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Exercise 1

Let $AB = BA$ and v is eigen vector of A and λ is eigen value than we know that

$$Av = \lambda v$$

and Multiply both sides by B than

$$BAv = B\lambda v$$

and we know $AB = BA$ than

$$A(Bv) = \lambda(Bv)$$

So $B \cdot v$ is eigenvector and by defination we know that eigenvector $\neq 0$ than $Bv \neq 0$.

Exercise 2

a) Yes, eigenvalue of A^T are the same as those of A.

and characterstic polynomial is

$$\det(\lambda I - A^T)$$

$$= \det|(\lambda I - A)^T|$$

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$= \det|\lambda I - A|$

Because we know that $\det(A) = \det(A^T)$ is equal. So A and A^T have the same characterstic polynomial and hence they have same eigenvalue.

b) No, eigenvector of A^T is not equal to eigenvector of A. If it is not symmatric matrix

let $A = \begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix}$

eigen value is

$$= \det\left|\begin{bmatrix} 0 & -1 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right|$$

$$= \det\begin{vmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{vmatrix}$$

$$= -3\lambda + \lambda^2 + 2$$

$$= (\lambda - 1)(\lambda - 2)$$

eigen value is $1, 2$

eigenvector is $\lambda = 1$

$$\begin{pmatrix} -\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 2 & 2 & 0 \end{array}\right)$$

multiply row 1 by 2 and add row 2

$$= \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-x - y = 0$$
$$-x = y$$
$$x = -y$$

$\lambda = 1$ eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

eigenvector of $\lambda = 2$

$$= \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 2 & 1 & 0 \end{array}\right)$$

~~multiply row 1~~ by
add row 1 and 2

$$= \left(\begin{array}{cc|c} -2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-2x - y = 0$$

$$-2x = y$$

$$x = -\frac{y}{2}$$

eigenvector for $\lambda = 2$ is $\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

Eigenvector for A^T, $\lambda = 1$

$$= \left(\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right)$$

multiply row 1 by -1 and add row 2

$$= \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-x + 2y = 0$$

$$-x = -2y$$

$$x = 2y$$

eigenvector A^T for $\lambda = 1$ is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Eigen vector for $\lambda = 2$

$$= \left(\begin{array}{cc|c} -2 & 2 & 0 \\ -1 & 1 & 0 \end{array}\right)$$

multiply row 1 by $-\frac{1}{2}$ and add row 2

$$= \left(\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-2x + 2y = 0$$

$$x = y$$

eigenvector for $\lambda = 2$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

As we can see
eigenvector for

$A = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

$A^T = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

It is not equal.

Exercise 3

Let A ~~$v = \lambda v$~~ be an invertible matrix and we know that

$Av = \lambda v$ where λ is eigenvalue

~~now~~ and v is eigenvector of A

Now

$$Av = \lambda v$$

Multiply both side by A^{-1} than

$$A^{-1}Av = A^{-1}\lambda v$$

$$A^{-1}Av = \lambda A^{-1}v$$

And we know that $A^{-1}A = I$

$$Iv = \lambda A^{-1}v$$

Now Multiply bothside by λ^{-1}
than

$$\lambda^{-1}V = \lambda^{-1}\lambda A^{-1}V$$

$$\lambda^{-1}V = A^{-1}V$$

Than it is proved λ^{-1} is eigenvalue of A^{-1}

Exercise 4

Let $A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$

eigenvalue is $\det\begin{vmatrix} 2-\lambda & -3 \\ -1 & 4-\lambda \end{vmatrix}$

$$= \lambda^2 - 6\lambda + 5$$

$$= (x-1)(x-5)$$

eigenvalue of A is 1 and 5

$A^2 = 1, 25$

$A^{-1} = 1, \frac{1}{5}$

$A^{-2} = 1, \frac{1}{25}$

$3A = 3^2 \det A \Rightarrow 9, 45$

$A - 4I = \begin{bmatrix} 2-\lambda & -3 \\ -1 & 4-\lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} -2-\lambda & -3 \\ -1 & -\lambda \end{bmatrix}$$

$$= \lambda^2 + 2\lambda - 3$$

$$= (\lambda+3)(\lambda-1)$$

eigen value is $1, -3$

eigen vector is $\lambda = 1$

$$A = \left[\begin{matrix} 2-1 & -3 \\ -1 & 4-1 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

$$= \left[\begin{matrix} 1 & -3 \\ -1 & 3 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

~~Multiply~~ Add row 1 and 2

$$= \left[\begin{matrix} 1 & -3 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

$$x - 3y = 0$$

$$x = 3y$$

eigen vector of A is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

and $\lambda = \cancel{3}\ 5$

$$= \left[\begin{matrix} 2-5 & -3 \\ -1 & 4-5 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

$$= \left[\begin{matrix} -3 & -3 \\ -1 & -1 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

Multiply row 1 by $-\frac{1}{3}$ and add row 2

$$= \left(\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$-3x - 3y = 0$$
$$-3x = +3y$$
$$x = -y$$

eigenvector for $\lambda = 5$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvector of A^{-1} is

$$A^{-1} = \begin{bmatrix} 4/5 & 3/5 \\ 1/5 & 2/5 \end{bmatrix}$$

$\lambda = 1$

$$= \left[\begin{bmatrix} 4/5 - 1 & 3/5 \\ 1/5 & 2/5 - 1 \end{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix}\right]$$

$$= \left[\begin{array}{cc|c} -1/5 & 3/5 & 0 \\ 1/5 & -3/5 & 0 \end{array}\right]$$

Add row 1 and 2

$$= \left[\begin{array}{cc|c} -1/5 & 3/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-\frac{1}{5}x + \frac{3}{5}y = 0$$
$$x = 3y$$

eigenvector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

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eigen vector A^{-1} for $\lambda = \frac{1}{5}$

$$= \left[\begin{array}{cc|c} \frac{4}{5} - \frac{1}{5} & 3/5 & 0 \\ 1/5 & 2/5 - \frac{1}{5} & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \frac{3}{5} & 3/5 & 0 \\ 1/5 & 1/5 & 0 \end{array}\right]$$

Multiply row 1 by $-\frac{1}{3}$ and add row 2

$$= \left[\begin{array}{cc|c} 3/5 & 3/5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\frac{3}{5}x + \frac{3}{5}y = 0$$

$$x = -y$$

than $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvector for A^2

$$A^2 = \begin{bmatrix} 7 & -18 \\ -6 & 19 \end{bmatrix}$$

$\lambda = 1$

$$= \left[\begin{matrix} 7-1 & -18 \\ -6 & 19-1 \end{matrix}\right] \left[\begin{matrix} 0 \\ 0 \end{matrix}\right]$$

$$= \left[\begin{array}{cc|c} 6 & -18 & 0 \\ -6 & 18 & 0 \end{array}\right]$$

Add row 1 and 2

$$= \left[\begin{array}{cc|c} 6 & -18 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$6x - 18y = 0$$
$$x = 3y$$

that $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda = 25$

$$= \left[\begin{array}{cc|c} 7-25 & -18 & 0 \\ -6 \cancel{-25} & 19-25 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -18 & -18 & 0 \\ -6 & -6 & 0 \end{array}\right]$$

Multiply row 1 by $-\frac{1}{3}$ and add row 2

$$= \left[\begin{array}{cc|c} -18 & -18 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-18\cancel{x} - 18y = 0$$
$$x = -y$$

that is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

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Eigen vector for

A^{-2} is $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$

$3A$ is $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$

$A-4I$ is $\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$

That ~~mean~~ eigenvector of all ~~ma~~ is same $\begin{pmatrix}3\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$

b) Trace of Matrix $A = \begin{bmatrix}2 & -3\\-1 & 4\end{bmatrix}$

$$tr(A) = 2+4$$
$$= 6$$

Sum of eigenvalue is $1+5 = 6$

Product of eigenvalue is $1 \times 5 = 5$

Determinant of A = 2*4-(-1)(-3)
= 8-3
=5

tr(A) = sum of eigenvalue

Determinant of A = Product of eigenvalue

Exercise 6

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$$A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}$$

eigenvalue of A = $\det|A - \lambda I|$

$$= \det\begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= \lambda^2 + 6\lambda + 5$$

$$= (\lambda+1)(\lambda+5)$$

eigen value is -1 and -5

eigenvector for -1

$$= \left[\begin{array}{cc|c} -3+1 & 2 & 0 \\ 2 & -3+1 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right]$$

Add row 1 and 2

$$= \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$-2x + 2y = 0$$

$$x = y$$

than $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -5$

$$= \cancel{\det}\left[\begin{array}{cc|c} -3+5 & 2 & 0 \\ 2 & -3+5 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right]$$

Multiply row 1 by -1 and add row 2

$$= \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$2x + 2y = 0$$
$$x = -y$$

eigen vector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

eigen vector of A is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$S = \begin{bmatrix} -1 & +1 \\ 1 & 1 \end{bmatrix}$ is for standard basis

$$A' = (S^T)^{-1} A S^T$$

$$= \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 & -5/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix}$$

Exercise 2

(a)

Assume 0 is an eigenvalue. Thus there is some nontrivial solution to Ax = 0x = 0. By the invertible matrix theorem, if A was invertible there would only be the trivial solution. Since there in a nontrivial solution, it must be the case that A is not invertible.