Assignment 5

Exercises

P1 We can compute the algorithm with the arithmetic average or with the harmonic average.
$a\_avg(a, b) := (a + b)/2;$
$h\_avg(a, b) := 2 * a * b/(a + b);$
$x : 2;$
$k : 1;$
$\epsilon : 10^{-15};$

$a[0] : 10;$
$b[0] : x/a[0];$

$b[n] := x/a[n];$
$a[n] := a\_avg(a[n-1], b[n-1]);$
or we can use $a[n] := h\_avg(a[n-1], b[n-1]);$

$while\ \ abs(a[k] - b[k]) >= \epsilon\ do$
$(k : k + 1);$
$done;$

$display(k, float(a[k]));$
$x = 2$ and $a_0 = 10$ it needs 7 iterations, with $a_0 = 5$ it needs 6 iteration. When $a_0 = 1.4142$ it needs 2 iterations.
The choice of the average functions does not change the number of iteration needed to compute the root of x with the same accuracy.
However choosing different values for a_0 affects the result. The closer a_0 is to $\sqrt{x}$ the lesser iterations are required.

P2 We know that the distance of the top book from the edge of the table is found by $\frac{1}{2} \sum_{k=1}^{n} \frac{1}{k}$.
Therefore we just need to compute the sum until we reach 2.
Maxima will help us.
k:1;
x[n]:= (1/2)*sum(1/k,k,1,n);

while x[k] ¡ 2 do
(k: k+1);
done;

display(k,float(x[k]));
We need 31 books to have the top one be extended by twice is full length.

P3 $\sum\limits_{k=0}^{\infty} \frac{k}{2^k} = 2$

We can find the closed formula, which is

$$\sum_{k=0}^{n} \frac{k}{2^k} = \frac{2^{n+1} - (n+2)}{2^n}$$

Lets prove it by induction

Base case

$$\sum_{k=0}^{1} \frac{k}{2^k} = \frac{1}{2} = \frac{4-3}{2}$$

Let $n \in \mathbb{N}$ be an arbitrary element, we want to show that

$$\sum_{k=0}^{n+1} \frac{k}{2^k} = \frac{2^{n+2} - (n+3)}{2^{n+1}}$$

Lets start with

$$\sum_{k=0}^{n+1} \frac{k}{2^k} = \sum_{k=0}^{n} \frac{k}{2^k} + \frac{n+1}{2^{n+1}}$$

Now we can use the H.I,

$$\frac{2^{n+1} - (n+2)}{2^n} + \frac{n+1}{2^{n+1}} = \frac{2^{n+2} - 2n - 4 + n + 1}{2^{n+1}} = \frac{2^{n+2} - (n+3)}{2^{n+1}}$$

We just proved that

$$\sum_{k=0}^{n} \frac{k}{2^k} = \frac{2^{n+1} - (n+2)}{2^n}$$

holds for all n.

We are going to calculate the limit of the partial sums

$\lim \sum\limits_{k=0}^{n} \frac{k}{2^k} = \lim \frac{2^{n+1}-(n+2)}{2^n} = \lim 2 - \lim \frac{n+2}{2^n} = 2 - \lim \frac{n+2}{2^n}$

Now we can prove that the $\lim \frac{n+2}{2^n} = 0$
We can show that $2^n > (n+2)^2 \ \forall n \geq 7$ by induction.

Base case $128 = 2^7 > (7+2)^2 = 81$.

Let n be an arbitrary element of the natural numbers.
$2^{n+1} = 2 \cdot 2^n > 2(n+2)^2 = 2n^2 + 8n + 8 > n^2 + 8n + 8 = n^2 + 6n + 2n + 8$
we know that $n \geq 7$ therefore $n^2 + 6n + 14 + 8 > n^2 + 6n + 9 = (n+3)^2$.
Hence $2^{n+1} > (n+3)^2$
We just proved by induction that $2^n > (n+2)^2 \ \forall n \geq 7$
Therefore $\frac{1}{2^n} < \frac{1}{(n+2)^2} \ \forall n \geq 7$
Then
$\frac{n+2}{2^n} < \frac{n+2}{(n+2)^2} < \frac{1}{n+2} < \frac{1}{n}$.
We also know that
$\frac{1}{2^n} < \frac{n+2}{2^n}$ therefore
$\frac{1}{2^n} \leq \frac{n+2}{2^n} \leq \frac{1}{n} \quad \forall n \geq 7 \in \mathbb{N}$.

Since we have that
$\lim \frac{1}{2^n} = 0 = \lim \frac{1}{n}$ and $\frac{1}{2^n} \leq \frac{n+2}{2^n} \leq \frac{1}{n}$ $\forall n \geq 7 \in \mathbb{N}$ then taking $N \in \mathbb{N}$ such that $\forall n \geq N \geq 7$ by sandwich theorem $\lim \frac{n+2}{2^n} = 0$.

Hence $\lim \frac{2^{n+1}-(n+2)}{2^n} = \lim 2 - \lim \frac{n+2}{2^n} = 2 - \lim \frac{n+2}{2^n} = 2$.

P4 Lets prove that $\frac{9}{4}$ can be written as 2.24999... and 2.25000

$2.2499999... = \frac{2}{10^0} + \frac{2}{10^1} + \frac{4}{10^2} + \sum_{k=3}^{n} \frac{9}{10^k} = \frac{2}{1} + \frac{2}{10^1} + \frac{4}{10^2} + \frac{\frac{9}{10^3} + (\frac{1}{10})^{n+1}}{\frac{9}{10}} =$

For $n \to \infty$

$\frac{2}{1} + \frac{2}{10^1} + \frac{4}{10^2} + \frac{\frac{9}{10^3}}{\frac{9}{10}} = \frac{2}{1} + \frac{2}{10^1} + \frac{4}{10^2} + \frac{1}{10^2} = \frac{2}{1} + \frac{2}{10^1} + \frac{5}{10^2} = 2.25$