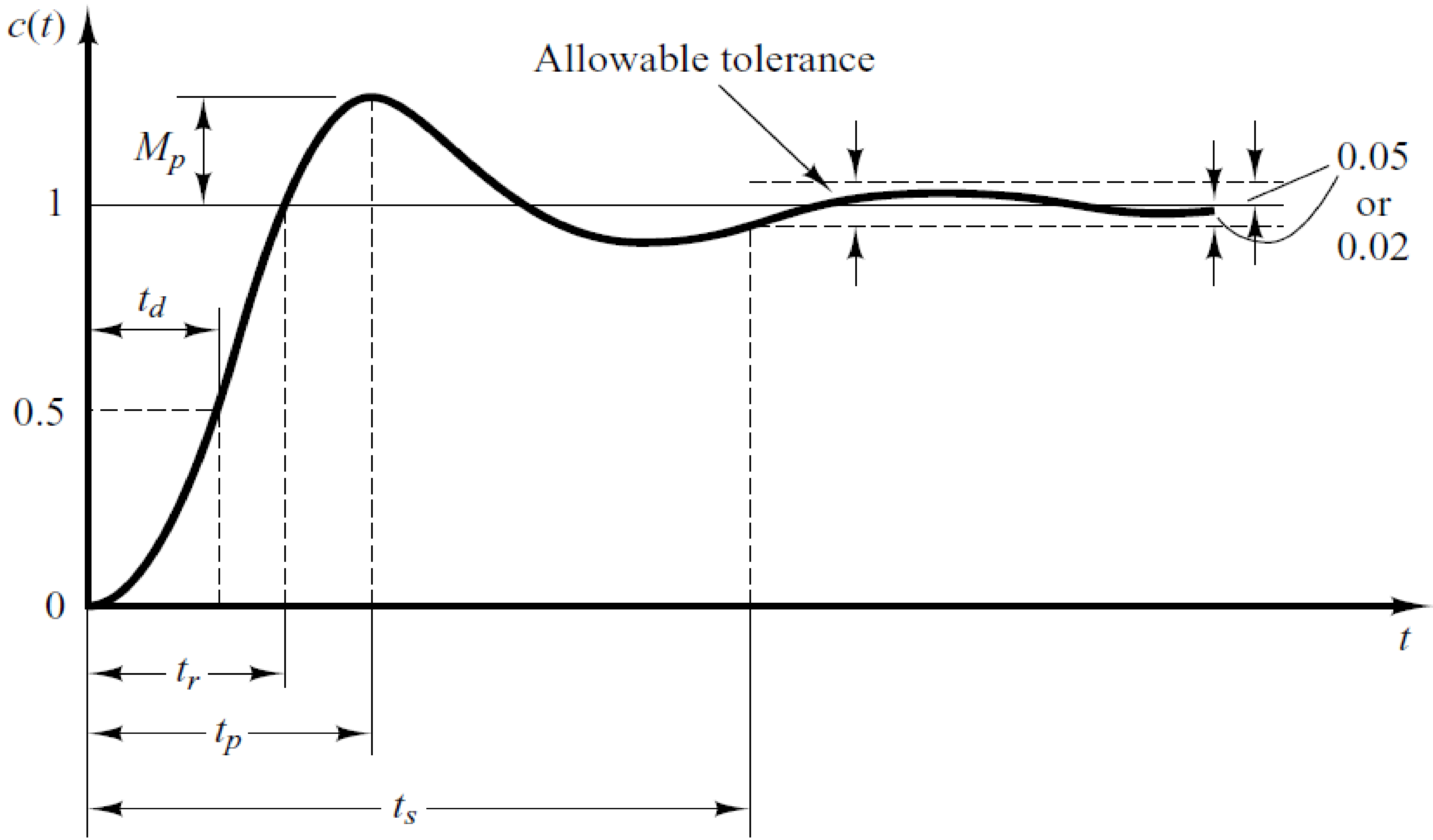
Time Domain Specifications of Underdamped system



Rise Time ($\boldsymbol{t_r}$)

Response of Second Order System for under damped Case

C(t) = 1 - $\frac{e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}$ sin(ω_dt + θ)

At t = $\boldsymbol{t_r}$, C($\boldsymbol{t_r}$) = 1

$\Rightarrow$C($\boldsymbol{t_r}$) = 1 - $\frac{e^{-\xi\omega_n \boldsymbol{t_r}}}{\sqrt{(1-\xi^2)}}$ sin($\omega_d \boldsymbol{t_r}$+ θ) = 1

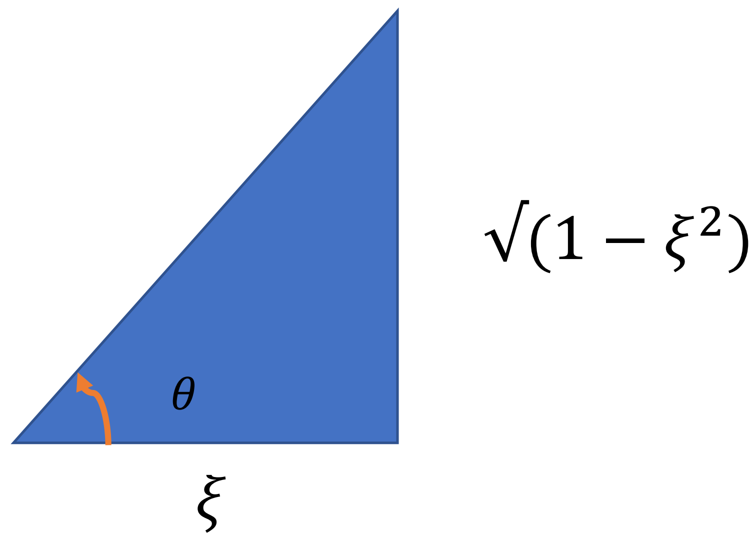
$\Rightarrow$ - $\frac{e^{-\xi\omega_n \boldsymbol{t_r}}}{\sqrt{(1-\xi^2)}}$ sin($\omega_d \boldsymbol{t_r}$+ θ) = 0

Rise Time ($\boldsymbol{t_r}$)

Since $\frac{e^{-\xi\omega_n \boldsymbol{t_r}}}{\sqrt{(1-\xi^2)}} \neq 0$

=> $\sin(\omega_d \boldsymbol{t_r} + \theta) = 0$

$\Rightarrow \omega_d \boldsymbol{t_r} + \theta = \pi$



=> $\boldsymbol{t_r} = \frac{\pi-\theta}{\omega_d}$ where $\theta = tan^{-1}\frac{\sqrt{(1-\xi^2)}}{\xi}$ and $\omega_d = \omega_n\sqrt{(1-\xi^2)}$

Peak Time (t_p)

Response of Second Order System for under damped Case

$$C(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}} \sin(\omega_d t + \theta)$$

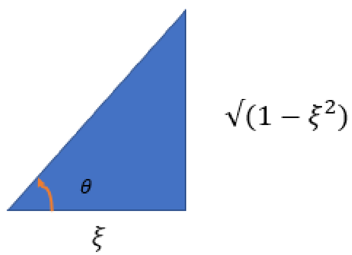
To get the Peak value, $\frac{dc(t)}{dt} |t = t_p = 0$

$$\frac{dc(t)}{dt} = \frac{-e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}} (-\xi\omega_n) \sin(\omega_d t + \theta) + \frac{-e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}} (\omega_d) \sin(\omega_d t + \theta)$$

Peak Time (t_p)

$$\frac{dc(t)}{dt} = \frac{-e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}(-\xi\omega_n)\sin(\omega_d t + \theta) + \frac{-e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}(\omega_d)\sin(\omega_d t + \theta)$$

Put $\omega_d = \omega_n \sqrt{(1-\xi^2)}$



$$\Rightarrow \frac{dc(t)}{dt} = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}\left[(\xi)\sin(\omega_d t + \theta) - \sqrt{(1-\xi^2)}\sin(\omega_d t + \theta) \right]$$

$$\Rightarrow \frac{dc(t)}{dt} = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}\left[\cos(\theta)\sin(\omega_d t + \theta) - \sin(\theta)\sin(\omega_d t + \theta) \right]$$

Peak Time ($\boldsymbol{t_p}$)

$$\Rightarrow \frac{dc(t)}{dt} = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}} [\sin(\omega_d t + \theta - \theta)] = \frac{\omega_n e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}} \sin(\omega_d t)$$

$$\frac{dc(t)}{dt} |t = \boldsymbol{t_p} = 0$$

Since $\frac{\omega_n e^{-\xi\omega_n \boldsymbol{t_p}}}{\sqrt{(1-\xi^2)}} \neq 0 \quad \Rightarrow \sin(\omega_d \, \boldsymbol{t_p}) = 0$

$\Rightarrow \omega_d \, \boldsymbol{t_p} = \pi$ $\qquad$ $\Rightarrow \boldsymbol{t_p} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{(1-\xi^2)}}$

Peak Overshoot ($\boldsymbol{M_p}$)

% **Peak Overshoot (**% $\boldsymbol{M_p}$**)** = $\frac{\boldsymbol{c}(\boldsymbol{t_p}) - \boldsymbol{c}(\propto)}{\boldsymbol{c}(\propto)} * 100$

From Response of Second Order System for under damped Case

C(t) = 1 - $\frac{e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}$ sin(ω_dt + θ)

=> C($\boldsymbol{t_p}$) = 1 - $\frac{e^{-\xi\omega_n \boldsymbol{t_p}}}{\sqrt{(1-\xi^2)}}$ sin($\omega_d \boldsymbol{t_p}$+ θ)

Peak Overshoot ($\boldsymbol{M_p}$)

Since $\boldsymbol{t_p} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{(1-\xi^2)}}$

$$\Rightarrow C(\boldsymbol{t_p}) = 1 - \frac{e^{-\xi \omega_n \frac{\pi}{\omega_n \sqrt{(1-\xi^2)}}}}{\sqrt{(1-\xi^2)}} \sin(\omega_d \frac{\pi}{\omega_d} + \theta)$$

$$C(\boldsymbol{t_p}) = 1 - \frac{e^{-\xi \frac{\pi}{\sqrt{(1-\xi^2)}}}}{\sqrt{(1-\xi^2)}} \sin(\pi + \theta)$$

But $\sin(\pi + \theta) = -\sin(\theta)$

Peak Overshoot ($\boldsymbol{M_p}$)

$$C(\boldsymbol{t_p}) = 1 + \frac{e^{-\xi\frac{\pi}{\sqrt{(1-\xi^2)}}}}{\sqrt{(1-\xi^2)}}\sin(\theta)$$

$$C(\boldsymbol{t_p}) = 1 + \frac{e^{-\xi\frac{\pi}{\sqrt{(1-\xi^2)}}}}{\sqrt{(1-\xi^2)}}\sqrt{(1-\xi^2)}$$

$$C(\boldsymbol{t_p}) = 1 + e^{-\xi\frac{\pi}{\sqrt{(1-\xi^2)}}}$$

And $C(\propto) = 1 - \frac{e^{-\xi\omega_n\propto}}{\sqrt{(1-\xi^2)}}\sin(\omega_d \propto + \theta) = 1 - 0 = 1$

Peak Overshoot (M_p)

% Peak Overshoot (% M_p) = $\frac{c(t_p)-c(\propto)}{c(\propto)} * 100$

$$= \frac{1 + e^{-\xi\frac{\pi}{\sqrt{(1-\xi^2)}}} - 1}{1} * 100$$

% M_p = $e^{-\xi\frac{\pi}{\sqrt{(1-\xi^2)}}} * 100$

Settling Time ($\boldsymbol{t_s}$)

The response of the second order system has two components

1. Decaying Exponential Component, $\frac{e^{-\xi\omega_n t}}{\sqrt{(1-\xi^2)}}$
2. Sinusoidal component, sin(ω_dt + θ)

For 2 % tolerance error band at t = $\boldsymbol{t_s}$, $\frac{e^{-\xi\omega_n \boldsymbol{t_s}}}{\sqrt{(1-\xi^2)}}$ = 0.02

For Least Values of ξ , Denominator = 1

=> $e^{-\xi\omega_n \boldsymbol{t_s}}$ = 0.02 => $-\xi\omega_n \boldsymbol{t_s}$ = ln(0.02) = - 4

Settling Time ($\boldsymbol{t_s}$)

$\Rightarrow -\xi\omega_n \boldsymbol{t_s}$ = - 4

$$\Rightarrow \boldsymbol{t_s} = \frac{4}{\xi\omega_n}$$

Similarly for 5% error,

$e^{-\xi\omega_n \boldsymbol{t_s}}$ = 0.05 => $-\xi\omega_n \boldsymbol{t_s}$ = ln(0.05) = - 3

$$=> \boldsymbol{t_s} = \frac{3}{\xi\omega_n}$$

Summary of Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta \omega_n}$$

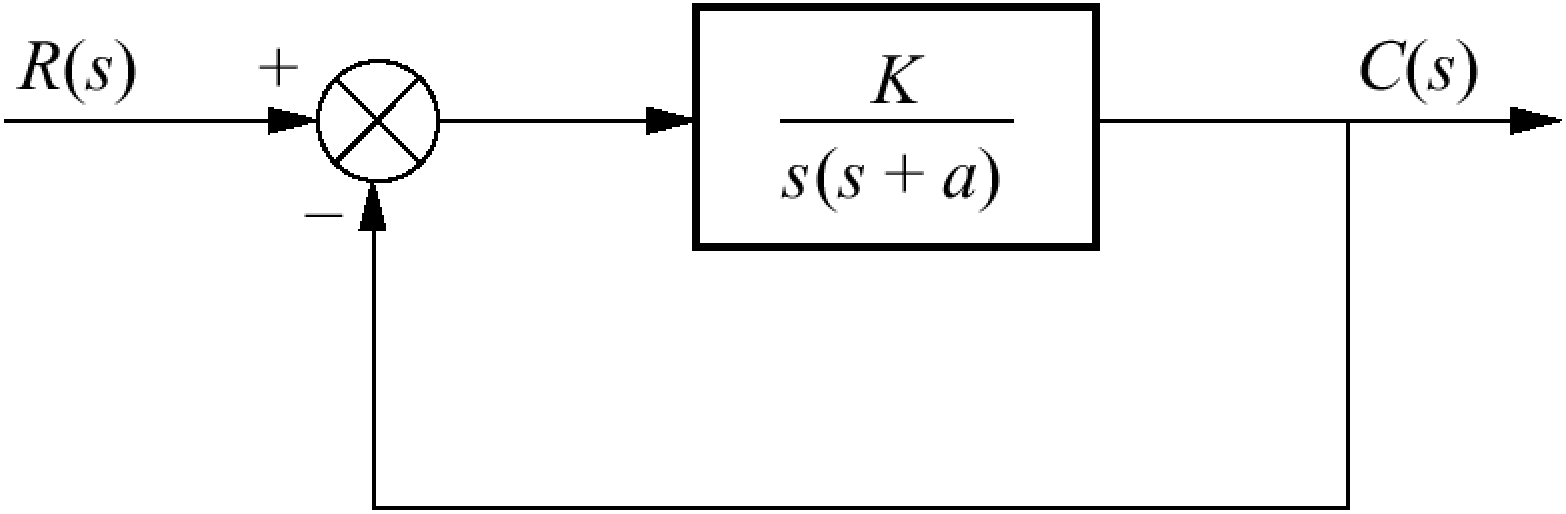
$$t_s = 3T = \frac{3}{\zeta \omega_n}$$

Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Second-order feedback control system



The closed loop transfer function is $$T(s) = \frac{K}{s^2 + as + K}$$

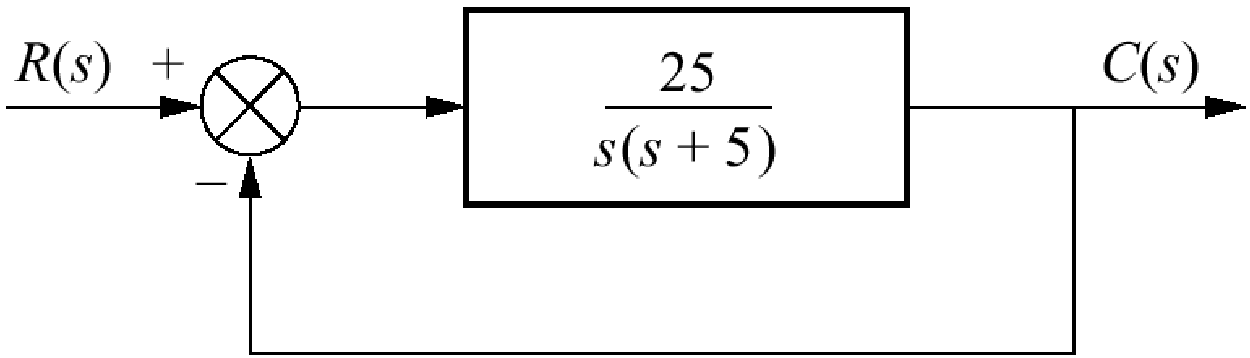
Note K is the amplifier gain, As K varies, the poles move through the three ranges of operations OD, CD, and UD

$0<K<a^2/4$ system is over damped

$K = a^2/4$ system is critically damped

$K > a^2/4$ system is under damped

Finding transient response Example



Problem: For the system shown, find peak time, percent overshot, and settling time.

Solution: The closed loop transfer function is $T(s) = \dfrac{25}{s^2 + 5s + 25}$

And $\omega_n = \sqrt{25} = 5$

$2\xi\omega_n = 5 \quad \text{so} \quad \xi = 0.5$

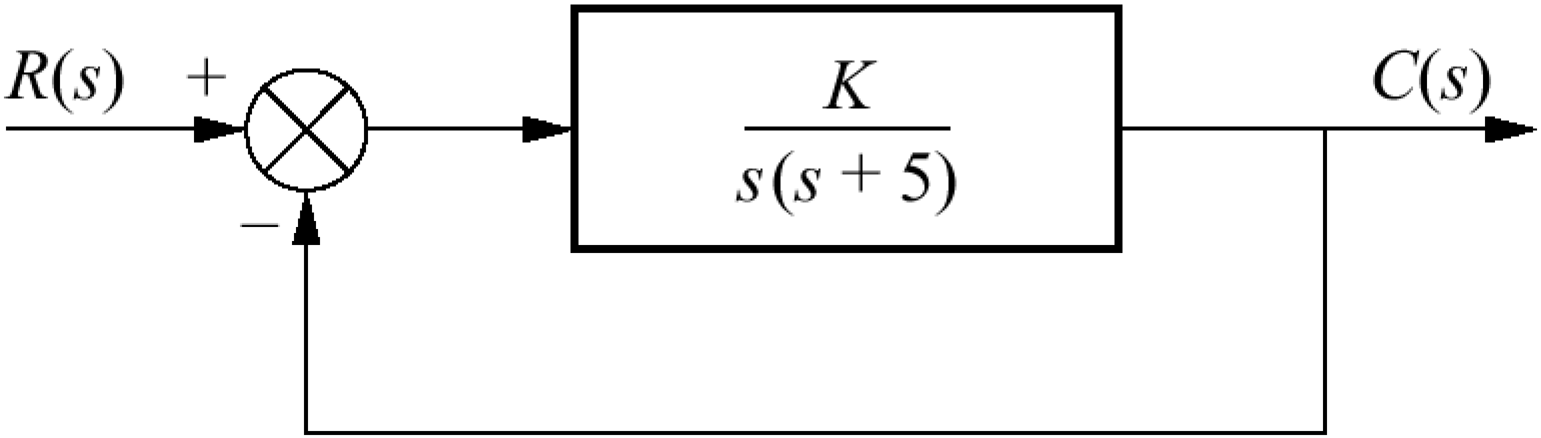
using values for ξ and ω_n and equation in chapter 4 we find

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\xi^2}} = 0.726 \quad \text{sec}$$

$$\%OS = e^{-\xi\pi/\sqrt{1-\xi^2}} X\ 100 = 16.303$$

$$T_s = \frac{4}{\xi\omega_n} = 1.6 \quad \text{sec}$$

Gain design for transient response Example



Problem: Design the value of gain K, so that the system will respond with a 10% overshot.

Solution: The closed loop transfer function is $T(s) = \frac{K}{s^2 + 5s + K}$

$$\omega_n = \sqrt{K} \quad \text{and} \quad 2\xi\omega_n = 5 \quad \text{thus} \quad \xi = \frac{5}{2\sqrt{K}}$$

For 10% OS we find $\xi = 0.591$

We substitute this value in previous equation to find K = 17.9

Example

A unity feedback system has

$$G(s) = \frac{16}{s(s+5)}$$

If a step input is given calculate

1. Damping Ratio
2. Overshoot
3. Settling Time

Solution: $G(s) = \dfrac{16}{s(s+5)}$ $\qquad H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\dfrac{16}{s(s+5)}}{1+\dfrac{16}{s(s+5)}} = \frac{16}{s^2 + 5s + 16}$$

Example cont……

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + \underline{2\xi\omega_n s} + \underline{\omega_n^2}} = \frac{16}{s^2 + \underline{5s} + \underline{16}}$$

Compare denominators of both

Natural Frequency;

$$\omega_n^2 = 16 \qquad \therefore \omega_n = 4 \ \ rad / \sec$$

Damping Ratio;

$$2\xi\omega_n s = 5s \qquad \therefore \xi = \frac{5}{2 \times \omega_n} = \frac{5}{2 \times 4} = 0.625$$

Settling Time;

$$T_s = \frac{4}{\xi\omega_n} = \frac{4}{(0.625)\times(4)} = 1.6 \sec$$

Example cont……

Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.625)\pi}{\sqrt{1-(0.625)^2}}\right\}} \times 100$$

$$\%M_p = 8.08\%$$

Example

A unity feedback system has

$$G(s) = \frac{100}{s(s+5)}$$

If it is subjected to unit step input determine;

1. Damped frequency of oscillations
2. Time for first overshoot
3. Settling Time
4. Maximum Peak Overshoot

Example cont……

Solution: $G(s) = \dfrac{100}{s(s+5)}$ $\qquad H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{100}{s(s+5)}}{1+\frac{100}{s(s+5)}} = \frac{100}{s^2+5s+100}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{100}{s^2+5s+100}$$

Compare denominators of both

Natural Frequency;

$$\omega_n^2 = 100 \qquad \therefore \omega_n = 10 \ \ rad/\sec$$

Example cont.....

Damping Ratio;

$$2\xi\omega_n s = 5s \qquad \therefore \xi = \frac{5}{2\times\omega_n} = \frac{5}{2\times10} = 0.25$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 10\sqrt{1-(0.25)^2} = 9.68\ rad\ /\sec$$

Time for first overshoot (Peak Time);

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{9.68} = 0.324\sec$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.25)\times(10)} = 1.6\ \sec$$

Example cont……

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.25)\pi}{\sqrt{1-(0.25)^2}}\right\}} \times 100$$

$$\%M_p = 44.48\%$$

Example

For the unity feedback control system having open loop transfer function

$$G(s) = \frac{10}{s(s+4)}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Example cont……

Solution: $G(s) = \dfrac{10}{s(s+4)}$ $H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\dfrac{10}{s(s+4)}}{1+\dfrac{10}{s(s+4)}} = \frac{10}{s^2+4s+10}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{10}{s^2+4s+10}$$

Compare denominators of both

Natural Frequency;

$$\omega_n^2 = 10 \qquad \therefore \omega_n = 3.16 \quad rad/\sec$$

Example cont……

Damping Ratio;

$$2\xi\omega_n s = 4s \qquad \therefore \xi = \frac{4}{2\times\omega_n} = \frac{4}{2\times 3.16} = 0.633$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 3.16\sqrt{1-(0.633)^2} = 2.44 \ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.633)}{3.16} = 0.457 \quad \sec$$

Example cont......

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.633)^2}}{(0.633)} = 0.885 \quad \text{rad}$$

$$T_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-0.885}{(0.244)} = 0.92 \quad \text{sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.44} = 1.273 \quad \text{sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.633)\times(3.16)} = 1.997 \quad \text{sec}$$

Example cont……

Maximum Peak Overshoot

$$\% M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\% M_p = e^{-\left\{\frac{(0.633)\pi}{\sqrt{1-(0.633)^2}}\right\}} \times 100$$

$$\% M_p = 7.66\%$$

Example

A second order servo system has a unity feedback,

$$G(s) = \frac{500}{s(s+15)}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Example cont……

Solution: $G(s) = \frac{500}{s(s+15)}$ $H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{500}{s(s+15)}}{1+\frac{500}{s(s+15)}} = \frac{500}{s^2+15s+500}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{500}{s^2+15s+500}$$

By comparing denominators of both

Natural Frequency;

$$\omega_n^2 = 500 \qquad \therefore \omega_n = 22.36 \quad rad/\sec$$

Example cont……

Damping Ratio;

$$2\xi\omega_n s = 15s \qquad \therefore \xi = \frac{15}{2\times\omega_n} = \frac{15}{2\times 22.36} = 0.335$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 22.36\sqrt{1-(0.335)^2} = 21.06\ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.335)}{22.36} = 0.055\sec$$

Example cont.....

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.335)^2}}{(0.335)} = 1.229 \quad \text{rad}$$

$$T_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-1.229}{(21.06)} = 0.091 \quad \text{sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{21.06} = 32.73 \text{ sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.335)\times(22.36)} = 0.534 \quad \text{sec}$$

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.335)\pi}{\sqrt{1-(0.335)^2}}\right\}} \times 100$$

$$\%M_p = 32.75\%$$

Example

The open loop transfer function of a unity feedback system is,

$$G(s) = \frac{4}{s(s+1)}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Example cont……

Solution: $G(s) = \dfrac{4}{s(s+1)}$ $H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\dfrac{4}{s(s+1)}}{1+\dfrac{4}{s(s+1)}} = \frac{4}{s^2+s+4}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{4}{s^2+s+4}$$

By comparing denominators of both

Natural Frequency;

$$\omega_n^2 = 4 \qquad \therefore \omega_n = 2 \ rad/\sec$$

Example cont......

Damping Ratio;

$$2\xi\omega_n s = s \qquad \therefore \xi = \frac{1}{2\times\omega_n} = \frac{1}{2\times 2} = 0.25$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 2\sqrt{1-(0.25)^2} = 1.936\ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.25)}{2} = 0.587\ \sec$$

Example cont......

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-(0.25)^2}}{(0.25)} = 1.310 \quad \text{rad}$$

$$T_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-1.310}{(1.936)} = 0.945 \quad \text{sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{1.936} = 1.622 \text{ sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.25)\times(2)} = 8 \text{ sec}$$

Example cont……

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.25)\pi}{\sqrt{1-(0.25)^2}}\right\}} \times 100$$

$$\%M_p = 43.26\%$$

Example

The open loop transfer function of a unity feedback system is,

$$G(s) = \frac{25}{s(s+5)}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Example cont……

Solution: $G(s) = \dfrac{25}{s(s+5)}$ $H(s) = 1$

Determine the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{25}{s(s+5)}}{1+\frac{25}{s(s+5)}} = \frac{25}{s^2+5s+25}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{25}{s^2+5s+25}$$

By comparing denominators of both

Natural Frequency;

$$\omega_n^2 = 25 \qquad \therefore \omega_n = 5 \ \ rad/\sec$$

Example cont……

Damping Ratio;

$$2\xi\omega_n s = 5s \qquad \therefore \xi = \frac{5}{2\times\omega_n} = \frac{5}{2\times 5} = 0.5$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 5\sqrt{1-(0.5)^2} = 4.33\ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.5)}{5} = 0.27\ \sec$$

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\left[\frac{\sqrt{1-(0.5)^2}}{(0.5)}\right] = 1.24 \quad \text{rad}$$

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - 1.04}{(4.330)} = 0.485 \quad \text{sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{4.330} = 0.725 \quad \text{sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.5)\times(5)} = 1.6 \quad \text{sec}$$

Example cont……

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.5)\pi}{\sqrt{1-(0.5)^2}}\right\}} \times 100$$

$$\%M_p = 16.30\%$$

Example

The closed loop transfer function of a unity feedback system is,

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 4s + 5}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Example cont……

Solution:

The given closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 4s + 5}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{10}{s^2 + 4s + 5}$$

By comparing denominators of both

Natural Frequency;

$$\omega_n^2 = 5 \qquad \therefore \omega_n = 2.23 \quad rad / \sec$$

Example cont……

Damping Ratio;

$$2\xi\omega_n s = 4s \qquad \therefore \xi = \frac{4}{2\times\omega_n} = \frac{4}{2\times 2.23} = 0.896$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 2.23\sqrt{1-(0.896)^2} = 0.99 \ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.896)}{2.23} = 0.72 \quad \sec$$

Example cont……

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\left[\frac{\sqrt{1-(0.896)^2}}{(0.896)}\right] = 0.46 \quad \text{rad}$$

$$T_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-0.46}{(0.99)} = 2.70 \quad \text{sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{0.99} = 2.1515 \text{ sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.896)\times(2.23)} = 2.00 \quad \text{sec}$$

Example cont……

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.896)\pi}{\sqrt{1-(0.896)^2}}\right\}} \times 100$$

$$\%M_p = 0.17\%$$

Example

The closed loop transfer function of a unity feedback system is,

$$\frac{C(s)}{R(s)} = \frac{100}{s^2 + 15s + 100}$$

Determine;

1. Delay Time
2. Rise Time
3. Peak Time
4. Settling Time
5. Maximum Peak Overshoot

Solution:

The given closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{100}{s^2 + 15s + 100}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{100}{s^2 + 15s + 100}$$

By comparing denominators of both

Natural Frequency;

$$\omega_n^2 = 100 \qquad \therefore \omega_n = 10 \quad rad/\sec$$

Example cont……

Damping Ratio;

$$2\xi\omega_n s = 15s \qquad \therefore \xi = \frac{15}{2\times\omega_n} = \frac{15}{2\times 10} = 0.75$$

Damped frequency of oscillations;

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \therefore \omega_d = 10\sqrt{1-(0.75)^2} = 6.61\ rad/\sec$$

Delay Time;

$$T_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.75)}{10} = 0.135\sec$$

Example cont……

Rise Time;

$$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\left[\frac{\sqrt{1-(0.75)^2}}{(0.75)}\right] = 0.722 \quad \text{rad}$$

$$T_r = \frac{\pi-\beta}{\omega_d} = \frac{\pi-0.722}{(6.61)} = 0.365 \quad \text{sec}$$

Peak Time;

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{6.61} = 0.474 \text{ sec}$$

Settling Time;

$$T_s = 4T = \frac{4}{\xi\omega_n} = \frac{4}{(0.75)\times(10)} = 0.533 \quad \text{sec}$$

Maximum Peak Overshoot

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

$$\%M_p = e^{-\left\{\frac{(0.75)\pi}{\sqrt{1-(0.75)^2}}\right\}} \times 100$$

$$\%M_p = 2.83\%$$

Example

The closed loop transfer functions of certain second order unity feedback control system is,

$$\frac{C(s)}{R(s)} = \frac{8}{s^2 + 3s + 8}$$

Determine the type of damping in the system.

Example cont……

Solution:

The given closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{8}{s^2 + 3s + 8}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{8}{s^2 + 3s + 8}$$

By comparing denominators of both

Natural Frequency; $\omega_n^2 = 8$ $\quad \therefore \omega_n = 2.82 \quad rad/\sec$

Damping Ratio; $2\xi\omega_n s = 3s$ $\quad \therefore \xi = \frac{3}{2 \times \omega_n} = \frac{3}{2 \times 2.82} = 0.53$

Since $\xi < 1$, it is an underdamped system.

Example

The closed loop transfer functions of certain second order unity feedback control system is,

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 4s + 2}$$

Determine the type of damping in the system.

Example cont.....

Solution:

The given closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 4s + 2}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{2}{s^2 + 4s + 2}$$

By comparing denominators of both

Natural Frequency; $\omega_n^2 = 2$ $\quad \therefore \omega_n = 1.414 \quad rad/\sec$

Damping Ratio; $2\xi\omega_n s = 4s$ $\quad \therefore \xi = \frac{4}{2 \times \omega_n} = \frac{4}{2 \times 1.414} = 1.41$

Since $\xi > 1$, it is an overdamped system.

Example

The closed loop transfer functions of certain second order unity feedback control system is,

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 2s + 1}$$

Determine the type of damping in the system.

Example cont……

Solution:

The given closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 2s + 1}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{2}{s^2 + 2s + 1}$$

By comparing denominators of both

Natural Frequency; $\omega_n^2 = 1$ $\quad \therefore \omega_n = 1 \ rad/\sec$

Damping Ratio; $2\xi\omega_n s = 2s$ $\quad \therefore \xi = \frac{2}{2 \times \omega_n} = \frac{2}{2 \times 1} = 1$

Since $\xi = 1$, it is an critically damped system.

Example

The closed loop transfer functions of certain second order unity feedback control system is,

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 4}$$

Determine the type of damping in the system.

Solution:

The given closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{2}{s^2 + 4}$$

Compare closed loop TF with standard form of second order system

$$\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{2}{s^2 + 4}$$

By comparing denominators of both

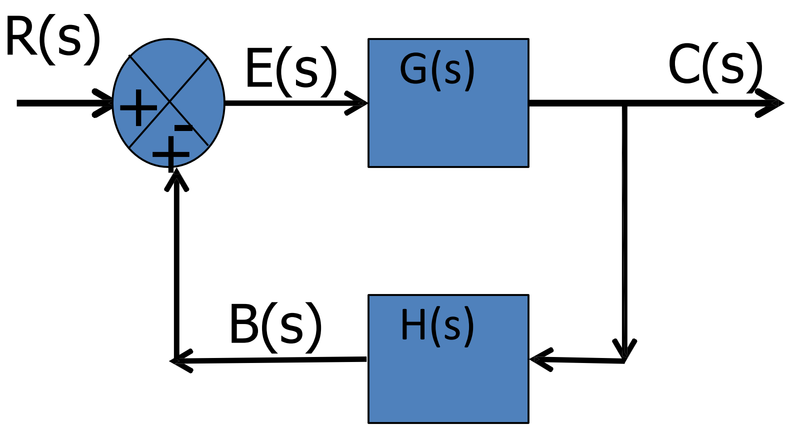
Natural Frequency; $\omega_n^2 = 4 \quad \therefore \omega_n = 2 \; rad/\sec$

Damping Ratio; $2\xi\omega_n s = 0 \quad \therefore \xi = 0$

Since $\xi = 0$, it is an undamped system.

Derivation of Steady State Error

The steady state response is important to judge the accuracy of the output. The difference between the steady state response and desired reference gives the steady state error.



For given figure,

$$E(s) = R(s) - B(s)$$

But

$$B(s) = C(s).H(s)$$

$$E(s) = R(s) - C(s).H(s)$$

But

$$C(s) = G(s).E(s)$$

$$E(s) = R(s) - G(s).E(s).H(s)$$

$$R(s) = E(s) + G(s).E(s).H(s)$$

$$R(s) = E(s)\{1 + G(s).H(s)\}$$

$$E(s) = \frac{R(s)}{1 \pm G(s).H(s)}$$

Derivation of Steady State Error

In time domain,

$$e(\mathrm{t}) = \mathrm{L}^{-1} \; E(\mathrm{s})$$

and is the expression of error valid for all time. Steady state error is defined as,

$$e_{ss}(\mathrm{t}) = \lim_{t \to \infty} e(t)$$

From the final value theorem in Laplace transform,

$$e_{ss}(\mathrm{t}) = \lim_{s \to 0} sE(\mathrm{s})$$

Steady state error,

$$e_{ss}(\mathrm{t}) = \lim_{s \to 0} sE(\mathrm{s})$$

$$\boxed{e_{ss}(\mathrm{t}) = \lim_{s \to 0} \frac{sR(\mathrm{s})}{1 + G(\mathrm{s})\;\mathrm{H(s)}}}$$

Steady state error for step input:

A step input of magnitude A is applied,

$$r(t) = A.\,u(t) \qquad t>0$$
$$= 0 \qquad t<0$$

Taking Laplace transform,

$$R(s) = L\{r(t)\} = L\{A\} = \frac{A}{s}$$

Steady state error,

$$e_{ss}(t) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)\ H(s)}$$

Steady state error and Standard Signals

Steady state error for step input:

$$e_{ss}(t) = \lim_{s \to 0} \frac{s \frac{A}{s}}{1 + G(s)\ H(s)}$$

$$e_{ss}(t) = \lim_{s \to 0} \frac{A}{1 + G(s)\ H(s)}$$

$$e_{ss}(t) = \frac{A}{1 + \lim_{s \to 0} G(s)\, H(s)}$$

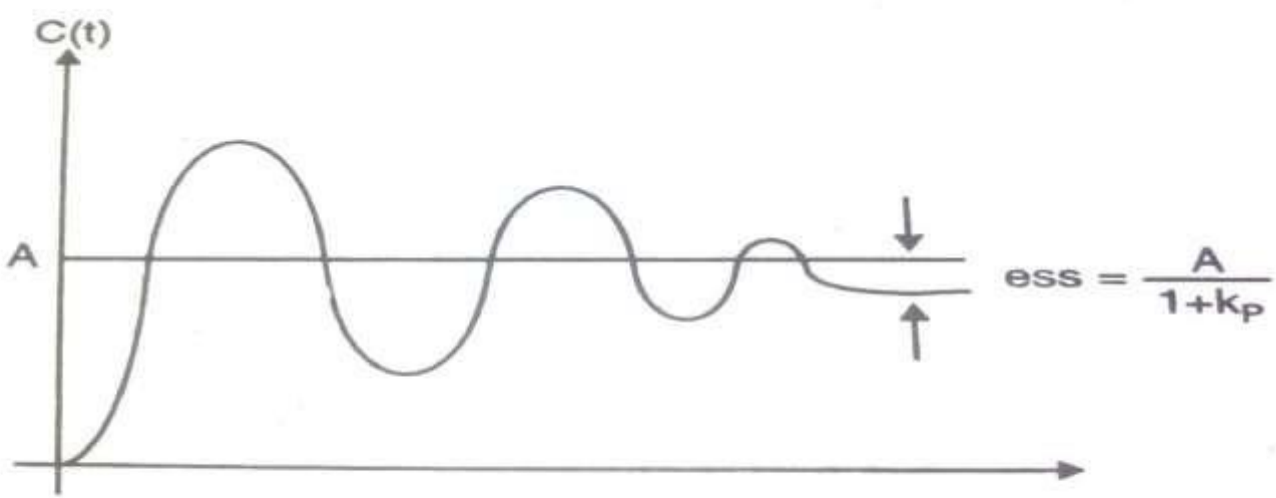
$$e_{ss}(t) = \frac{A}{1 + K_p}$$

Steady state error for step input:

The position error constant Kp of a system is defined as,

$$K_p = \lim_{s \to 0} G(s).H(s)$$

When a step input of magnitude A is given, in response to this gives $e_{ss}(t) = \dfrac{A}{1 + K_p}$ steady state error



*Kp depends on type of system

Steady state error for ramp input:

A ramp input of slope A is applied,

$$r(t) = A.t \qquad t>0$$
$$= 0 \qquad t<0$$

Taking Laplace transform,

$$R(s) = L\{r(t)\} = L\{At\} = \frac{A}{s^2}$$

Steady state error,

$$e_{ss}(t) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)\ H(s)}$$

Steady state error and Standard Signals

Steady state error for ramp input:

$$e_{ss}(t) = \lim_{s \to 0} \frac{s\,\dfrac{A}{s^2}}{1 + G(s)\ H(s)}$$

$$e_{ss}(t) = \lim_{s \to 0} \frac{\dfrac{A}{s}}{1 + G(s)\ H(s)}$$

$$e_{ss}(t) = \lim_{s \to 0} \frac{A}{s + sG(s)H(s)}$$

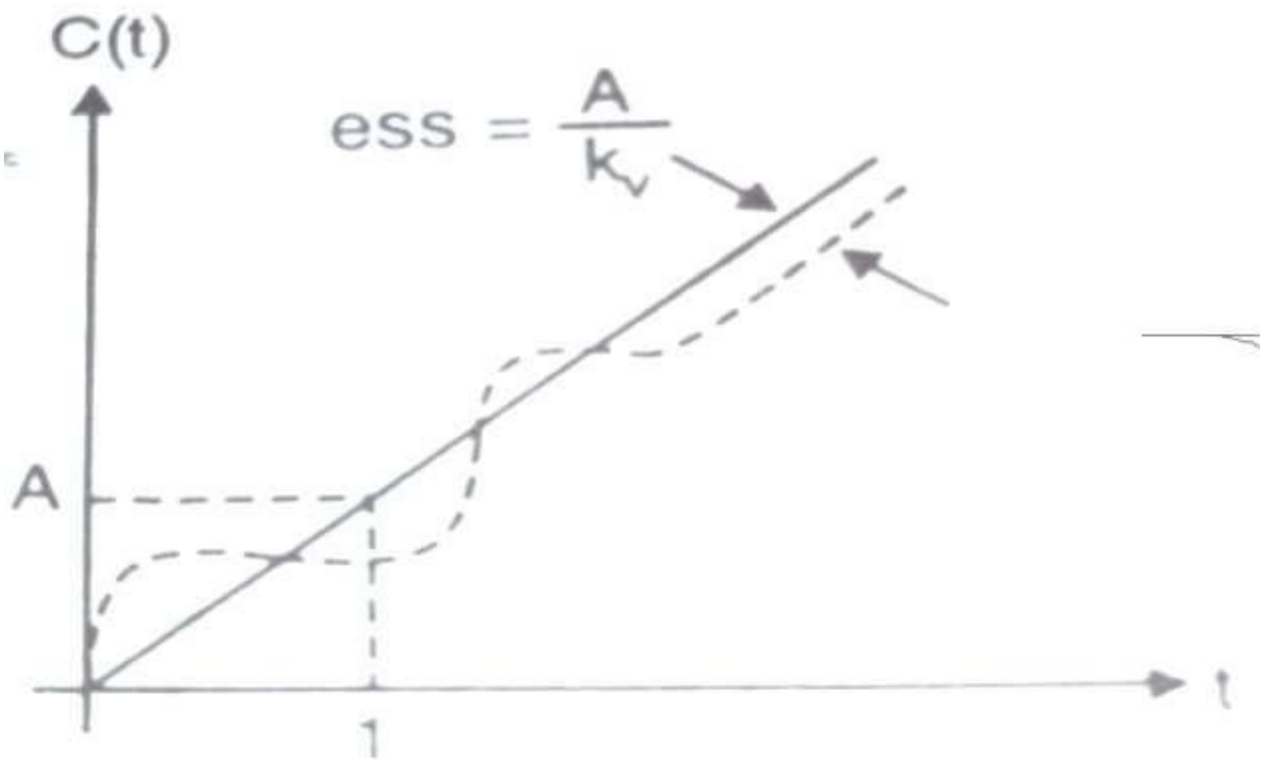
$$e_{ss}(t) = \frac{A}{0 + \lim_{s \to 0} sG(s)H(s)}$$

$$e_{ss}(t) = \frac{A}{K_v}$$

Steady state error for ramp input:

The velocity error constant Kv of a system is defined as,

$$K_v = \lim_{s \to 0} sG(s).\,H(s)$$



Steady state error for parabolic input:

A parabolic input of slope coefficient A/2 is applied,

$$r(t) = \frac{At^2}{2} \qquad t>0$$
$$= 0 \qquad t<0$$

Taking Laplace transform,

$$R(s) = L\{r(t)\} = L\{\frac{A}{2}\, t^2\} = \frac{A}{s^3}$$

Steady state error,

$$e_{ss}(t) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)\ H(s)}$$

Steady state error and Standard Signals

Steady state error for parabolic input:

$$e_{ss}(t) = \lim_{s \to 0} \frac{s \dfrac{A}{s^3}}{1 + G(s)\ H(s)}$$

$$e_{ss}(t) = \lim_{s \to 0} \frac{\dfrac{A}{s^2}}{1 + G(s)\ H(s)}$$

$$e_{ss}(t) = \lim_{s \to 0} \frac{A}{s^2 + s^2 G(s)\ H(s)}$$

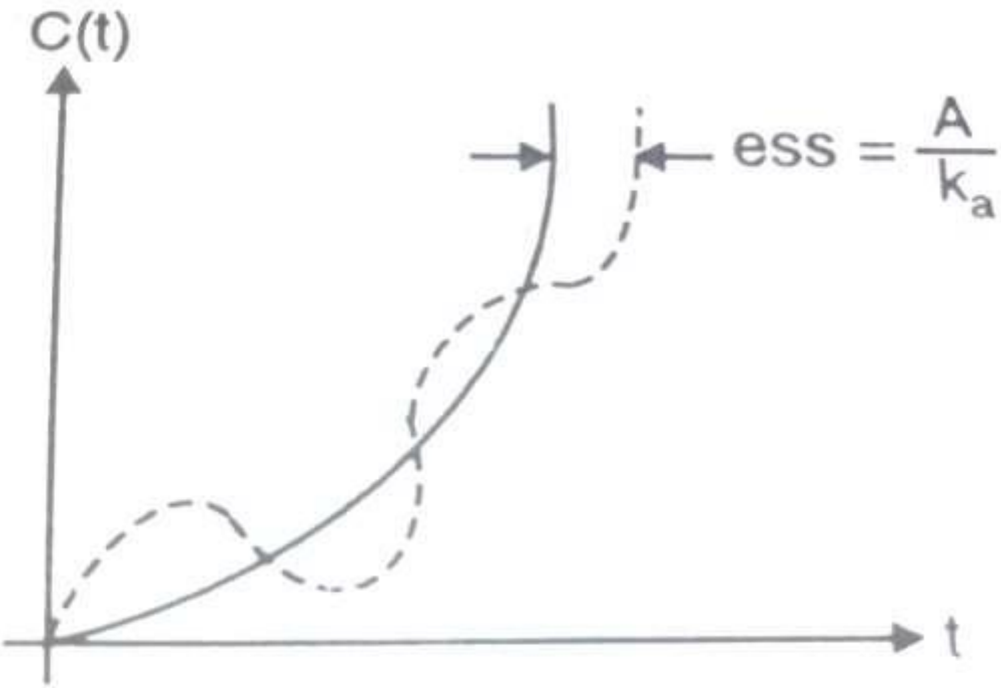
$$e_{ss}(t) = \frac{A}{0 + \lim_{s \to 0} s^2 G(s) H(s)}$$

$$e_{ss}(t) = \frac{A}{K_a}$$

Steady state error for parabolic input:

The acceleration error constant Ka of a system is defined as,

$$K_a = \lim_{s \to 0} s^2 G(s).H(s)$$



Steady state error and Standard Signals

Summary:

Sr. No.	Input Signal	Steady State Error	Constant	Constant Expression
1	Step Input	$e_{ss}(t) = \frac{A}{1+K_p}$	Position Error Constant	$K_p = \lim_{s\to 0} G(s).H(s)$
2	Ramp Input	$e_{ss}(t) = \frac{A}{K_v}$	Velocity Error Constant	$K_v = \lim_{s\to 0} sG(s).H(s)$
3	Parabolic Input	$e_{ss}(t) = \frac{A}{K_a}$	Acceleration Error Constant	$K_a = \lim_{s\to 0} s^2 G(s).H(s)$

Relation between steady state error and Type of system

The type of system means the number of poles G(s)H(s) at s=0. Consider the general form,

$$G(s).\ H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{s^n(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

Here there are n poles at s=0. Hence the type of system is n.

Steady state error for <u>Step input</u> for <u>Type 0</u> system

For type zero system, n=0

$$G(s).H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

The position error constant is given by,

$$K_p = \lim_{s\to 0} G(s).H(s)$$

$$K_p = \lim_{s\to 0} \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

$$K_p = \frac{K(1+T_1 0)(1+T_2 0)\ldots\ldots\ldots\ldots\ldots(1+T_m 0)}{(1+T_a 0)(1+T_b 0)\ldots\ldots\ldots\ldots\ldots(1+T_n 0)}$$

Steady state error for <u>Step input</u> for <u>Type 0</u> system

$$K_p = \frac{K(1)(1)\ldots\ldots\ldots\ldots\ldots(1)}{(1)(1)\ldots\ldots\ldots\ldots\ldots(1)}$$

$$K_p = K$$

The steady state error is given by,

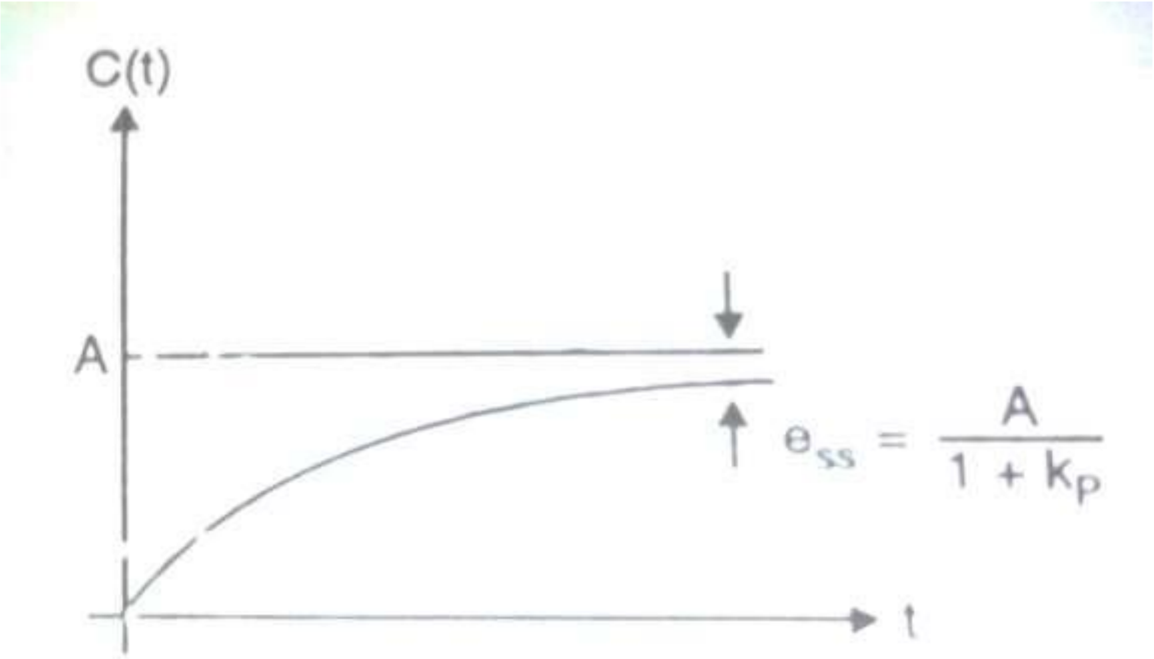
$$e_{ss}(t) = \frac{A}{1+K_p}$$

$$e_{ss}(t) = \frac{A}{1+K}$$

Steady state error for Step input for Type 0 system

$$e_{ss}(t) = \frac{A}{1 + K}$$

A type zero system has a finite steady state error to a step input ,



Steady state error for Step input for Type 1 system

For type one system, n=1

$$G(s).H(s) = \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s(1+T_a s)(1+T_b s)................(1+T_n s)}$$

The position error constant is given by,

$$K_p = \lim_{s\to 0} G(s).H(s)$$

$$K_p = \lim_{s\to 0} \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s(1+T_a s)(1+T_b s)................(1+T_n s)}$$

$$K_p = \frac{K(1+T_1 0)(1+T_2 0)................(1+T_m 0)}{0(1+T_a 0)(1+T_b 0)................(1+T_n 0)}$$

Steady state error for Step input for Type 1 system

$$K_p = \frac{K\,(1)(1)\ldots\ldots\ldots\ldots\ldots(1)}{0}$$

$$K_p = \infty$$

The steady state error is given by,

$$e_{ss}(\mathrm{t}) = \frac{A}{1 + K_p}$$

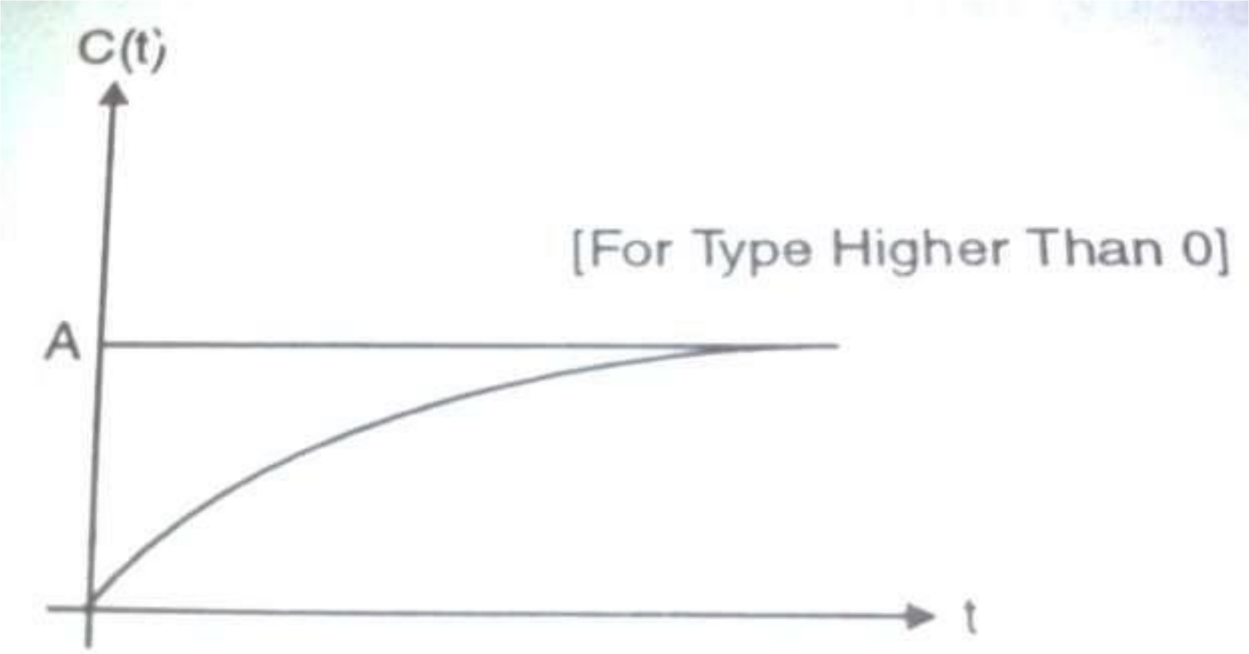
$$e_{ss}(\mathrm{t}) = \frac{A}{1 + \infty}$$

$$e_{ss}(\mathrm{t}) = 0$$

Steady state error for Step input for Type 1 system

$$e_{ss}(t) = 0$$

A type one system has a zero steady state error to a step input ,



Steady state error for Step input for Type 2 system

For type two system, n=2

$$G(s).\ H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{s^2(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

The position error constant is given by,

$$K_p = \lim_{s\to 0} G(s).\,H(s)$$

$$K_p = \lim_{s\to 0} \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{s^2(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

$$K_p = \frac{K(1+T_1 0)(1+T_2 0)\ldots\ldots\ldots\ldots\ldots(1+T_m 0)}{0(1+T_a 0)(1+T_b 0)\ldots\ldots\ldots\ldots\ldots(1+T_n 0)}$$

Steady state error for <u>Step input</u> for <u>Type 1</u> system

$$K_p = \frac{K\,(1)(1)\ldots\ldots\ldots\ldots\ldots(1)}{0}$$

$$K_p = \infty$$

The steady state error is given by,

$$e_{ss}(\mathrm{t}) = \frac{A}{1 + K_p}$$

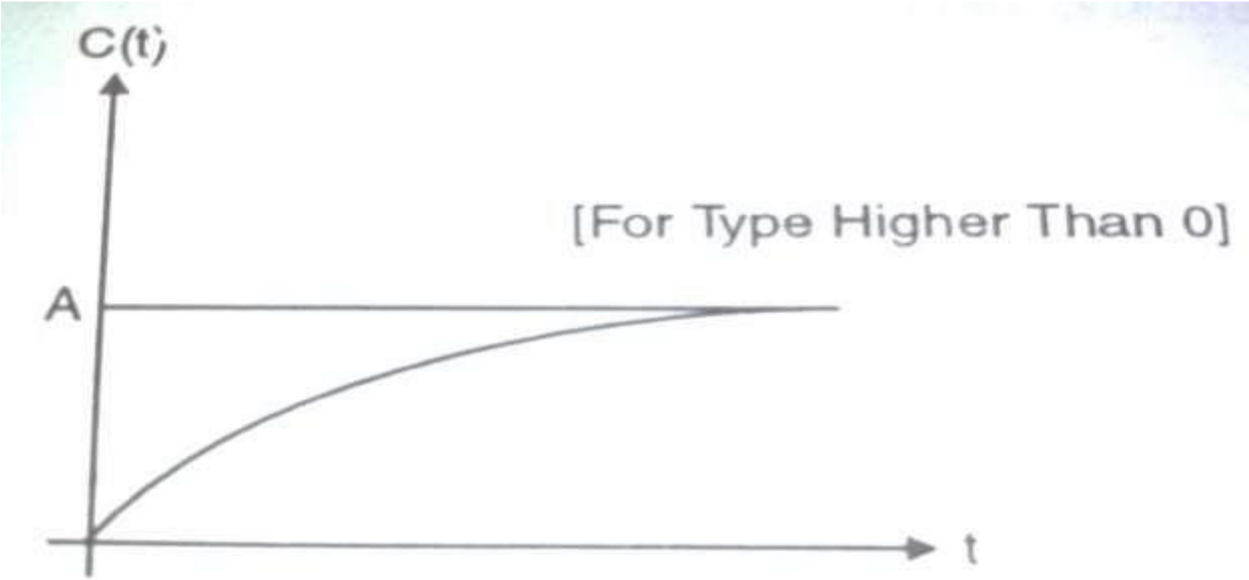
$$e_{ss}(\mathrm{t}) = \frac{A}{1 + \infty}$$

$$e_{ss}(\mathrm{t}) = 0$$

Steady state error for Step input for Type 1 system

$$e_{ss}(t) = 0$$

A type two system has a zero steady state error to a step input ,



It is clear that all higher type systems except type zero have zero steady state error.

Steady state error for Ramp input for Type 0 system

For type zero system, n=0

$$G(s).H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

The velocity error constant is given by,

$$K_v = \lim_{s\to 0} sG(s).H(s)$$

$$K_v = \lim_{s\to 0} s\left\{\frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}\right\}$$

$$K_v = 0\times\left\{\frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}\right\}$$

Steady state error for Ramp input for Type 0 system

$$K_v = 0$$

The steady state error is given by,

$$e_{ss}(t) = \frac{A}{K_v}$$

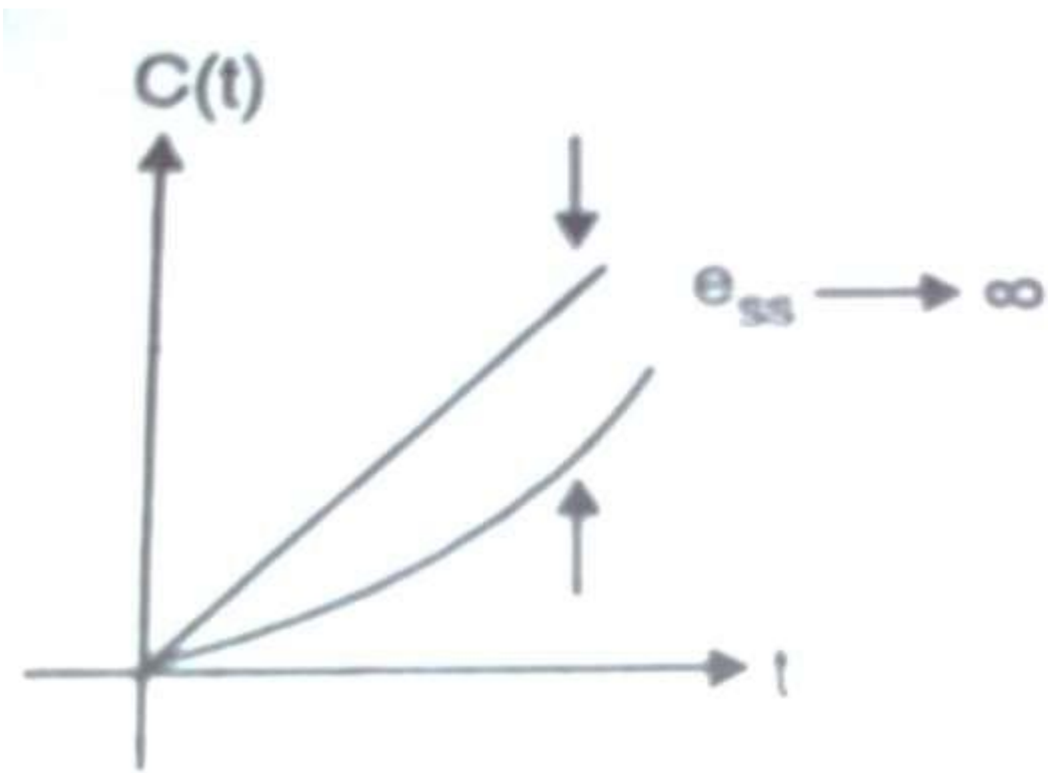
$$e_{ss}(t) = \frac{A}{0}$$

$$e_{ss}(t) = \infty$$

Steady state error for <u>Ramp input</u> for <u>Type 0</u> system

$$e_{ss}(t) = \infty$$

The error increase continuously hence type zero system fails to track a ramp input successfully.



Steady state error for Ramp input for Type 1 system

For type one system, n=1

$$G(s).H(s) = \frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{s(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}$$

The velocity error constant is given by,

$$K_v = \lim_{s\to 0} sG(s).H(s)$$

$$K_v = \lim_{s\to 0} s\left\{\frac{K(1+T_1 s)(1+T_2 s)\ldots\ldots\ldots\ldots\ldots(1+T_m s)}{s(1+T_a s)(1+T_b s)\ldots\ldots\ldots\ldots\ldots(1+T_n s)}\right\}$$

$$K_v = \frac{K(1+T_1 0)(1+T_2 0)\ldots\ldots\ldots\ldots\ldots(1+T_m 0)}{(1+T_a 0)(1+T_b 0)\ldots\ldots\ldots\ldots\ldots(1+T_n 0)}$$

Steady state error for <u>Ramp input</u> for <u>Type 1</u> system

$$K_v = K$$

The steady state error is given by,

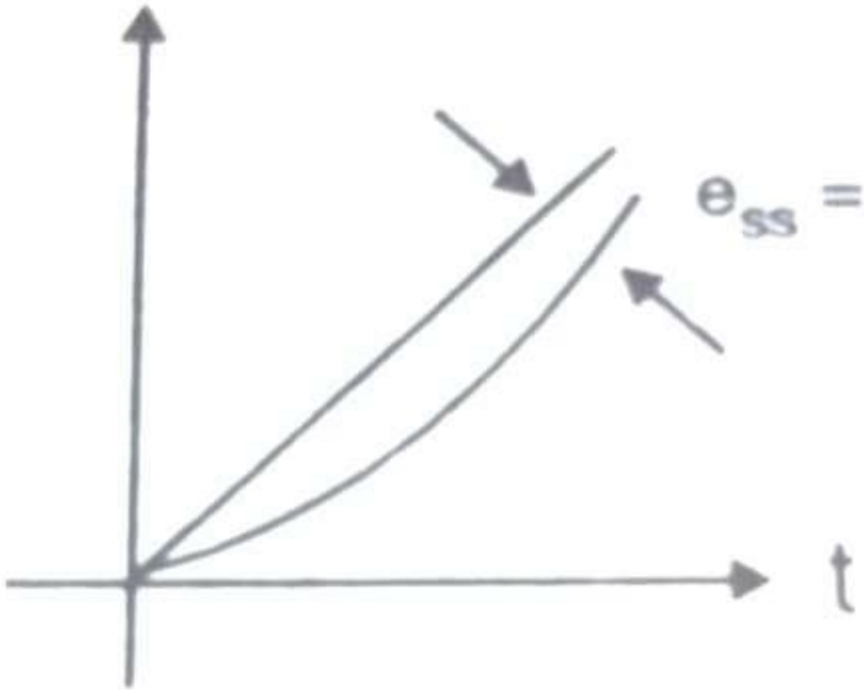
$$e_{ss}(t) = \frac{A}{K_v}$$

$$e_{ss}(t) = \frac{A}{K}$$

Steady state error for <u>Ramp input</u> for <u>Type 1</u> system

$$e_{ss}(t) = \frac{A}{K}$$

This indicates finite steady state error for type one system for ramp input



Steady state error for Ramp input for Type 2 system

For type two system, n=2

$$G(s).\ H(s) = \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s^2(1+T_a s)(1+T_b s)................(1+T_n s)}$$

The velocity error constant is given by,

$$K_v = \lim_{s \to 0} sG(s).\,H(s)$$

$$K_v = \lim_{s \to 0} s\left\{\frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s^2(1+T_a s)(1+T_b s)................(1+T_n s)}\right\}$$

$$K_v = \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s(1+T_a s)(1+T_b s)................(1+T_n s)}$$

Steady state error for <u>Ramp input</u> for <u>Type 2</u> system

$$K_v = \infty$$

The steady state error is given by,

$$e_{ss}(t) = \frac{A}{K_v}$$

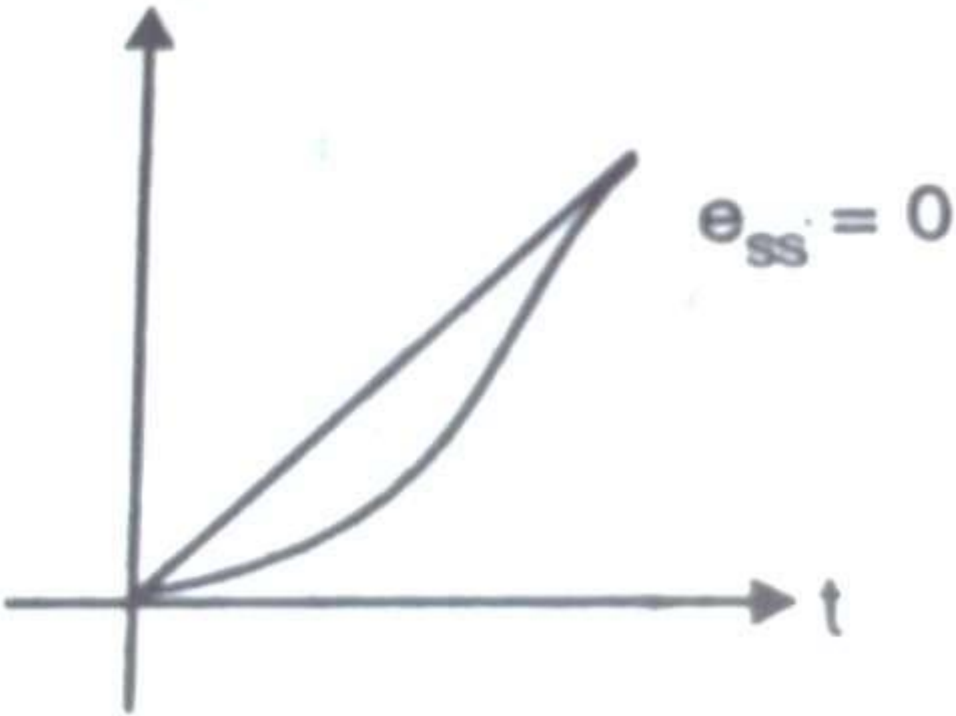
$$e_{ss}(t) = \frac{A}{\infty}$$

$$e_{ss}(t) = 0$$

Steady state error for Ramp input for Type 2 system

$$e_{ss}(t) = 0$$

There is no steady state error for a ramp input for type two system



Steady state error for Parabolic input for Type 0 system

For type zero system, n=0

$$G(s).H(s) = \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{(1+T_a s)(1+T_b s)................(1+T_n s)}$$

The acceleration error constant is given by,

$$K_a = \lim_{s\to 0} s^2 G(s).H(s)$$

$$K_a = \lim_{s\to 0} s^2 \left\{ \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{(1+T_a s)(1+T_b s)................(1+T_n s)} \right\}$$

$$K_a = 0 \times \left\{ \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{(1+T_a s)(1+T_b s)................(1+T_n s)} \right\}$$

Steady state error for Parabolic input for Type 0 system

$$K_a = 0$$

The steady state error is given by,

$$e_{ss}(t) = \frac{A}{K_a}$$

$$e_{ss}(t) = \frac{A}{0}$$

$$e_{ss}(t) = \infty$$

Steady state error for <u>Parabolic input</u> for <u>Type 0</u> system

$$e_{ss}(t) = \infty$$

There is infinite steady state error indicating failure to track a parabolic input in type zero system

Steady state error for Parabolic input for Type 1 system

For type one system,   n=1

$$G(s).H(s) = \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s(1+T_a s)(1+T_b s)................(1+T_n s)}$$

The acceleration error constant is given by,

$$K_a = \lim_{s\to 0} s^2 G(s).H(s)$$

$$K_a = \lim_{s\to 0} s^2 \left\{ \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s(1+T_a s)(1+T_b s)................(1+T_n s)} \right\}$$

$$K_a = 0 \times \left\{ \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{(1+T_a s)(1+T_b s)................(1+T_n s)} \right\}$$

Steady state error for Parabolic input for Type 1 system

$$K_a = 0$$

The steady state error is given by,

$$e_{ss}(t) = \frac{A}{K_a}$$

$$e_{ss}(t) = \frac{A}{0}$$

$$e_{ss}(t) = \infty$$

Steady state error for <u>Parabolic input</u> for <u>Type 1</u> system

$$e_{ss}(t) = \infty$$

There is infinite steady state error indicating failure to track a parabolic input in type one system

Steady state error for Parabolic input for Type 2 system

For type two system, n=2

$$G(s).\ H(s) = \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s^2(1+T_a s)(1+T_b s)................(1+T_n s)}$$

The acceleration error constant is given by,

$$K_a = \lim_{s\to 0} s^2 G(s).\,H(s)$$

$$K_a = \lim_{s\to 0} s^2 \left\{ \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{s^2(1+T_a s)(1+T_b s)................(1+T_n s)} \right\}$$

$$K_a = \left\{ \frac{K(1+T_1 s)(1+T_2 s)................(1+T_m s)}{(1+T_a s)(1+T_b s)................(1+T_n s)} \right\}$$

Steady state error for Parabolic input for Type 2 system

$$K_a = K$$

The steady state error is given by,

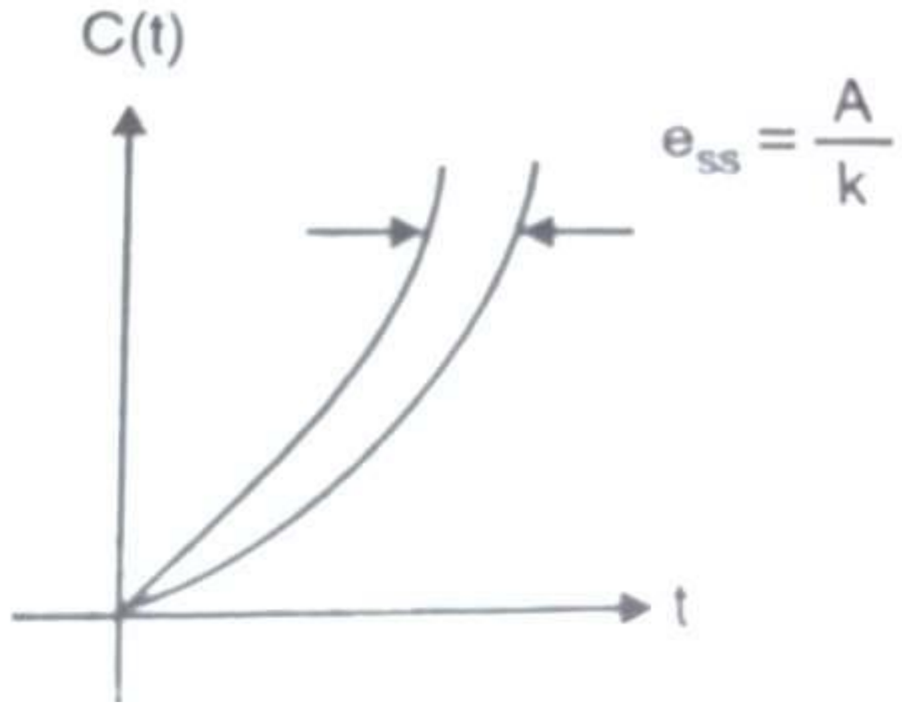
$$e_{ss}(t) = \frac{A}{K_a}$$

$$e_{ss}(t) = \frac{A}{K}$$

Steady state error for Parabolic input for Type 2 system

$$e_{ss}(t) = \frac{A}{K}$$

There is finite steady state error for type two system



Relation between steady state error and Type of system

Summary:

Sr. No.	Type of System	Step Input		Ramp Input		Parabolic Input	
		K_p	e_{ss}	K_v	e_{ss}	K_a	e_{ss}
1	Zero	K	$\frac{A}{1+K}$	0	∞	0	∞
2	One	∞	0	K	$\frac{A}{K}$	0	∞
3	Two	∞	0	∞	0	K	$\frac{A}{K}$

Example

The control system having unity feedback has,

$$G(s) = \frac{20}{s(1 + 4s)(1 + s)}$$

Determine

1. Different static error coefficients.

2. Steady State error if input $r(t) = 2 + 4t + \frac{t^2}{2}$

Example cont……

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{20}{s(1+4s)(1+ \quad s)}$$

$$K_p = \frac{20}{0(1+4s)(1+ \quad s)}$$

$$K_p = \infty$$

Example cont……

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{20}{s(1+4s)(1+s)}\right]$$

$$K_v = \frac{20}{(1+4s)(1+s)}$$

$$K_v = 20$$

Example cont……

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 [\frac{20}{s(1+4s)(1+s)}]$$

$$K_a = 0[\frac{20}{s(1+4s)(1+s)}]$$

$$K_a = 0$$

Example cont……

Steady state error, for $r(t) = 2 + 4t + \frac{t^2}{2}$

$$R(s) = L\{r(t)\} = \frac{2}{s} + \frac{4}{s^2} + \frac{1}{s^3}$$

Steady state error is given by,

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{s[\frac{2}{s} + \frac{4}{s^2} + \frac{1}{s^3}]}{1 + \frac{20}{s(1+4s)(1+s)}}$$

$$e_{ss} = \infty$$

Example

The control system having unity feedback has,

$$G(s) = \frac{50(s+5)}{s^2}$$

Determine

1. Different static error coefficients.

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{50(s+5)}{s^2}$$

$$K_p = \frac{50(s+5)}{(0)^2}$$

$$K_p = \infty$$

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{50(s+5)}{s^2}\right]$$

$$K_v = \lim_{s \to 0} \frac{50(s+5)}{s}$$

$$K_v = \frac{50(s+5)}{0}$$

$$K_v = \infty$$

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 [\frac{50(s+5)}{s^2}]$$

$$K_a = \lim_{s \to 0} 50(s+5)$$

$$K_a = 250$$

Example

The control system having,

$$G(s) = \frac{20}{s(s^2 + 2s + 5)} \qquad H(s) = \frac{10}{(s+4)}$$

Determine

1. Different static error coefficients.
2. Steady State error if input $r(t) = 5 + 10t + \frac{t^2}{2}$

Example cont……

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{20}{s(s^2 + 2s + 5)} \times \frac{10}{(s + 4)}$$

$$K_p = \infty$$

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{20}{s(s^2+2s+5)} \times \frac{10}{(s+4)}\right]$$

$$K_v = \frac{200}{20}$$

$$K_v = 10$$

Example cont……

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} \quad s^2 [\frac{20}{s(s^2 + 2s + 5)} \times \frac{10}{(s+4)}]$$

$$K_a = 0[\frac{20}{(s^2 + 2s + 5)} \times \frac{10}{(s+4)}]$$

$$K_a = 0$$

Example cont……

Steady state error, for $r(t) = 5 + 10t + \frac{t^2}{2}$

$$R(s) = L\{r(t)\} = \frac{5}{s} + \frac{10}{s^2} + \frac{1}{s^3}$$

Steady state error is given by,

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{s\left[\frac{5}{s} + \frac{10}{s^2} + \frac{1}{s^3}\right]}{1 + \left[\frac{20}{s(s^2 + 2s + 5)} \times \frac{10}{(s+4)}\right]}$$

$$e_{ss} = \infty$$

Example

The control system having unity feedback has,

$$G(s) = \frac{20(1+s)}{s^2(2+s)(4+s)}$$

Determine

1. Different static error coefficients.

2. Steady State error if input $r(t) = 40 + 2t + 5t$
   2

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s) H(s)$$

$$K_p = \lim_{s \to 0} \frac{20(1+s)}{s^2 (2+s)(4+s)}$$

$$K_p = \frac{20(1+s)}{0^2 (2+s)(4+s)}$$

$$K_p = \infty$$

Example cont……

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{20(1+s)}{s^2(2+s)(4+s)}\right]$$

$$K_v = \lim_{s \to 0}\left[\frac{20(1+s)}{s(2+s)(4+s)}\right]$$

$$K_v = \infty$$

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 \left[\frac{20(1+s)}{s^2 (2+s)(4+s)} \right]$$

$$K_a = \lim_{s \to 0} \left[\frac{20(1+s)}{(2+s)(4+s)} \right]$$

$$K_a = \frac{5}{2}$$

Example cont……

Steady state error, for $r(t) = 40 + 2t + 5t^2$

$$R(s) = L\{r(t)\} = \frac{40}{s} + \frac{2}{s^2} + \frac{10}{s^3}$$

Steady state error is given by,

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{s\left[\frac{40}{s} + \frac{2}{s^2} + \frac{10}{s^3}\right]}{1 + \frac{20(1+s)}{s^2(2+s)(4+s)}}$$

$$e_{ss} = 4$$

Example

The control system having unity feedback has,

$$G(s) = \frac{20(s+1)}{s(s+2)(s^2+2s+2)}$$

Determine

1. Different static error coefficients.

2. Steady State error if input $r(t) = 10 + 20t$

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{20(s+1)}{s(s+2)(s^2+2s+2)}$$

$$K_p = \frac{20(s+1)}{0(s+2)(s^2+2s+2)}$$

$$K_p = \infty$$

Example cont……

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{20(s+1)}{s(s+2)(s^2+2s+2)}\right]$$

$$K_v = \lim_{s \to 0}\left[\frac{20(s+1)}{(s+2)(s^2+2s+2)}\right]$$

$$K_v = 5$$

Example cont......

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 \left[\frac{20(s+1)}{s(s+2)(s^2+2s+2)} \right]$$

$$K_a = \lim_{s \to 0} s \left[\frac{20(s+1)}{(s+2)(s^2+2s+2)} \right]$$

$$K_a = 0$$

Example

The control system having unity feedback has,

$$G(s) = \frac{20(s+4)}{s(s+2)(s^2+2s+2)}$$

Determine

1. Different static error coefficients.

2. Steady State error if input $r(t) = 6t + \frac{3}{2}t^2$

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{20(s+4)}{s(s+2)(s^2+2s+2)}$$

$$K_p = \frac{20(s+4)}{0(s+2)(s^2+2s+2)}$$

$$K_p = \infty$$

Example cont……

Velocity error constant,

$$K_v = \lim_{s\to 0} sG(s)H(s)$$

$$K_v = \lim_{s\to 0} s\left[\frac{20(s+4)}{s(s+2)(s^2+2s+2)}\right]$$

$$K_v = \lim_{s\to 0}\left[\frac{20(s+4)}{(s+2)(s^2+2s+2)}\right]$$

$$K_v = 20$$

Example cont……

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 [\frac{20(s+4)}{s(s+2)(s^2+2s+2)}]$$

$$K_a = \lim_{s \to 0} s[\frac{20(s+4)}{(s+2)(s^2+2s+2)}]$$

$$K_a = 0$$

Example cont……

Steady state error, for $r(t) = 6t + \frac{3}{2}t^2$

$$R(s) = L\{r(t)\} = \frac{6}{s^2} + \frac{3}{s^3}$$

Steady state error is given by,

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{s[\frac{6}{s^2} + \frac{3}{s^3}]}{1 + \frac{20(s+4)}{s(s+2)(s^2 + 2s + 2)}}$$

$$e_{ss} = \infty$$

Example

The open loop transfer function of servo system with unity feedback is,

$$G(s) = \frac{10}{s(0.1s+1)}$$

Determine

1. Different static error coefficients.

2. Steady State error if input $r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{10}{s(0.1s+1)}$$

$$K_p = \frac{10}{0(0.1s+1)}$$

$$K_p = \infty$$

Example cont……

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s\left[\frac{10}{s(0.1s+1)}\right]$$

$$K_v = \lim_{s \to 0}\left[\frac{10}{(0.1s+1)}\right]$$

$$K_v = 10$$

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 [\frac{10}{s(0.1s+1)}]$$

$$K_a = \lim_{s \to 0} s[\frac{10}{(0.1s+1)}]$$

$$K_a = 0$$

Example cont......

Steady state error, for $r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$

$$R(s) = L\{r(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}$$

Steady state error is given by,

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)}$$

$$e_{ss} = \lim_{s \to 0} \frac{s\left[\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3}\right]}{1 + \frac{10}{s(0.1s+1)}}$$

$$e_{ss} = \infty$$

Example

The open loop transfer function of servo system with unity feedback is,

$$G(s) = \frac{10}{s^2(1+s)}$$

Determine

1. Different static error coefficients.

2. Steady State error if input $r(t) = a_0 + a_1 t + a_2 t^2$

Example cont……

Solution:

Position error constant,

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$K_p = \lim_{s \to 0} \frac{10}{s^2 (1 + \quad s)}$$

$$K_p = \frac{10}{0^2 (1 + \quad s)}$$

$$K_p = \infty$$

Example cont……

Velocity error constant,

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$K_v = \lim_{s \to 0} s[\frac{10}{s^2(1+s)}]$$

$$K_v = \lim_{s \to 0}[\frac{10}{s(1+s)}]$$

$$K_v = \infty$$

Acceleration error constant,

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$K_a = \lim_{s \to 0} s^2 [\frac{10}{s^2(1+s)}]$$

$$K_a = \lim_{s \to 0} [\frac{10}{(1+s)}]$$

$$K_a = 10$$

Example cont……

Steady state error, for $r(t) = a_0 + a_1 t + a_2 t^2$

$$R(s) = L\{r(t)\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3}$$

Steady state error is given by,

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}$$

$$e_{ss} = \lim_{s\to 0} \frac{s\left[\frac{a_0}{s} + \frac{a_1}{s^2} + \frac{2a_2}{s^3}\right]}{1+\frac{10}{s^2(1+s)}}$$

$$e_{ss} = \frac{a_2}{5}$$

Example

A unity feedback system is characterized by the open loop transfer function is,

$$G(s) = \frac{1}{s(0.5s+1)(0.2s+1)}$$

Determine steady state errors for unit step input, unit ramp input and unit acceleration input.

Solution:

Position error constant, $K_p = \lim_{s \to 0} G(s)H(s)$

$$K_p = \lim_{s \to 0} \frac{1}{s(0.5s+1)(0.2s+1)}$$

$$K_p = \infty$$

Steady state error for unit step input is given by,

$$e_{ss}(t) = \frac{1}{1+K_p}$$

$$e_{ss}(t) = \frac{1}{1+\infty}$$

$$e_{ss}(t) = 0$$

Example cont......

Velocity error constant, $K_v = \lim_{s \to 0} sG(s)H(s)$

$$K_v = \lim_{s \to 0} s[\frac{1}{s(0.5s+1)(0.2s+1)}]$$

$$K_v = 1$$

Steady state error for unit ramp input is given by,

$$e_{ss}(t) = \frac{1}{K_v}$$

$$e_{ss}(t) = \frac{1}{1}$$

$$e_{ss}(t) = 1$$

Example cont......

Acceleration error constant, $K_a = \lim_{s \to 0} s^2 G(s) H(s)$

$$K_a = \lim_{s \to 0} s^2 [\frac{1}{s(0.5s+1)(0.2s+1)}]$$

$$K_a = 0$$

Steady state error for unit parabolic input is given by,

$$e_{ss}(t) = \frac{1}{K_a}$$

$$e_{ss}(t) = \frac{1}{0}$$

$$e_{ss}(t) = \infty$$