SATISH KUMAR

Exercise 1

$$f:[1,1] \rightarrow \mathbb{R}, \quad x \rightarrow x^2+3x+2$$

$$f(x) = x^2+3x+2$$
$$= x^2+2x+1x+2$$
$$= x(x+2)+1(x+2)$$
$$= (x+2)(x+1)$$

and if $x,y \in \mathbb{R}$ and $x<y$ than $f(x)<f(y)$

~~Since we know~~

Suppose $f(x) < f(y)$

$$\cancel{x^2+3x+2}$$
$$(x+2)(x+1) < (y+2)(y+1)$$
$$x^2+2x+x+2 < y^2+2y+y+2$$
$$x^2+3x+2 < y^2+3y+2$$
$$x^2+3x+2-y^2-3y-2 < 0$$
$$x^2+3x-3y-y^2 < 0$$
$$x^2-y^2+3(x-y) < 0$$

As we know $x<y$ and ~~y~~ $x-y<0$ than we proved that is true. And it is strictly increasing.

Exercise 2

let consider $f:[-3,2] \to \mathbb{R}$ with

$$f(x) = x^2$$

than $f(-3)=9$ and $f(2)=4$ therfore

$f^{-1}[9,4] \to \mathbb{R}$ but it means that

$9 \leq 4$ which is incorrect. Since it is not strictly increasing we cannot say that $a<x<b$ than $f(a)<f(x)<f(b)$.

when we consider $f:[a,b] \to \mathbb{R}$ is strictly increasing but not continous than f^{-1} will not necessarily increasing and continous. we cannot apply the intermediate theorm and therefore we cannot say if exist $c \in (a,b)$ with $f(c)=y$ and therefore cannot take ~~inverse~~ $f^{-1}(y)=c$.

Exercise 3

$$f: \mathbb{R}^+ \to \mathbb{R}, \quad x \to \sqrt{x}$$

Let we have $\varepsilon > 0$ and $\delta = \varepsilon^2$ and

we know $|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|$ for $x, y \in \mathbb{R}$

hence if $|x - y| < \delta$ than

$$|\sqrt{x} - \sqrt{y}| \leq |\sqrt{x} + \sqrt{y}|$$

Multiply both sides by $(\sqrt{x} - \sqrt{y})$

$$(|\sqrt{x} - \sqrt{y}|)(\sqrt{x} - \sqrt{y}) \leq (|\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$$

$$|\sqrt{x} - \sqrt{y}|^2 \leq |x - y| < \delta$$

$$|\sqrt{x} - \sqrt{y}|^2 \leq \delta$$

$$|\sqrt{x} - \sqrt{y}|^2 < \varepsilon^2$$

Square root both sides.

$$|\sqrt{x} - \sqrt{y}| < \varepsilon$$

than we proved function is uniform continous.

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Exercise 4

Let $a > 0$, $x_1 = \sqrt{a}$, $x_{n+1} = \sqrt{x_n}$ and $y_n = 2^n(x_n - 1)$

Base case $x_1 = \sqrt{a^{1/2}}$, $x_2 = \sqrt{a^{1/2}}$

Let $n \in \mathbb{N}$ be arbitrary and $x_n = a^{\frac{1}{2^n}}$

$x_{n+1} = x^{\frac{1}{2^n}}_n$

Induction hypothesis

$= \sqrt{a^{1/2n}} = a^{\frac{1}{2^{n+1}}}$

Therefore $x_n = a^{1/2n}$ for $\forall n \in \mathbb{N}$

than $y_n = 2^n(a^{\frac{1}{2n}} - 1)$

$= \dfrac{a^{\frac{1}{2n}} - 1}{2^{1/n}}$

Hence $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \dfrac{a^{\frac{1}{2n}} - 1}{2^{\frac{1}{n}}}$ which we can written $\frac{1}{2n} = x$

$= \lim_{x\to 0} \dfrac{a^x - 1}{x}$

we can proceed by substitution

$a^x - 1 = y$ than $a^x = y + 1 \Rightarrow x = \log_a(y+1)$

$\lim_{x\to 0} \dfrac{y+1-1}{\log_a(y+1)} = \lim_{x\to 0} \dfrac{y}{\log_a(y+1)} = \lim_{x\to 0} \dfrac{1}{\frac{\log_a(y+1)}{y}}$

$= \lim_{x\to 0} \dfrac{1}{\log_a(y+1)^{1/y}}$ and since $\lim_{z\to 0}(z+1)^{\frac{1}{z}} = e$

because with $z = \frac{1}{x}$ it is $\lim_{x\to\infty}\left(\frac{1}{x} + 1\right)^x = e$

than $\dfrac{1}{\log_a e} = \dfrac{\log_a e}{\log_a e} = \log_e a$.