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Exercise 1

$$x_n = \frac{n}{n+1} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\} \quad n \in \mathbb{N}$$

$$y_n = \frac{n+1}{n} = \left\{2, \frac{3}{2}, \frac{4}{3}, \cdots\right\}, n \in \mathbb{N}$$

and the $\lim x_n = \lim y_n$ that is 1

$$\lim x_n = 1$$

$$\lim y_n = 1$$

And ~~p~~ than $\lim x_n \nless \lim y_n$

Exercise 2

$$x_n = \sqrt{\frac{1}{(n^2+1)}}$$

Let $\epsilon > 0$ and $\alpha = \epsilon^2 > 0$ than $\epsilon = \sqrt{\alpha}$. And assume that $n \in \mathbb{N}$ such that $\frac{1}{n^2+1} < \alpha$ for any $n^2+1 > n$

then square root both sides and we have $\sqrt{\frac{1}{n^2+1}} < \sqrt{\alpha}$ as we know $\sqrt{\alpha} = \epsilon$ than we can say $\sqrt{\frac{1}{n^2+1}} < \epsilon$ and it is convergent.

Limit $x_n = \sqrt{\frac{1}{n^2+1}}$ square root of 1 is $\sqrt{1} = 1$

Multipy with $\frac{1}{n^2}$

$$x_n = \sqrt{\frac{\frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}} \Rightarrow \sqrt{\frac{\frac{1}{n^2}}{1 + \frac{1}{n^2}}}$$

$$\lim x_n = \lim \sqrt{\frac{1/n^2}{1 + \frac{1}{n^2}}}$$

$$\lim x_n = \sqrt{\frac{\lim\left(\frac{1}{n} \cdot \frac{1}{n}\right)}{\lim(1) + \lim\left(\frac{1}{n} \cdot \frac{1}{n}\right)}}$$

as we know $\lim\left(\frac{1}{n}\right) = 0$

$$\lim x_n = 0$$

(b) $$y_n = \frac{2 - 3n^2 + n^3}{1 + 2n - 3n^3} \text{ for } n \in \mathbb{N}$$

equation divide by n^3

$$= \frac{\frac{2}{n^3} - \frac{3n^2}{n^3} + \frac{n^3}{n^3}}{\frac{1}{n^3} + \frac{2n}{n^3} - \frac{3n^3}{n^3}}$$

$$= \frac{\frac{2}{n^3} - \frac{3}{n} + 1}{\frac{1}{n^3} + \frac{2}{n^2} - 3}$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(\frac{2/n^3 - \frac{3}{n} + 1}{\frac{1}{n^3} + \frac{2}{n^2} - 3}\right)$$

$$= \frac{2\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\cdot\frac{1}{n}\right) - 3\lim_{n\to\infty}\left(\frac{1}{n}\right) + \lim_{n\to\infty}(1)}{\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\cdot\frac{1}{n}\right) + 2\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1}{n}\right) - \lim_{n\to\infty}(3)}$$

$$= \frac{2\lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right) - 3\lim_{n\to\infty}\left(\frac{1}{n}\right) + \lim_{n\to\infty}(1)}{\lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right) + 2\lim_{n\to\infty}\left(\frac{1}{n}\right)\lim_{n\to\infty}\left(\frac{1}{n}\right) - \lim_{n\to\infty}(3)}$$

As we know $\lim(\frac{1}{n}) = 0$

then.

$$= \frac{2.0 - 3.0 + 1}{0 + 2.0 - 3}$$

$$= \frac{1}{-3} \Rightarrow -\frac{1}{3}$$

c) Suppose Z_n is converges let $m \in R$ be bound for Z_n then there exist $n \in N$ such that $n > m$. And we know that $2^n > n > m$ ~~by~~ but $2^n = Z_n$ a contradiction. Therefore Z_n is divergent and 2^n is also divergent for $n \in N$

Exercise 3

$x_1 = 1, \; x_2 = 2, \; x_3 = 1.5$

$x_n = \{1, 2, 1.5 \, ...\}$

then firstly we prove

$$0 < x_n < x_{n+2} \leq 2$$

than $x_{n+2} \leq 2$ then prove by induction

$$x_{n+3} = \frac{1}{2}(x_{n+1} + x_{n+2}) > x_{n+1}$$

since $\frac{1}{2}(x_{n+1} + x_{n+2}) \leq 2$, ~~then~~ and

$x_{n+2} > 0$, we also know $x_{n+3} > 0$

then $0 < x_n < x_{n+2} < x_{n+3} \leq 2$

hold by induction.

then $x = \lim_n x_n$ then after four element the value is near $1.6***$

$$x_{n+3} x_{n+2} = x_n + 2$$

$2x_n^2 = x_n + 2 \Rightarrow x_n = \sqrt{2}$ that is $1.647..$

Bonus Exercise :

Let $m > 0$ be such that $|x_n| \leq m$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ and let $n_1 \in \mathbb{N}$ be such that for all $n > \cancel{N}\, n_1$ we have $|x_n - a| < \epsilon/2$.

Let $n \cancel{N} > 2(M+a)n_1/\epsilon$ for any $n > n_1$

$$|y_n - a| = \left|\frac{(x_1 - a) + \cdots + (x_n - a)}{n}\right| \leq \frac{|x_1 - a|}{n} + \cdots + \frac{|x_n - a|}{n}$$

finally $|x_i - a| \leq \cancel{m+a}\; m + a$ when $\cancel{i \leq N_1}$ $i \leq n_1$ and $|x_i - a| < \epsilon/2$ when $i > n_1$ then.

$$|y_n - a| \leq \frac{n_1(m+a)}{n} + \left(\frac{n - n_1}{n}\right)\epsilon \leq \frac{n_1(m+a)}{n} + \left(\frac{n - n_1}{n}\right)\epsilon$$

$\leq \epsilon$ for all $n > N$