Assignment 2

Exercises

P1 $a, b, c \in \mathbb{Q}$ and $a \neq 0$

$$ax = b \to x = ba^{-1}$$

$axa^{-1} = ba^{-1}$ property in slide ($xz = yz \to x = y,\ z \neq 0$)
$aa^{-1}x = ba^{-1}$ commutativity
$1x = ba^{-1}$ identity
$x = ba^{-1}$

In order to prove the uniqueness we will proceed by contradiction.
Assume there are two different solutions, x and y.
$ax = b$ and $ay = b$ therefore $ax = ay$
Now using the field axioms and properties we can prove it.
$axa^{-1} = aya^{-1}$ property in slide ($xz = yz \to x = y,\ z \neq 0$)
$aa^{-1}x = aa^{-1}y$ commutativity
$1x = 1y$ identity
$x = y$

We have just showed that the solution is unique.

P2 For all $x, y \in \mathbb{Q}$ if $0 < x < y$ then $x^2 < y^2$.
Knowing that $0 < x < y$ we can use the order axiom number 5 (O5).
Starting from $x < y$ we can multiply both members by x one time and a second one by y.
$xx < yx$ and $xy < yy$ by O5 therefore
$xx < yx = xy < yy$ commutativity
Since

$$x^2 = xx < yy = y^2$$

we proved

$$x^2 < y^2$$

P3 Base case n=1.
$|x_1| \leq |x_1|$ is true.
Let n be an arbitrary element of $\mathbb{N}$, assume $P(n)$ which is

$$|x_1 + x_2 + ... + x_n| \leq |x_1| + |x_2| + ... + |x_n|$$

and prove $P(n+1)$.
$P(n+1)$ is

$$|x_1 + x_2 + ... + x_n + x_{n+1}| \leq |x_1| + |x_2| + ... + |x_n| + |x_{n+1}|$$

We can write $P(n+1)$ as $|x_1 + x_2 + ... + x_n + x_{n+1}| = |(x_1 + x_2 + ... + x_n) + x_{n+1}|$
Now we are going to prove that $|x + y| \leq |x| + |y|$.
Since $x \leq |x|, y \leq |y|$ and $-x \leq |x|, -y \leq |y|$ we can say $x + y \leq |x| + |y|$ and

$-(x+y) = -x-y \leq |x|+|y|$ which means $|x+y| \leq |x|+|y|$.
Therefore if we call $x = (x_1 + x_2 + ... + x_n)$ and $y = x_{n+1}$ then

$$|(x_1 + x_2 + ... + x_n) + x_{n+1}| = |x+y|$$

therefore

$$|x+y| \leq |x|+|y|$$

which is

$$|(x_1 + x_2 + ... + x_n) + x_{n+1}| \leq |(x_1 + x_2 + ... + x_n)| + |x_{n+1}|$$

Using the induction hypothesis

$$|(x_1 + x_2 + ... + x_n)| + |x_{n+1}| \leq |x_1| + |x_2| + ... + |x_n| + |x_{n+1}|.$$

We have just showed that $\forall n \in \mathbb{N}\ P(n)$ holds.

P4 Bonus Exercise
Given $a, b, c \in \mathbb{R}$ which satisfy the equation

$$a+b=c.$$

Is also true that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$?
$a, b, c \neq 0$
$\frac{1}{a} + \frac{1}{b} = \frac{1}{c} \Rightarrow$
$\Rightarrow \frac{b+a}{ab} = \frac{1}{a+b} \Rightarrow$ $\qquad a+b=c$
$\Rightarrow \frac{(a+b)^2}{ab} = 1 \Rightarrow$ $\qquad a, b, c \neq 0$
$\Rightarrow (a+b)^2 = ab$ which is $(a+b)^2 - ab = 0 \Rightarrow$
$\Rightarrow a^2 + ab + b^2 = 0$
$\Delta = \sqrt{b^2 - 4b^2} = \sqrt{-3b^2}$
knowing that $\forall x \in \mathbb{R}, x \geq 0$ and $x^2 = 0 \Leftrightarrow x = 0$
Since $\Delta = \sqrt{-3b^2}$ and $-3b^2$ is a negative number the equation has no solutions.