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Exercise 1

a) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

~~det / B ≠~~ $\cancel{\begin{pmatrix} \cos\theta, & -\sin\theta \\ \sin\theta & \cos \end{pmatrix}}$

we know that

$$\det(A \cdot B \cdot C) = \det(A)\det(B)\det(C)$$

$$\det(A) = \det\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \det\begin{pmatrix} 3 & 0 \\ 0 & 7 \end{pmatrix} \det\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= (\cos^2\theta + \sin^2\theta)(3 \times 7 - 0)(\cos^2\theta + \sin^2\theta)$$

$$= 1 \cdot 21 \cdot 1$$

$$= 21 \quad \text{Ans}$$

b) Eigenvalue of A

~~As we know that~~ $\cancel{B^{-1} A B}$

~~$(S^T)^{-1} = \frac{1}{\det(S^T)}$ and eigenvector is also~~

~~$\frac{1}{S^T}$ and then~~

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$= \begin{pmatrix} \cancel{\cos\theta} \end{pmatrix} = B^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \det(B^{-1} C B) - \lambda I = \det(B^{-1}) \det(C) \det(B) - \det(I\lambda)$

$= \det(C - \lambda I)$

then the eigen ~~vectore~~ is $\det|(C - \lambda I)|$

and we have $(S^T)^{-1} C (S^T)$ than

eigenvalue is $\det \left| \begin{pmatrix} 3-\lambda & 0 \\ 0 & 7-\lambda \end{pmatrix} \right|$ that is

equal to $(3-\lambda)(7-\lambda)$ and eigenvalue

is 3, and 7

c) According to diagonalizing a matrix we know that $A = S \Lambda S^{-1}$ and Λ is eigen value of a and S is the eigenvector of A and S is inverse matrix. and then

the eigen vector is $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

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Exercise 2

$$A = \begin{pmatrix} s & -1 & -1 \\ -1 & s & -1 \\ -1 & -1 & s \end{pmatrix}$$

The characterstic polynomial of A is

$$= \det \begin{vmatrix} s-\lambda & -1 & -1 \\ -1 & s-\lambda & -1 \\ -1 & -1 & s-\lambda \end{vmatrix}$$

$$= (s-\lambda)\begin{vmatrix} s-\lambda & -1 \\ -1 & s-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & s-\lambda \end{vmatrix} + (-1)\begin{vmatrix} -1 & s-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (s-\lambda)\left((s-\lambda)^2 - 1\right) + \left((\lambda-s)-1\right) - 1\left(1-(-1)(s-\lambda)\right)$$

$$= (s-\lambda)^3 - s+\lambda+\lambda-s-1-1-s+\lambda$$

$$= (s-\lambda)^3 - 3(s-\lambda) - 2$$

$$= \left((s-\lambda)-2\right)\left((s-\lambda)+1\right)^2$$

than eigen value is

$\lambda_1 = s-2$

$\lambda_2 = s+1$

$\lambda_3 = s+1$

They all are positive when $s > 2$

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$$B = \begin{pmatrix} t & 3 & 1 \\ 3 & t & 0 \\ 1 & 0 & t \end{pmatrix}$$

$$= \det \begin{vmatrix} t-\lambda & 3 & 1 \\ 3 & t-\lambda & 0 \\ 1 & 0 & t-\lambda \end{vmatrix}$$

$$= (t-\lambda)\begin{vmatrix} t-\lambda & 0 \\ 0 & t-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 1 & t-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & t-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (t-\lambda)^3 - 3(3t-3\lambda) + 1(-t+\lambda)$$

$$= (t-\lambda)^3 - 10t + 10\lambda$$

$$= (t-\lambda)^3 - 10(t-\lambda)$$

$$= (t-\lambda)\left((t-\lambda)^2 - 10\right)$$

$$= (t-\lambda)\left((t-\lambda)+\sqrt{10}\right)\left((t-\lambda)-\sqrt{10}\right)$$

eigenvalue of B is

$t, \; t+\sqrt{10}, \; t-\sqrt{10}$

then they are positive when $t > \sqrt{10}$

Exercise 3

Bonus!

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Exercise 4

$$A = \begin{pmatrix} 0.6 & -0.4 & 0 \\ -0.4 & 0.6 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

eigenvalue of A

$$= \det \begin{vmatrix} 0.6-\lambda & -0.4 & 0 \\ -0.4 & 0.6-\lambda & 0 \\ 0 & 0 & 0.5-\lambda \end{vmatrix}$$

$$= (0.6-\lambda) \begin{vmatrix} 0.6-\lambda & 0 \\ 0 & 0.5-\lambda \end{vmatrix} - (-0.4) \begin{vmatrix} -0.4 & 0 \\ 0 & 0.5-\lambda \end{vmatrix} + 0$$

$$= (0.6-\lambda)^2 (0.5-\lambda) - (-0.4)(-0.4)(0.5-\lambda)$$

$$= (0.5-\lambda)\left((0.6-\lambda)^2 - 0.16\right)$$

$$= (0.5-\lambda)\left(0.36+\lambda^2-1.2\lambda-0.16\right)$$

$$= (0.5-\lambda)\left(\lambda^2-1.2\lambda+0.20\right)$$

$$= (0.5-\lambda)\left(x-\frac{1}{5}\right)(x-1)$$

Eigen value of A is $0.5, 0.2, 1$

eigenvector for $\lambda = 0.5$

$$= \left(\begin{array}{ccc|c} 0.6-0.5 & -0.4 & 0 & 0 \\ -0.4 & 0.6-0.5 & 0 & 0 \\ 0 & 0 & 0.5-0.5 & 0 \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} 0.1 & -0.4 & 0 & 0 \\ -0.4 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Multiply row 1 by $\frac{4}{10}$ and add row 2

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$$= \left(\begin{array}{ccc|c} 0.1 & -0.4 & 0 & 0 \\ 0 & 1.6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$z = 1 \qquad = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

eigenvector for $\lambda = 0.2$

$$= \left(\begin{array}{ccc|c} 0.6-0.2 & -0.4 & 0 & 0 \\ -0.4 & 0.6-0.2 & 0 & 0 \\ 0 & 0 & 0.5-0.2 & 0 \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} 0.4 & -0.4 & 0 & 0 \\ -0.4 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{array}\right)$$

Add row 1 and 2

$$= \left(\begin{array}{ccc|c} 0.4 & -0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{array}\right)$$

than $y = 1$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

eigenvector for $\lambda = 1$

$$= \left(\begin{array}{ccc|c} 0.6-1 & -0.4 & 0 & 0 \\ -0.4 & 0.6-1 & 0 & 0 \\ 0 & 0 & 0.5-1 & 0 \end{array}\right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left(\begin{array}{ccc|c} -0.4 & -0.4 & 0 & 0 \\ -0.4 & -0.4 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

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Multiply row 1 by -1 and add row 2

$$= \left(\begin{array}{ccc|c} -0.4 & -0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 \end{array}\right)$$

$y = 1$ than $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

eigenvector of A is

$$S = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

As we know with diagonalising matrix

$$A^p = S \Lambda^p S^{-1}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{pmatrix}^p \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} (0.5)^p & 0 & 0 \\ 0 & (0.2)^p & 0 \\ 0 & 0 & 1^p \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

As we know that

$$\lim_{p \to \infty} (0.5)^p = \lim_{p \to \infty} \frac{1}{2^p} = 0$$

and

$$\lim_{p \to \infty} (0.2) = \lim_{p \to \infty} \frac{1}{5^p} = 0$$

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$$= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ -1/2 & 1/2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ Ans