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Exercise 1

1) $\underline{b} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

$$P = \frac{a^T b}{a^T a} \cdot a$$

$$= \frac{\begin{pmatrix} 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$= \frac{10}{10} \cdot \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

$$e = b - p$$

$$= \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$$

2) $\underline{b} = \begin{pmatrix} -3 \\ -2 \\ 4 \\ 1 \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} 3 \\ 2 \\ -4 \\ -1 \end{pmatrix}$

$$P = \frac{a^T b}{a^T a} \cdot a$$

$$= \frac{\begin{pmatrix} 3 & 2 & -4 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 4 \\ 1 \end{pmatrix}}{\begin{pmatrix} 3 & 2 & -4 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -4 \\ -1 \end{pmatrix}} \cdot \begin{pmatrix} 3 \\ 2 \\ -4 \\ -1 \end{pmatrix}$$

$$= \frac{-30}{30} \cdot \begin{pmatrix} 3 \\ 2 \\ -4 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} -3 \\ -2 \\ 4 \\ 1 \end{pmatrix}$$

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$e = b - p$

$= \begin{pmatrix} -3 \\ -2 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ -2 \\ 4 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -3+3 \\ -2+2 \\ 4-4 \\ 1-1 \end{pmatrix} = 0$

3) $\underline{b} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \quad \underline{a} = \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$

$P = \dfrac{a^T b}{a^T a} \cdot a$

$= \dfrac{\begin{pmatrix} 2 & -6 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}}{\begin{pmatrix} 2 & -6 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}} \cdot \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$

$= \dfrac{0}{49} \cdot \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$e = b - p$

$= \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$

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Exercise 2

$$a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \text{and } b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$P = A((A^TA)^{-1}(A^Tb))$ is projection vector

Projection matrix is

$$m = A((A^TA)^{-1} \cdot A^T)$$

$$(A^TA)^{-1} = \left(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \right)^{-1}$$

$$= \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}^{-1}$$

$$= \left(\begin{array}{cc|cc} 3 & 3 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array} \right)$$

Multiply row 1 by $\frac{1}{3}$

$$= \left(\begin{array}{cc|cc} 1 & 1 & 1/3 & 0 \\ 3 & 5 & 0 & 1 \end{array} \right)$$

Multiply row 1 by -3 and add row 2

$$= \left(\begin{array}{cc|cc} 1 & 1 & 1/3 & 0 \\ 0 & 2 & -1 & 1 \end{array} \right)$$

Multiply row 2 by $\frac{1}{2}$

$$= \left(\begin{array}{cc|cc} 1 & 1 & 1/3 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array} \right)$$

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Multiply row 2 by -1 and add row 1

$$= \left(\begin{array}{cc|cc} 1 & 0 & 5/6 & -1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$$

$$m = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \left(\begin{pmatrix} 5/6 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix}$$

Projection vector is

$P = mb$

$$= \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/6 & 1/3 & 5/6 \end{pmatrix} \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

Ans

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Exercise 3

1) $M = \{\underline{x} \in \mathbb{R}^2 \mid 2x_2 + 3x_1 = 0\}$

In equation form

$$2x_2 + 3x_1 = 0$$
$$x_2 = \frac{-3}{2} x_1$$

$$M = \text{span}\left\{\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}\right\}$$

$$M^\perp = \left\{ c \cdot \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

2) $M = \{0\}$

Every vector in space is perpendicular to zero vector

$$M^\perp = \mathbb{R}^n$$

3) $M = \{x \in \mathbb{R}^3 \mid x_1 = 0\}$

In equation we have

$$x_1 + x_2 + x_3 = 0$$
$$0 + x_2 + x_3 = 0$$
$$x_2 = -x_3$$

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$$M = span\left\{\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}$$

$$M^{\perp} = \left\{c\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} \,\middle|\, c \in \mathbb{R}\right\}$$

4) $M = \{\underline{x} \in \mathbb{R}^3 \mid 2x_1 - x_2 = 0 \wedge x_3 = 0\}$

In equation

$$2x_1 - x_2 = 0$$
$$2x_1 = +x_2$$
$$x_1 = \frac{x_2}{2}$$

$$M = span\left\{\begin{pmatrix}1/2\\1\\0\end{pmatrix}\right\}$$

$$M^{\perp} = \left\{c\cdot\begin{pmatrix}-2\\1\\0\end{pmatrix} \,\middle|\, c \in \mathbb{R}\right\}$$

5) $M = \{\underline{x} \in \mathbb{R}^4 \mid 2x_2 - x_4 = 0 \wedge 2x_3 - x_2 = 0 \wedge 2x_4 - x_1 = 0\}$

In equation

$$2x_2 - x_4 = 0$$
$$2x_2 = x_4$$
$$x_2 = \frac{x_4}{2}$$

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$$2x_3 - x_2 = 0$$
$$2x_3 = x_2$$
$$x_3 = \frac{x_2}{2}$$

$$2x_4 - x_1 = 0$$
$$2x_4 = x_1$$
$$x_4 = \frac{x_1}{2}$$

$$x_2 = \frac{x_4}{2} = \frac{\frac{x_1}{2}}{2} = \frac{x_1}{4}$$

$$x_3 = \frac{x_2}{2} = \frac{\frac{x_1}{4}}{2} = \frac{x_1}{8}$$

$$M = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1/4 \\ 1/8 \\ 1/2 \end{pmatrix} \right\}$$

$$M^{\perp} = \left\{ c \cdot \begin{pmatrix} 1 \\ -2 \\ -8 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

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Exercise 4

1) $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$QQ^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2) Let $u = (1, 0)$
$v = (0, 0)$
are orthogonal and not independent.
Non zero vectors are independent.

3) $q_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$q_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$q_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Prove those vector are orthonormal.

$$\|q_1\| = \sqrt{\left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-1}{\sqrt{6}}\right)^2} = \sqrt{\frac{4+1+1}{6}} = \sqrt{\frac{6}{6}} = \sqrt{1} = 1$$

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$$\|q_2\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1+1+1}{3}} = \sqrt{1} = 1$$

$$\|q_3\| = \sqrt{(0)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1+1}{2}} = \sqrt{1} = 1$$

$$\langle q_1, q_2 \rangle = \frac{2}{\sqrt{6}} \cdot \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{18}} - \frac{2}{\sqrt{18}} = 0$$

$$\langle q_1, q_3 \rangle = \frac{2}{\sqrt{6}} \cdot 0 + \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{12}} - \frac{1}{\sqrt{12}} = 0$$

$$\langle q_2, q_3 \rangle = \frac{1}{\sqrt{3}} \cdot 0 - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} = 0$$

That's proved

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Exercise 6

$$V_1 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix}, \quad V_2 = \begin{pmatrix} -3 \\ 5 \\ 0 \\ 1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 \\ -5 \\ 3 \\ 0 \end{pmatrix}$$

① let $V_1' = V_1 = \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix}$

2) $$V_2' = V_2 - \frac{V_1'^T V_2}{V_1'^T V_1'} \cdot V_1'$$

$$= \begin{pmatrix} -3 \\ 5 \\ 0 \\ 1 \end{pmatrix} - \frac{\begin{pmatrix} 0 & 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 & 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix}} \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 11/3 \\ -4/3 \\ 7/3 \end{pmatrix}$$

$$V_3' = V_3 - \frac{V_1'^T V_3}{V_1'^T V_1'} \cdot V_1' - \frac{\cancel{V_2}'^T V_3}{V_2'^T V_2'} \cdot V_2'$$

$$= \begin{pmatrix} 0 \\ -5 \\ 3 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 0 & 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 & 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix}} \cdot \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix} - \frac{\begin{pmatrix} -3 & 11/3 & -4/3 & 7/3 \end{pmatrix} \begin{pmatrix} 0 \\ -5 \\ 3 \\ 0 \end{pmatrix}}{\begin{pmatrix} -3 & 11/3 & -4/3 & 7/3 \end{pmatrix} \begin{pmatrix} -3 \\ 11/3 \\ -4/3 \\ 7/3 \end{pmatrix}} \cdot \begin{pmatrix} -3 \\ 11/3 \\ -4/3 \\ 7/3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -5 \\ 3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2/3 \\ 2/3 \\ -2/3 \end{pmatrix} - \begin{pmatrix} -603/250 \\ 737/250 \\ -268/250 \\ 469/250 \end{pmatrix}$$

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$$= \begin{pmatrix} \cancel{-17/3} \\ -17/3 \\ 7/3 \\ -2/3 \end{pmatrix} - \begin{pmatrix} -603/250 \\ 737/250 \\ -268/250 \\ 469/250 \end{pmatrix}$$

$$= \begin{pmatrix} +603/250 \\ -6461/750 \\ 2554/750 \\ -1907/750 \end{pmatrix}$$

$$q_1 = \frac{V_1'}{\|V_1'\|} = \frac{1}{\sqrt{(0)^2+(3)^2+(3)^2+(-3)^2}} \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{27}} \begin{pmatrix} 0 \\ 3 \\ 3 \\ -3 \end{pmatrix}$$

$$q_2 = \frac{V_2'}{\|V_2'\|} = \frac{1}{\sqrt{(-3)^2+\left(\frac{11}{3}\right)^2+\left(-\frac{4}{3}\right)^2+\left(\frac{7}{3}\right)^2}} \begin{pmatrix} -3 \\ 11/3 \\ -4/3 \\ 7/3 \end{pmatrix} = \frac{1}{\sqrt{\frac{250}{9}}} \begin{pmatrix} -3 \\ 11/3 \\ -4/3 \\ 7/3 \end{pmatrix}$$

$$q_3 = \frac{V_3'}{\|V_3'\|} = \frac{1}{\sqrt{\left(\frac{603}{250}\right)^2+\left(\frac{-6461}{750}\right)^2+\left(\frac{2554}{750}\right)^2+\left(\frac{-1907}{750}\right)^2}} \cdot \begin{pmatrix} 603/250 \\ -6461/750 \\ 2554/750 \\ -1907/750 \end{pmatrix}$$

$$= \frac{1}{\sqrt{\frac{52994913}{562500}}} \begin{pmatrix} 603/250 \\ -6461/750 \\ 2554/750 \\ -1907/750 \end{pmatrix}$$