Assignment 3

Exercises

P1 (a)

$S = \{1, 4, 9, 16, 25\}$
Def:
$b \in \mathbb{R}$ upper bound if $\forall x \in S\ x \leq b$
$a \in \mathbb{R}$ lower bound if $\forall x \in S\ a \leq x$
25 and 26 are upper bounds because they are greater than or equal to 1,4,9,16 and 25.
1 and 0 are lower bounds because they are lesser than or equal to 1,4,9,16 and 25.
Therefore the set is bounded from above and from below.

(b)
$S = \{x \in \mathbb{R}, x < 0\}$
$b \in \mathbb{R}$ upper bound if $\forall x \in S\ x \leq b$
$a \in \mathbb{R}$ lower bound if $\forall x \in S\ a \leq x$
25 and 26 are upper bounds because they are greater than or equal to 0.
There is no lower bounds because $x \in \mathbb{R}$ and $\mathbb{R}$ has an infinite number of negative elements.
Therefore the set is bounded from above.

(c)
$S = \{1 + \frac{1}{2^n} : n \in \mathbb{N}\}$
$b \in \mathbb{R}$ upper bound if $\forall x \in S\ x \leq b$
$a \in \mathbb{R}$ lower bound if $\forall x \in S\ a \leq x$
25 and 26 are upper bounds because they are greater than or equal to $\frac{3}{2}$.
1 and 0 are lower bounds because they are lesser than or equal to every element of $\{1 + \frac{1}{2^n} : n \in \mathbb{N}\}$ because $1 + \frac{1}{2^n}$ it is a decreasing sequence and $\frac{1}{2^n}$ is > 0.
Therefore the set is bounded from above and from below.

(d)
$S = \{1 + x : x \in \mathbb{R}\}$
$b \in \mathbb{R}$ upper bound if $\forall x \in S\ x \leq b$
$a \in \mathbb{R}$ lower bound if $\forall x \in S\ a \leq x$
No upper bound and no lower bound because $\{x + 1 : x \in \mathbb{R}\}$ has an infinite number of elements. Therefore the set is not bounded.

(e)
$[0, 2] \cup (3, 4)$
$b \in \mathbb{R}$ upper bound if $\forall x \in S\ x \leq b$
$a \in \mathbb{R}$ lower bound if $\forall x \in S\ a \leq x$
5 and 6 are upper bounds by definition because $[0, 2] \cup (3, 4)$ is $\{0 \leq x \leq 2 \wedge 3 < x < 4 : x \in \mathbb{R}\}$
-1 and -2 are lower bounds by definition.
Therefore the set is bounded from above and from below.

P2 (a)
$S = \{1, 4, 9, 16, 25\}$
Def max and min:
$b \in S$ maximum if $\forall x \in S$ $x \leq b$
$a \in S$ minimum if $\forall x \in S$ $a \leq x$
By definition 1 is the minimum and 25 is the maximum.
Def of inf:
$i \in \mathbb{R}$ infimum if $\forall i' \in$ {all upper bounds of S} $i \leq i'$
The infimum is 1 since it satisfies the definition.

(b)
$S = \{x \in \mathbb{R}, x < 0\}$
Def max and min:
$b \in S$ maximum if $\forall x \in S$ $x \leq b$
$a \in S$ minimum if $\forall x \in S$ $a \leq x$
There is no max, because there is no $b \in S$ such that $\forall x \in S$ $x \leq b$.
There is no min, because $\mathbb{R}$ has an infinite number of negative elements.
Def of inf:
$i \in \mathbb{R}$ infimum if $\forall i' \in$ {all upper bounds of S} $i \leq i'$
There is no infimum since there is no element $i \in \mathbb{R}$ that satisfies the definition.. The reason being that the set diverges to negative infinity and that therefore it has no lower bounds.

(c)
$S = \{1 + \frac{1}{2^n} : n \in \mathbb{N}\}$
Def max and min:
$b \in S$ maximum if $\forall x \in S$ $x \leq b$
$a \in S$ minimum if $\forall x \in S$ $a \leq x$
$\frac{3}{2}$ is the max, because $1 + \frac{1}{2^n}$ it is a decreasing sequence and the first element is $\frac{3}{2}$.
There is no min, because there is no element $a \in S$ such that $a \leq x$ $\forall x \in S$.
Def of inf:
$i \in \mathbb{R}$ infimum if $\forall i' \in$ {all upper bounds of S} $i \leq i'$
The infimum is 1 since the elements of the decreasing sequence approach 1 without ever reaching it. Therefore it satisfies the definition.

(d)
$S = \{1 + x : x \in \mathbb{R}\}$
Def max and min:
$b \in S$ maximum if $\forall x \in S$ $x \leq b$
$a \in S$ minimum if $\forall x \in S$ $a \leq x$
No max and no min because $\{x + 1 : x \in \mathbb{R}\}$ have no elements such that they satisfy the definitions of min and max.
Def of inf:
$i \in \mathbb{R}$ infimum if $\forall i' \in$ {all upper bounds of S} $i \leq i'$
There is no infimum since there is no element $i \in \mathbb{R}$ that satisfies the definition because an infinite set has no lower bounds.

(e)
$[0, 2] \cup (3, 4)$
Def max and min:
$b \in S$ maximum if $\forall x \in S$ $x \leq b$
$a \in S$ minimum if $\forall x \in S$ $a \leq x$

There is no max because there is no element $b \in S$ which satisfies the definition of maximum.
There is a min which is 0 since it satisfies the definition.
Def of inf:
$i \in \mathbb{R}$ infimum if $\forall i' \in \{\text{all upper bounds of S}\}$ $i \leq i'$
The infimum is 0 since it satisfies the definition. The reason being that the set which is a union of two intervals is bounded from below by 0 by the definition of the half open interval and that 0 is greater than every other lower bound.

P3 Using
a[0]:1;
a[1]:2;
for n:0 thru 100 do
(a[n+2]:(a[n+1]+a[n])/2);

for n:0 thru 18 do
(display(float(a[n])));
We can try and guess it will somehow go to 1.666666.. therefore just to be sure

display(float(a[50]));
display(rat(float(a[50])));
we can see that the limit is $\frac{5}{3}$.

P4 Bonus
Given a rectangle, A_0
the area is $S_0 = l \cdot w = 1m^2$, where l is the length and w is the witdth.
A_1 is the rectangle obtained by dividing A_0 in two identical part.
The length/width ratio of A1 is the same as that of A0.

$$\begin{cases} lw = 1 \\ \frac{l}{w} = \frac{w}{\frac{l}{2}} \end{cases} \rightarrow \begin{cases} l = \frac{1}{w} \\ l^2 = 2w^2 \end{cases} \rightarrow \begin{cases} l = \frac{1}{w} \\ \frac{1}{w}^2 = 2w^2 \end{cases} \rightarrow \begin{cases} l = \frac{1}{w} \\ 1 = 2w^4 \end{cases}$$

Therefore $w = \sqrt[4]{\frac{1}{2}}$ and $l = \sqrt[4]{2}$. These are the values for the width and length of A_0.
The area of the rectangle after four cuts is $S_0 = 2S_1 = 2(2S_2) = 2(2(2S_3)) = 2(2(2(2S_4))) = 16S_4$.
Then $S_4 = \frac{S_0}{16}$ knowing that $S_0 = 1$ we have $S_4 = \frac{1}{16}m^2$
Having the same length/width ratio means that $\frac{l_4}{w_4} = \frac{l}{w}$.
Therefore $w \cdot l_4 = l \cdot w_4$ From the area of A_4 and A_0 we know that $l_4 = \frac{1}{16w_4}$, $l = \frac{1}{w}$
Therefore $w \cdot \frac{1}{16w_4} = w_4 \cdot \frac{1}{w}$
$w^2 = 16{w_4}^2$. Then $w = 4w_4$ and therefore $l = 4l_4$.
Since $w = \sqrt[4]{\frac{1}{2}}$ and $l = \sqrt[4]{2}$
$w_4 = \frac{\sqrt[4]{\frac{1}{2}}}{4} = \sqrt[4]{\frac{1}{512}}$ and $l_4 = \frac{\sqrt[4]{2}}{4} = \sqrt[4]{\frac{2}{256}} = \sqrt[4]{\frac{1}{128}}$
Yes this rectangle looks familiar. Rectangle A_4 represents the industry standard A_4 paper format used for printing and document representation.