Calculus - Assignment 8

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1. (a)

$$f : S \to \mathbb{R}, f(x) = \sqrt{9+x}$$
$$g : T \to \mathbb{R}, g(x) = x^2$$
$$dom(f) = S = [-9, \infty)$$
$$dom(g) = T = \mathbb{R}$$

i.

$$f + g := f(x) + g(x) = \sqrt{9+x} + x^2$$
$$dom(f+g) = dom(\sqrt{9+x} + x^2) = [-9, \infty)$$

ii.

$$f \cdot g := f(x) \cdot g(x) = (\sqrt{9+x}) \cdot x^2 = [-9, \infty)$$

(b) i.

$$(f \circ g)(x) := f(g(x)) = f(x^2)$$

x	$g(x) = x^2$	$f(g(x)) = f(x^2) = \sqrt{9+x^2}$
0	0	$\sqrt{9+0} = \sqrt{9} = 3$
-4	16	$\sqrt{9+16} = \sqrt{25} = 5$
4	16	$\sqrt{9+16} = \sqrt{25} = 5$

ii.

$$(g \circ f)(x) := g(f(x)) = g(\sqrt{9+x})$$

x	$f(x) = \sqrt{9+x}$	$g(f(x)) = g(\sqrt{9+x}) = (\sqrt{9+x})^2$
0	$\sqrt{9+0} = \sqrt{9} = 3$	9
-4	$\sqrt{9+(-4)} = 2\sqrt{5}$	5
4	$\sqrt{9+4} = \sqrt{13}$	13

(c) No.

$$f \circ j := f(g(x)) = f(x^2) = \sqrt{9 + x^2}$$

$$g \circ f := g(f(x)) = g(\sqrt{9 + x}) = \sqrt{(9 + x)}^2 = 9 + x$$

so,

$$f \circ g \neq g \circ f$$

2. In this problem, we want to find the fixed points of $f(x) = exp(x - 2)$. A fixed point is an x, such that $f(x) = x$. And we know that a fixed point of f is a root of the function $g(x) := f(x) - x$. Therefore, we if we find a root for $g(x)$, we have just found also a fixed point for $f(x)$.

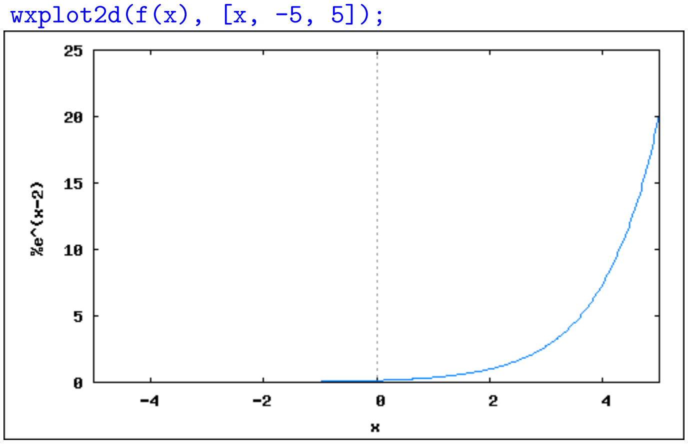
(%i1) `f(x) := exp(x - 2);`

(%o1) $\mathrm{f}(x) := \exp(x - 2)$

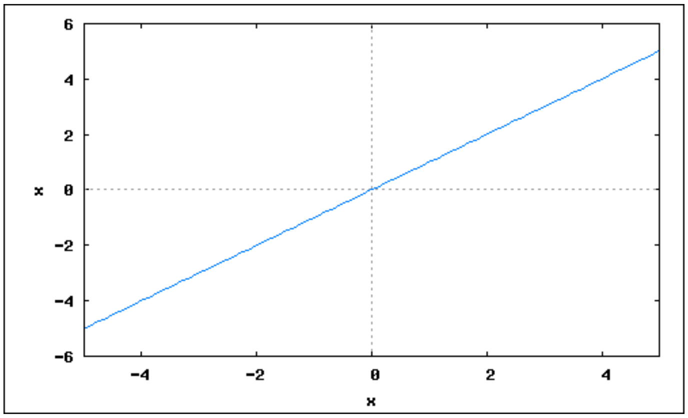
(%i2) `g(x) := x;`

(%o2) $\mathrm{g}(x) := x$

(%i3)
```
/* Plotting f(x) */

wxplot2d(f(x), [x, -5, 5]);
```

(%t3) 

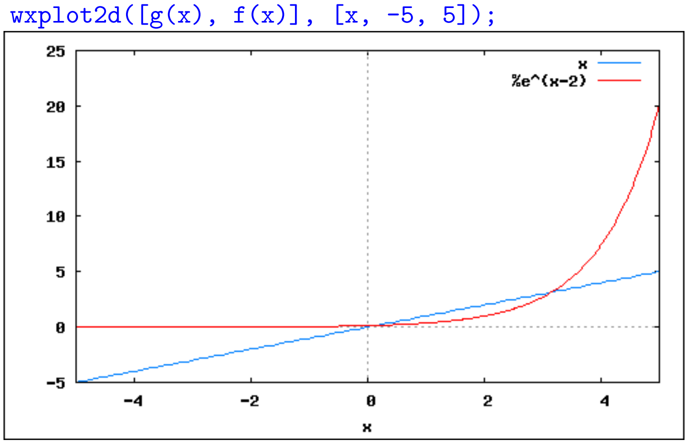
(%i4)
```
/* Plotting g(x) */

wxplot2d(g(x), [x, -5, 5]);
```

(%t4) 

(%o4)

(%i5) `/* This shows up intersection between f(x) and g(x) */`

```
wxplot2d([g(x), f(x)], [x, -5, 5]);
```

(%t5) 

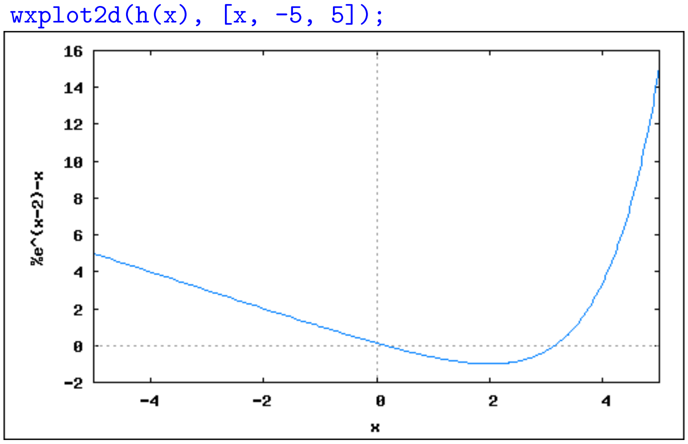
(%o5)

(%i7) `h(x) := f(x) - x;`

(%o7) $\mathrm{h}(x) := \mathrm{f}(x) - x$

(%i8) `/* This is the function we need to study */`

```
wxplot2d(h(x), [x, -5, 5]);
```

(%t8) 

Since we are more familiar with Python (and because we have had some problems with Maxima), we decided to implement the *bisection algorithm* directly with Python. I think you have no problems understanding Python (even if you don't know it), because it's like pseudo code. The definition of a function starts with the keyword "def". Under the signature (heading) of each function, you have a description of the function and of the parameters (variables) the function requires. What ever starts with # and that is between triple quotes is a comment. If you need any help, just contact the authors.

The following is the code we have used:

```python
import math as m

def h(x):
    """Function of which we want to find the root"""
    return m.exp(x - 2) - x


def same_signs(a, b):
    """Returns True if a and b have the same sign"""
    return a * b > 0


def bisect(f, a, b, tol=0.00001):
    """
    func: reference to the function
    a: first x of the domain for f (f(a) should have different sign of f(b))
    b: second x of the domain for f (f(b) should have different sign of f(a))
    tol: maximum acceptable error or tolerance (epsilon)
    this parameter is useful because not always we manage
    to obtain an root easily exactly, but we need an approximation
    """

    # print(f(a), "\n", f(b))

    e = (a + b) / 2 # point between a and b (estimation)

    # Distance between e and our root is at most (b - a) / 2
    max_error = (b - a) / 2

    while max_error > tol:
        # repeat loop until (b - a) / 2 <= tol

        if f(e) == 0:
            break

        if not same_signs(f(a), f(e)):
            b = e

        elif not same_signs(f(b), f(e)):
            a = e

        # Estimating a new point between a and b
        e = (a + b) / 2

        # Distance between e and our root is at most (b - a) / 2
        max_error = (b - a) / 2
```

```
    return e

print(bisect(h, 1, 5))
```

The output when $a = 1$ and $b = 5$ is 3.1461868286132812. If, for example, $a = 0$ and $b = 1$, we obtain: 0.15859222412109375

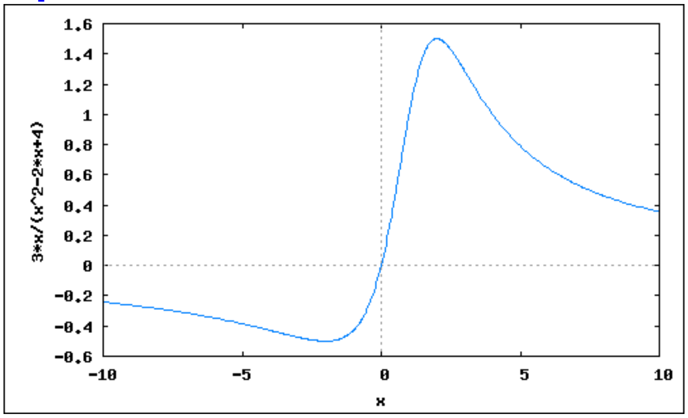
3. (%i1) /* Defining f(x) := (3x)/(x^2 - 2*x + 4) */

```
f(x) := (3*x)/(x^2 - 2*x + 4);
```

(%o1) $\mathrm{f}(x) := \frac{3\,x}{x^2 - 2\,x + 4}$

(%i2) /* Plotting f(x) := (3x)/(x^2 - 2*x + 4) */

```
wxplot2d(f(x), [x, -10, 10]);
```



(%t2)

(%o2)

--> %/* Finding the minimum (which should be -1/2 at -2) and the maximum (which should be 3/2 at
of f(x) := (3x)/(x^2 - 2*x + 4) */;

(%i3) diff(f(x), x);

(%o3) $\frac{3}{x^2 - 2\,x + 4} - \frac{3\,x\,(2\,x - 2)}{(x^2 - 2\,x + 4)^2}$

(%i4) solve(%, x);
(%o4) $[x = -2, x = 2]$

(%i5) f(-2);

(%o5) $-\frac{1}{2}$

(%i6) f(2);

(%o6) $\frac{3}{2}$

We can also observe the minimum and the maximum of $f(x)$ by iterating through some x values and see their corresponding y values.

```
(%i7) for x : -4 thru 4 do
      print([x, f(x)]);
```

$[-4, -\frac{3}{7}], [-3, -\frac{9}{19}], [-2, -\frac{1}{2}], [-1, -\frac{3}{7}], [0, 0], [1, 1], [2, \frac{3}{2}], [3, \frac{9}{7}], [4, 1]$

This sequence of pairs "$x, f(x)$" values are probably not enough to show that $\frac{3}{2}$ is really the maximum at 2 and that $-\frac{1}{2}$ is really the minimum at -2, so let's try to do it.

Let's first replace $f(x)$ with y in order to make some calculations depending also on the variable y. Let's split the problem in two cases $y = 0$, then we know that $x = 0$:

$$0 = \frac{3x}{x^2 - 2x + 4}$$

And the second case is $y \neq 0$.

In this case, we have:

$$y(x^2 - 2x + 4) = 3x$$
$$yx^2 + (-2y - 3)x + 4y = 0$$

Considering the *discriminant* ($b^2 - 4ac$) of this quadratic equation which depends on y (and we know that $y \neq 0$), we want that:

$$(-2y - 3)^2 - 4 \cdot y \cdot (4y) \geq 0$$
$$4y^2 + 12y + 9 - 16y^2 \geq 0$$
$$12y + 9 - 12y^2 \geq 0$$

Now, we multiply both sides by -1 in order to have a positive coefficient for the x^2 term:

$$12y^2 - 9 - 12y \leq 0$$
$$4y^2 - 4y - 3 \leq 0$$
$$(2y - 3)(2y + 1) \leq 0$$

Therefore, the solution is:

$$-\frac{1}{2} \leq y \leq \frac{3}{2}$$
$$-\frac{1}{2} \leq f(x) \leq \frac{3}{2}$$

We have the equalities in the cases:

$$f(-2) = -\frac{1}{2}$$

and

$$f(2) = \frac{3}{2}$$