

Convex Formulation

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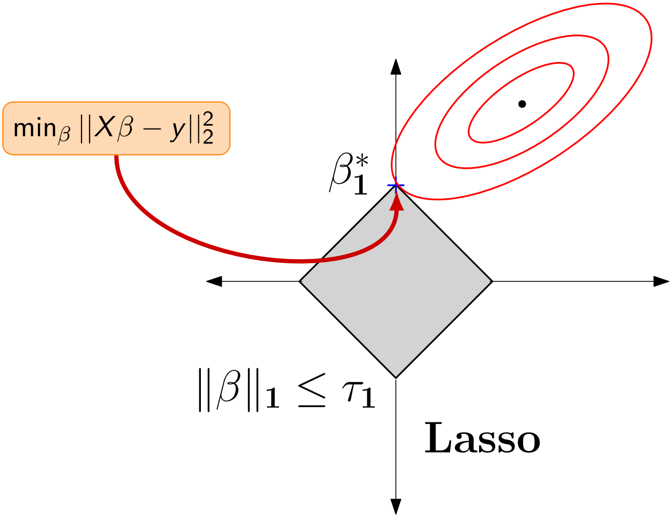
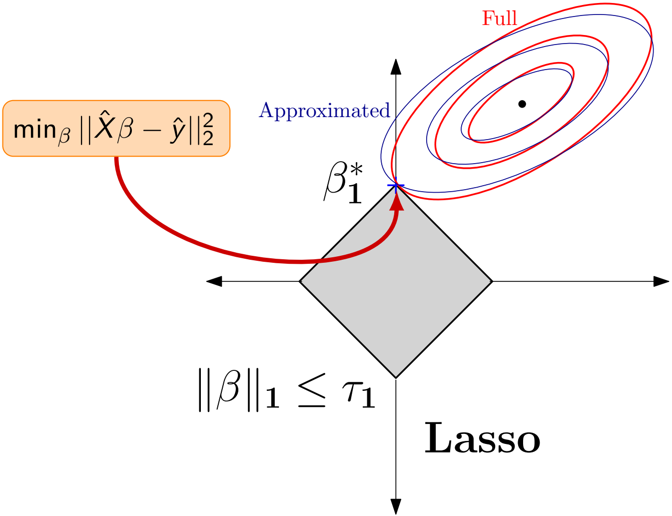
Goal: Extend to f sparse linear

Sparse f: Feature Selection

Look at constrained formulations



Look at constrained formulations



ED–S

$$\min_{\mu \in [0,1]^n} \log\det \left(\sum_{i=1}^{n} \mu_i x_i x_i^T \right)^{-1} + \lambda R_\Gamma(\Gamma_\mu) \quad \text{s.t.} \quad c^T \mu \leq B$$

ED–S

Linearity

$$\min_{\mu \in [0,1]^n} \log\det \left(\sum_{i=1}^{n} \mu_i x_i x_i^T \right)^{-1} + \lambda R_\Gamma(\Gamma_\mu) \quad \text{s.t.} \quad c^T \mu \leq B$$

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Regularization Parameter

Curvature

ED–S

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- Γ denotes the set of eigenvectors of the full set.
- R measures the distance between the eigenvectors of the full and the selected subset.
- Explicitly...

ED–S

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ED–Spectral (ED–S) with the top eigenvector γ

$$\min_{\mu,u} \log\det\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1} + \lambda||\gamma - u||_2^2 + u^T\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1} u$$

$$\text{s.t. } 0 \le \mu \le 1, \quad c^T\mu \le B, \quad ||u||_2^2 = 1$$

ED–Spectral (ED–S) with the top eigenvector γ

Quadratic over linear

$$\min_{\mu,u} \log\det\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} + \lambda||\gamma - u||_2^2 + u^T\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} u$$

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Nonconvex

ED–S More generally

$$\min_{\mu,U} \log\det\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1} + \sum_{j=1}^{k}\lambda_j||\gamma_j - u_j||_2^2 + \mathrm{tr}\left(U^T\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1}U\right)$$

$$\text{s.t. } 0 \le \mu \le 1, \quad c^T\mu \le B, \quad U^TU = I$$

ED–S More generally

Convex Optimization with Orthogonality Constraints

$$\min_{\mu,U} \log\det\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1} + \sum_{j=1}^{k}\lambda_j||\gamma_j - u_j||_2^2 + \mathrm{tr}\left(U^T\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1}U\right)$$

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ED–S Algorithm

Alternating Minimization or Batch Coordinate Descent Algorithm

$$\min_{\mu,U} \log\det\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1} + \sum_{j=1}^{k}\lambda_j||\gamma_j - u_j||_2^2 + \mathrm{tr}\left(U^T\left(\sum_{i=1}^{n}\mu_i x_i x_i^T\right)^{-1}U\right)$$

$$\text{s.t. } 0 \le \mu \le 1, \quad c^T\mu \le B, \quad U^TU = I$$

(i) Fix U, update μ

(ii) Fix μ, update U

Practical implications

- Formulated geometric intuition into a well studied optimization problem.

Synopsis

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- # Decision variables increased to $pk + n$

Practical implications

- Formulated geometric intuition into a well studied optimization problem.
- # Decision variables increased to $pk + n$
- Still nonconvex! Can we do better?

Restricted Isometry Property (RIP)

Definition

Let $X \in \mathbb{R}^{n \times p}$. For $s \geq 0$, the s-restricted isometry constant δ_s of X is the smallest nonnegative number δ such that $(1-\delta)||\beta||_2 \leq ||X\beta||_2 \leq (1+\delta)||\beta||_2$ for all $s-$sparse β, i.e., $||\beta||_0 \leq s$. If $\delta_s < 1$, then X is a $s-$restricted isometry ($s-$RIP).

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Theorem (Candes et al.)

Suppose X has $4s-$RIP constant $\delta_{4s} \leq \frac{1}{4}$. Let $\beta_0 \in \arg\min_\beta\{||\beta||_0 : X\beta = y\}$ and $\beta_1 \in \arg\min_\beta\{||\beta||_1 : X\beta = y\}$. If $||\beta_0||_0 \leq s$, then $\beta_1 = \beta_0$.

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Checking RIP is NP Hard! (Bandeira et al.)

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Relax, RIP is only sufficient

"Certainly, the RIP is not the only property of a design (sensing) matrix X which allows to obtain good error bounds for ℓ_1-recovery of sparse signals"–Nemirovski

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Coherence:

$$\max_{i=1,\ldots,n} \operatorname{diag}\left(X\left(X^TX\right)^{-1}X^T\right)$$

ED–Incoherent (ED–I)

$$\min_{\mu} \log\det \left(\sum_{i=1}^{n} \mu_i x_i x_i^T \right)^{-1} + \lambda \max_{i} \left(\mu_i e_i^T X \left(\sum_{i=1}^{n} \mu_i x_i x_i^T \right)^{-1} X^T e_i \right)$$

$$\text{s.t. } \mu \in \{0,1\}^n, \quad c^T \mu \leq B$$

ED–I

$$\min_{\mu} \log\det\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} + \lambda \max_{i}\left(\mu_i^2 e_i^T X \left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} X^T e_i\right)$$

$$\text{s.t. } \mu \in \{0,1\}^n, \quad c^T\mu \le B$$

ED–I

Set $\mu_i e_i^T X = z_i$

$$\min_{\mu} \log\det\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} + \lambda \max_{i}\left(z_i^T\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} z_i\right)$$

$$\text{s.t. } \mu \in \{0,1\}^n, \quad c^T\mu \le B$$

ED–I

Quadratic over linear

$$\min_{\mu} \log\det\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} + \lambda \max_i \left(z_i^T \left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} z_i \right)$$

$$\text{s.t. } \mu \in [0,1]^n, \quad c^T\mu \leq B, \quad \mu_i e_i^T X = z_i \ \forall\ i = 1, \ldots, n$$

Relax

ED–I

$$\min_{\mu} \log\det\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} + \lambda \max_{i}\left(\mu_i^2 e_i^T X \left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} X^T e_i\right)$$

$$\text{s.t.} \;\; \mu \in [0,1]^n, \quad c^T \mu \leq B$$

Convex Optimization Problem with no extra variables!

ED–I Coordinate Descent Algorithm

$$\min_{\mu} \log\det\left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} + \lambda \max_i \left(\mu_i^2 e_i^T X \left(\sum_{i=1}^{n} \mu_i x_i x_i^T\right)^{-1} X^T e_i\right)$$

$$\text{s.t. } \mu \in [0,1]^n, \quad c^T \mu \le B$$

for $t = 1, 2, ..., T$ do
    for $k = 1, 2, ..., n$ do
        Pick $i \in \{0, ..., n\}$, $\mu_i \leftarrow \mu_i - \eta \nabla_{\mu_i} f$
    end for
    Project μ
end for
Output: μ^*

How do we round (the fractional) μ^* with guarantees?

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Is Pipage Rounding scheme applicable?

- For any feasible μ, need a vector ν, $\delta, \tau > 0$ s.t. $\mu + \delta\nu$ or $\mu - \tau\nu$ is more integral.
- f must be convex in the direction ν.
- Need μ s.t. $f(\mu) \leq \kappa \cdot \text{opt}$

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Convexity

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- f must be convex in the direction ν. ✓
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Choose $e_{i_f} - e_{j_f}$

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- f must be convex in the direction ν. ✓
- Need μ s.t. $f(\mu) \leq \kappa \cdot \text{opt}$ ✓

Efficiently implementable/Derive approximation ratios. ✓

Experiments

Evaluations setup

- Evaluate the consistency of both the models: Do ED-I and ED-S agree?
- Sensitivity analysis with respect to the budget.
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Need f to be sparse linear – use standard datasets.

Standard Dataset

prostate

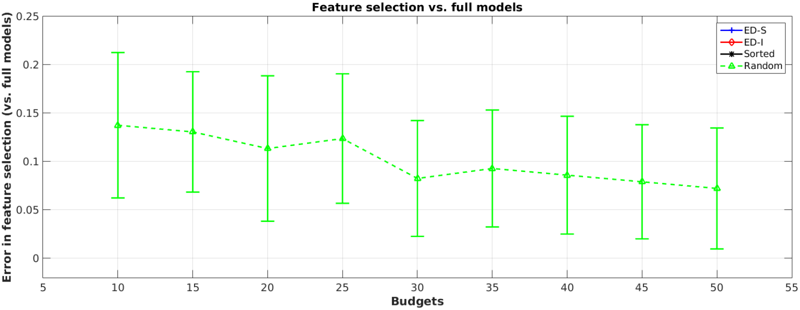
x : cancer volume, prostate weight, age, Gleason score etc.

y : prostate specific antigen
n : 97

Standard Dataset

prostate

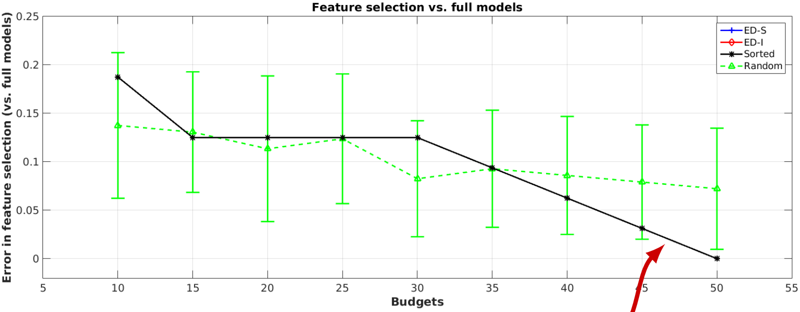
Sensitivity with Budgets



Standard Dataset

prostate

Sensitivity with Budgets

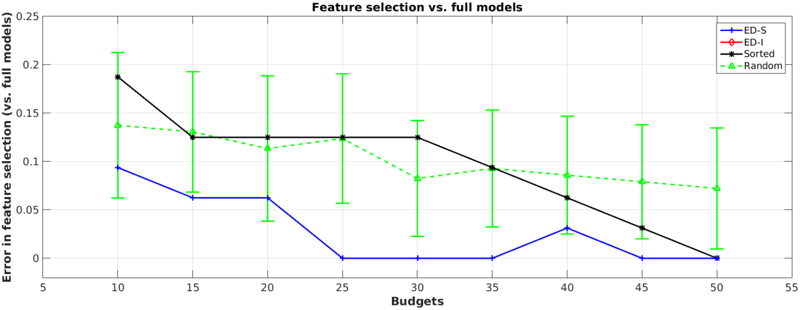


sorted $\approx$ $k-$means

Standard Dataset

prostate

Sensitivity with Budgets



prostate

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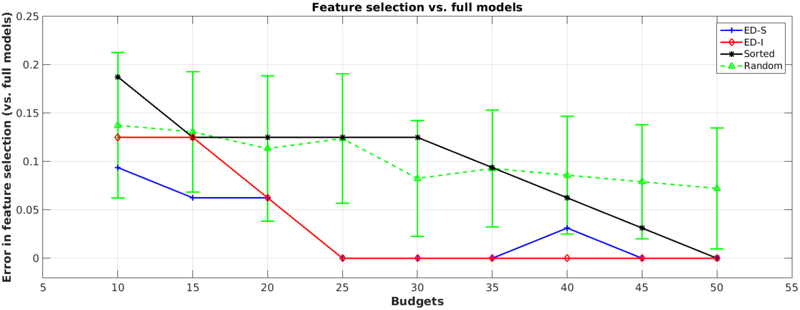
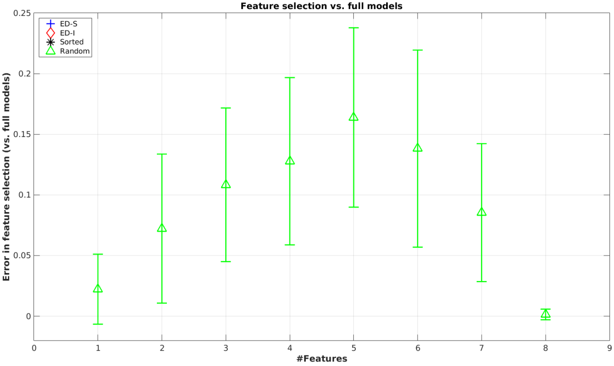


Standard Dataset

prostate

Feature selection errors



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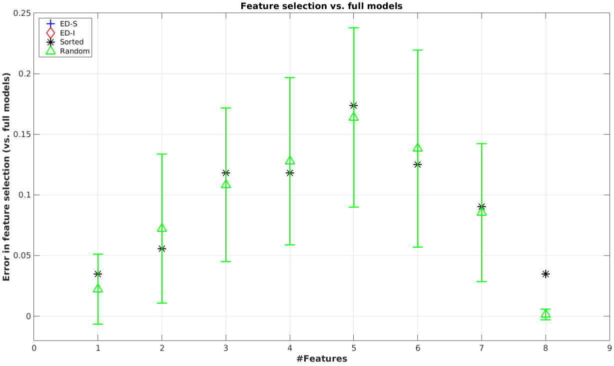

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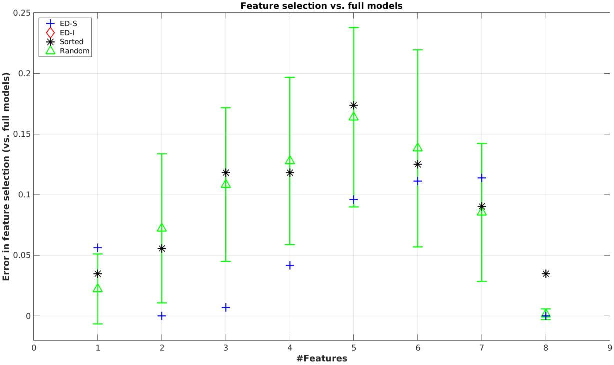
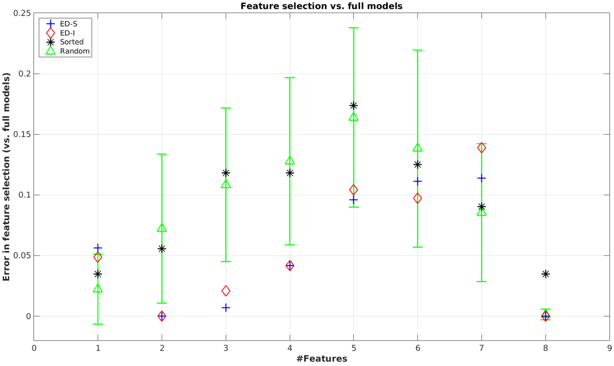


Standard Dataset

prostate

Feature selection errors

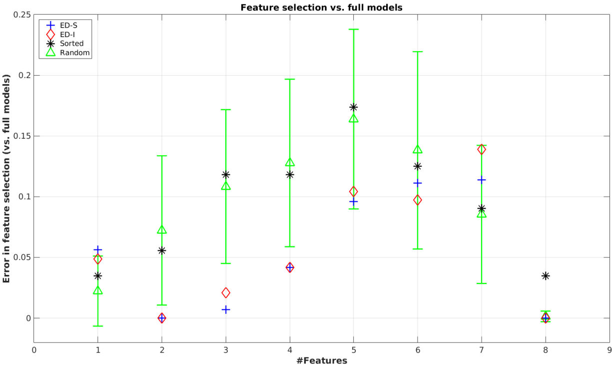


Standard Dataset

Similar results on lars dataset

prostate

Feature selection errors



Neuroimaging Dataset

ADNI: $x_i \in \mathbb{R}^{118}$, $n = 985$

x : Image derived ROI summaries from PET

y : 1. ADAS, 2. CDR

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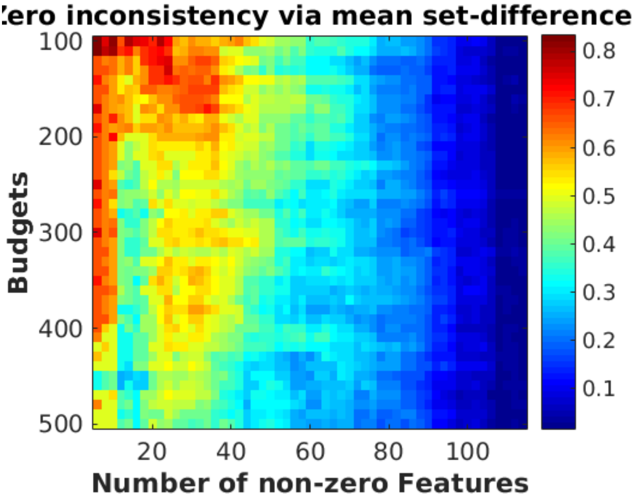

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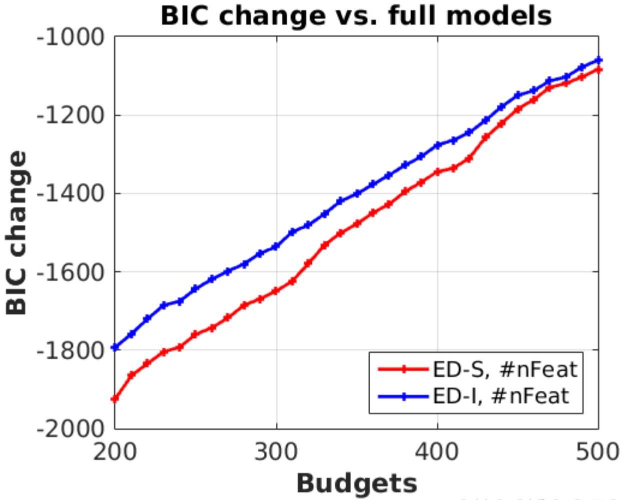

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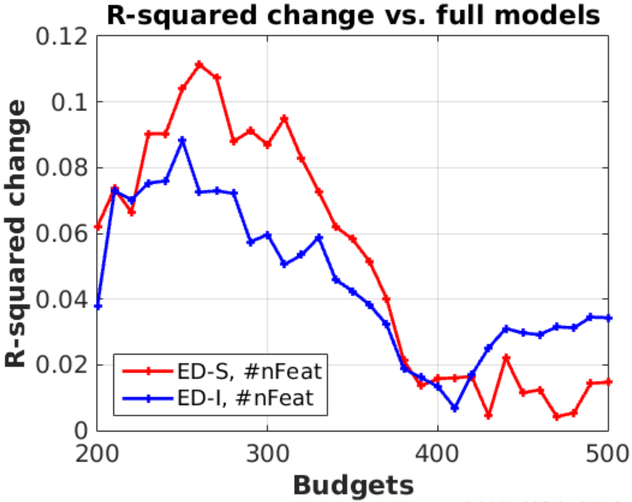
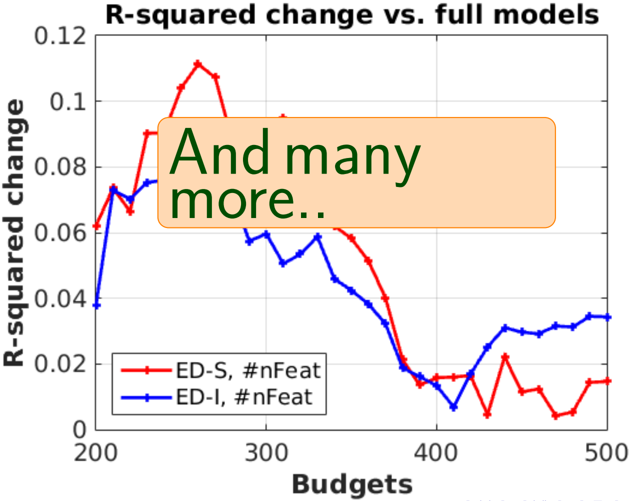


ADNI



And many more..

Conclusions

Conclusions

Power calculations → Baseline Data (N) → Followup Data (N) → Analyze

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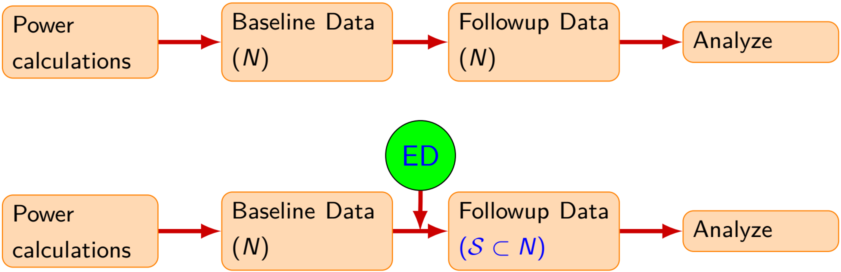
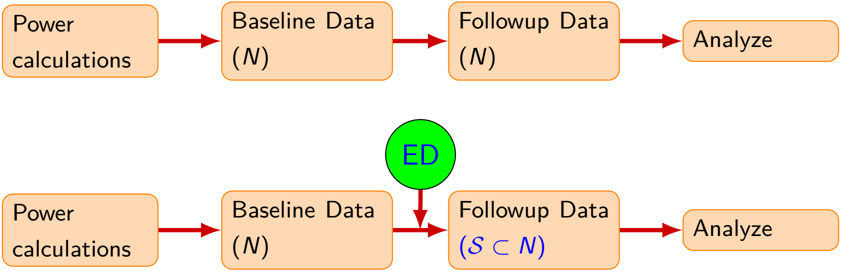


Conclusions



- Proposed algorithmically efficient formulations for Experimental Design for Sparse Linear Model which are effective practically as well.
- Future direction: Theory for adaptive statistical power calculations using this framework.

The end...

Thank you!
Questions?

Code is publicly available at https://github.com/sravi-uwmadison/Exp_design_sparse.
See you at the poster session (today 3 pm - 7 pm)!

We are supported by NIH R01 AG040396 (VS), NSF CAREER RI 1252725, NIH R01 AG027161 (SCJ), NSF CCF 1320755, UW ADRC (P50 AG033514) and UW CPCP (U54 AI117924).