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Exercise 1

$$f(x) = \log(x) \qquad f(2) = \log(2)$$

$$f'(x) = \frac{1}{x} \qquad f'(2) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{2^2}$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(2) = \frac{2}{2^3}$$

$$f^4(x) = -\frac{(2)(3)}{x^4} \qquad f^4(2) = -\frac{(2)\cdot(3)}{2^4}$$

$$f^5(x) = \frac{(2)(3)(4)}{x^5} \qquad f^5(2) = \frac{(2)(3)(4)}{2^5}$$

$$f^n(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$$

Taylor series is

$$= \log(2) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\, 2^n}(x-2)^n$$

The maximum difference between f ~~and~~ and T_f is ∞ because the function of $\log(x)$ is asymptotical at $x=0$. So it goes to $-\infty$ and partial sums have all defined value in $\mathbb{R}$ for $x = 0$

Exercise 2

i) Let consider the uniform partition that is

$$\Delta_n : 0 < \frac{1}{n} < \frac{2}{n} < \dots < \frac{n-1}{n} < 1$$

$$x_k = \left\{\frac{x}{n}\right\} \text{ for } k = 0, \dots, n$$

SATISH KUMAR

Now ~~assume~~ we have

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = f(x_{k-1}) = \frac{(k-1)^2}{n^2}$$

and

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = f(x_k) = \frac{k^2}{n^2}$$

and

$$U_{\Delta n} = \sum_{k=1}^{n} \frac{k^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6n^3}$$

and

$$L_{\Delta n} = \sum_{k=1}^{n} \frac{(k-1)^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{n \cdot (n-1)(2n-1)}{6n^3}$$

ii) Limit of $U_{\Delta n}$ is

$$= \lim_{n \to \infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

Limit of $L_{\Delta n}$ is

$$= \lim_{n \to \infty} \frac{2n^3 - 3n^2 - n}{6n^3} = \frac{1}{3}$$

iii) But $L(f) \leq U(f)$ for any bounded function [1, theorem 32.4]. We conclude

$$L(f) = U(f) = \frac{1}{3}$$

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Exercise 3

Let $f(x) = 2$ and $g(x) = 3$ and

$(f \cdot g)(x) = 6$

According to fundamental theorm of calculus $F'(x) = f(x)$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$\int_0^1 (f \cdot g)(x)\,dx = 6x\Big|_0^1 = 6(1) - 6(0) = 6$$

$$\int_0^1 f(x)\,dx = 2x\Big|_0^1 = 2(1) - 2(0) = 2$$

$$\int_0^1 g(x)\,dx = 3x\Big|_0^1 = 3(1) - 2(0) = 3$$

and,

$$\int_0^1 f(x)\,dx \cdot \int_0^1 g(x)\,dx$$

$$= 2 \cdot 3 = 6$$

And its proved.

SATISH KUMAR

Bonus

Let $|f(x)-f(y)| < \varepsilon/(b-a)$ for all $x, y \in [a,b]$ and if the partition Δ is a chosen sufficiently dense so that $x_k - x_{k-1} < \delta$ for $k = 1 \dots n$ an som $\delta > 0$. for example $x_k = \frac{k}{n}$ with $n \in N$ such that $N > \frac{1}{\delta}$ than it is clear that $M_k - m_k \leq \varepsilon/b-a$ which

$$U_\Delta(f) - L_\Delta(f) = \sum_{k=1}^{n} (x_k - x_{k-1})(M_k - m_k)$$

$$\leq \sum_{k=1}^{n} (x_k - x_{k-1}) \frac{\varepsilon}{b-a}$$

$$= \frac{\varepsilon}{(b-a)} \sum_{k=1}^{n} (x_k - x_{k-1}) = \frac{\varepsilon}{(b-a)} (x_n - x_0)$$

$$= \varepsilon$$

hence $U(f) \leq U_\Delta(f) \leq L_\Delta(f) + \varepsilon \leq L(f) + \varepsilon$

$\leq U(f) + \varepsilon$

so implies that $U(f) = L(f)$