Assignment 4

Exercises

P1 $(x_n) = \frac{1}{n}$ and $(y_n) = -(x_n) = -\frac{1}{n}$
$y_n < x_n,\ \forall n \in \mathbb{N}$ because $x_n > 0,\ \forall n \in \mathbb{N}$ and $y_n < 0,\ \forall n \in \mathbb{N}$
Both sequences converge to the same limit 0.
$(x_n) \to 0$ and $(y_n) = -1 \cdot (x_n) \to 0$ then $(y_n) \to 0$.
Therefore $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$

P2 (a) $(x_n),\ x_n = \frac{1}{\sqrt{n^2+1}}$
By corollary 1.20 $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t $\frac{1}{N} < \epsilon$ and so $\forall n \geq N$ we can say $\frac{1}{n} \leq \frac{1}{N} < \epsilon$
Since $\sqrt{n^2+1} > \sqrt{n^2} = n > N > 0$ then $0 < \frac{1}{\sqrt{n^2+1}} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$
therfore $0 < \frac{1}{\sqrt{n^2+1}} < \epsilon$ which means $|\frac{1}{\sqrt{n^2+1}}| < \epsilon$.
$\forall \epsilon > 0\ \exists N \in \mathbb{N}$ s.t $\forall n \geq N\ |\frac{1}{\sqrt{n^2+1}}| < \epsilon$ which is the definition of convergence.

(b) $(y_n)\, y_n = \frac{2-3n^2+n^3}{1+2n-3n^3}$
We are going to divide by n^3
$y_n = \frac{\frac{2}{n^3} - \frac{3}{n} + 1}{\frac{1}{n^3} + \frac{2}{n^2} - 3}$
We know from $\lim \frac{1}{n} = 0$ that $\lim \frac{1}{n}\frac{1}{n}\frac{1}{n} = 0$ which is $\lim \frac{1}{n^3} = 0$
$\lim \frac{2}{n^3} - \frac{3}{n} + 1 = \lim \frac{2}{n^3} + \lim \frac{-3}{n} + \lim 1 = \lim 1 \lim \frac{1}{n^3} + \lim -3 \lim \frac{1}{n} + \lim 1 = 0+0+1=1$
Now we are going to calculate the second part.
$\lim \frac{1}{n^3} + \frac{2}{n^2} - 3 = \lim \frac{1}{n^3} + \lim \frac{2}{n^2} + \lim -3 = \lim \frac{1}{n^3} + \lim 2 \lim \frac{1}{n^2} + \lim -3 = 0 + 0 - 3$
$= -3$
Since we need $\forall n > N$ we can simply take $N > 1$ therefore $\frac{1}{n^3} + \frac{2}{n^2} - 3 \neq 0$ and we can apply theorem 2.4 point 3.
$\lim \frac{\frac{2}{n^3} - \frac{3}{n} + 1}{\frac{1}{n^3} + \frac{2}{n^2} - 3} = \frac{\lim \frac{2}{n^3} - \frac{3}{n} + 1}{\lim \frac{1}{n^3} + \frac{2}{n^2} - 3} = \frac{1}{-3} = -\frac{1}{3}\ \forall n \geq N$

(c) $(z_n) = 2^n$ for $n \in \mathbb{N}$ Let assume (z_n) has a limit a, it then follows from definition 2.2 that for $\epsilon = \frac{1}{2} > 0$ there exists some $N \in \mathbb{N}$ such that $|a - z_n| < \frac{1}{2}$ for all $n \geq N$.
Since $|z_n - z_{n+1}| = |2^n - 2^{n+1}| = |2^n(-1)| = |2^n|$ and $n \geq N > 1$ then $|2^n| > 2$
Now using the triangle inequality, we can conclude
$2 < |z_n - z_{n+1}| = |(z_n - a + a - z_{n+1}| \leq |a - z_n| + |a - z_{n+1}| \leq \frac{1}{2} + \frac{1}{2} = 1$
Since $2 < 1$ is a contradiction the sequence diverges.

P3 $x_1 = 1$
$x_2 = 2$
$x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$

We can rewrite $x_{n+2} = x_{n+1} + (\frac{-1}{2})^n = x_n + (\frac{-1}{2})^{n-1} + (\frac{-1}{2})^n = x_1 + \sum_{k=0}^{n} (\frac{-1}{2})^k = x_1 + \frac{1}{\frac{2}{3}} =$
for $n \to \infty$ it is $= 1 + \frac{3}{2} = \frac{5}{3}$

P4 $y_n = \frac{x_1+x_2+....+x_n}{n}$

We want to show that
$\forall \epsilon > 0\ \exists N$ s.t. $\forall n \geq \mathbb{N}\ |y_n - L| < \epsilon$
We know that
$\forall \epsilon > 0\ \exists N$ s.t. $\forall n \geq \mathbb{N}\ |x_n - L| < \epsilon$

$|\frac{x_1+x_2+...+x_n}{n} - L| = |\frac{x_1+x_2+...+x_n-mL}{n}| = |\frac{x_1-L+x_2-L+...+x_n-L}{n}|$
$|\frac{x_1-L+x_2-L+...+x_n-L}{n}| \leq |\frac{x_1-L+x_2-L+...+x_{N-1}-L}{n}| + |\frac{x_1-L+x_2-L+...+x_n-L}{n}|$

Since $|\frac{x_1-L+x_2-L+...+x_{N-1}-L}{n}|$ is a constant $a \in \mathbb{R}$

$lim|\frac{x_1-L+x_2-L+...+x_{N-1}-L}{n}| = lim\frac{|a|}{n} = 0$

$\leq \frac{|a|}{n} + \frac{|x_N-L|}{n} + \frac{|x_{N+1}-L|}{n} + \ ... \ + \frac{|x_n-L|}{n}$

$\forall \delta > 0\ \exists N$ s.t. $\forall n \geq \mathbb{N}\ |y_n - L| < \delta$

$< \frac{|a|}{n} + \frac{|\delta|}{n} + \frac{|\delta|}{n} + \ ... \ + \frac{|\delta|}{n}$