Assignment 8

Exercises

P1 Let $f : [-1, 1] \to \mathbb{R}$ and $f(x) = x^2 + 3x + 2$
If $x, y \in \mathbb{R}\ \ x < y$ then $f(x) < f(y)$
We can prove it by contradiction, assuming that $f(x) \geq f(y)$.
Since $y > x$ we can write $y = x + \epsilon$ for some $\epsilon \in \mathbb{R}, \epsilon > 0$.
$f(x) = x^2 + 3x + 2 \geq (x + \epsilon)^2 + 3(x + \epsilon) + 2 = f(y)$
$x^2 + 3x + 2 \geq x^2 + 2x\epsilon + \epsilon^2 + 3x + 3\epsilon + 2$
$0 \geq 2x\epsilon + \epsilon^2 + 3\epsilon$
$0 \geq \epsilon^2 + \epsilon(2x + 3)$
Since $\epsilon > 0$ and $-1 \leq x \leq 1$
$0 \geq \epsilon^2 + \epsilon(2x + 3) > 0$ therefore $0 > 0$.
Which is a contradiction therefore if $x < y$ then $f(x) < f(y)$ (Strictly increasing).

P2 Lets consider $f : [-2, 1] \to \mathbb{R}$ with $f(x) = x^2$
Then $f(-2) = 4$ and $f(1) = 1$ therefore $f^{-1} : [4, 1] \to \mathbb{R}$ but it means that $4 \leq 1$ which is clearly incorrect. Since is not strictly increasing we cannot say that $a < x < b$ then $f(a) < f(x) < f(b)$.
When we consider $f : [a, b] \to \mathbb{R}$ is strictly increasing but not continuous then f^{-1} will not be necessarily increasing and continuous. We cannot apply the intermediate theorem, and therefore we cannot say if there exists always a $c \in (a, b)$ with $f(c) = y$ and therefore cannot take $f^{-1}(y) = c$.

P3 We want to show that $\forall\epsilon > 0, \exists\delta > 0,\ \forall x, y \in D$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
Let $\epsilon > 0$, we need to find $\delta > 0$ such that $|x - y| < \delta$.
Consider $\delta = \epsilon^2$. Since $\sqrt{x}, \sqrt{y} \geq 0$ then $|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|$ and since $|xy| = |x||y|$ then $|x - y||x + y| = |(x - y)(x + y)| = |x^2 - y^2|$.
Hence, $|\sqrt{x} - \sqrt{y}|^2 = |\sqrt{x} - \sqrt{y}||\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y| < \delta = \epsilon^2$
Then $|\sqrt{x} - \sqrt{y}| < \epsilon$, therefore we have proved the definition of uniform continuity.

P4 Let $a > 0$, $x_1 = \sqrt{a}$, $x_{n+1} = \sqrt{x_n}$ and $y_n = 2^n(x_n - 1)$
We can prove by induction that $x_n = a^{\frac{1}{2^n}}$
Base case: $x_1 = a^{\frac{1}{2}}, x_2 = \sqrt{a^{\frac{1}{2}}} = a^{\frac{1}{2^2}}$
Let $n \in \mathbb{N}$ be arbitrary and $x_n = a^{\frac{1}{2^n}}$.
$x_{n+1} = x_n^{\frac{1}{2}} = \text{I.H} = \sqrt{a^{\frac{1}{2^n}}} = a^{\frac{1}{2^{n+1}}}$
Therefore $x_n = a^{\frac{1}{2^n}}\ \forall n \in \mathbb{N}$.
We can rewrite $y_n = 2^n(a^{\frac{1}{2^n}} - 1) = \frac{a^{\frac{1}{2^n}} - 1}{\frac{1}{2^n}}$. Hence $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{a^{\frac{1}{2^n}} - 1}{\frac{1}{2^n}}$ which can be written as $\lim_{x\to 0} \frac{a^x - 1}{x}$ with $\frac{1}{2^n} = x$.
To solve $\lim_{x\to 0} \frac{a^x - 1}{x}$ we can proceed by substitution
$a^x - 1 = y$ then $a^x = y + 1 \Rightarrow x = \log_a (y + 1)$ and since
$\lim_{x\to 0} \frac{y+1-1}{\log_a (y+1)} = \lim_{x\to 0} \frac{y}{\log_a (y+1)} = \lim_{x\to 0} \frac{1}{\frac{\log_a (y+1)}{y}} = \lim_{x\to 0} \frac{1}{\frac{1}{y} \log_a (y+1)} = \lim_{x\to 0} \frac{1}{\log_a (y+1)^{\frac{1}{y}}}$
and since $\lim_{z\to 0}(z + 1)^{\frac{1}{z}} = e$ because with $z = \frac{1}{x}$ it is $\lim_{x\to\infty}(\frac{1}{x} + 1)^x = e$ then $= \frac{1}{\log_a e} = \frac{\log_e e}{\log_a e} = \log_e a$.