Assignment 11

Exercise 1

$$f(x) = \log 2$$

As we showed in class, the general formula for $f^{(k)}(x) = \log 2$ is

$$f^{(k)}(x) = \frac{(-1)^{k+1}(k-1)!}{x^k}$$

$$f^{(k)}(2) = \frac{(-1)^{k+1}(k-1)!}{2^k}$$

$$T_f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k$$

But because our formula for $f^{(k)}(x)$ does not work for $k = 0$ we have to consider that case, so our T_f has to be rewrite as

$$T_f(x) = f^{(0)}(2) + \sum_{k=1}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2)^k$$

$$T_f(x) = \log 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(k-1)!}{2^k k!}(x-2)^k$$

$$T_f(x) = \log 2 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k k}(x-2)^k$$

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(%i1) f[x] := log(2) + sum(((-1)^(k+1)*(x-2)^k)/(k*2^k), k, 1, 2), simpsum$
      g[x] := log(2) + sum(((-1)^(k+1)*(x-2)^k)/(k*2^k), k, 1, 4), simpsum$
      h[x] := log(2) + sum(((-1)^(k+1)*(x-2)^k)/(k*2^k), k, 1, 6), simpsum$
      t[x] := log(x)$
      wxplot2d([f[x], g[x], h[x], t[x]], [x,-1/2,4], [y,-3,3]);
      sf[x] := abs(t[x] - f[x])$
      sg[x] := abs(t[x] - g[x])$
      sh[x] := abs(t[x] - h[x])$
      wxplot2d([sf[x], sg[x], sh[x]], [x,-1/2,4], [y,-3,3]);
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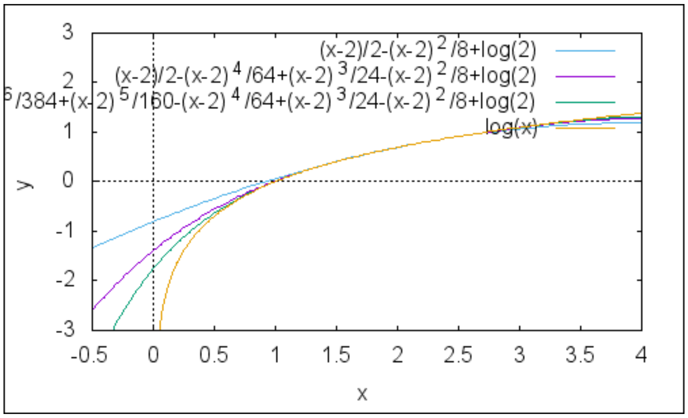
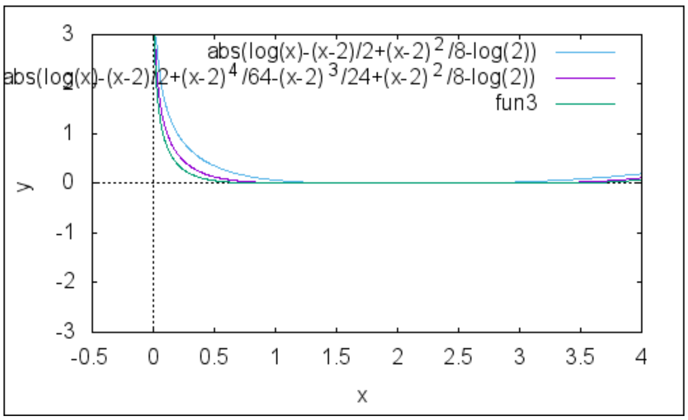


(%t5)



(%t9)

In the last plot it is possible to see the difference between f and the partial sums of T_f over the interval $[-1/2, 4]$. As we know, $\log x$ tends to $-\infty$ for $x \to 0$ and the partial sums have all a defined value in $\mathbb{R}$ for $x = 0$, so we can conclude that the maximum difference between f and T_f is ∞.

Exercise 2

From example 95 we know that

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = f(x_{k-1}) = cx_{k-1} = \frac{c(k-1)}{n}$$

and

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = f(x_k) = cx_k = \frac{ck}{n},$$

We consider the function $f : [0, 1] \to \mathbb{R}, f(x) = x^2$, so we can write

$$m_k = \inf_{x \in [x_{k-1}, x_k]} f(x) = f(x_{k-1}) = x_{k-1} = \frac{(k-1)^2}{n^2}$$

and

$$M_k = \sup_{x \in [x_{k-1}, x_k]} f(x) = f(x_k) = x_k = \frac{k^2}{n^2},$$

By example 95 we know that hence

$$L_{\Delta_n} = \sum_{k=1}^{n} (x_k - x_{k-1}) m_k = \sum_{k=1}^{n} \frac{1}{n} \frac{(k-1)^2}{n^2} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{1}{n^3} \sum_{k=1}^{n-1} k^2 = \frac{1}{n^3} \frac{n(n-1)(2n-1)}{6} =$$

$$= \frac{1}{6} \frac{2n^2 - 3n + 1}{n^2} = \frac{1}{6} \frac{2n^2}{n^2} - \frac{3n+1}{n^2} = \frac{1}{3} - \frac{3n+1}{n^2}$$

By (ii) we have to calculate the limit for $n \to \infty$

$$\lim_{n \to \infty} \frac{1}{3} - \frac{3n+1}{n^2} = \frac{1}{3}$$

$$U_{\Delta_n} = \sum_{k=1}^{n} (x_k - x_{k-1}) M_k = \sum_{k=1}^{n} \frac{1}{n} \frac{k^2}{n^2} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{1}{6} \frac{(n+1)(2n+1)}{n^2} =$$

$$\frac{1}{6} \frac{2n^2 + 3n + 1}{n^2} = \frac{1}{6} \frac{2n^2}{n^2} + \frac{3n+1}{n^2}$$

By (ii) we have to calculate the limit for $n \to \infty$

$$\lim_{n \to \infty} \frac{1}{3} + \frac{3n+1}{n^2} = \frac{1}{3}$$

$$L(f) = \sup_{\Delta} L_\Delta(f) \geq \lim_{n \to \infty} L_{\Delta_n} = \frac{1}{3}$$

and

$$U(f) = \inf_{\Delta} U_\Delta(f) \leq \lim_{n \to \infty} U_{\Delta_n} = \frac{1}{3}.$$

But as $L(f) \leq U(f)$ for any bounded function [1, Theorem 32.4], we then conclude $L(f) = U(f) = \frac{1}{3}$.

Exercise 3

Consider the functions $f(x) = 2$ and $g(x) = 5$. With these functions, we have $(f \cdot g)(x) = 10$. We can then calculate $F(x) = 2x + c$, $G(x) = 5x + c$, and $(F \cdot G)(x) = 10x + c$. Therefore, we have

$$\int_0^1 f(x) = 2 \cdot 1 = 2$$

$$\int_0^1 g(x) = 5 \cdot 1 = 5$$
$$\int_0^1 (f \cdot g)(x) = 10 \cdot 1 = 10$$
$$\int_0^1 f(x) \cdot \int_0^1 g(x) = 2 \cdot 5 = 10$$

Therefore we've shown what we wanted to show.

Bonus Exercise

Consider a monotonically increasing function $f(x) : [a, b] \to \mathbb{R}$.

Let's divide the interval in n parts, so that $x_k - x_{k-1} = \dfrac{b-a}{n}$. Also, since f is monotonically increasing, we have that $m_k = f(x_{k-1})$ and $M_k = f(x_k)$.

Now we can write the lower and upper Darboux sums as follows:

$$L_\Delta(f) = \sum_{k=1}^{n} \frac{b-a}{n} f(x_{k-1})$$

$$U_\Delta(f) = \sum_{k=1}^{n} \frac{b-a}{n} f(x_k)$$

And we can show that they are equal:

$$U_\Delta(f) - L_\Delta(f) =$$

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k) - \sum_{k=1}^{n} \frac{b-a}{n} f(x_{k-1}) =$$

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k) - \sum_{k=0}^{n-1} \frac{b-a}{n} f(x_k) =$$

$$\frac{b-a}{n} \left(\sum_{k=1}^{n} f(x_k) - \sum_{k=0}^{n-1} f(x_k) \right) =$$

$$\frac{b-a}{n} (f(x_n) - f(x_0)) =$$

$$\frac{b-a}{n} (f(b) - f(a)) =$$

$$\frac{(b-a)(f(b)-f(a))}{n} =$$

Now, considering smaller and smaller intervals, $(n \to \infty, \Delta \to 0)$, we have:

$$\lim_{n\to\infty} \frac{(b-a)(f(b)-f(a))}{n} = 0$$

Therefore we've shown that the upper and lower darboux sums have the same values, meaning that the function is integrable on that interval.