Assignment 6

Exercises

P1 $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$. To prove that F_{3n} is even. ($F_{3n} = 2k\ k \in \mathbb{Z}$)
Base case: $n = 1$, $F_3 = F_2 + F_1 = 1 + 1 = 2$ is even.
Inductive Hypothesis: for some $n \in \mathbb{N}$ F_{3n} is even.
Inductive Step:
$F_{3(n+1)} = F_{3n+3} = F_{3n+2} + F_{3n+1} = F_{3n} + 2F_{3n+1}$ by I.H $\exists h \in \mathbb{Z}$ s.t $F_{3n} = 2h$, therefore $2h + 2(F_{3n+1}) = 2(h + F_{3n+1})$ where $h, F_{3n+1} \in \mathbb{Z}$ because F_n belongs to the natural number, $\mathbb{N}$ is closed under addition.
Hence $h + F_{3n+1} \in \mathbb{Z}$ and $F_3 n + 1$ is even. By induction we have proven that F_{3n} is even $\forall n \in \mathbb{N}$.

P2 (a)
We can use a geometric series we already know very well, therefore

$$\sum_{k=0}^{\infty} \frac{4}{3^k} = \lim \sum_{k=0}^{n} \frac{4}{3^k} = 4\frac{1}{1-\frac{1}{3}} = 4\frac{3}{2} = 6$$

(b)

$$\sum_{k=1}^{\infty} \frac{k^4}{2^k}$$

Let's prove using the ratio test that the sum converges.
$\lim \frac{(k+1)^4}{2^{k+1}} \frac{2^k}{k^4} = \lim \frac{1}{2} \frac{(k+1)^4}{k^4} = \frac{1}{2}$
$\lim \frac{k+1}{k} = \lim 1 + \frac{1}{k} = 1$ therefore $\lim \underbrace{\frac{k+1}{k} \cdot ... \cdot \frac{k+1}{k}}_{4} = \lim \frac{k+1}{k} \cdot \lim \frac{k+1}{k} \cdot \lim \frac{k+1}{k} \cdot$ $\lim \frac{k+1}{k} = 1 \cdot 1 \cdot 1 \cdot 1 = 1$ Hence the series converges. We don't know to what. (Using maxima we find out that the limit is 150)

(c)

$$\sum_{k=1}^{\infty} \frac{k(-1)^k}{k+2}$$

We will show that it diverges using the divergence test (not verifying that $(x_k) \to 0$)
$\lim \frac{k(-1)^k}{k+2} = \lim(-1)^k \cdot \frac{k}{k+2}$.
We know that $\lim \frac{k}{k+2} = \lim \frac{1}{\frac{k+2}{k}} = \lim \frac{1}{1+\frac{2}{k}} = 1$
Since we can find two subsequences which converge to different limits, the sequence does not converge.
The subsequences are (x_{2h}) s.t $\exists h \in \mathbb{Z}$ and $\lim(-1)^{2h} \cdot \frac{2h}{2h+2} = 1$ and (x_{2j+1}) s.t $\exists j \in \mathbb{Z}$ and $\lim(-1)^{2j+1} \cdot \frac{2j+1}{(2j+1)+2} = -1$, therefore there is no limit for $(-1)^k \cdot \frac{k}{k+2}$.
Hence, the necessary condition is not respected. The series is divergent.

P3 $t : 1;$
$x[n] := sum(1/k!, k, 0, n);$

$while\ x[t] < 2.718do$
$(t : t + 1);$
$done;$
$display(t, float(x[t]));$

after 6 iterations we get 2.178055555555

$t : 1;$
$y[n] := (1 + 1/n)^n;$

$while\ y[t] < 2.718do$
$(t : t + 1);$
$done;$
$display(t, float(y[t]));$

after 4822 iterations we get 2.718000019543363

$t : 1;$
$z[n] := n/(n!)^(1/n);$
$while\ bfloat(z[t]) < 2.718\ do$
$(t : t + 1);$
$done;$
$display(t, bfloat(z[t]));$

It is very slow. It requires 62084 iterations to compute $2.7180000031790 12b0$.

P4 Let (a_n) be a bounded sequence of real numbers. That means
$\exists\ d = \max\{|\inf\{(a_n) \mid n \in \mathbb{N}\}|, |\sup\{(a_n) \mid n \in \mathbb{N}\}|\}$,
therefore we will use the comparison test to prove that $\sum_{k=1}^{n} a_k x^k$ converges.

We can choose $y_n = d|x|^n$, $d|x|^n \geq 0$ because d and $|x|^n$ have absolute values.

Since $|a_n| \leq d\ \forall n \in \mathbb{N}$, then $|a_n x^n| \leq d|x|^n\ \forall n \in \mathbb{N}$ with $|x| < 1$.
Since $\sum_{k=1}^{n} d|x|^k = d \sum_{k=1}^{n} |x|^k = d\frac{|x|-|x|^{n+1}}{1-|x|}$ then
$\lim \sum_{k=1}^{n} d|x|^k = \lim d \cdot \lim \frac{|x|-|x|^{n+1}}{1-|x|} = d\frac{|x|}{1-|x|}$, because $|x| < 1$ it converges.
By the comparison test, $\sum_{k=1}^{n} a_k x^k$ converges.