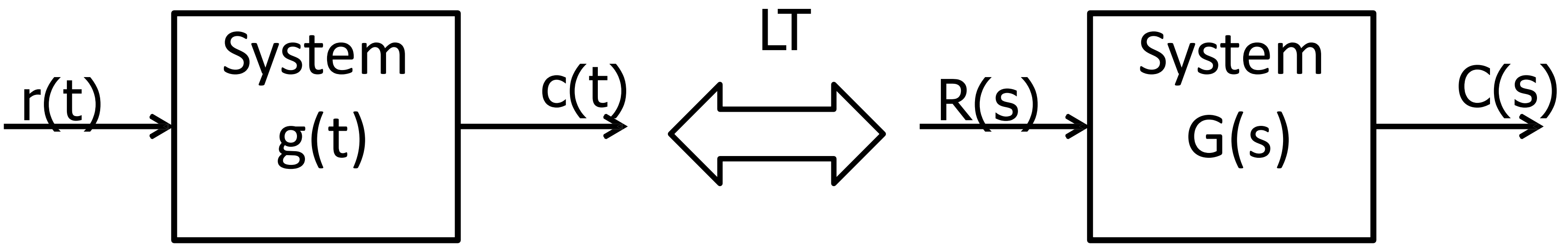
Transfer Function

- The relationship between input & output of a system is given by the transfer function.
- **<u>Definition:</u>** The ratio of Laplace transform of the output to the Laplace transform of the input under the assumption of zero initial conditions is defined as **<u>" Transfer Function"</u>**.

Transfer Function



For the system shown,

c(t)= output
r(t)= input
g(t)= System function

L{c(t)}= C(s)
L{r(t)}= R(s)
L{g(t)}= G(s)

Therefore transfer function G(s) for above system is given by,

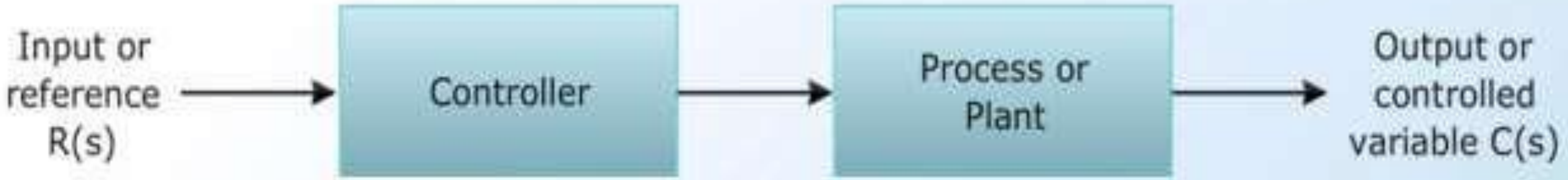
$$G(s) = \frac{\text{Laplace of output}}{\text{Laplace of input}} = \frac{C(s)}{R(s)}$$

Transfer Function

Transfer Function of Open-loop System

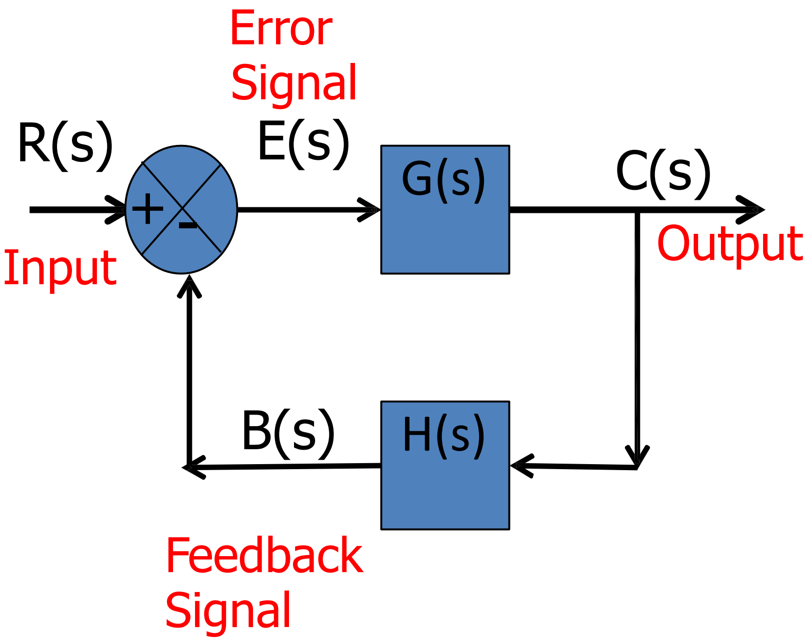
- It is activated by a single signal at the input provided that the system is SISO
- No provision within the system for supervision of output
- Transfer function = C(s)/R(s)

  = G(s)



Block diagram of Open-loop Control System

Transfer Function of closed loop system



Error signal is given by;

$$E(s) = R(s) - B(s) ------(1)$$

$$\therefore R(s) = E(s) + B(s)$$

Gain of feedback network is given by;

$$H(s) = \frac{B(s)}{C(s)}$$

$$B(s) = H(s).C(s) ------(2)$$

Gain for CL system is given by;

$$G(s) = \frac{C(s)}{E(s)}$$

$$\therefore C(s) = G(s).\,E(s) ------(3)$$

Substitute value of E(s) from eq. 1 to 3

$$C(s) = G(s).(R(s) - B(s))$$

$$\therefore C(s) = G(s).R(s) - G(s).B(s) ------(4)$$

Substitute value of B(s) from eq. 2 to 4

$$C(s) = G(s)R(s) - G(s).H(s).C(s)$$

$$G(s).R(s) = C(s) + G(s).H(s).C(s)$$

$$G(s).R(s) = C(s)(1 + G(s).H(s))$$

Transfer function is given by;

$$\text{T.F.} = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s).H(s)}$$

Effect of Feedback

- "Feedback" as a means of automatic regulation and control, is inherent in the nature of a control system
- The Benefits are:
  - The controlled variable accurately follows the desired value
  - Effect of external disturbances other than those associated with the feedback are greatly reduced
  - Feedback in the control loop allows accurate control of the output, even when the controlled plant parameters are not known accurately
  - It greatly improves the speed

Effect of Feedback

Effect of Feedback on Overall Gain

- The overall transfer function of open loop system is

$$\frac{C(s)}{R(s)} = G(s)$$

- The overall transfer function of closed loop system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

- For feedback, the gain G(s) is reduced by a factor 1/(1+G(s)H(s)). So, due to feedback, "overall gain" of the system gets reduced

Effect of Feedback

Effect of Feedback on Stability

- Consider the open loop system with transfer function

$$G(s) = \frac{K}{s+T}$$

Pole is located at S = -T

- Now consider closed loop system with unity feedback
- Thus feedback controls the time response by adjusting the location of the poles. Thus the closed-loop system is faster than the open-loop system
- The stability depends upon the location of the pole. Thus the feedback also affects the stability
- The closed-loop T.F. is

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)} = \frac{K}{S+(T+K)}$$

Its pole is shifted to s = -(T + K)

Effect of Feedback

Effect of Feedback on Sensitivity

- The changes in parameters due to disturbances are called parameter variations. Such parameter variations affect the system performance adversely.
- Such an effect in the system performance due to parameter variations can be studied mathematically defining the term "sensitivity"

$$For \quad Open \quad -loop \qquad S_G^T = \frac{G(s)\partial G(s)}{G(s)\partial G(s)}$$

- The sensitivity of T(s) with respect to G(s) for an open loop system is unity

$$For \quad Closed \quad -loop \qquad S_G^T = \frac{1}{1+G(s)H(s)}$$

- The sensitivity due to the feedback get reduced by the factor 1/[1+G(s) H(s)] compared to an open loop system

$$S_G^T = \frac{\partial T}{\partial G}\frac{G}{T}$$

Do partial differentiation with respect to G on both sides of Equation

$$\frac{\partial T}{\partial G} = \frac{\partial}{\partial G}\left(\frac{G}{1+GH}\right) = \frac{(1+GH).1-G(H)}{(1+GH)^2} = \frac{1}{(1+GH)^2}$$

and $\frac{G}{T} = 1 + GH$

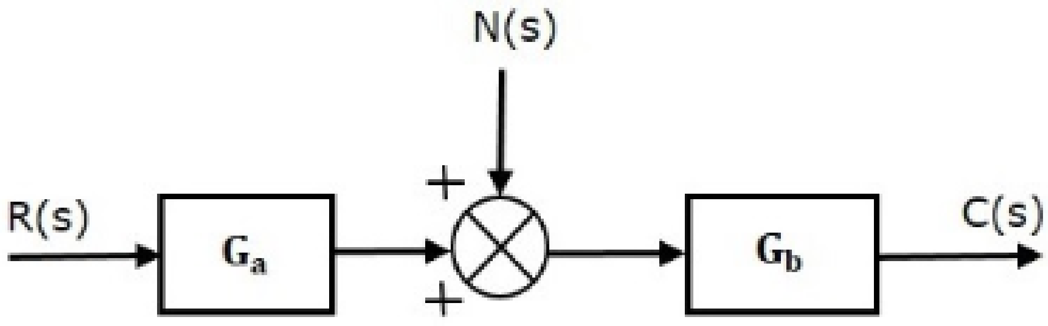
Substitute

$$S_G^T = \frac{1}{(1+GH)^2}(1+GH) = \frac{1}{1+GH}$$

Effect of Feedback on Noise

To know the effect of feedback on noise, let us compare the transfer function relations with and without feedback due to noise signal alone.

Consider an **open loop control system** with noise signal as shown below.

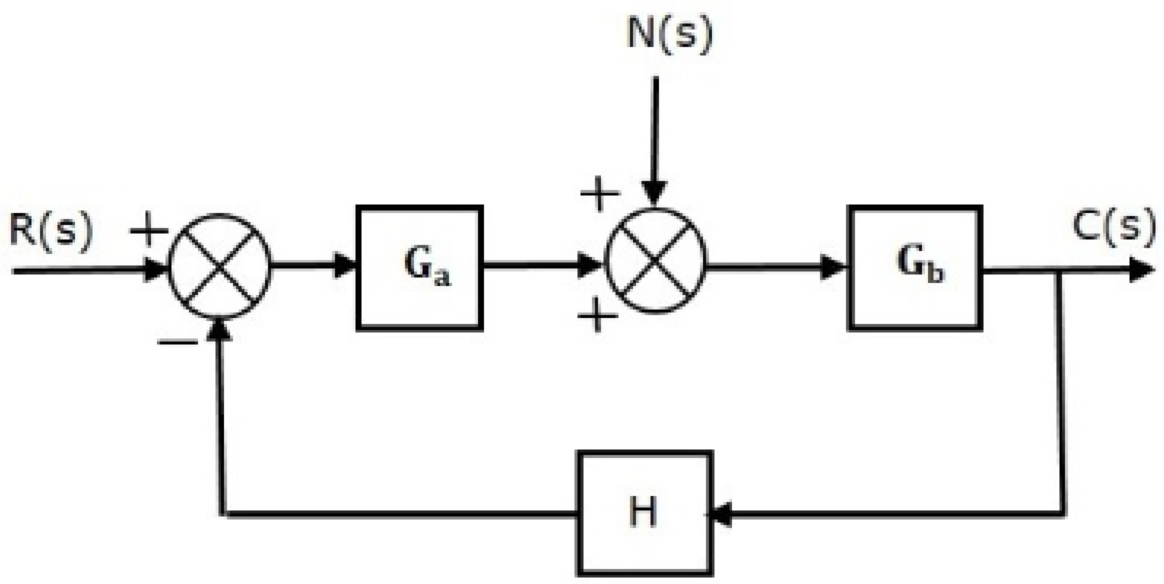


The **open loop transfer function** due to noise signal alone is

$$\frac{C(s)}{N(s)} = G_b$$

It is obtained by making the other input $R(s)$ equal to zero.

Consider a **closed loop control system** with noise signal as shown below.



The **closed loop transfer function** due to noise signal alone is

$$\frac{C(s)}{N(s)} = \frac{G_b}{1+G_aG_bH}$$

It is obtained by making the other input $R(s)$ equal to zero.

Laplace Transform of Passive Element (R,L & C)

- The Laplace transform can be used independently on different circuit elements, and then the circuit can be solved entirely in the S Domain (Which is much easier).
- Let's take a look at some of the circuit elements

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the resistor is given by Ohm's law i-e

$$v_R(t) = i_R(t)R$$

- The Laplace transform of the above equation is

$$V_R(s) = I_R(s)R$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the Capacitor is given as:

$$v_c(t) = \frac{1}{C}\int i_c(t)dt$$

- The Laplace transform of the above equation (assuming there is no charge stored in the capacitor) is

$$V_c(s) = \frac{1}{Cs} I_c(s)$$

Basic Elements of Electrical Systems

- The time domain expression relating voltage and current for the inductor is given as:

$$v_L(t) = L\frac{di_L(t)}{dt}$$

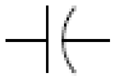
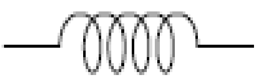
- The Laplace transform of the above equation (assuming there is no energy stored in inductor) is

$$V_L(s) = LsI_L(s)$$

Voltage, Current, Charge Relationship for Capacitor, Resistor, and Inductor.

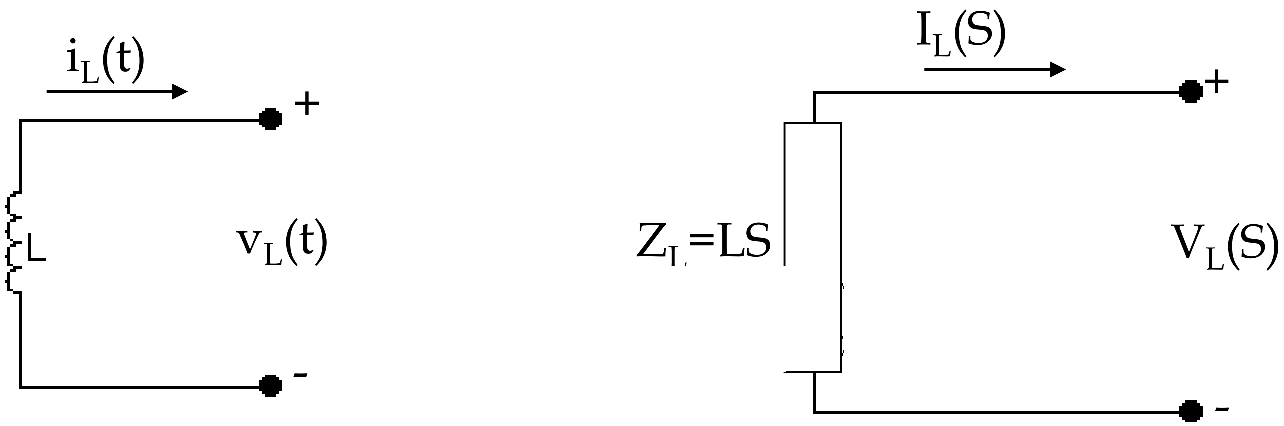
Component	Symbol	V-I Relation	I-V Relation
Resistor		$v_R(t) = i_R(t)R$	$i_R(t) = \dfrac{v_R(t)}{R}$
Capacitor		$v_c(t) = \dfrac{1}{C}\int i_c(t)dt$	$i_c(t) = C\dfrac{dv_c(t)}{dt}$
Inductor		$v_L(t) = L\dfrac{di_L(t)}{dt}$	$i_L(t) = \dfrac{1}{L}\int v_L(t)dt$

Voltage, Current, Charge Relationship for Capacitor, Resistor, and Inductor.

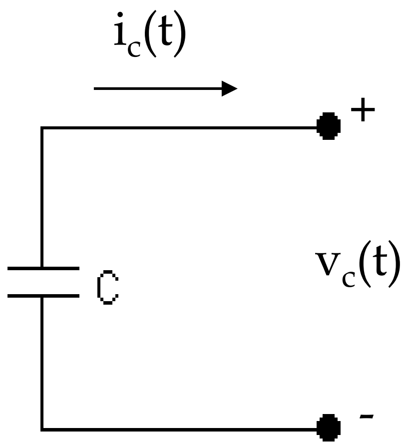
Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

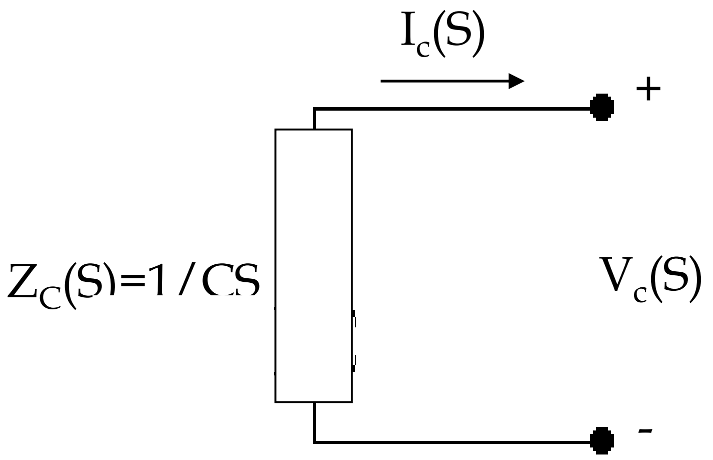
Transform Impedance (Resistor)

Transform Impedance (Inductor)



Transform Impedance (Capacitor)



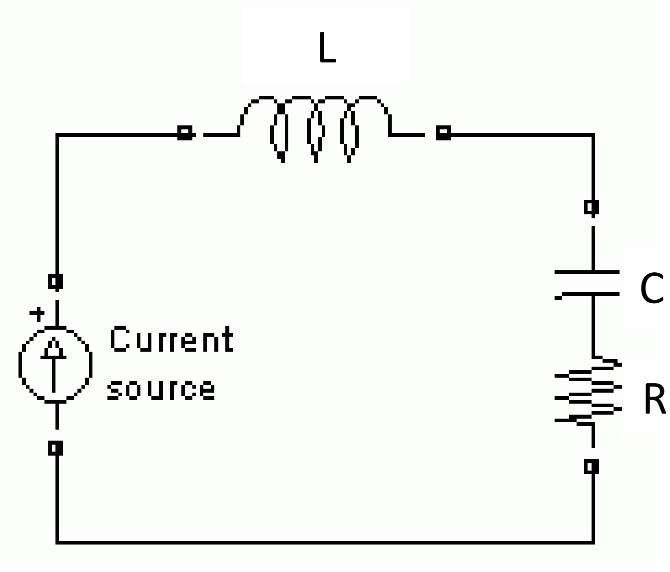


Equivalent Transform Impedance (Series)

Consider following arrangement, find out equivalent transform impedance.

$$Z_T = Z_R + Z_L + Z_C$$

$$Z_T = R + Ls + \frac{1}{Cs}$$

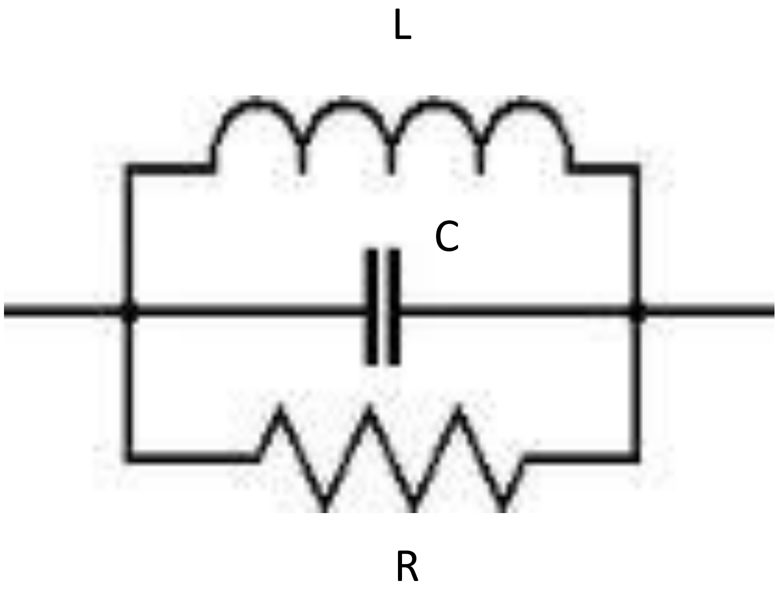


Equivalent Transform Impedance (Parallel)

- Consider following arrangement, find out equivalent transform impedance.

$$\frac{1}{Z_T} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C}$$

$$\frac{1}{Z_T} = \frac{1}{R} + \frac{1}{Ls} + \frac{1}{\frac{1}{Cs}}$$



Kirchhoff’s Law

- Basic laws governing electrical circuits are Kirchhoff’s current law and voltage law.
- Kirchhoff’s current law (node law) states that the algebraic sum of all currents entering and leaving a node is zero.
- Kirchhoff’s voltage law (loop or mesh law) states that at any given instant the algebraic sum of the voltages around any loop in an electrical circuit is zero.
- A mathematical model of an electrical circuit can be obtained by applying one or both of Kirchhoff’s laws to it.

Method-I (Kirchhoff's Law or Differential Equation Approach): Three Steps to get the Transfer Function of Electrical System

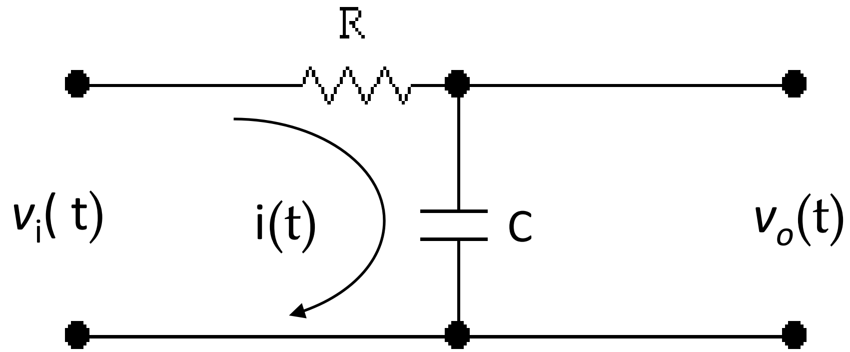
1. Apply Kirchhoff’s law (Node or Loop Law) and write the differential equations for the circuit.

2. Then take the Laplace transforms of the differential equations.

3. Finally solve the equations for the transfer function.

Example :

The two-port network shown in the following figure has $v_i(t)$ as the input voltage and $v_o(t)$ as the output voltage. Find the transfer function $V_o(s)/V_i(s)$ of the network.



$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

Example :

$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt \qquad v_o(t) = \frac{1}{C}\int i(t)dt$$

Taking Laplace transform of both equations, considering initial conditions to zero.

$$V_i(s) = I(s)R + \frac{1}{Cs}I(s) \qquad V_o(s) = \frac{1}{Cs}I(s)$$

Re-arrange both equations as:

$$V_i(s) = I(s)(R + \frac{1}{Cs}) \qquad CsV_o(s) = I(s)$$

Example:

$$V_i(s) = I(s)(R + \frac{1}{Cs})$$

$$CsV_o(s) = I(s)$$

Substitute *I(s)* in equation on left

$$V_i(s) = CsV_o(s)(R + \frac{1}{Cs})$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{Cs(R + \frac{1}{Cs})}$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}}$$

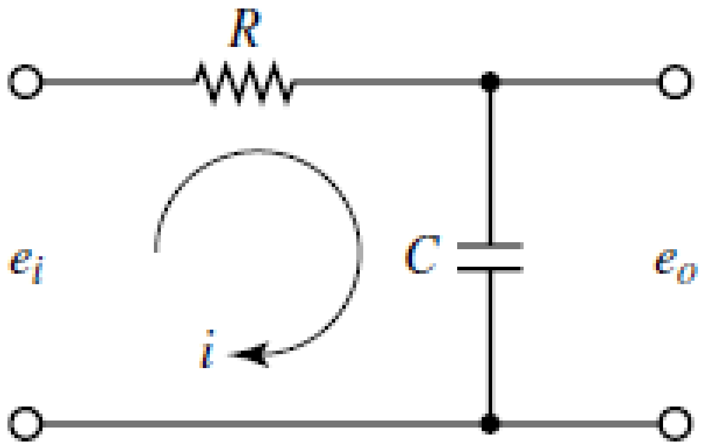
Example :

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

The system has one pole at

$$1 + RCs = 0 \qquad \Rightarrow s = -\frac{1}{RC}$$

Example: Obtain the transfer function of the RC Circuit.



- The equations of this RC circuits are;

$$i = \frac{e_i - e_o}{R}$$

$$e_o = \frac{\int i\,dt}{C}$$

- The above Equations give a mathematical model of the RC circuit.

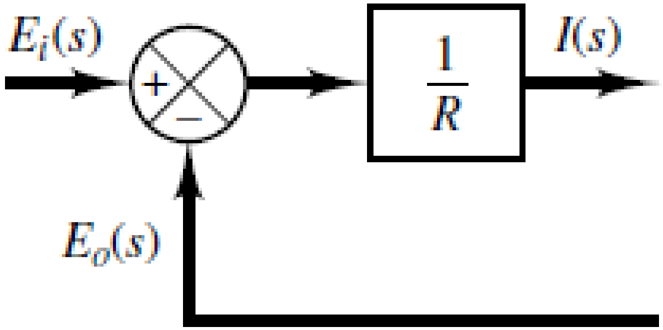
- The Laplace transform of these equation are;

$$I(s) = \frac{E_i(s) - E_o(s)}{R}$$

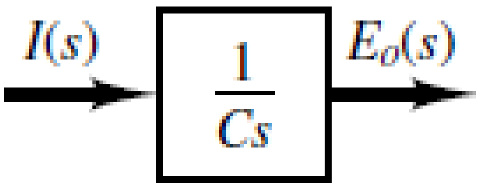
$$E_o(s) = \frac{I(s)}{Cs}$$

Block diagrams of these equations are;

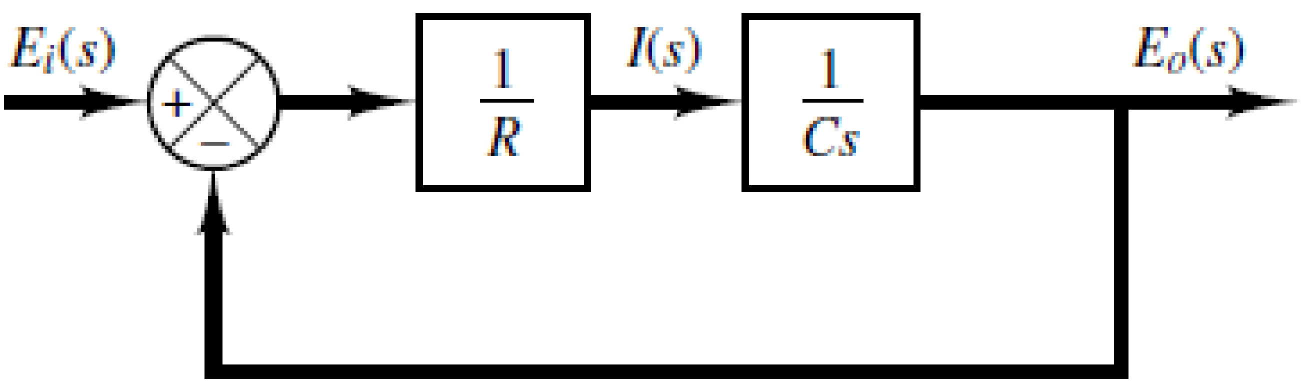
$$I(s) = \frac{E_i(s) - E_o(s)}{R}$$



$$E_o(s) = \frac{I(s)}{Cs}$$



Combining the above two blocks we get



The transfer function *T* of this unit feedback system or RC circuit is

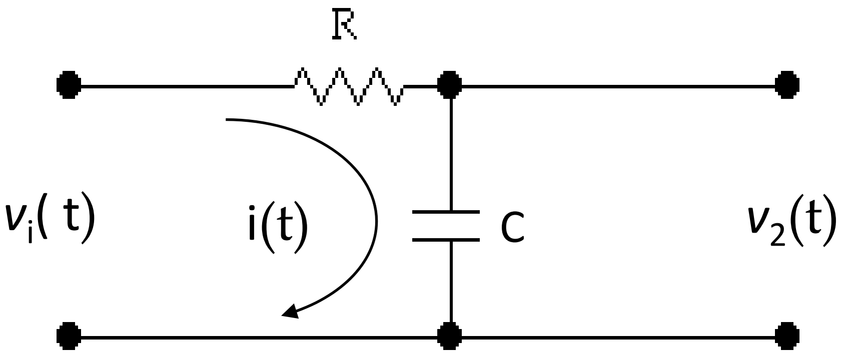
$$T = Eo(s)/Ei(s) = \frac{1/RCS}{1 + 1/RCS}$$

$$\boxed{T = Eo(s)/Ei(s) = \frac{1}{RCS + 1}}$$

Example :

Design an Electrical system that would place a pole at -3 if added to another system.

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1+RCs}$$



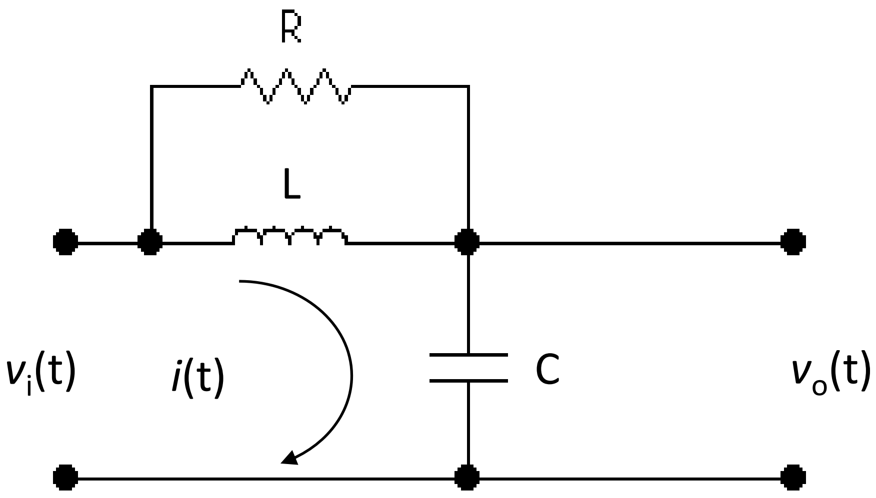
System has one pole at

$$1+RCs=0 \qquad \Rightarrow s=-\frac{1}{RC}$$

Therefore,

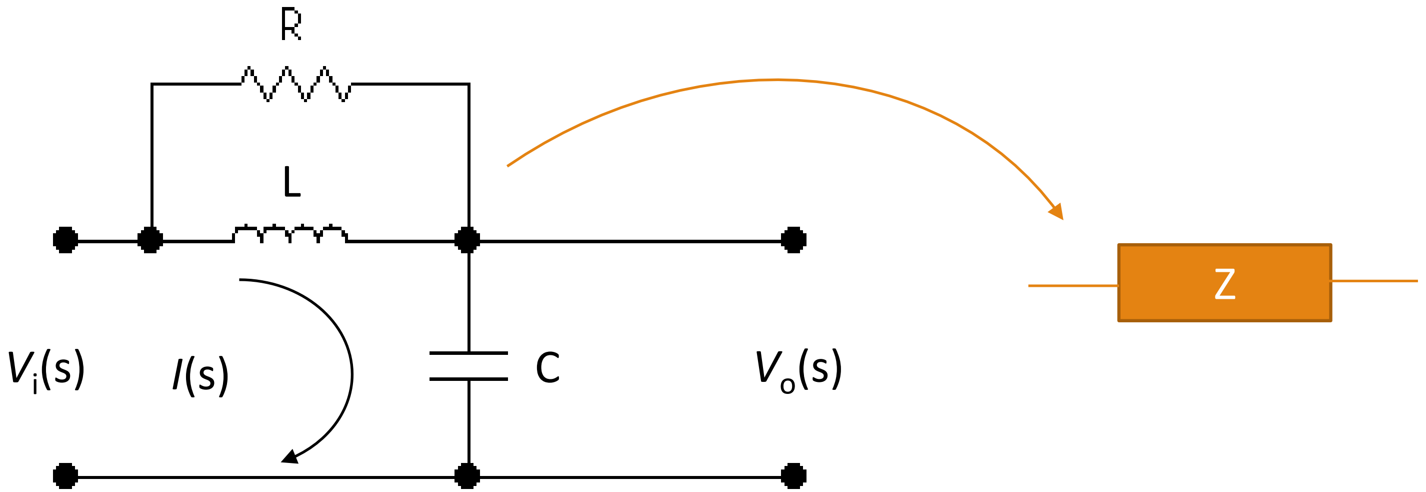
$$-\frac{1}{RC}=-3 \qquad if \qquad R=1\ M\Omega \quad and \quad C=333\ pF$$

Example : Find the transfer function G(S) of the following two port network.

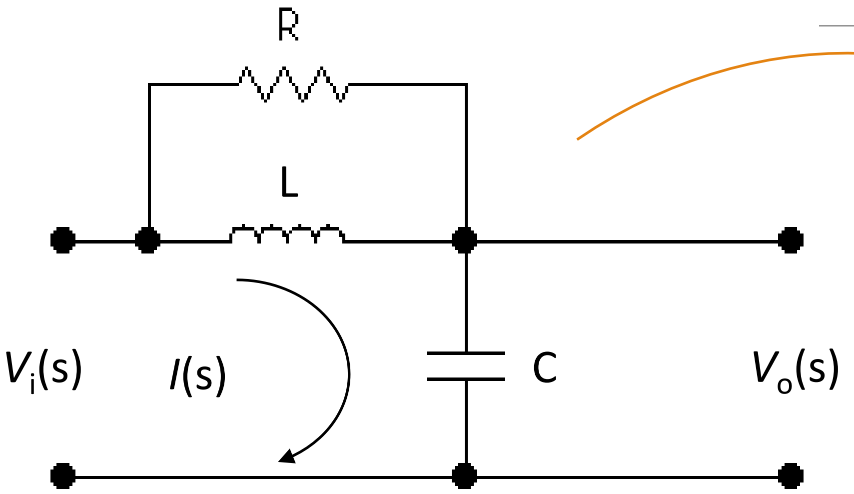


Example :

Simplify network by replacing multiple components with their equivalent transform impedance.



Example :



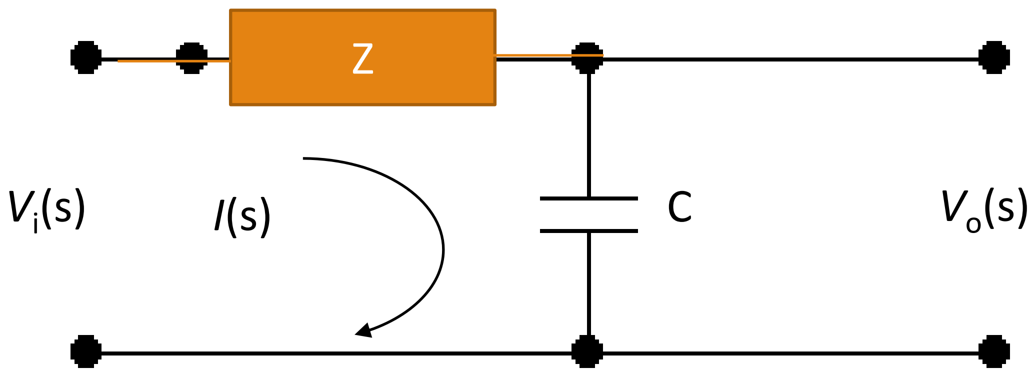
$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_L}$$

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{Ls}$$

$$Z = \frac{RLs}{R + Ls}$$

Example :

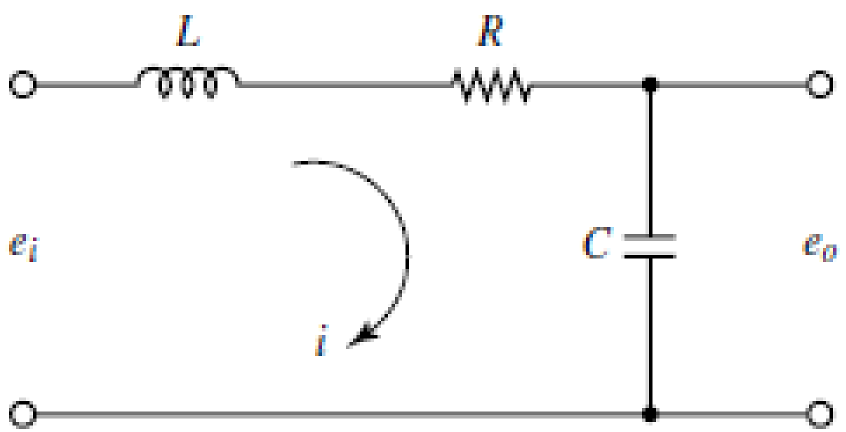
$$Z = \frac{RLs}{R + Ls}$$



$$V_i(s) = I(s)Z + \frac{1}{Cs}I(s) \qquad V_o(s) = \frac{1}{Cs}I(s)$$

$$\boxed{\frac{V_o(s)}{V_i(s)} = \frac{1}{ZCs + 1}}$$

Example: Obtain the transfer function of the given RLC Circuit.



Applying Kirchhoff's voltage law to the system, we obtain the following equations:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e_i \longrightarrow \text{(a)}$$

$$\frac{1}{C}\int i\,dt = e_o \longrightarrow \text{(b)}$$

A transfer-function model of the circuit can be obtained by taking the Laplace transforms of Equations (a) and (b) with the assumption of zero initial condition, we obtain

$$LsI(s) + RI(s) + \frac{1}{C}\frac{1}{s}I(s) = E_i(s) \longrightarrow \text{(c)}$$

$$\frac{1}{C}\frac{1}{s}I(s) = E_o(s) \longrightarrow \text{(d)}$$

The transfer function, $T = E_o(s)/E_i(s)$, of this RLC circuit can be obtain as;

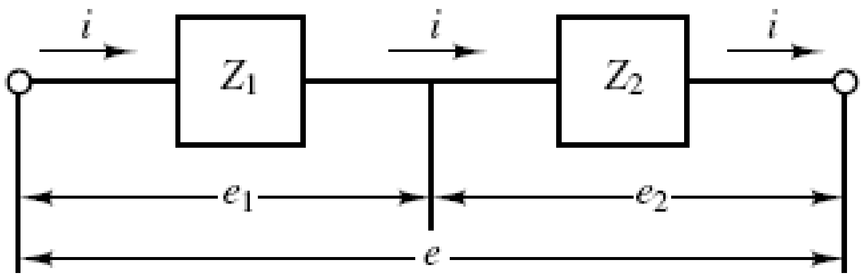
- Taking the *I(s)* common in equation (c), will get equation (e),
- Divide equation (d) by (e),
- Finally, Multiply and divided by *CS.*

Hence, the transfer function, $T = E_o(s)/E_i(s)$, of the RLC circuit after simplification is

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}}$$

Voltage Divider Rule

- Consider the series circuit shown below. Assume that the voltages e_i and e_o are the input and output of the circuit, respectively.



- Then the transfer function of this circuit is

$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)}}$$

Complex Impedance Approach to get Transfer Function

- In deriving transfer functions for electrical circuits, it is convenient to write the Laplace-transformed equations directly, without writing the differential equations.
- Remember that the impedance approach is valid only if the initial conditions involved are all zeros.
- Since the transfer function requires zero initial conditions, the impedance approach can be applied to obtain the transfer function of the electrical circuit.
- This approach greatly simplifies the derivation of transfer functions of electrical circuits.

Method-II (Complex Impedance Approach): Three Steps to get the Transfer Function of Electrical System

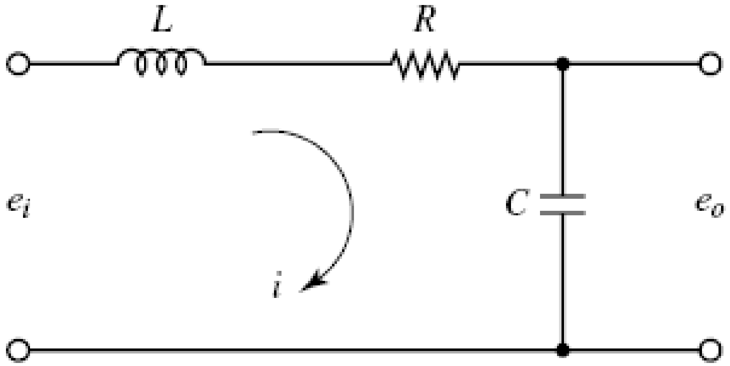
1. The first step is to transform the circuit into the equivalent impedance form.

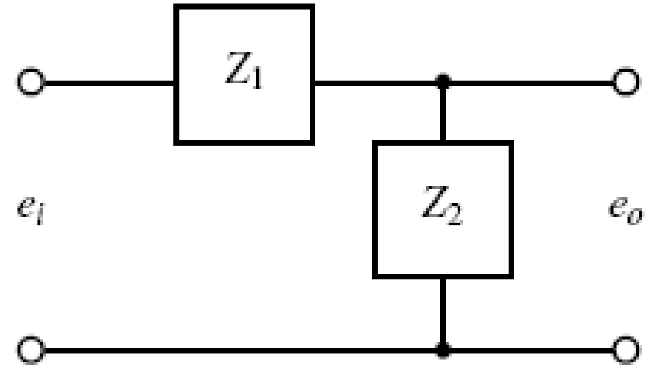
2. Apply the voltage or current divider rule to find the output voltage.

3. Finally solve the equations for the transfer function.

Example: repeat previousexample and obtain the TF using the Impedance Approach.

- The first step is to transform this RLC circuit into the equivalent impedance form,





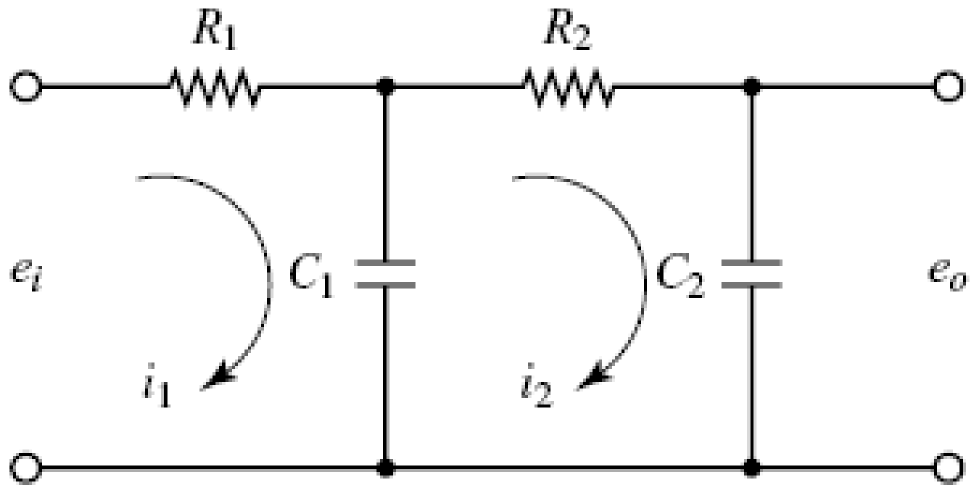
$$Z_1 = Ls + R, \qquad Z_2 = \frac{1}{Cs}$$

- The transfer function, $E_o(s)/E_i(s)$, can be obtain by applying the voltage-divider rule, hence

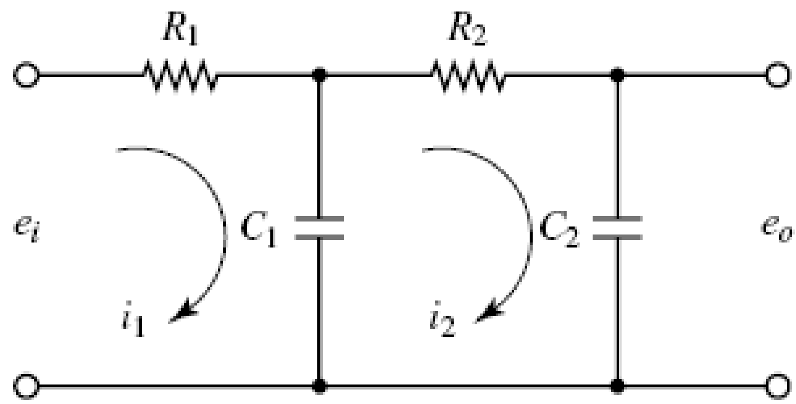
$$\boxed{\frac{E_o(s)}{E_i(s)} = \frac{\frac{1}{Cs}}{Ls + R + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}}$$

Example: Obtain the transfer function of cascaded elements using Kirchhoff's law.

- Consider the system shown below. Assume that e_i is the input and e_o is the output.
- The capacitances $C1$ and $C2$ are not charged initially.
- It will be shown that the second stage of the circuit ($R2C2$ portion) produces a loading effect on the first stage ($R1C1$ portion).



Example: Continue.



- The equations for this system are;

$$\frac{1}{C_1}\int (i_1 - i_2)\,dt + R_1 i_1 = e_i \longrightarrow \text{(a)}$$

$$\frac{1}{C_1}\int (i_2 - i_1)\,dt + R_2 i_2 + \frac{1}{C_2}\int i_2\,dt = 0 \longrightarrow \text{(b)}$$

$$\frac{1}{C_2}\int i_2\,dt = e_o \longrightarrow \text{(c)}$$

- Taking the Laplace transforms of Equations (a), (b) and (c), using zero initial conditions, we obtain

$$\frac{1}{C_1 s}[I_1(s) - I_2(s)] + R_1 I_1(s) = E_i(s) \longrightarrow \text{(d)}$$

$$\frac{1}{C_1 s}[I_2(s) - I_1(s)] + R_2 I_2(s) + \frac{1}{C_2 s} I_2(s) = 0 \longrightarrow \text{(e)}$$

$$\frac{1}{C_2 s} I_2(s) = E_o(s) \longrightarrow \text{(e)}$$

Example: Continue.

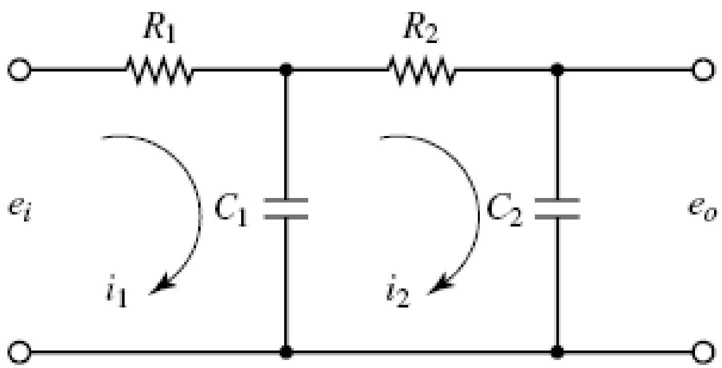
- Eliminating $I_1(s)$ from Equations (d) and (e) and writing $E_i(s)$ in terms of $I_2(s)$, we find the transfer function between $E_o(s)$ and $E_i(s)$ to be

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{(R_1C_1s + 1)(R_2C_2s + 1) + R_1C_2s}$$

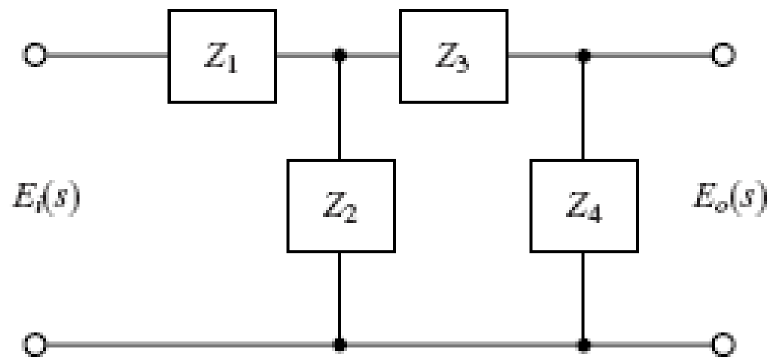
$$= \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_1C_2)s + 1}$$

Example: repeat example using the Impedance Approach.

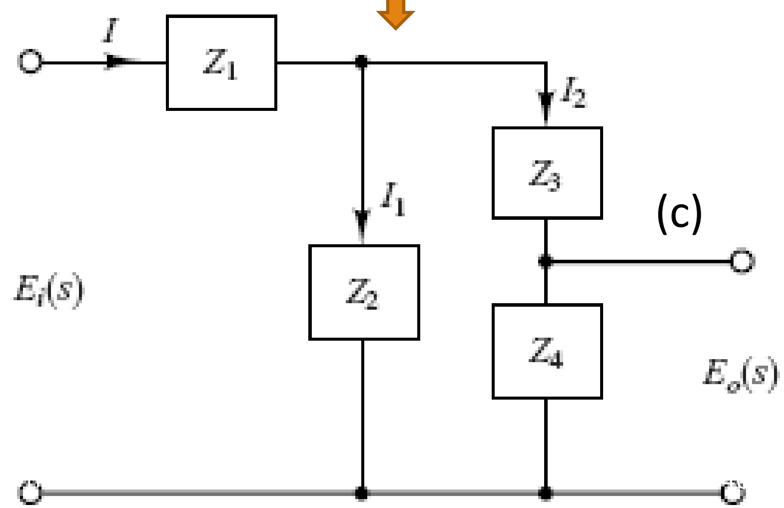
- Obtain the transfer function $E_o(s)/E_i(s)$ by use of the complex impedance approach. (Capacitors $C1$ and $C2$ are not charged initially.)



(a)

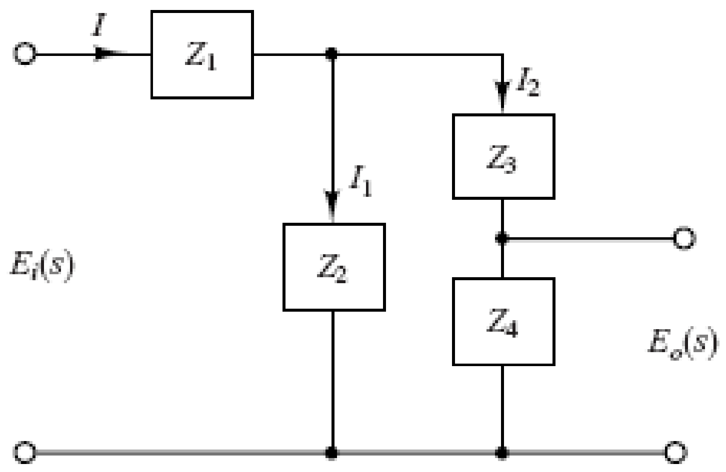


(b)



- The circuit shown in Figure (a) can be redrawn as that shown in Figure (b), which can be further modified to Figure (c).

Example: Continue.



- In the system shown in Figure 3–10(b) the current I is divided into two currents $I1$ and $I2$. Noting that

$$Z_2 I_1 = (Z_3 + Z_4) I_2, \qquad I_1 + I_2 = I$$

$$I_1 = \frac{Z_3 + Z_4}{Z_2 + Z_3 + Z_4} I, \qquad I_2 = \frac{Z_2}{Z_2 + Z_3 + Z_4} I$$

$$E_i(s) = Z_1 I + Z_2 I_1 = \left[Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4} \right] I \qquad E_o(s) = Z_4 I_2 = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4} I$$

$$\frac{E_o(s)}{E_i(s)} = \frac{Z_2 Z_4}{Z_1(Z_2 + Z_3 + Z_4) + Z_2(Z_3 + Z_4)}$$

Example: Continue.

- Substituting Z1 = R1, Z2 = 1/(C1S), Z3 = R2 , and Z4 = 1/(C2S) into this last equation, we get

$$\frac{E_o(s)}{E_i(s)} = \frac{\dfrac{1}{C_1 s}\dfrac{1}{C_2 s}}{R_1\left(\dfrac{1}{C_1 s} + R_2 + \dfrac{1}{C_2 s}\right) + \dfrac{1}{C_1 s}\left(R_2 + \dfrac{1}{C_2 s}\right)}$$

$$= \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

Example: Find transfer function $V_{out}(s)/V_{in}(s)$ of the following electrical network using Kirchhoff's law and verify the result using impedance approach.