UNIT-II

TIME RESPONSE ANALYSIS

UNIT – II

Time domain analysis: Standard test signals – step, ramp, parabolic and impulse response function – characteristic polynomial and characteristic equations of feedback systems – transient response of first order and second order systems to standard test signals. Time domain specifications - steady state response – steady state error and error constants. Effect of adding poles and zeros on overshoot, rise time, band width – dominant poles of transfer functions.

Stability analysis in the complex plane: Absolute, relative, conditional, bounded input –bounded output, zero input stability, conditions for stability, Routh –Hurwitz criterion.

Time Response

- In time domain analysis, time is the independent variable. When a system is given an excitation, there is a response (output).
- **<u>Definition:</u>** The response of a system to an applied excitation is called "**Time Response**" and it is a function of c(t).

Time Response

Generally speaking, the response of any system thus has two parts

(i) Transient Response

(ii) Steady State Response

Transient Response

- That part of the time response that goes to zero as time becomes very large is called as "**Transient Response**"

i.e. $$\underset{t \to \infty}{Lt}\; c(\mathrm{t}) = 0$$

- As the name suggests that transient response remains only for some time from initial state to final state.

Transient Response

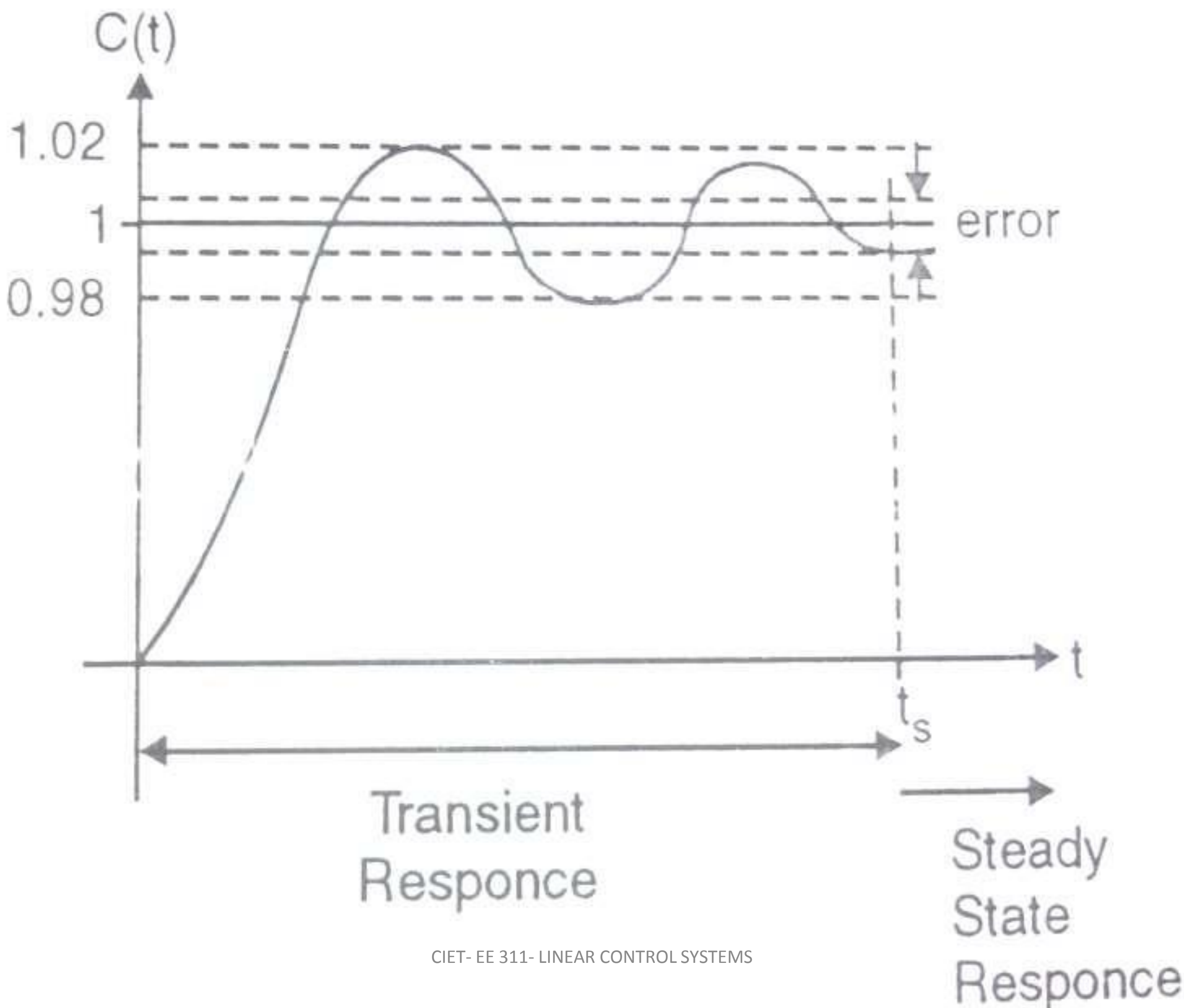
From the transient response we can know;

- ✓ When system begins to respond after an input is given.
- ✓ How much time it takes to reach the output for the first time.
- ✓ Whether the output shoots beyond the desired value & how much.
- ✓ Whether the output oscillates about its final value.
- ✓ When does it settle to the final value.

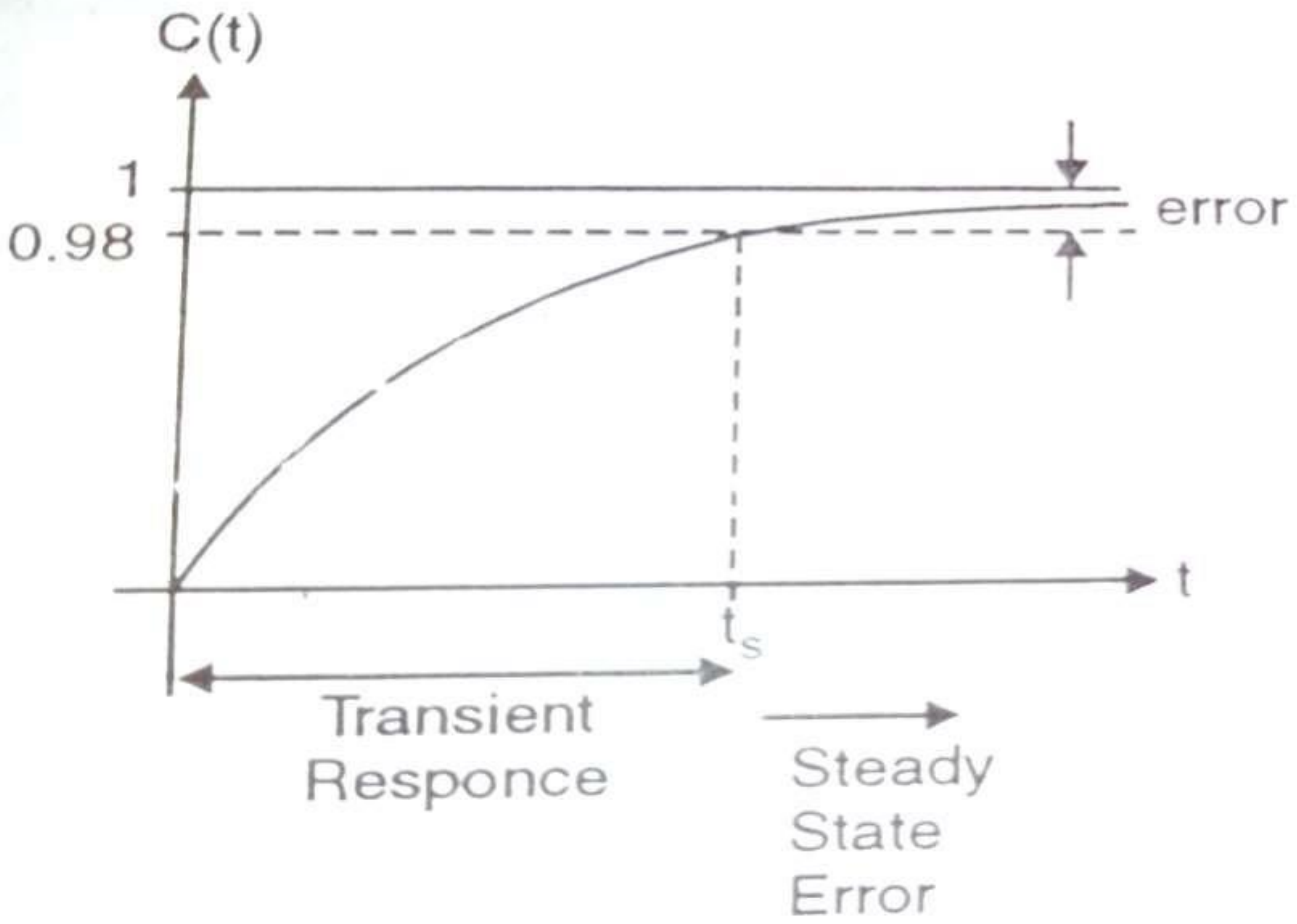
Steady State Response

- That part of the response that remains after the transients have died out is called "**Steady State Response**".
- **From the steady state we can know;**
  - ✓ How long it took before steady state was reached.
  - ✓ Whether there is any error between the desired and actual values.
  - ✓ Whether this error is constant, zero or infinite i.e. unable to track the input.

Steady State Response



Steady State Response



Poles & Zeros of Transfer Function

The transfer function is given by,

$$G(s) = \frac{C(s)}{R(s)}$$

Both C(s) and R(s) are polynomials in s

$$\therefore G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots\ldots\ldots\ldots + b_o}{s^n + a_{n-1} s^{n-1} + \ldots\ldots\ldots\ldots\ldots\ldots + a_n}$$

$$= \frac{K(s-b_1)(s-b_2)(s-b_3)\ldots\ldots\ldots\ldots(s-b_m)}{(s-a_1)(s-a_2)(s-a_3)\ldots\ldots\ldots\ldots(s-a_n)}$$

Where, K= system gain
n= Type of system

Poles

- The values of ‘s’, for which the transfer function magnitude $|G(s)|$ becomes infinite after substitution in the denominator of the system are called as **“Poles”** of transfer function.

Example 1

Determine the poles of given transfer function.

$$G(s) = \frac{s(s+2)(s+4)}{s(s+3)(s+4)}$$

Solution: The poles can be obtained by equating denominator with zero

$$s(s+3)(s+4) = 0$$

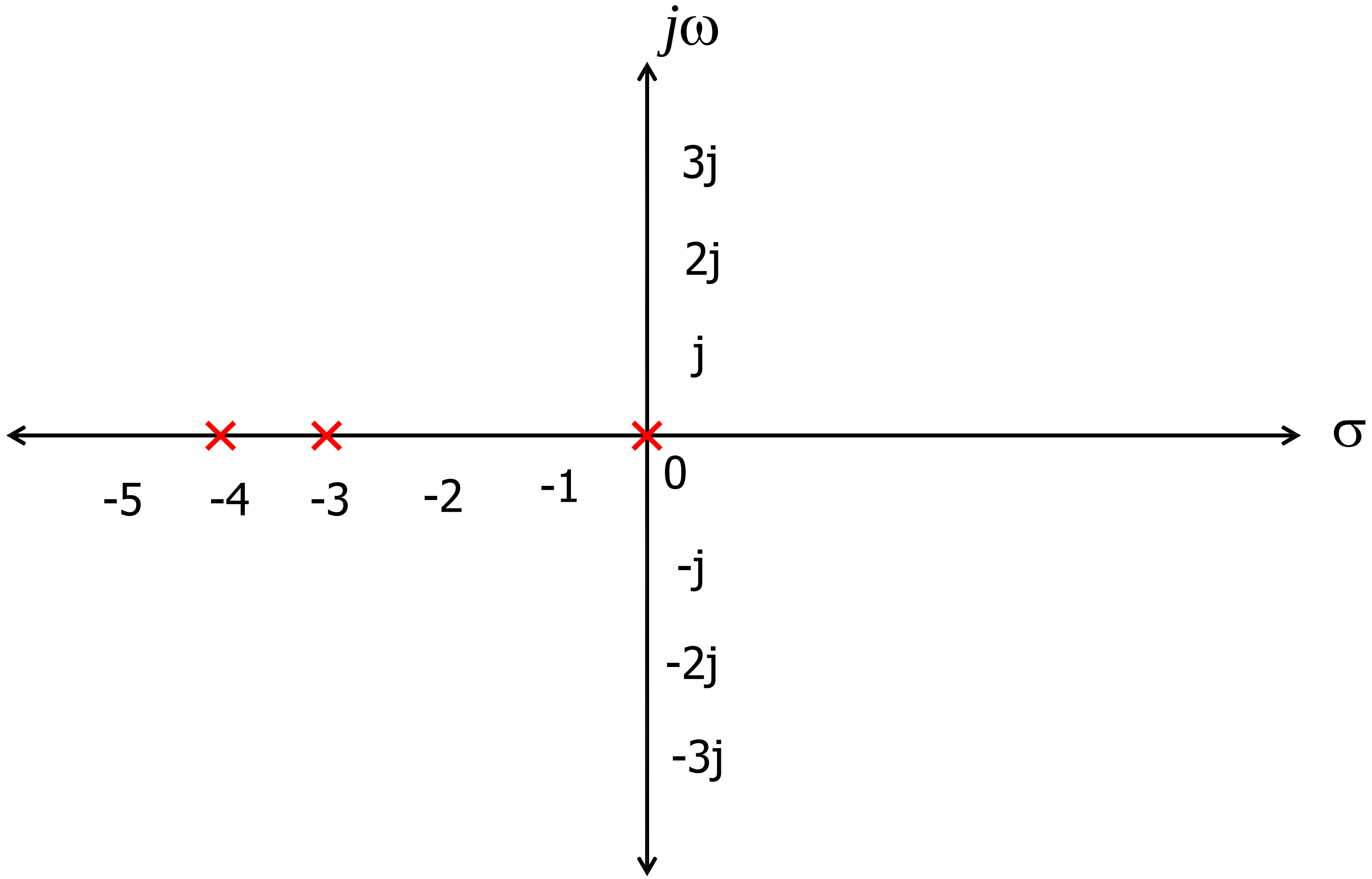
$$\therefore s = 0$$

$$\therefore s+3 = 0 \qquad \therefore s = -3$$

$$\therefore s+4 = 0 \qquad \therefore s = -4$$

The poles are s=0, -3, -4

S-plane Representation of Poles



Zeros

- The values of 's', for which the transfer function magnitude |G(s)| becomes zero after substitution in the numerator of the system are called as "**Zeros**" of transfer function.

Example 2

Determine the zeros of given transfer function.

$$G(s) = \frac{s(s+2)(s+4)}{s(s+3)(s+4)}$$

Solution: The zeros can be obtained by equating numerator with zero

$$s(s+2)(s+4) = 0$$

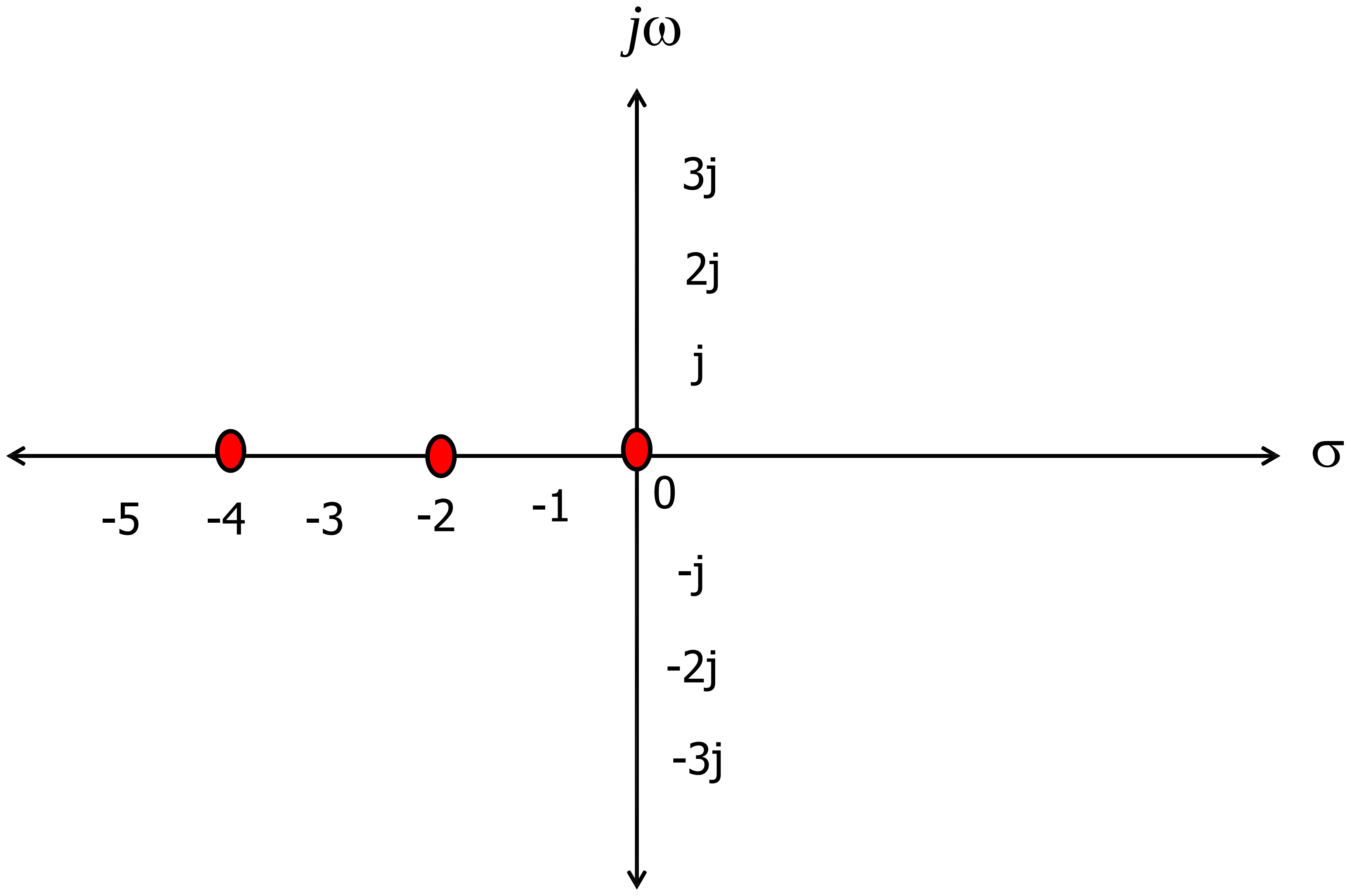
$$\therefore s = 0$$

$$\therefore s+2 = 0 \qquad \therefore s = -2$$

$$\therefore s+4 = 0 \qquad \therefore s = -4$$

The poles are s=0, -2, -4

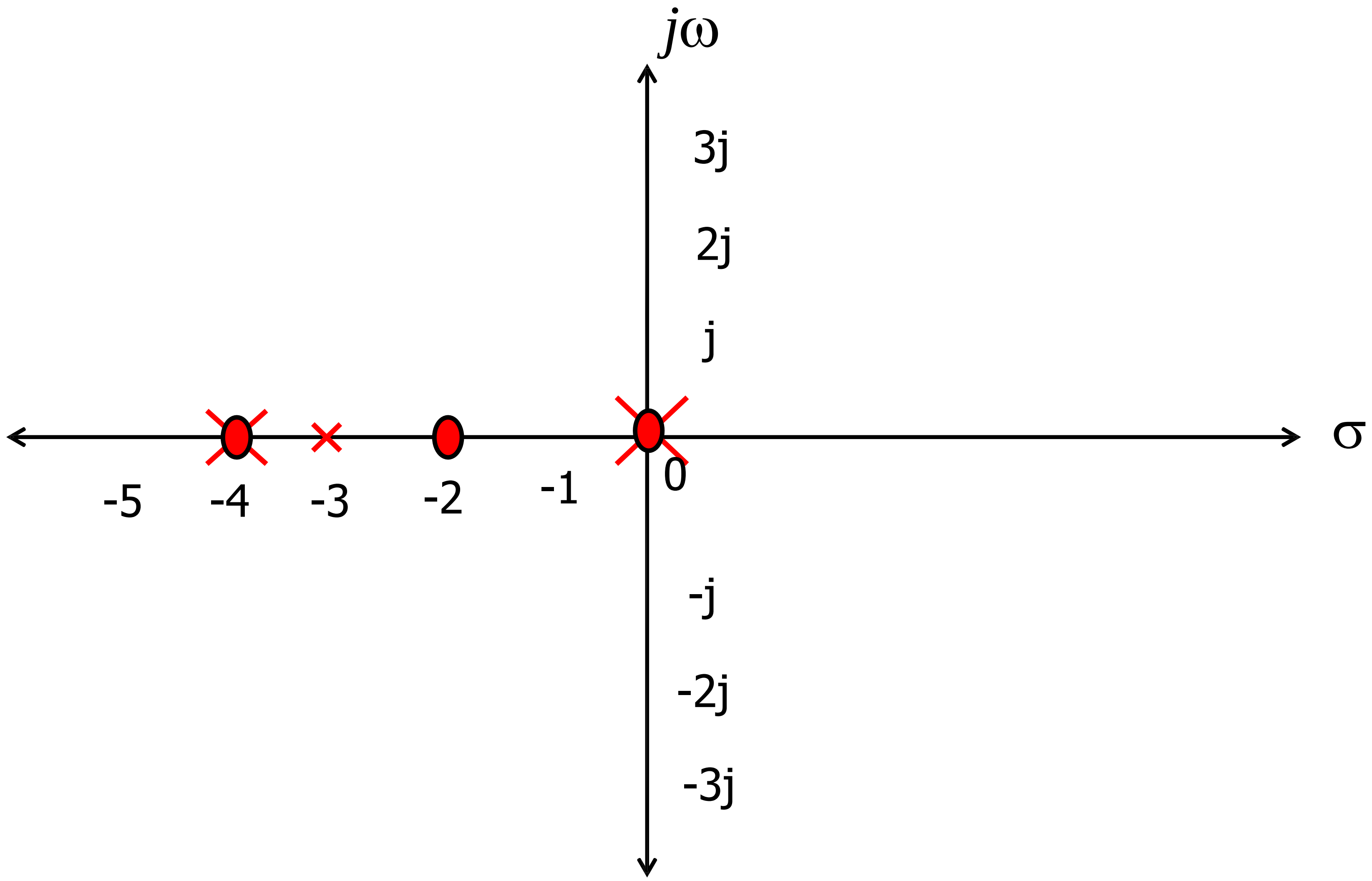
S-plane Representation of Zeros



Pole- Zero Plot

- The diagram obtained by locating all poles and zeros of the transfer function in the s-plane is called as “Pole-zero plot”.
- The s-plane has two axis real and imaginary. Since $s = \sigma + j\omega$, the X-axis stands for real axis and shows a value of σ .
- Similarly, Y-axis stands for $j\omega$ and represents the imaginary axis.

Pole- Zero Plot for Example 1 and 2



Characteristics Equation

Definition: The equation obtained by equating the denominator polynomial of a transfer function to zero is called as the "**Characteristics Equation**"

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots\ldots\ldots\ldots\ldots + a_n$$

Example 3

For the given transfer function,

$$T.F. = \frac{K(s+6)}{s(s+2)(s+5)(s^2+7s+12)}$$

Find: (i) Poles (ii)Zeros
(iii) Pole-zero Plot (iv) Characteristics Equation

Solution: (i)Poles

The poles can be obtained by equating denominator with zero

$$s(s+2)(s+5)(s^2+7s+12) = 0$$

$$\therefore s = 0$$

$$\therefore s+2=0 \qquad \therefore s=-2$$

$$\therefore s+5=0 \qquad \therefore s=-5$$

Example 3 Cont....

$$s(s+2)(s+5)\underline{(s^2+7s+12)}=0$$

$$(s^2+7s+12)=(s+3)(s+4)$$

$$\therefore s+3=0 \qquad \therefore s=-3$$

$$\therefore s+4=0 \qquad \therefore s=-4$$

The poles are s=0, -2, -3, -4, -5

(ii) Zeros:

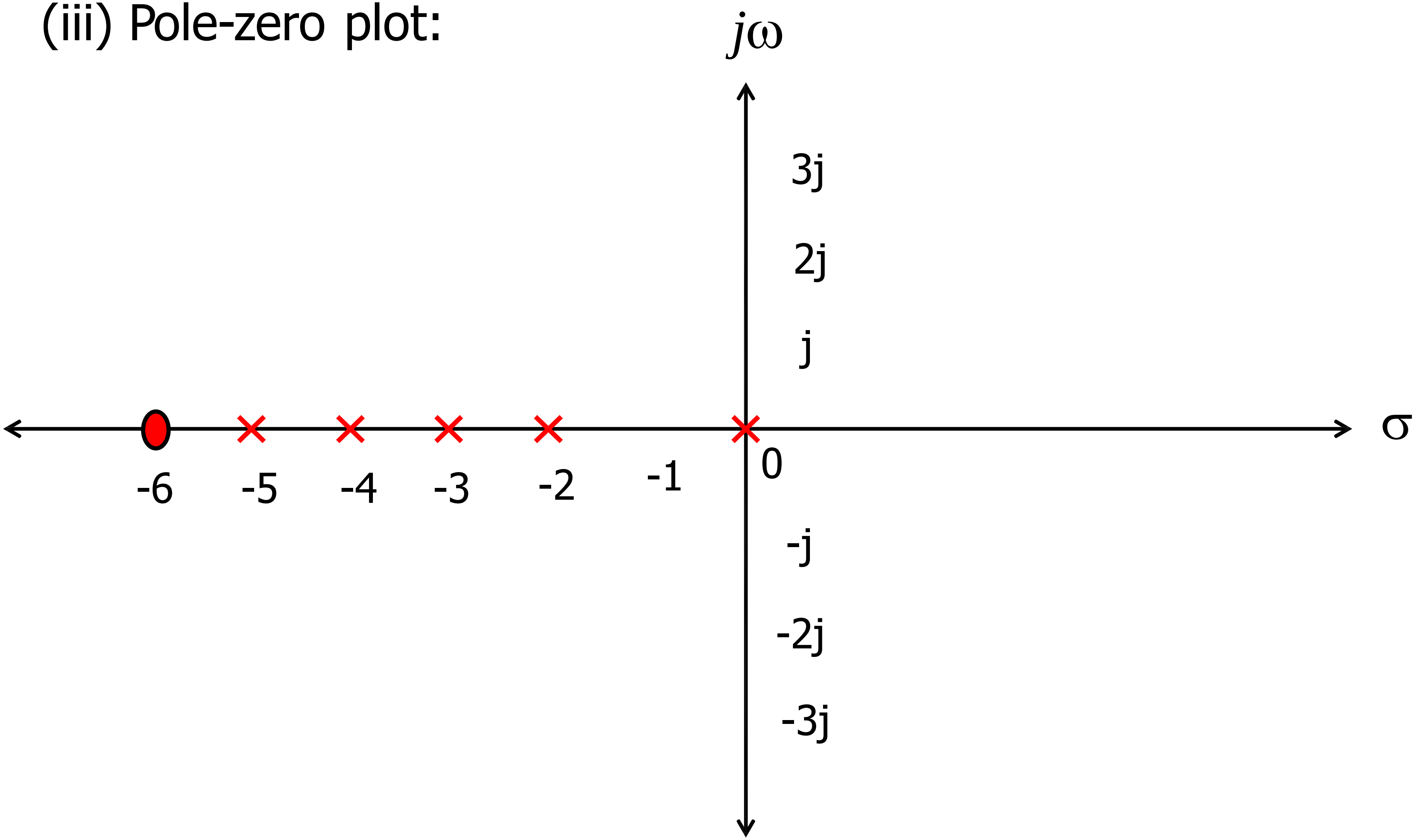
The zeros can be obtained by equating numerator with zero

$$s+6=0 \qquad \therefore s=-6$$

The zeros are s=-6

Example 3 Cont....

(iii) Pole-zero plot:



(iv) Characteristics Equation:

$$s(s+2)(s+5)(s^2+7s+12)=0$$

$$s(s^2+7s+10)(s^2+7s+12)=0$$

$$\therefore (s^3+7s^2+10s)(s^2+7s+12)=0$$

$$\therefore s^5+7s^4+12s^3+7s^4+49s^3+84s^2+10s^3+70s^2+120s=0$$

$$\therefore s^5+14s^4+71s^3+154s^2+120s=0$$

Example 4

For the given transfer function,

$$\frac{C(s)}{R(s)} = \frac{(s+2)}{s(s^2+2s+2)(s^2+7s+12)}$$

Find: (i) Poles (ii)Zeros
(iii) Pole-zero Plot (iv) Characteristics Equation

Solution: (i)Poles

The poles can be obtained by equating denominator with zero

$$s(s^2+2s+2)(s^2+7s+12) = 0$$

$$\therefore s = 0$$

$$\therefore s+3 = 0 \qquad \therefore s = -3$$

$$\therefore s+4 = 0 \qquad \therefore s = -4$$

$$s(s^2 + 2s + 2)(s^2 + 7s + 12) = 0$$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore s = -1 + j$$

$$\therefore s = -1 - j$$

The poles are s=0, -3, -4, -1+j,-1-j

(ii) Zeros:

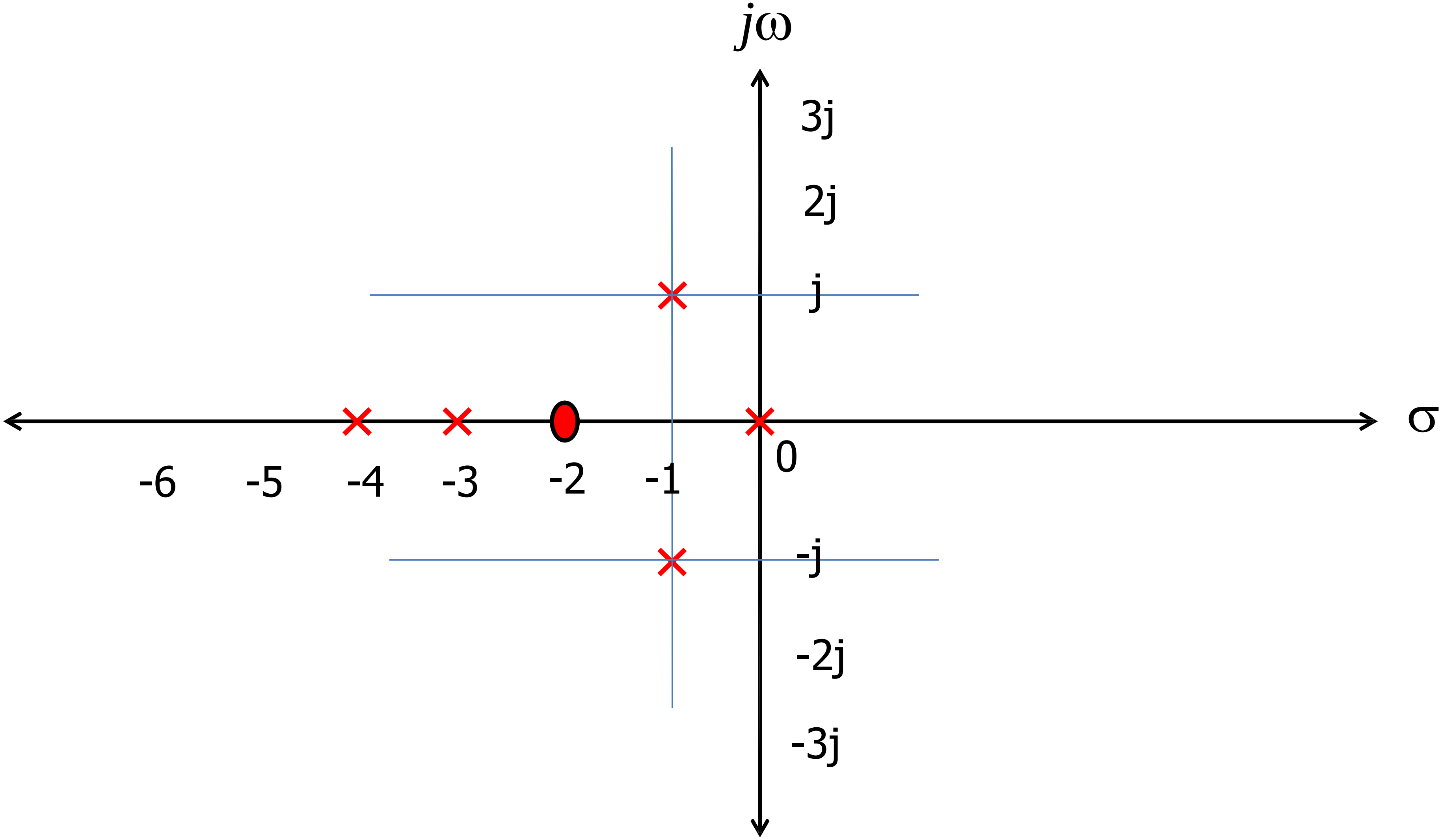
The zeros can be obtained by equating numerator with zero

$$s + 2 = 0 \qquad \therefore s = -2$$

The zeros are s=-2

Example 4 Cont....

(iii) Pole-zero plot:



Example 4 Cont….

(iv) Characteristics Equation:

$$s(s^2+2s+2)(s^2+7s+12)=0$$

$$\therefore (s^3+2s^2+2s)(s^2+7s+12)=0$$

$$\therefore s^5+7s^4+12s^3+2s^4+14s^3+24s^2+2s^3+14s^2+24s=0$$

$$\therefore s^5+9s^4+28s^3+38s^2+24s=0$$

Example 5

For the given transfer function,

$$T.F. = \frac{(s+2)}{s(s+4)(s^2+6s+25)}$$

Find: (i) Poles (ii)Zeros

(iii) Pole-zero Plot (iv) Characteristics Equation

Solution: (i)Poles

The poles can be obtained by equating denominator with zero

$$s(s+4)(s^2+6s+25)=0$$

$$\therefore s=0$$

$$\therefore s+4=0 \qquad \therefore s=-4$$

Example 5 Cont....

$$s(s+4)(s^2+6s+25)=0$$

$$roots = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore s=-3+j4$$

$$\therefore s=-3-j4$$

The poles are s= 0, -4, -3+j4, -3-j4

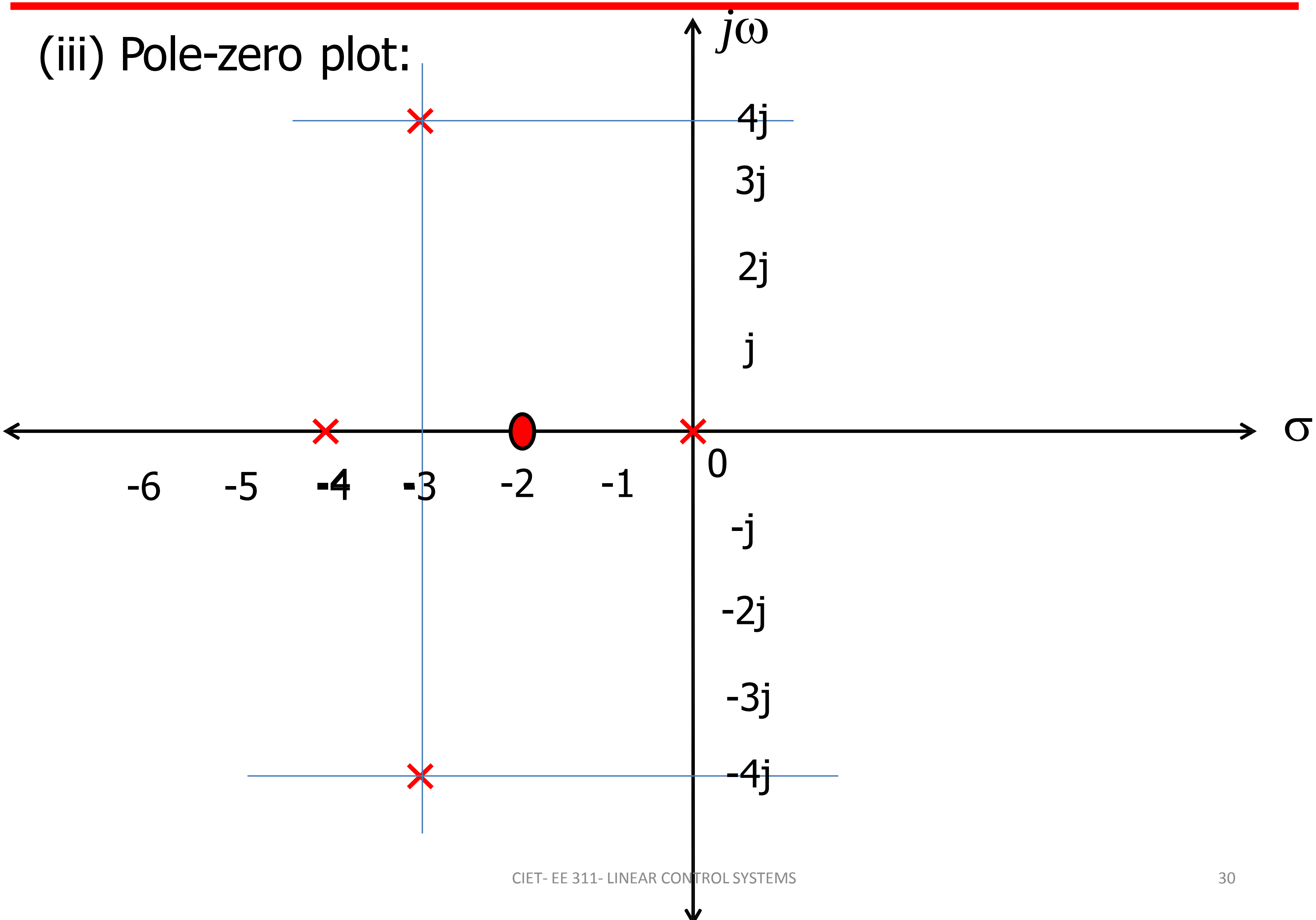
(ii) Zeros:

The zeros can be obtained by equating numerator with zero

$$s+2=0 \qquad \therefore s=-2$$

The zeros are s=-2

(iii) Pole-zero plot:



(iv) Characteristics Equation:

$$s(s+4)(s^2+6s+25)=0$$

$$(s^2+4s)(s^2+6s+25)=0$$

$$\therefore s^4+6s^3+25s^2+4s^3+24s^2+100s=0$$

$$\therefore s^4+10s^3+49s^2+100s=0$$

Order of System

- The order of control system is defined as the highest power of s present in denominator of closed loop transfer function G(s) of unity feedback system.

System Order and Proper System

- Highest power of s present in denominator of closed loop transfer function is called as "Order of System".
- A **proper system** is a system where the degree of the denominator is larger than or equal to the degree of the numerator polynomial.

Type-0 (Zero) System

Definition: A control system with no integration in the open loop transfer function and no pole of transfer function G(s) at the origin of s-plane is designated as "**Type-0** " system.

$$G(s) = \frac{K(1+Tz1s)(1+Tz2s)\ldots\ldots\ldots\ldots\ldots\ldots}{(1+Tp1s)(1+Tp2s)\ldots\ldots\ldots\ldots\ldots} \quad \text{(Standard form)}$$

An amplifier type control system is a practical example of Type-0 system

Type-1 (One) System

Definition: A control system with one integration in the open loop transfer function and one pole of transfer function G(s) at the origin of s-plane is designated as "**Type-1** " system.

$$G(s) = \frac{K(1+T_{z1}s)(1+T_{z2}s)\ldots\ldots\ldots\ldots\ldots\ldots}{s(1+T_{p1}s)(1+T_{p2}s)\ldots\ldots\ldots\ldots\ldots}$$

(Standard form)

An pneumatic type control system is a practical example of Type-1 system

Type-2 (Two) System

Definition: A control system with two integration in the open loop transfer function and two pole of transfer function G(s) at the origin of s-plane is designated as “**Type-2** ” system.

$$G(s) = \frac{K(1+T_{z1}s)(1+T_{z2}s)\ldots\ldots\ldots\ldots\ldots\ldots}{s^2(1+T_{p1}s)(1+T_{p2}s)\ldots\ldots\ldots\ldots\ldots}$$ (Standard form)

A mechanical displacement system is a practical example of Type-2 system

Example1: Determine order of given system

$$TF = G(s) = \frac{s(s+2)}{s^4 + 7s^3 + 10s^2 + 5s + 5}$$

Example1: Determine order of given system

$$TF = G(s) = \frac{s(s+2)}{s^4 + 7s^3 + 10s^2 + 5s + 5}$$

Answer: The highest power of equation in denominator

of given transfer function is '4'.

Hence the order of given system is fourth

Example 2 : Determine order of given system

$$G(s) = \frac{(s+5)(s+2)}{s(s+3)(s+4)}$$

Example 2 : Determine order of given system

$$G(s) = \frac{(s+5)(s+2)}{s(s+3)(s+4)}$$

Solution: To obtain highest power of denominator, Simplify denominator polynomial.

$$s(s+3)(s+4) = 0$$

$$s(s^2+7s+12) = 0$$

$$s^3+7s^2+12s = 0$$

The highest power of equation in denominator of given transfer function is '3'. Hence given system is **"Third Order system"**. The degree of denominator is larger than the numerator hence system is **"Proper System"**

Example 3 : Determine order of given system

$$G(s) = \frac{K(s+5)}{s^3(7s^2 + 12s + 5)}$$

Example 3 : Determine order of given system

$$G(s) = \frac{K(s+5)}{s^3(7s^2+12s+5)}$$

Solution: To obtain highest power of denominator, Simplify denominator polynomial.

$$s^3(7s^2+12s+5)=0$$

$$7s^5+12s^4+5s^3=0$$

The highest power of equation in denominator of given transfer function is '5'. Hence given system is **"Fifth Order system"**. The degree of denominator is larger than the numerator hence system is **"Proper System"**

Need of Standard Test Signal

- **Thus from such types of inputs we can expect a system in general to get an input which may be;**
    a) A sudden change
    b) A momentary shock
    c) A constant velocity
    d) A constant acceleration
- **Hence these signals form standard test signals. The response to these signals is analyzed. The above inputs are called as,**
    a) Step input - Signifies a sudden change
    b) Impulse input – Signifies momentary shock
    c) Ramp input – Signifies a constant velocity
    d) Parabolic input – Signifies constant acceleration

Standard Test Signals

- The following are the standard test signals:
  - Step Signal
  - Ramp Signal
  - Parabolic Signal
  - Impulse Signal

Standard Test Signals

Step Signal

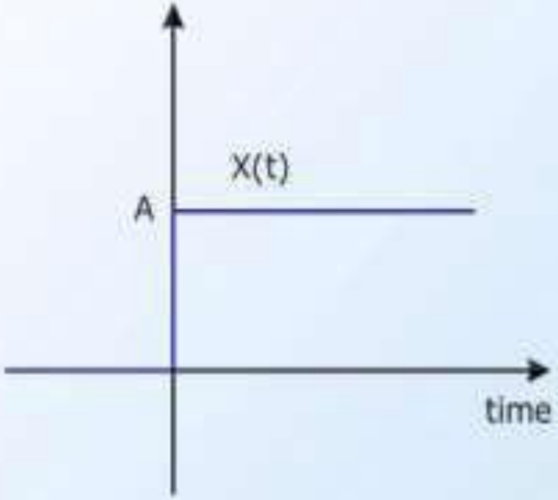
- Signal whose value changes from one level to another level (A) in zero time

$$x(t) = Au(t)$$

$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

- In Laplace Transform

$$X(s) = \frac{A}{s}$$



Step Signal

Standard Test Signals

Ramp Signal

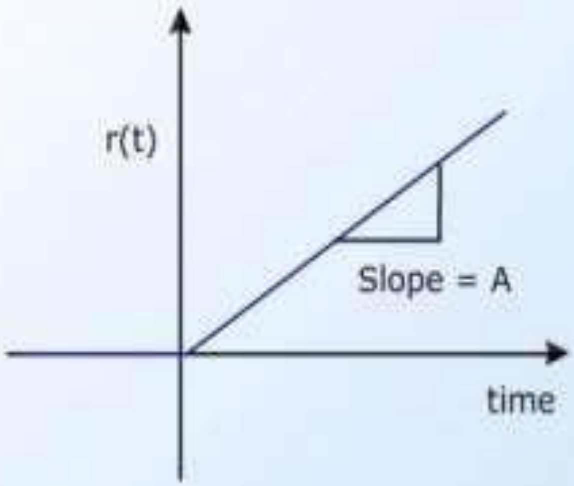
- Signal which starts at zero value and increases linearly with time

$$r(t) = \begin{cases} 0 & t < 0 \\ At & t \geq 0 \end{cases}$$

- In Laplace transform

$$R(s) = \frac{A}{s^2}$$

- Integration of step signal results in a ramp signal



Ramp Signal

Standard Test Signals

Parabolic Signal

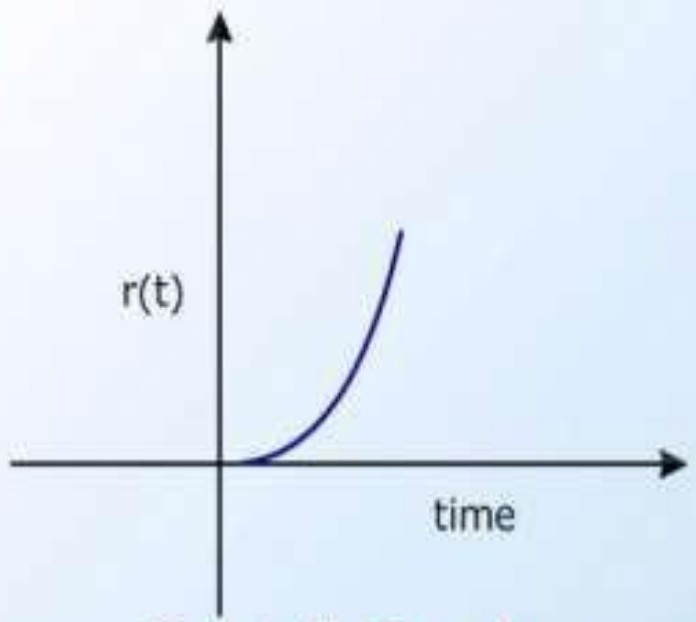
- The instantaneous value of a parabolic signal varies as square of time from an initial value of zero at t=0

$$r(t)=\begin{cases}0 & t<0\\ At^2/2 & t>0\end{cases}$$

- In the Laplace transform

$$R(s)=\frac{A}{s^3}$$

- Integration of ramp signal results in a parabolic signal



Parabolic Signal

Standard Test Signals

Impulse Signal

- Impulse signal is otherwise called shock input that occurs for a small interval of time.

  Mathematically it is expressed as

$$x(t) = \delta(t)\ \ for\ 0 < t < \frac{1}{H}\ where\ H \to \infty$$

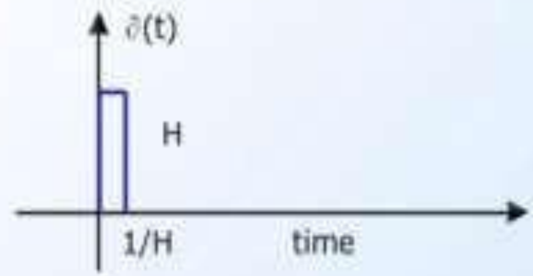
- For Unit impulse

$$\delta(t) = 1;\ t = 0$$
$$= 0;\ t \neq 0$$

- The Laplace transform of a

  Unit Impulse is:

$$L[\delta(t)] = 1 = R(s)$$



Impulse Signal

Time Response of First Order Systems

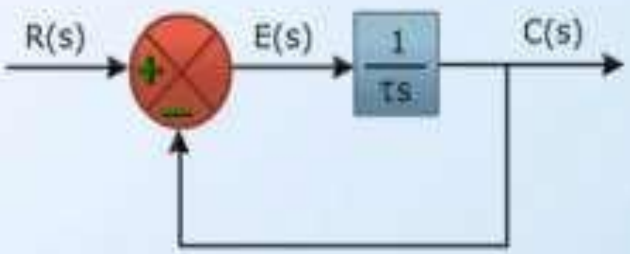
Step Response of a First Order System

- Transfer function of a first order system without zeros can be represented as:

$$\frac{C(s)}{R(s)} = \frac{1}{\tau s + 1}$$

- Given a step input i.e., R(s) = 1/s, then the system output (called step response in this case) is

$$C(s) = \frac{1}{s(\tau s + 1)}$$



First Order System

TIME RESPONSE OF FIRST ORDER SYSTEM

STEP RESPONSE OF FIRST ORDER SYSTEM

$$\boldsymbol{C(S)} = \frac{(\frac{\mathbf{1}}{\boldsymbol{\tau}})}{\boldsymbol{S}(\boldsymbol{S}+\frac{\mathbf{1}}{\boldsymbol{\tau}})} = \frac{\boldsymbol{A}}{\boldsymbol{S}} + \frac{\boldsymbol{B}}{(\boldsymbol{S}+\frac{\mathbf{1}}{\boldsymbol{\tau}})}$$

Calculating

$\Rightarrow$ A = 1, B = -1

$$\boldsymbol{C(S)} = \frac{\mathbf{1}}{\boldsymbol{S}} - \frac{\mathbf{1}}{(\boldsymbol{S}+\frac{\mathbf{1}}{\boldsymbol{\tau}})}$$

Applying Inverse Laplace Transform

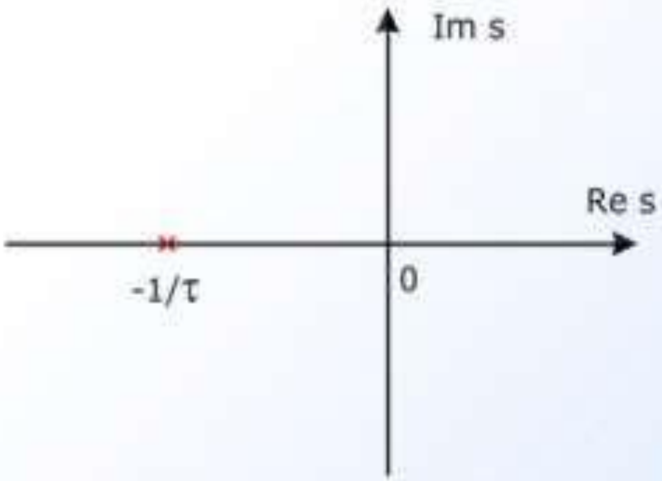
C(t) = 1 - $e^{-t/\tau}$

Time Response of First Order Systems

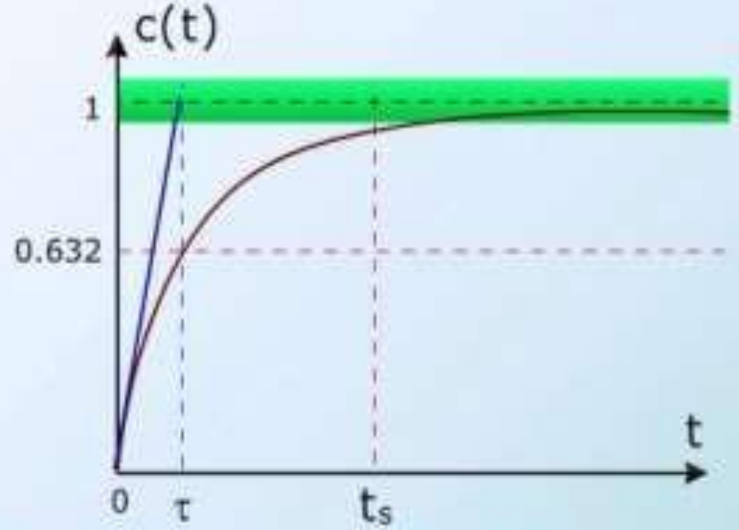
Step Response of a First Order System

- At t = τ, the step response is

$$C(t) = 1 - 0.37 = 0.632$$



Pole-Zero Plot of First Order System



Time Response of First Order System

RAMP RESPONSE OF FIRST ORDER SYSTEM

Transfer Function of First Order System is given by

$$\frac{C(S)}{R(S)} = \frac{1}{(\tau S+1)} \quad => \quad C(S) = \frac{1}{(\tau S+1)}R(S)$$

For Unit Ramp Input r(t) = t , In S-Domain R(S) = $\frac{1}{S^2}$

$$C(S) = \frac{(\frac{1}{\tau})}{S^2(S+\frac{1}{\tau})} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{(S+\frac{1}{\tau})}$$

Calculating

=> A =-τ, B=1, C=τ

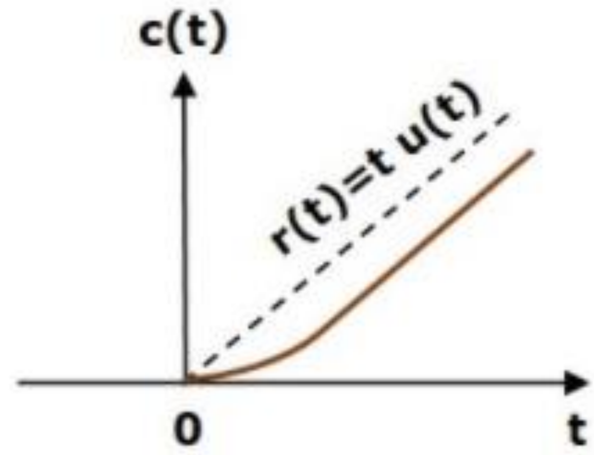
TIME RESPONSE OF FIRST ORDER SYSTEM

RAMP RESPONSE OF FIRST ORDER SYSTEM

$$\boldsymbol{C(S) = \frac{-\tau}{S} + \frac{1}{S^2} + \frac{\tau}{(S+\frac{1}{\tau})}}$$

Applying Inverse Laplace Transform

$$C(t) = -\tau + t + \tau e^{-t/\tau}$$



PARABOLIC RESPONSE OF FIRST ORDER SYSTEM

Transfer Function of First Order System is given by

$$\frac{C(S)}{R(S)} = \frac{1}{(\tau S+1)} \quad => \quad C(S) = \frac{1}{(\tau S+1)}R(S)$$

For Unit Ramp Input r(t) = $\frac{t^2}{2}$, In S-Domain R(S) = $\frac{1}{S^3}$

$$C(S) = \frac{(\frac{1}{\tau})}{S^3(S+\frac{1}{\tau})} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S^3} + \frac{D}{(S+\frac{1}{\tau})}$$

Calculating

=> A $=\tau^2$, B$= -\tau$, C=1 , $D =$ - τ^2

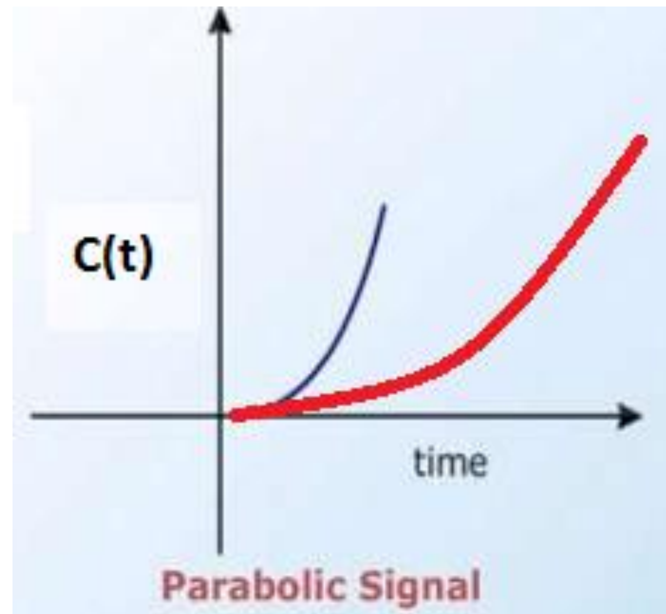
TIME RESPONSE OF FIRST ORDER SYSTEM

PARABOLIC RESPONSE OF FIRST ORDER SYSTEM

$$\boldsymbol{C(S)} = \frac{\tau^2}{\boldsymbol{S}} - \frac{\tau}{\boldsymbol{S}^2} + \frac{\mathbf{1}}{\boldsymbol{S}^3} - \frac{\tau^2}{(\boldsymbol{S}+\frac{\mathbf{1}}{\boldsymbol{\tau}})}$$

Applying Inverse Laplace Transform

$$\mathrm{C(t)} = \tau^2 - \tau t + \frac{\boldsymbol{t}^{\mathbf{2}}}{\mathbf{2}} - \tau^2 e^{-t/\tau}$$



TIME RESPONSE OF FIRST ORDER SYSTEM

IMPULSE RESPONSE OF FIRST ORDER SYSTEM

Transfer Function of First Order System is given by

$$\frac{C(S)}{R(S)} = \frac{1}{(\tau S+1)} \qquad => \qquad C(S) = \frac{1}{(\tau S+1)}R(S)$$

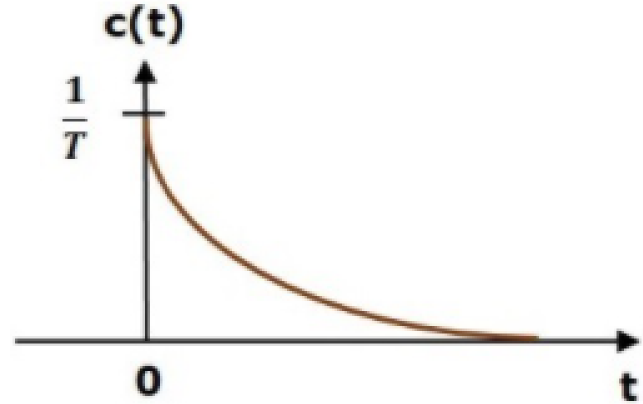
For Impulse Input r(t) = $\delta(t)$, In S-Domain R(S) = 1

$$C(S) = \frac{(\frac{1}{\tau})}{(S + \frac{1}{\tau})}$$

TIME RESPONSE OF FIRST ORDER SYSTEM

Applying Inverse Laplace Transform

$$C(t) = \frac{1}{\tau} e^{-t/\tau}$$



Characteristic Equation of Feedback Control Systems

- A general second order system is characterized by the following transfer function:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b}$$

- We can re-write the above transfer function in the following form:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where ω_n = Undamped natural frequency

ξ = Damping ratio

- Damping ratio determines how much the system oscillates as the response decays toward steady state or it is a measure of system's ability to oppose oscillatory response
- The denominator in the transfer function of a second order system is called the **"characteristic equation of feedback control system"**

Damping

Every system has a tendency to oppose the oscillatory behavior of the system which is known as **“Damping”**.

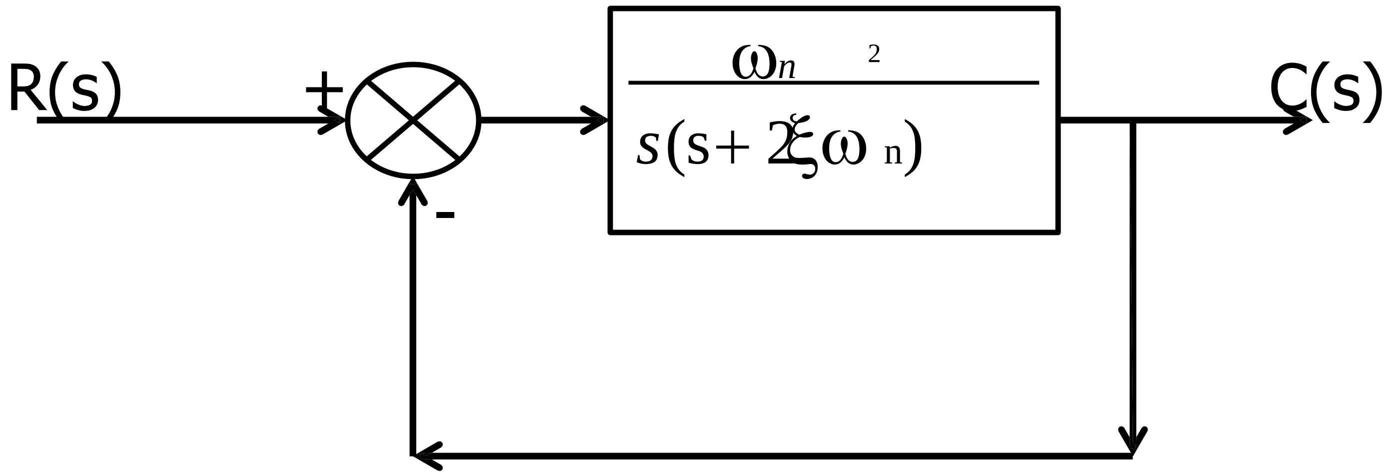
Damping Factor (ξ)

The damping in any system is measured by a factor or ratio which is known as damping ratio.

It is denoted by ξ (Zeta)

Analysis of second order system for Step input

Consider a second order system as shown;



Here $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$ and $H(s) = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Analysis of second order system for Step input

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

This is the standard form of the closed loop transfer function

These poles of transfer function are given by;

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\therefore s = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4(\omega_n)^2}}{2}$$

$$= -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Analysis of second order system for Step input

The poles are;

(i) Real and Unequal if $\sqrt{\xi^2 - 1} > 0$

i.e. $\xi > 1$ They lie on real axis and distinct

(ii) Real and equal if $\sqrt{\xi^2 - 1} = 0$

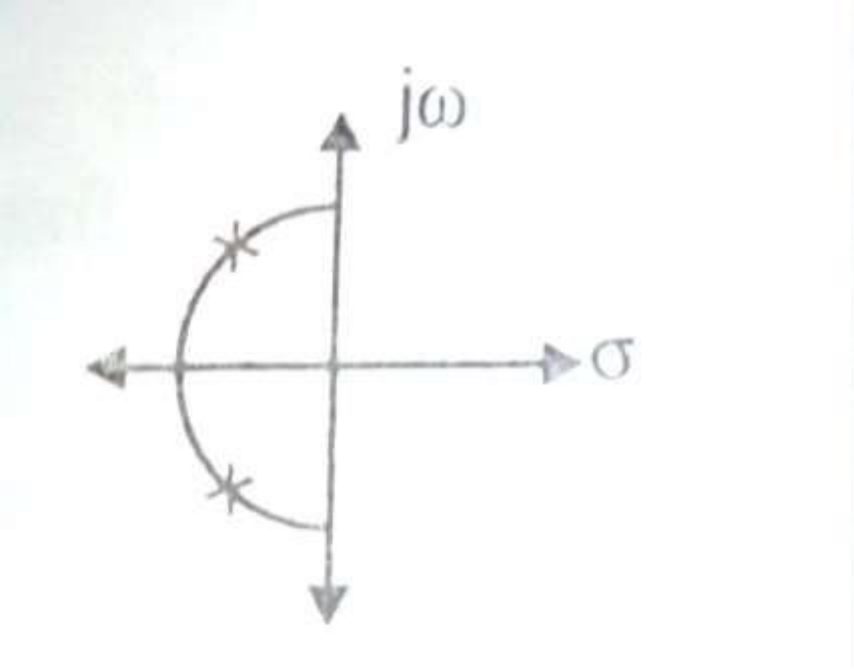
i.e. $\xi = 1$ They are repeated on real axis

(iii) Complex if $\sqrt{\xi^2 - 1} < 0$

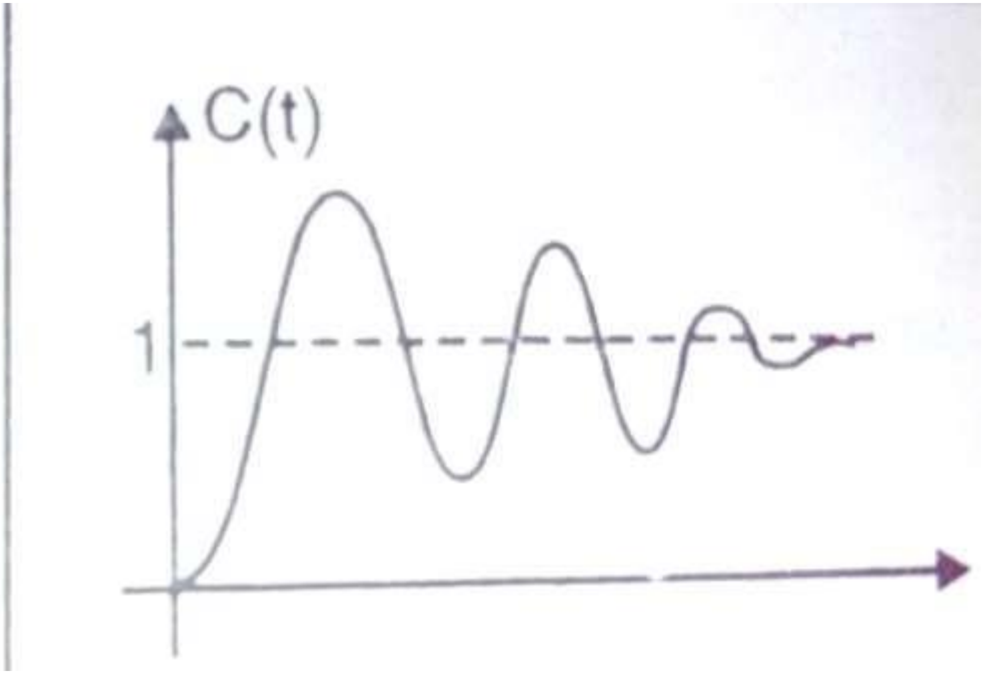
i.e. $\xi < 1$ Poles are in second and third quadrant

Relation between ξ and pole locations

(i) $0 < \xi < 1$ Under damped



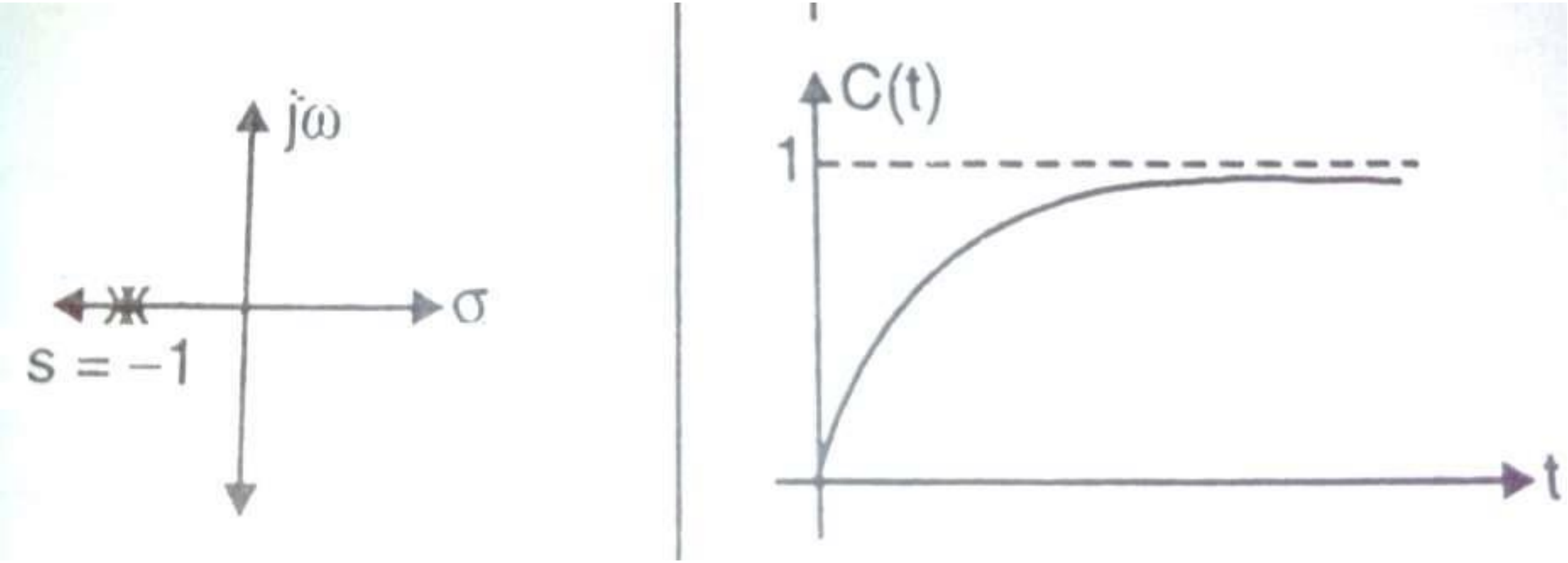
Pole Location



Step Response c(t)

Relation between ξ and pole locations

(ii) $\xi = 1$ Critically damped

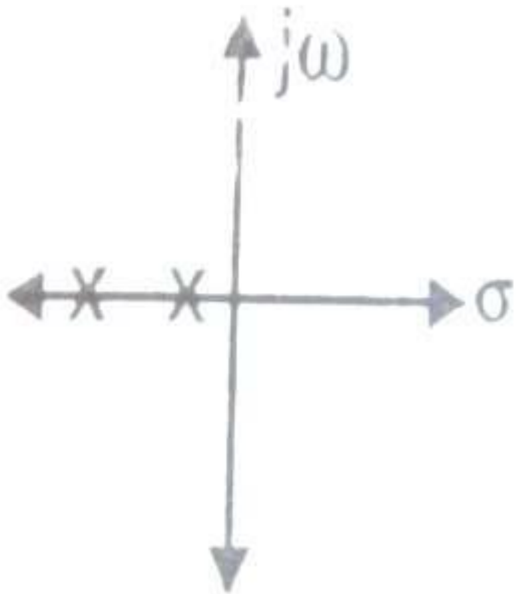


Pole Location

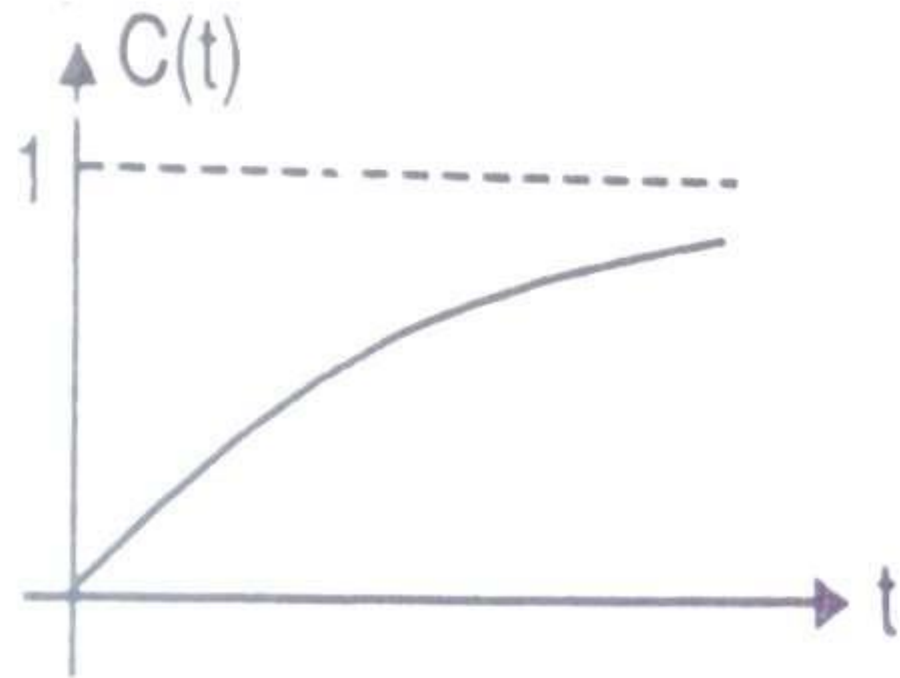
Step Response c(t)

Relation between ξ and pole locations

(iii) $\xi > 1$ over damped



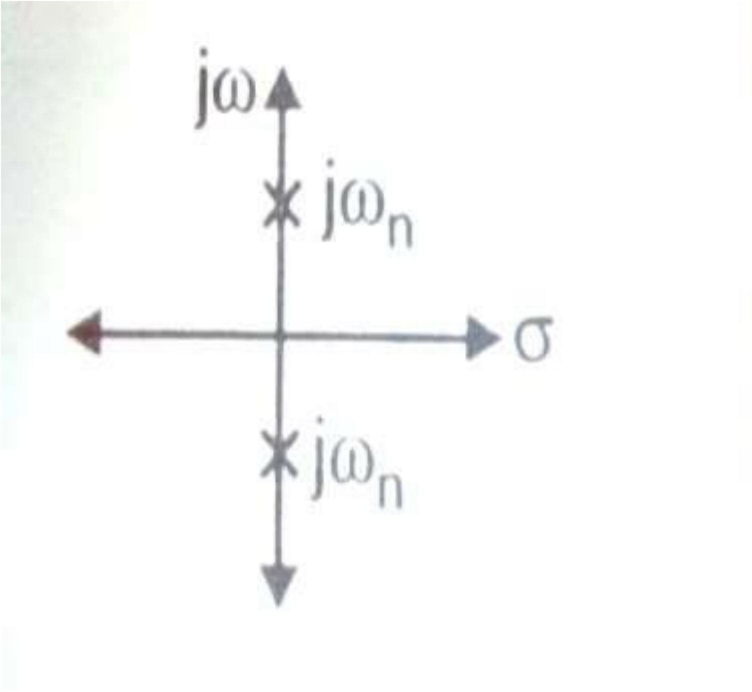
Pole Location

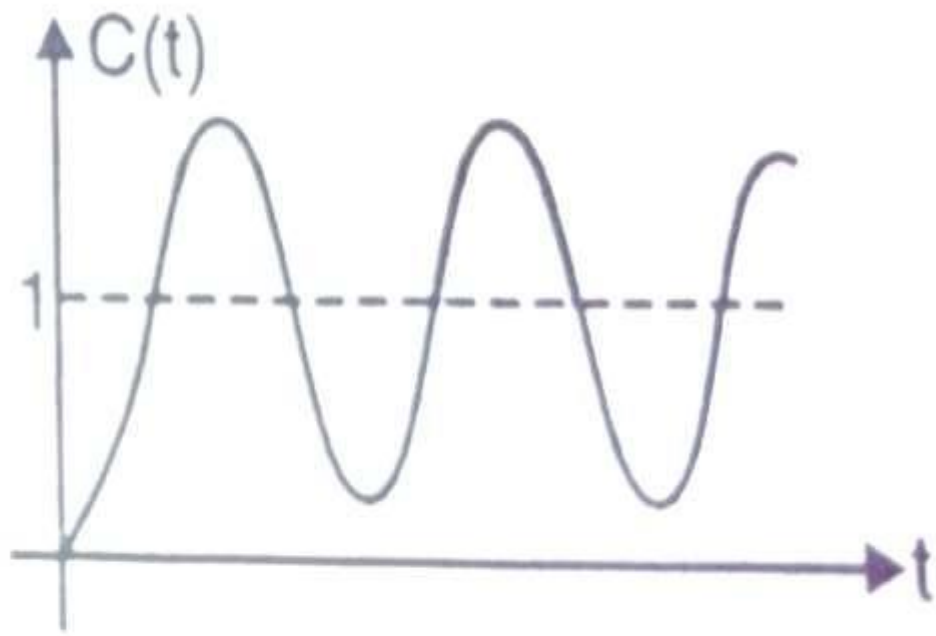


Step Response c(t)

Relation between ξ and pole locations

(iv) $\xi = 0$



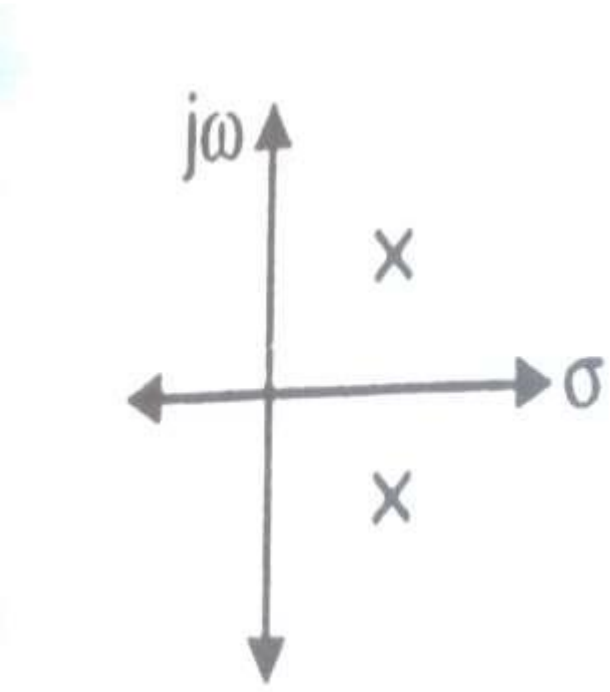


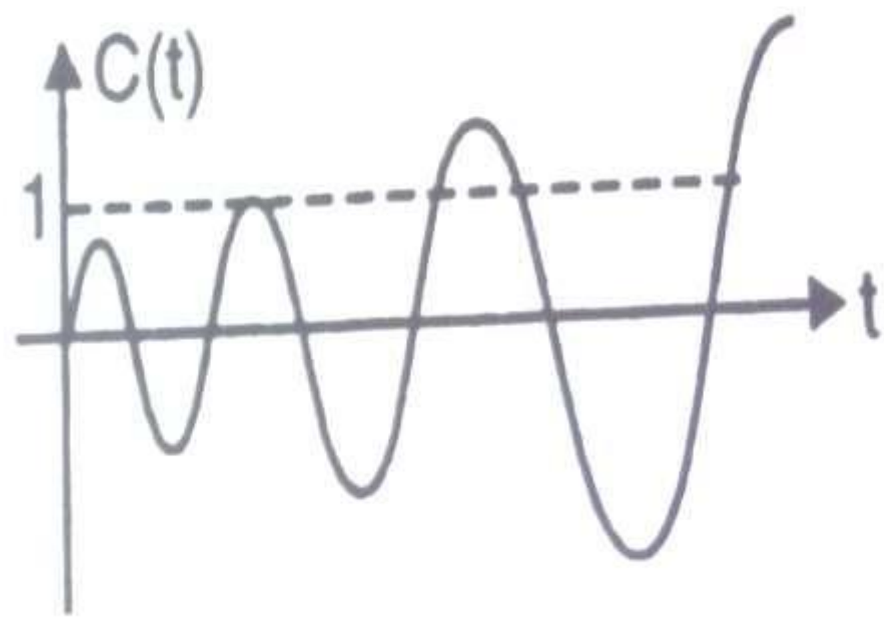
Pole Location

Step Response c(t)

Relation between ξ and pole locations

(v) $0 > \xi > \quad -1$



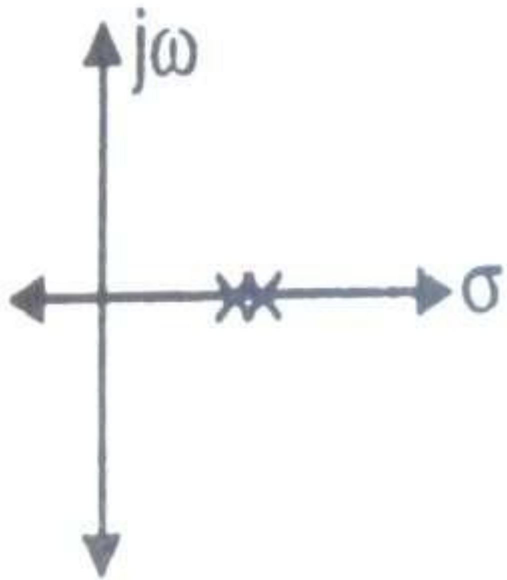


Pole Location

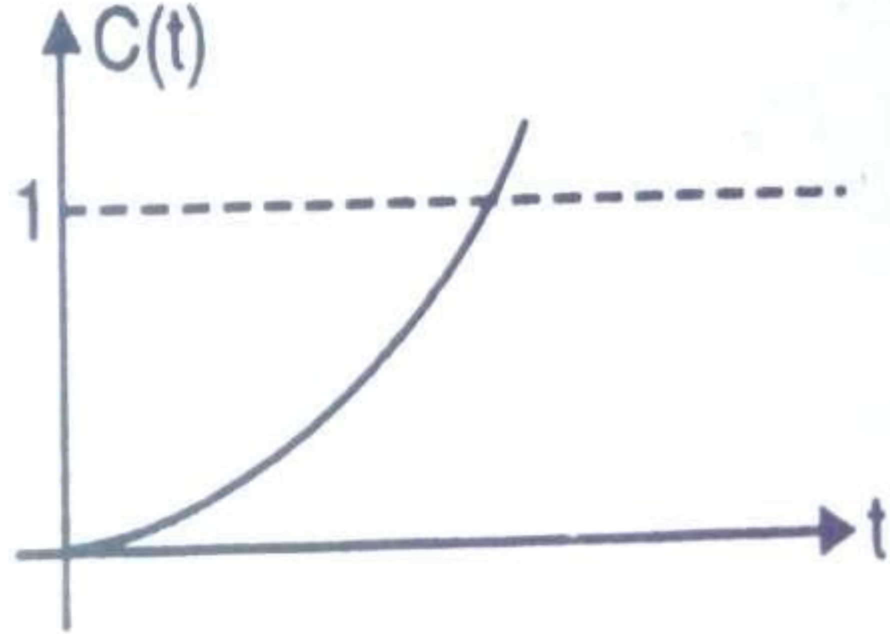
Step Response c(t)

Relation between ξ and pole locations

(vi) $\xi = \quad -1$



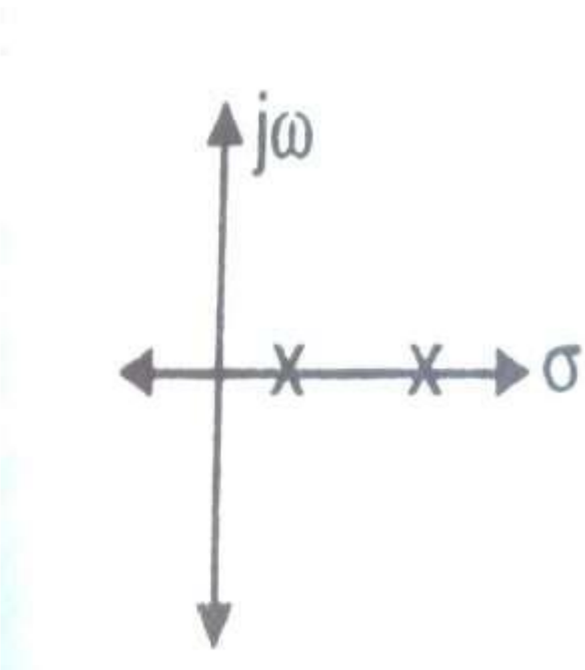
Pole Location

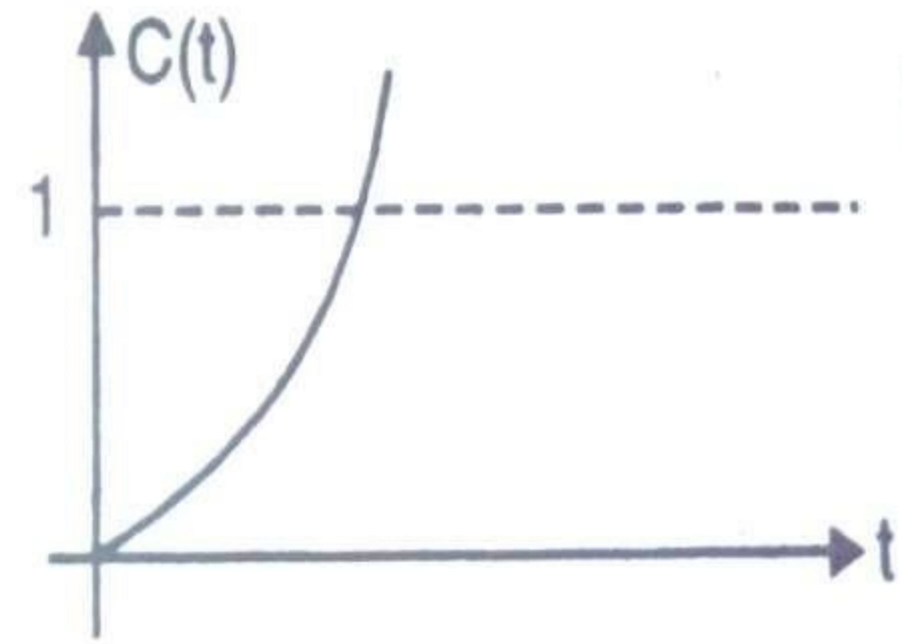


Step Response c(t)

Relation between ξ and pole locations

(vii) $\xi < \quad -1$





Pole Location

Step Response c(t)

Transient Response of Second Order Systems

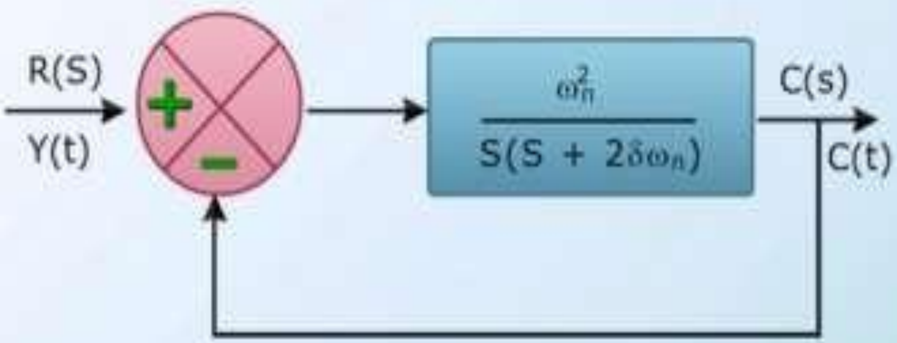
Time-response of Second-order Control System

- The order of a control system is defined as the highest derivative present in the differential equation of the system
- In the s-domain, the higher power of "s" in the characteristic equation 1 + G(s)H(s), 0 is the "order"
- Consider

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

$$= \frac{-2\delta\omega_n \pm \sqrt{4\delta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$S_{1,2} = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$



Standard Form of Second Order System

Transient Response of Second Order Systems

Case 1: Under damped (δ <1)

- The conditional frequency the two roots are said to be "complex conjugates"

$$S_{1,2} = -\delta\omega_n \pm j\omega_n\sqrt{\delta^2 - 1}$$

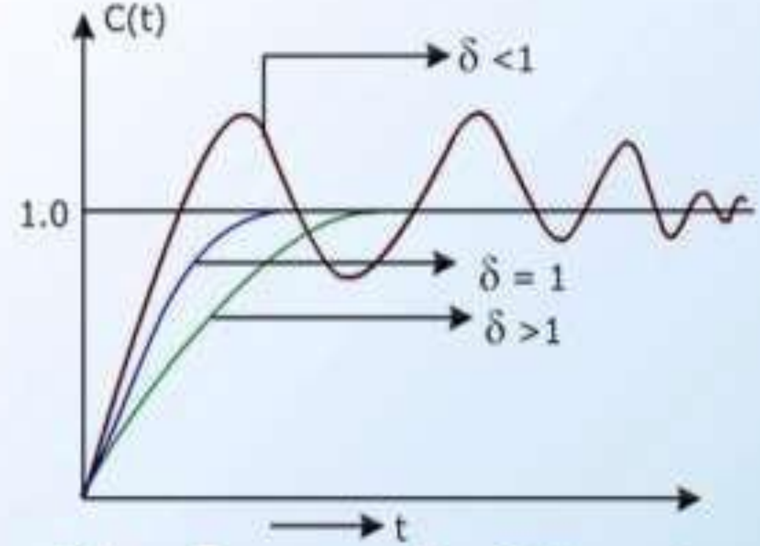
$$S_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2} = -\xi\omega_n + j\omega_d$$

$$S_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2} = -\xi\omega_n - j\omega_d$$

Case 2: Over damped (δ >1)

- The two roots are real and unequal
- The nature of the response is non-oscillatory

$$S_{1,2} = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$



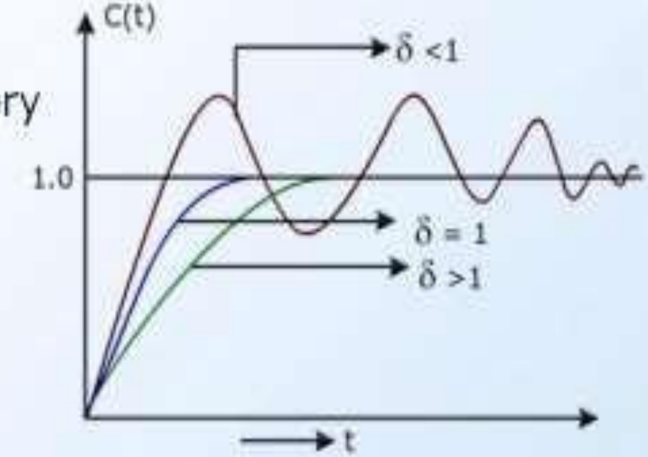
Time-Response for Different Ranges of δ

Transient Response of Second Order Systems

Case 3: Critically damped (δ =1)

- The two roots are real and equal
- The response is on the range of becoming oscillatory

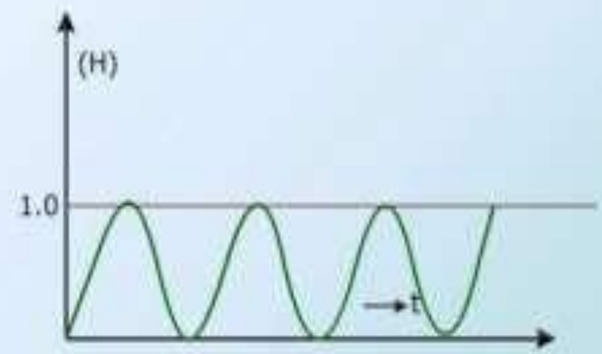
$$S_{1,2} = -\xi\omega_n$$



Time-Response for Different δ

Case 4: Undamped (δ =0)

- The response is oscillatory with a frequency of "ω_n" rad/sec
- The oscillations sustain without any change in the amplitude



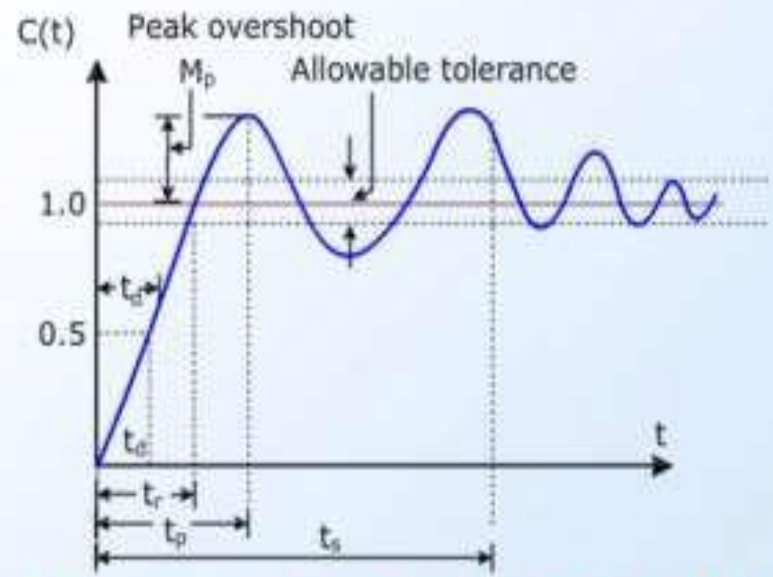
Time-Response for δ =0

Transient Response of Second Order Systems

Time-response for unit step input r(t) = u(t) and R(s)= 1/s

- Many control systems are generally under-damped in nature and their roots are "complex conjugates"

$$C(s) = R(s).\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$



Time Response of Second-order System for the Under Damped

Step Response of Second Order System

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$R(s)=1/s$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: δ = 0

Substitute, δ=0 in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Step Response of Second Order System

Substitute, $R(s)=1/s$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2+\omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

Apply partial transform and inverse Laplace transform on both the sides.

$$c(t) = \left(1 - \cos(\omega_n t)\right)$$

So, the unit step response of the second order system when /*delta*=0 will be a continuous time signal with constant amplitude and frequency.

Step Response of Second Order System

Case 2: δ = 1

Substitute, /*delta*=1 in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right) R(s)$$

Substitute, $R(s)=1/s$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Step Response of Second Order System

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1, -1 \textit{and} -\omega n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Step Response of Second Order System

Case 3: 0 < δ < 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}\right) R(s)$$

Substitute, $R(s)=1/s$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)\right)}$$

Step Response of Second Order System

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs + C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C
as $1, -1 and -2\delta\omega n$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)} - \frac{\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2}\right)$$

Step Response of Second Order System

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2+\omega_d^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_d}{(s+\delta\omega_n)^2+\omega_d^2}\right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}}e^{-\delta\omega_n t}\sin(\omega_d t)\right)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right)$$

Step Response of Second Order System

If $\sqrt{1-\delta^2}=\sin(\theta)$, then 'δ' will be $\cos(\theta)$. Substitute these values in the above equation.

$$c(t)=\left(1-\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t)+\cos(\theta)\sin(\omega_d t))\right)$$

$$\Rightarrow c(t)=\left(1-\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t+\theta)\right)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Step Response of Second Order System

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows −

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= \left(s + \delta\omega_n\right)^2 - \omega_n^2\left(\delta^2 - 1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}\right)R(s)$$

Step Response of Second Order System

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2} \right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$= \frac{A}{s} + \frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}} + \frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively. Substitute these values in above partial fraction expansion of $C(s)$.

Step Response of Second Order System

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\left(\frac{1}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}}\right) - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\right)\left(\frac{1}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}}\right)$$

Apply inverse Laplace transform on both the sides.

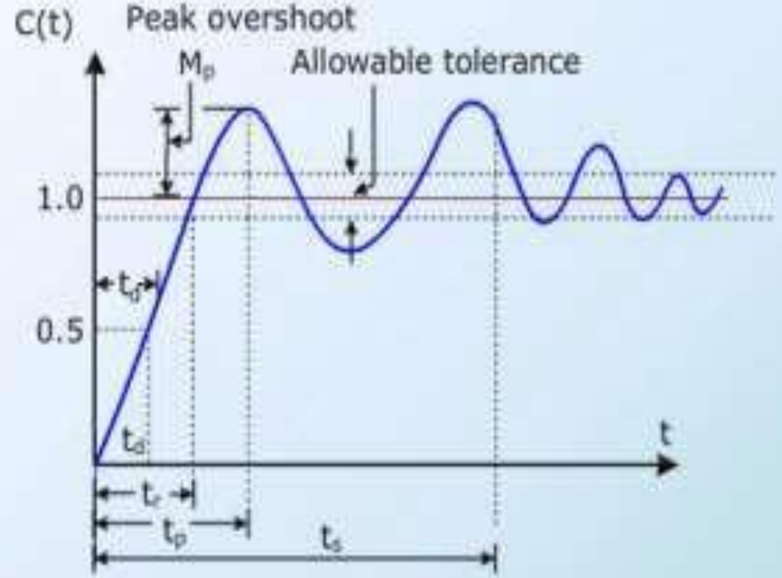
$$c(t) = \left(1 + \left(\frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\right)e^{-(\delta\omega_n + \omega_n\sqrt{\delta^2 - 1})t} - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\right)e^{-(\delta\omega_n - \omega_n\sqrt{\delta^2 - 1})t}\right)$$

Since it is over damped, the unit step response of the second order system when δ > 1 will never reach step input in the steady state.

Time domain specifications

Definitions of Specifications

- The following are the time domain specifications:
  - Delay Time t_d
  - Rise Time t_r
  - Peak Time t_p
  - Peak Overshoot or Max Overshoot M_p
  - Settling Time t_s
  - Steady-State Error



Time domain specifications

Time Response Specifications

✓ **Delay Time (t_d):**

It is time required for the response to reach 50% of the final value in the first attempt.

$$t_d = \frac{1 + 0.7\xi}{\omega_n}$$

Time Response Specifications

✓ **Rise Time (t_r):**

It is time required for the response to rise from 10% to 90% of the final value for overdamped systems.

(It is 0 to 100% for under damped systems)

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where, $$\beta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

and $$\omega_d = \omega_n\sqrt{1-\xi^2}$$

✓ Peak Overshoot (M_p):

The maximum overshoot is the maximum peak value of the response curve measured from unity. It is therefore largest error between input and output during the transient period.

$$\%M_p = e^{-\left\{\frac{\xi\pi}{\sqrt{1-\xi^2}}\right\}} \times 100$$

Time Response Specifications

✓ **Peak Time (t_p):**

It is the time required for the response to reach the first peak.

$$T_p = \frac{\pi}{\omega_d}$$

Time Response Specifications

- ✓ **Settling Time (t_s):**

It is the time required for the response curve to reach and stay within a specified percentage (usually 2% or 5%) of the final value.

$$T_s = 4T = \frac{4}{\xi \omega_n}$$