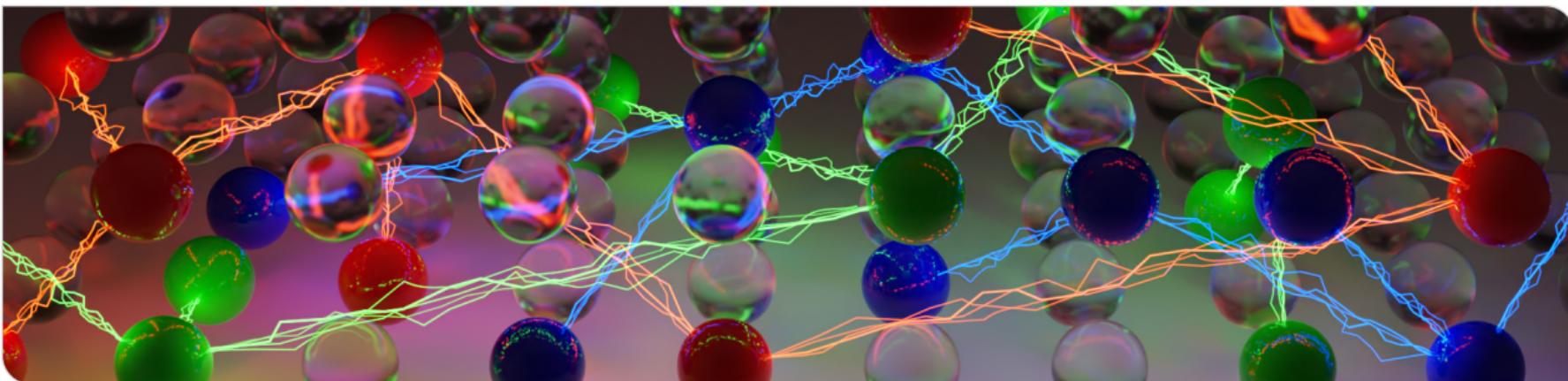


# Practical SAT Solving

## Lecture 6

Markus Iser, Dominik Schreiber, Tomáš Balyo | May 27, 2024



# Overview

Recap.

## Lecture 4: Classic Heuristics and Modern SAT Solving 1:

- Decision Heuristics, Restart Strategies, Phase Saving
- Modern SAT Solving 1: Conflict Analysis / Clause Learning

## Lecture 5: Parallel SAT Solving 1:

- To be continued on June 10, 2024: Parallel SAT Solving 2

Today's Topic: Modern SAT Solving 2

- Efficient Unit Propagation
- Clause Forgetting
- Modern Decision Heuristics
- Preprocessing

# Conflict-driven Clause Learning (CDCL) Algorithm

## Last Time

- Classic Decision Heuristics
- Restart Strategies
- Clause Learning
- Non-Chronological Backtracking

## Today

- Efficient Unit Propagation
- Clause Forgetting
- Modern Decision Heuristics
- Preprocessing

---

### Algorithm 1: CDCL(CNF Formula $F$ , &Assignment $A \leftarrow \emptyset$ )

---

```

1 if not PREPROCESSING then return UNSAT
2 while  $A$  is not complete do
3   UNIT PROPAGATION
4   if  $A$  falsifies a clause in  $F$  then
5     if decision level is 0 then return UNSAT
6     else
7       (clause, level)  $\leftarrow$  CONFLICT-ANALYSIS
8       add clause to  $F$  and backtrack to level
9       continue
10      if RESTART then backtrack to level 0
11      if CLEANUP then forget some learned clauses
12      BRANCHING
13 return SAT

```

---

# Unit Propagation

## Hot Paths in CDCL Solvers

heat	$\emptyset$ per sec. <sup>a</sup>	
Clause Access		Unpredictable memory access: most expensive
Iterate Occurrences		Predictable memory access: array of pointers (hardware prefetching)
<b>Propagation</b>	$\sim 10^6$	<b>Access occurrence-list of yet unpropagated literal</b>
Decision	$\sim 10^3$	
Conflict	$\sim 10^3$	<i>Learn a clause → more to check for propagation</i>
Restart	$\sim 10^{-1}$	
Cleanup		<i>Forget some learned clauses → less to check for propagation</i>

<sup>a</sup>Order of magnitude of average event count per second (in runs of Cadical on a large combined benchmark set)

# Unit Propagation

Example: Unit Propagation with Full Occurrence Lists

Trail			Occurrence Lists		Formula	
level	value	reason	idx.	occurrences	addr.	clause
1	a	$\perp$	a	*1	*1	$a \quad b \quad c$
			$\neg a$	*2 *3	*2	$\neg a \quad b \quad \neg c$
			b	*1 *2	*3	$\neg a \quad \neg b \quad c$
			$\neg b$	*3		
			c	*3 *1		
			$\neg c$	*2		

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level	value	reason
1	a	$\perp$

Occurrence Lists

idx.	occurrences
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b	*1 *2
$\neg b$	*3
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Formula

addr.	clause
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1	a	⊥	a	*1	*1	$a \quad b \quad c$
2	c	⊥	¬a	*2 *3	*2	$\neg a \quad b \quad \neg c$
			b	*1 *2	*3	$\neg a \quad \neg b \quad c$
			¬b	*3		
			c	*3 *1		
			¬c	*2		

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level	value	reason	idx.	occurrences	addr.	clause	
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2	c	⊥	¬a	*2 *3	*2	¬a	¬c
2	b	*2	b	*1 *2	*3	¬a	¬b
			¬b	*3			
			c	*3 *1			
			¬c	*2			

# Unit Propagation: Two Watched Literals

## Motivation: Hot Path

heat	$\emptyset$ per sec. <sup>a</sup>	<p><b>Idea:</b> Reduced occurrence tracking by only keeping the following <b>invariant:</b></p> <p><b>Each yet unsatisfied clause is watched by, i.e., in the occurrence list of, two of its unassigned literals.</b></p> <p><b>Reasoning:</b> less literals watched → shorter occurrence lists → less clause accesses → fast unit propagation</p>
Clause Access		
Iterate Occurrences		
Propagation	$\sim 10^6$	

<sup>a</sup>Order of magnitude of average event count per second (in runs of Cadical on a large combined benchmark set)

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Clause Access		
Iterate Occurrences		
Propagation	$\sim 10^6$	

<sup>a</sup>Order of magnitude of average event count per second (in runs of Cadical on a large combined benchmark set)

# Unit Propagation

Example: Unit Propagation with Two Watched Literals

Trail

level	value	reason

Two Watched Literals

idx.	occurrences
$a$	*1
$\neg a$	*2 *3
$b$	*1 *2
$\neg b$	*3
$c$	
$\neg c$	

Formula

addr.	clause
*1	$a \quad b \quad c$
*2	$\neg a \quad b \quad \neg c$
*3	$\neg a \quad \neg b \quad c$

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# Unit Propagation

Example: Unit Propagation with Two Watched Literals

Trail

level	value	reason
1	a	$\perp$
2	c	$\perp$
2	b	*2

Two Watched Literals

idx.	occurrences
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c	*3
$\neg c$	*2

Formula

addr.	clause
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*3	c $\neg b$ $\neg a$

# Unit Propagation: Two Watched Literals

## Two Watched Literals: Optimizations

heat	$\emptyset$ per sec. <sup>a</sup>	<b>Invariant:</b> Each yet unsatisfied clause is watched by two of its unassigned literals. $\rightarrow$ Reduced Load in Occurrence Tracking
Clause Access		<b>Optimization 1:</b> Keep watched literals the first two in clause $\rightarrow$ Alternative: Store watched literals in other location Note: What happens if clauses are kept in shared memory for parallel solving?
Iterate Occurrences		<b>Optimization 2:</b> Also keep a literal of each clause directly in occurrence list $\rightarrow$ Skip clause access if that literal is satisfied
Propagation	$\sim 10^6$	

<sup>a</sup>Order of magnitude of average event count per second (in runs of Cadical on a large combined benchmark set)

# Recap

## Unit Propagation

- Hottest path in CDCL solvers
- Two watched literals per clause suffice for unit propagation (and conflict detection)
- Other optimizations: keep watched literals first in clause, keep a literal of each clause directly in occurrence list

## Next Up

Clause Forgetting

# Clause Forgetting

## Motivation

Clause learning is most important pruning strategy in CDCL solvers.<sup>a</sup>

### Problem:

- Slows down unit propagation
- Risk of running out of memory

### Solution:

- Periodically forget some learned clauses
- Keep only “the best” learned clauses

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<sup>a</sup>“Empirical Study of the Anatomy of Modern Sat Solvers”, Katebi et al., 2013

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Clause learning is most important pruning strategy in CDCL solvers.<sup>a</sup>

### Problem:

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### Solution:

- Periodically forget some learned clauses
- Keep only “the best” learned clauses
- **How to figure out which learned clauses are “the best”?**

---

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# Clause Forgetting

## Periodic Clause Forgetting: Heuristics

- **Clause Size**

Keep short clauses

- **Least Recently Used (LRU)**

Keep clauses which were reasons in recent conflicts: clause activity (moving average)

- **Literal Block Distance (LBD)**

Keep clauses with a low number of decision levels<sup>a</sup>

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<sup>a</sup>Predicting Learnt Clauses Quality in Modern SAT Solvers, Audemard & Simon (IJCAI 2009)

# Forgetting Heuristic: Literal Block Distance (LBD)

“Impact of Community Structure on SAT Solver Performance”, Newsham et al., SAT 2014

Take home: LBD correlates with number of touched communities

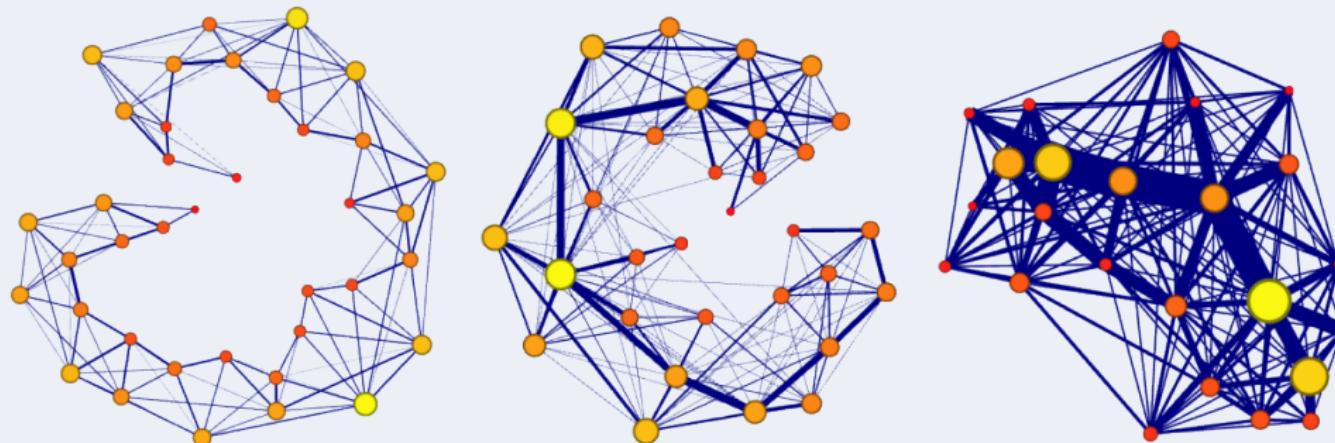


Image Source: “Community Structure in Industrial SAT Instances”, Ansotegui et al., AIJ 2019

# Clause Forgetting: Modern Hybrid Approach

Manage clauses differently in three tiers

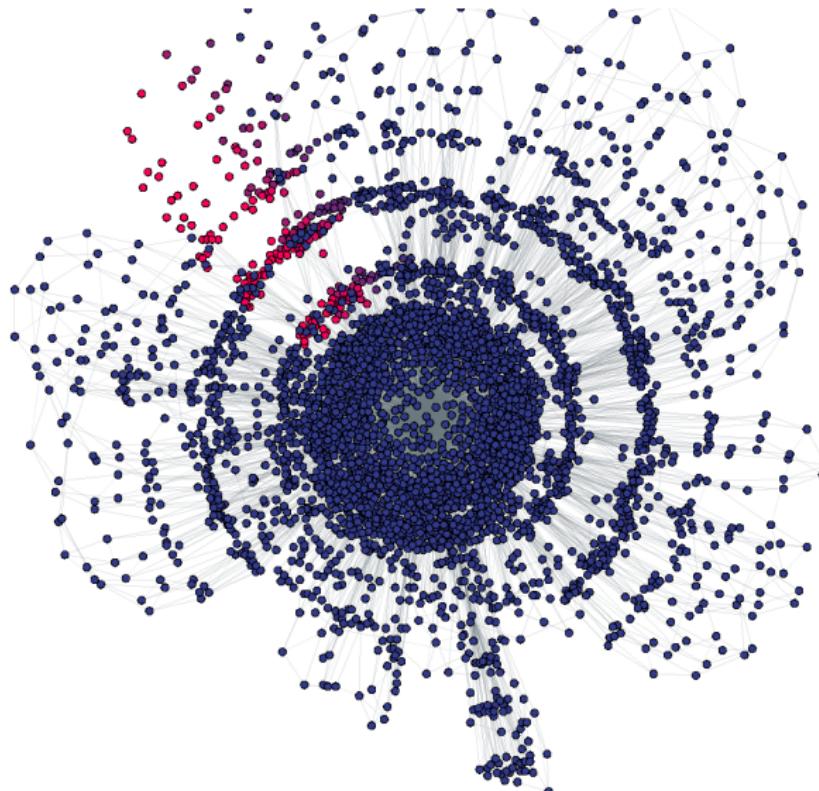
Tier	Strategy	Description
core	LBD	Permanently store clauses of $LBD \leq k$ (core-cut value, 3 in practice)
mid-tier	LRU	Clauses stay here if used in recent conflicts
local	LRU	Keep fixed number of clauses (say 5000) of highest activity

## History

- core and local tier introduced in SWDiA5BY (Chanseok Oh, 2014)
- mid-tier introduced in CoMinisatPS (Chanseok Oh, 2015)
- “Between SAT and UNSAT: The Fundamental Difference in CDCL SAT” (Chanseok Oh, 2015)
- Note: MapleCOMSPS (2016) is a CoMinisatPS fork

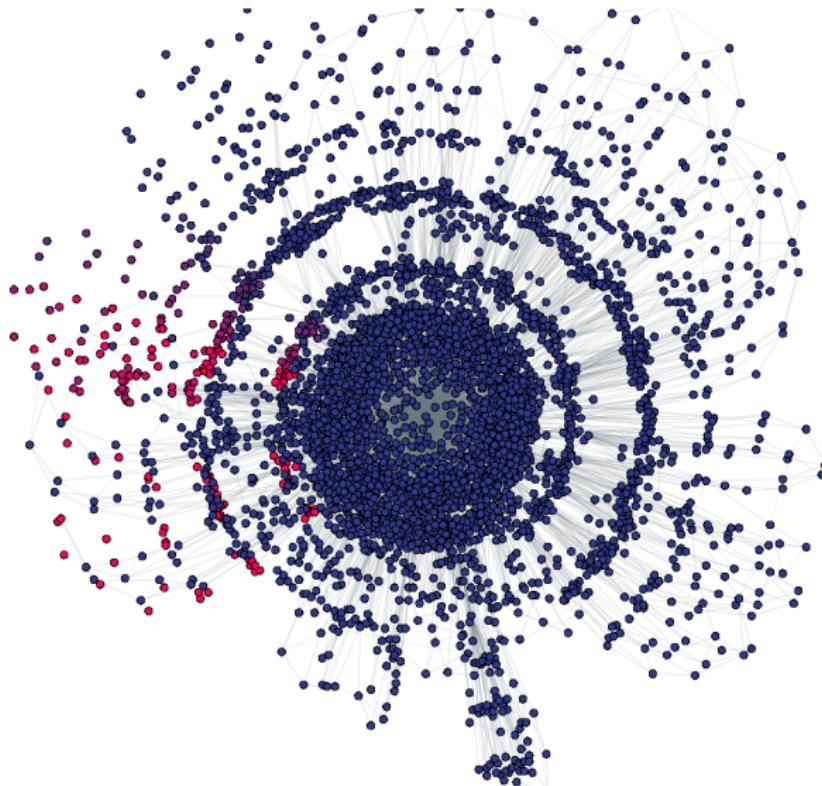
# Visualized Instance: Aprove (Termination Analysis, SAT)

initial layout, recently active variables after 1000 conflicts



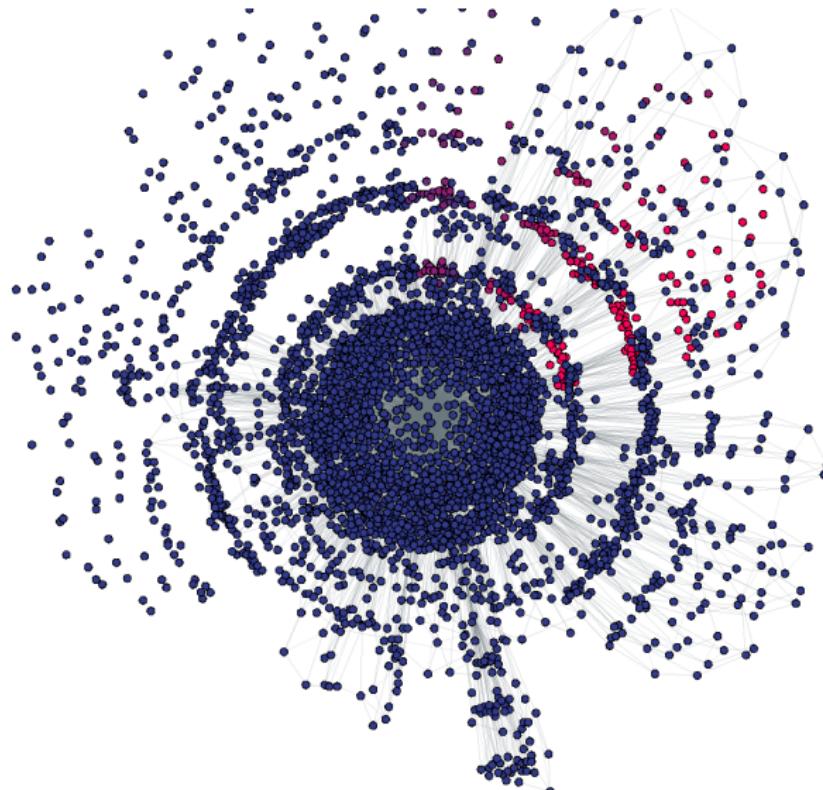
# Visualized Instance: Aprove (Termination Analysis, SAT)

initial layout, recently active variables after 1690 conflicts



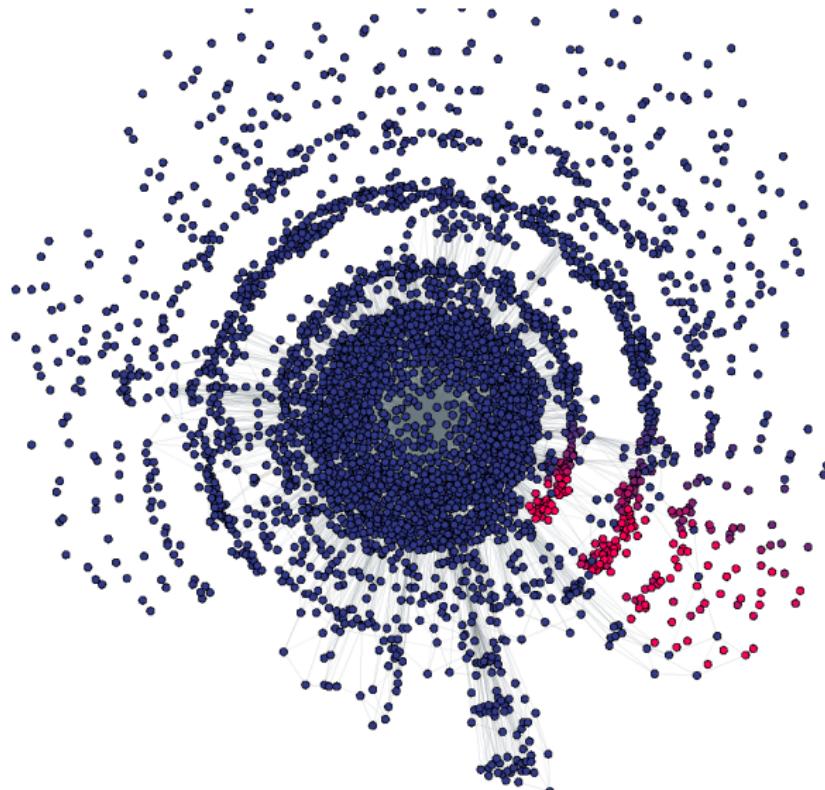
# Visualized Instance: Aprove (Termination Analysis, SAT)

initial layout, recently active variables after 3090 conflicts

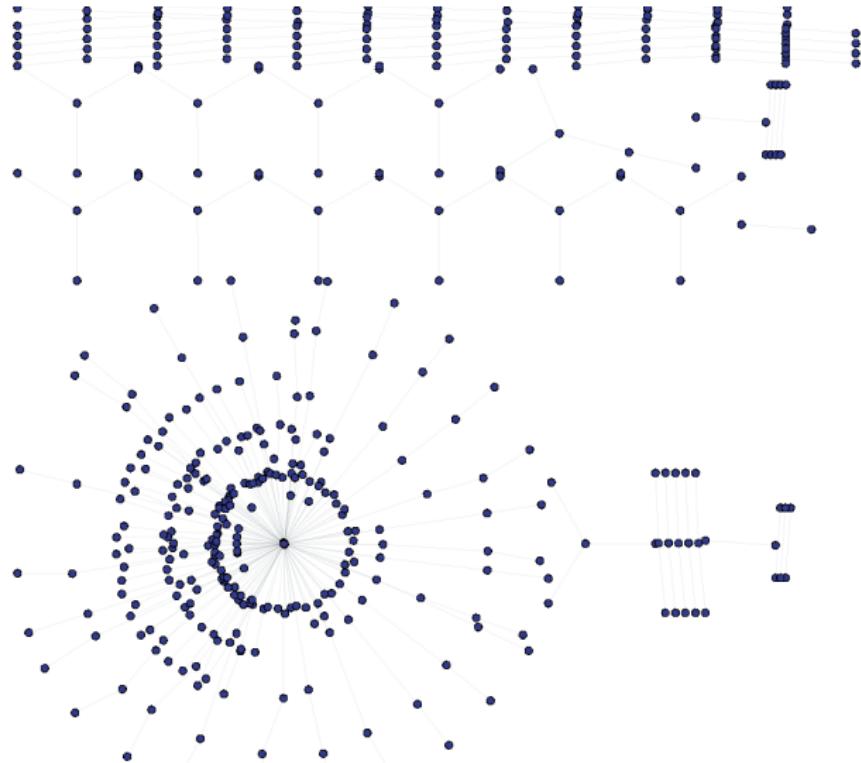


# Visualized Instance: Aprove (Termination Analysis, SAT)

initial layout, recently active variables after 5000 conflicts

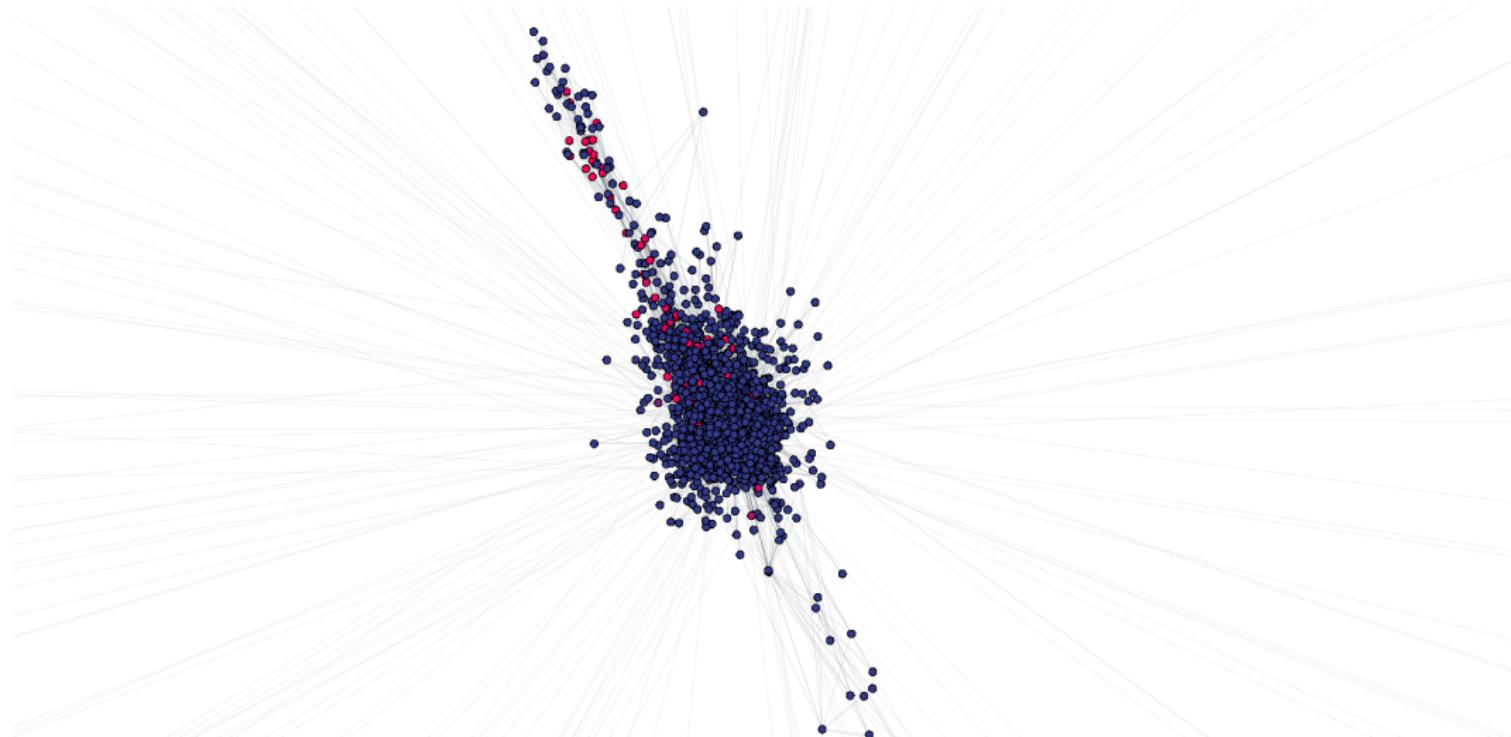


# Visualized Instance: Aprove (Termination Analysis, SAT) relayout after 6000 conflicts



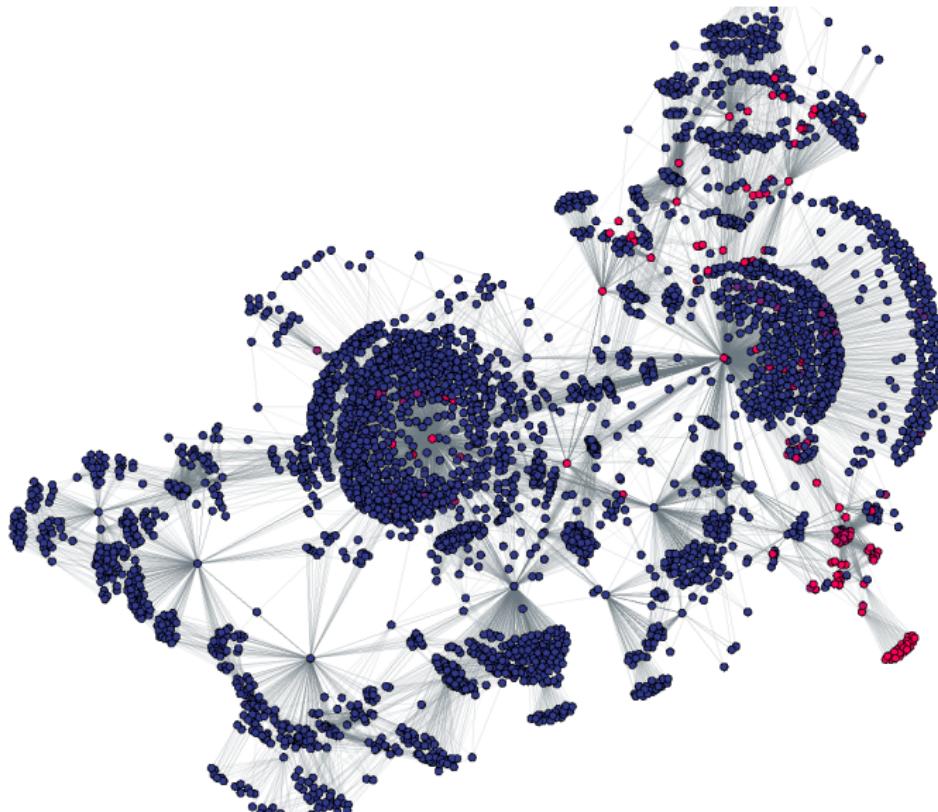
# Visualized Instance: Aprove (Termination Analysis, SAT)

core after 52500 conflicts

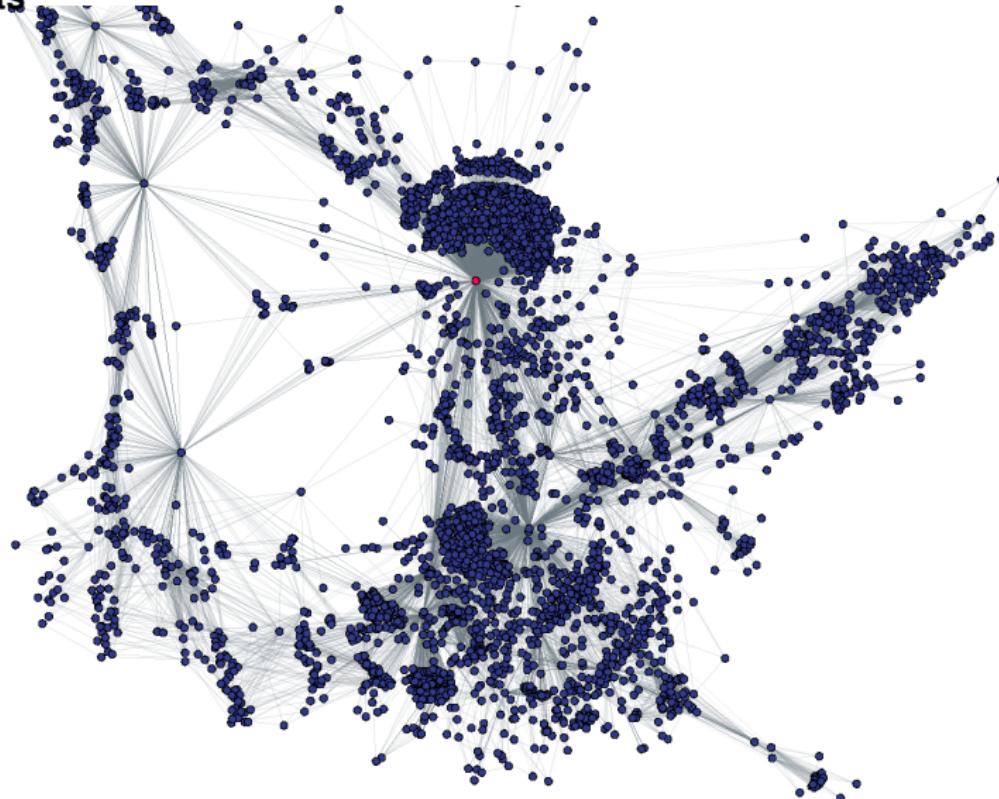


# Visualized Instance: Newton SMT (SV Competition, SAT)

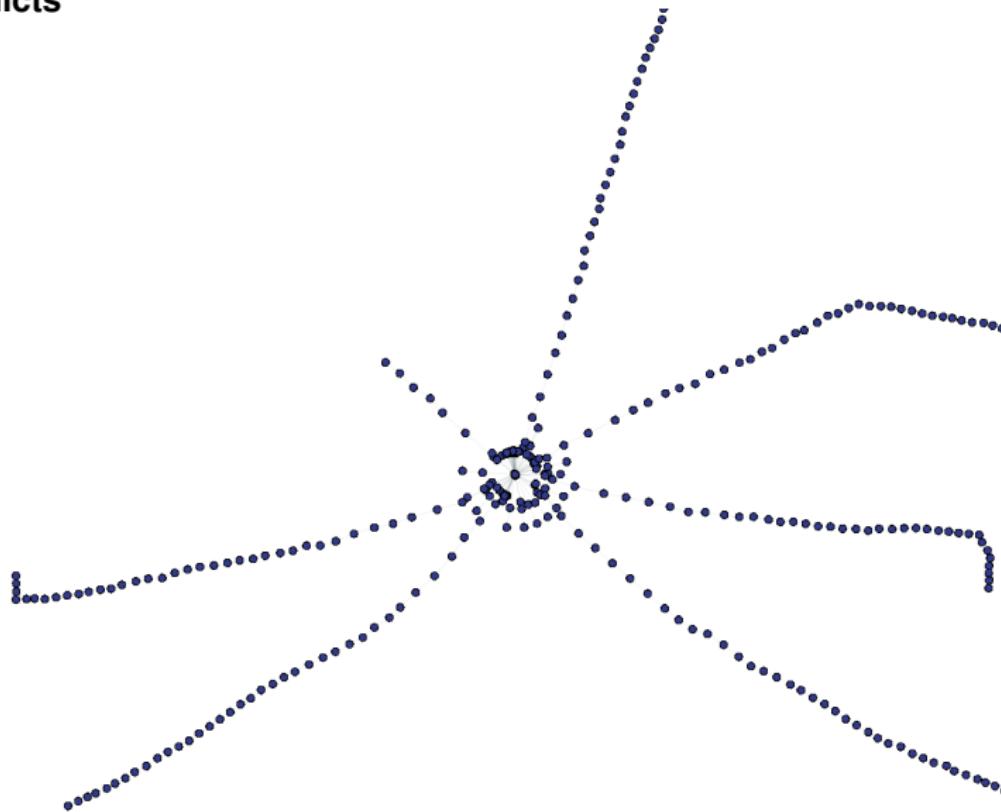
initial layout



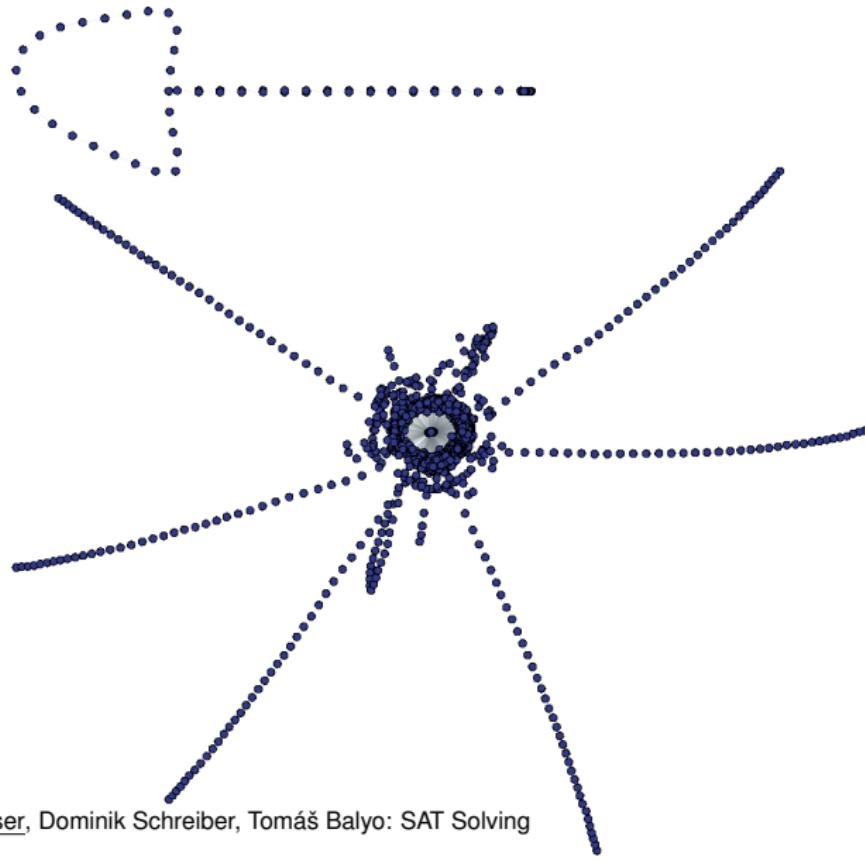
# Visualized Instance: Newton SMT (SV Competition, SAT) after 10000 conflicts



# Visualized Instance: Newton SMT (SV Competition, SAT) after 1000000 conflicts

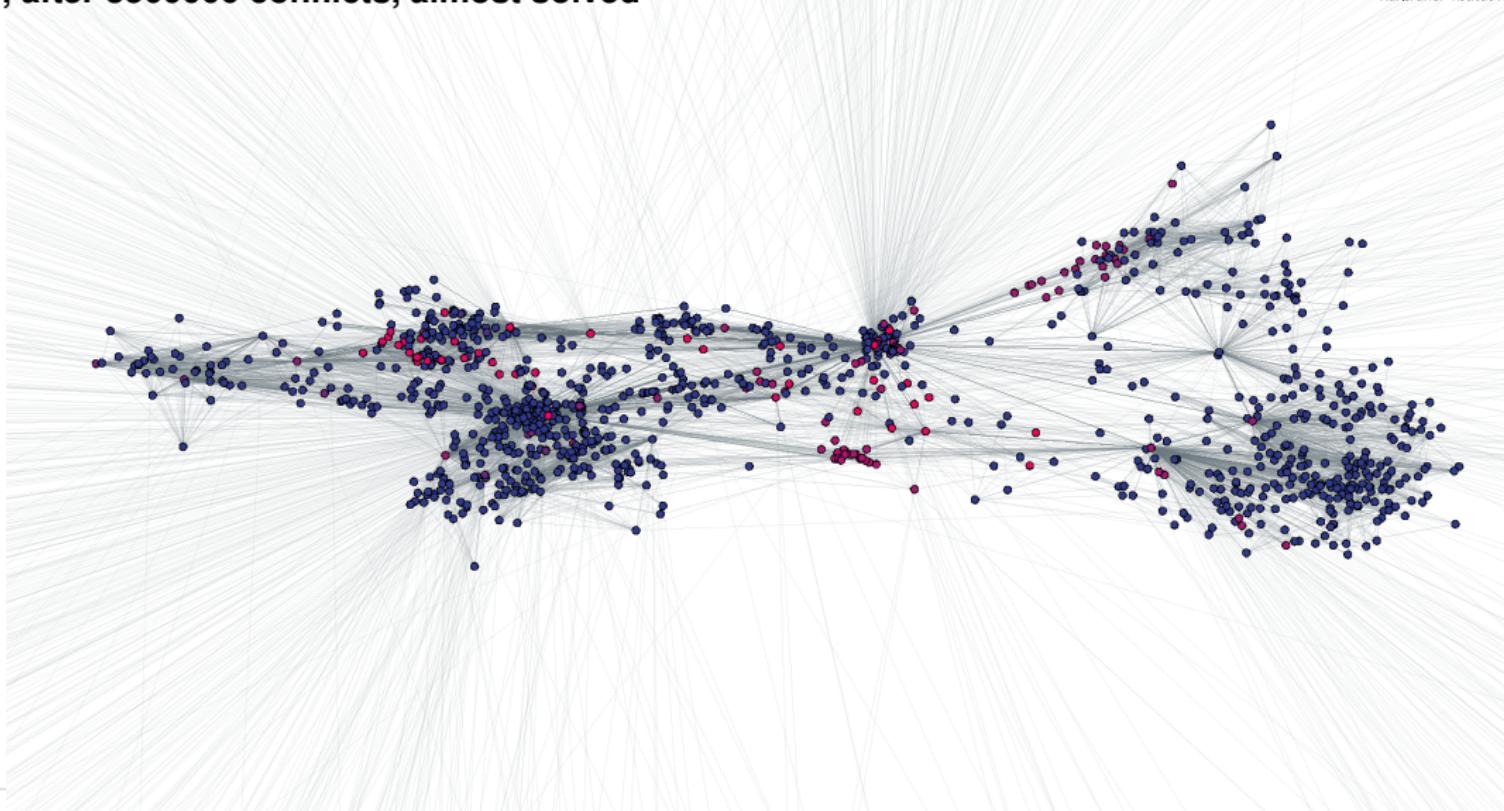


# Visualized Instance: Newton SMT (SV Competition, SAT) after 3000000 conflicts



# Visualized Instance: Newton SMT (SV Competition, SAT)

core, after 3500000 conflicts, almost solved



# Recap

## So far

- Efficient Unit Propagation
- Clause Forgetting Heuristics:
  - Size, LRU, LBD
  - Three-Tier Clause Management

## Next Up

Modern Decision Heuristics

# Variable State Independent Decaying Sum (VSIDS)

## VSIDS Heuristic

Implemented in most CDCL solvers. First presented in SAT solver Chaff.<sup>a</sup>

Always select variable with highest score for branching. Scores are updated after each conflict.

- Initialize variable score (with zero or use some static heuristic)
- New conflict clause  $c$ : score is incremented for all variables in  $c$
- Periodically, divide all scores by a constant

---

<sup>a</sup>Chaff: Engineering an efficient SAT solver (Moskewicz et al., 2001)

# Variable State Independent Decaying Sum (VSIDS)

## Example: Score Update after Conflict

### Formula:

$$\begin{aligned} & \{x_1, x_4\}, \{x_1, \overline{x}_3, \overline{x}_8\}, \{x_1, x_8, x_{12}\}, \{x_2, x_{11}\}, \\ & \{\overline{x}_7, \overline{x}_3, x_9\}, \{\overline{x}_7, x_8, \overline{x}_9\}, \{x_7, x_8, \overline{x}_{10}\} \\ & \{x_7, x_{10}, \overline{x}_{12}\} \end{aligned}$$

(new learned clause)

### Scores before:

4 : $x_8$	
3 : $x_1, x_7$	
2 : $x_3$	
1 : $x_2, x_4, x_9, x_{10}, x_{11}, x_{12}$	

### Scores after:

4 : $x_8, \cancel{x}_7$	
3 : $x_1$	
2 : $x_3, \cancel{x}_{10}, x_{12}$	
1 : $x_2, x_4, x_9, x_{11}$	

# Variable State Independent Decaying Sum (VSIDS)

## Example: Score Update after Conflict

Formula:	Scores before:	Scores after:
$\{x_1, x_4\}, \{x_1, \overline{x_3}, \overline{x_8}\}, \{x_1, x_8, x_{12}\}, \{x_2, x_{11}\},$	4 : $x_8$	4 : $x_8, \textcolor{red}{x_7}$
$\{\overline{x_7}, \overline{x_3}, x_9\}, \{\overline{x_7}, x_8, \overline{x_9}\}, \{x_7, x_8, \overline{x_{10}}\}$	3 : $x_1, x_7$	3 : $x_1$
$\{x_7, x_{10}, \overline{x_{12}}\}$	(new learned clause)	2 : $x_3, \textcolor{red}{x_{10}}, \textcolor{red}{x_{12}}$
		1 : $x_2, x_4, x_9, x_{10}, x_{11}, x_{12}$
		1 : $x_2, x_4, x_9, x_{11}$

- VSIDS leads to more “focused” search
- prefers variables that occurred in recent conflicts
- tends to find smaller unsatisfiable subsets

# Variable State Independent Decaying Sum (VSIDS)

Keep list of variables sorted by scores

Common implementation: Binary Heap

Heap Operation	Complexity	Callee
insert_with_priority	$\mathcal{O}(\log n)$	Backtracking
pull_highest_priority_element	$\mathcal{O}(\log n)$	Branching
increase_key / bump_variable	$\mathcal{O}(\log n)$	Conflict Analysis
decay	$\mathcal{O}(n)$	[Periodic] <sup>a</sup>

<sup>a</sup>Periodically divide scores to give priority to recently learned clauses

# Variable State Independent Decaying Sum (VSIDS)

## VSIDS Variants

### Chaff (2001)

- decay: half scores every 256 conflicts
- sort priority queue after each decay only

### Berkmin (2002)

- bump all literals in implication graph
- divide scores by 4

### Minisat (2003)

- Exponential VSIDS (EVSIDS)
- Reason-side Bumping

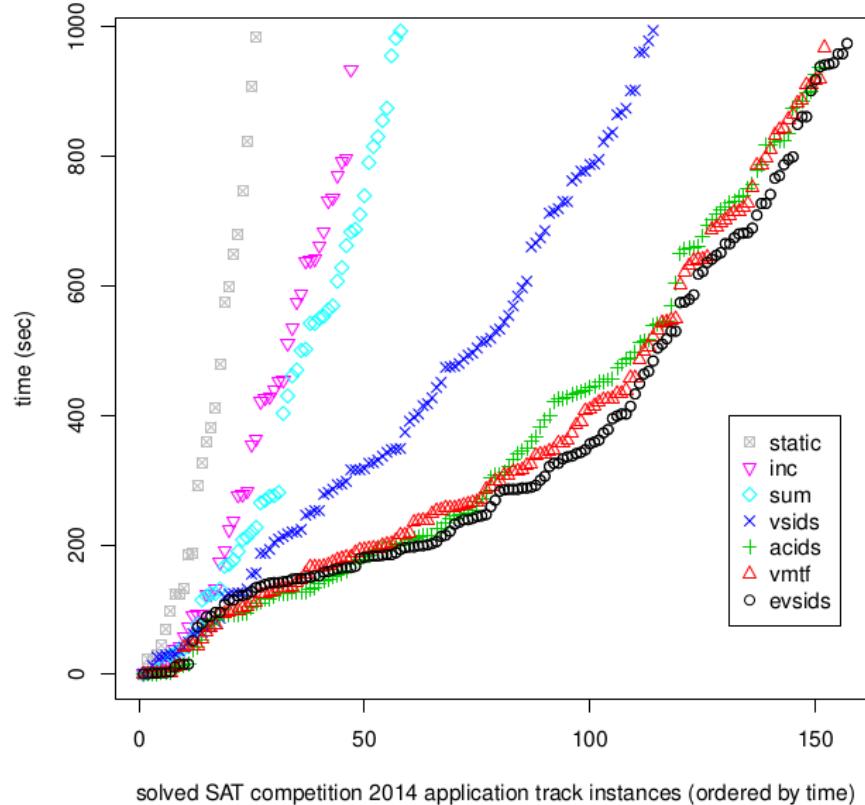
## Alternatives

**Siege (2004): Variable Move To Front (VMTF)**

**HaifaSAT (2008): Clause Move To Front (CMTF)**

# “Evaluating CDCL Variable Scoring Schemes”

Biere & Fröhlich, 2015



# Recent Hybrid Approaches

## Hybrid Approaches

- **Warmup Phase:**

- MapleCOMSPS (2016): use Learning Rate-based Branching (LRB) in *initial* period, then switch to VSIDS
- Maple\_LCM\_Dist (2017): use Distance Heuristic (Dist.) in *initial* period, then switch to VSIDS

- **Reinforcement Learning:** Kissat\_MAB (2021)

- Two-armed Bandit switches between VSIDS and Conflict History-Based (CHB) Heuristic
- Reward function favors variables that contribute to learning “good” clauses

# Recap

## So far

- Unit Propagation
- Clause Forgetting
- Modern Branching Heuristics
  - Mostly VSIDS
  - Hybrid approaches: warmup VSIDS scores, reinforcement learning

## Next Up

### Preprocessing

# Preprocessing

**Conjecture:** Smaller problems are easier to solve

⇒ Try to reduce the size of the input formula by (polynomial time) simplification procedures.

## Today: Basic Preprocessing

- Subsumption
- Self-subsuming Resolution
- (Bounded) Variable Elimination (BVE)
- Blocked Clause Elimination (BCE)

# Subsumption

A clause  $C$  is subsumed by  $D$  iff  $D \subseteq C$ .

Subsumed clauses can be removed from the formula without changing satisfiability.

$$\forall D \subseteq C, D \models C$$

## Example

$\{a, b\}$  subsumes  $\{a, b, c\}$  and  $\{a, b, d\}$

# Self-Subsuming Resolution

Applicable if the resolvent of  $C$  and another clause  $D$  subsumes  $C$ .

Let  $C, D$  be clauses and  $\otimes_x$  the resolution operator on variable  $x$ .

If  $C \otimes_x D \subseteq C$  then  $C$  can be replaced by  $C \otimes_x D$ .

## Example

$$C := \{\neg b, \neg e, f, \neg h\} \quad D := \{\neg b, \neg e, \neg f\} \quad E := C \otimes_f D = \{\neg b, \neg e, \neg h\}$$

→ Replace  $C$  by  $E$

## Bounded Variable Elimination (BVE)

Let  $S_x, S_{\bar{x}} \subset F$  be the sets of all clauses containing  $x$  resp.  $\bar{x}$ , and let  $R = \{C \otimes_x D \mid C \in S_x, D \in S_{\bar{x}}\}$  be the set of all resolvents on  $x$ .

**Variable Elimination:** Modify  $F$  such that  $F' := (F \setminus (S_x \cup S_{\bar{x}})) \cup R$

The formulas  $F$  and  $F'$  are **satisfiability equivalent** (why not equivalent?)

**Bounded Variable Elimination:** Eliminate variable only if **size of formula** decreases.

Most important simplification technique today

# Blocked Clause Elimination (BCE)

A clause  $\{l\} \cup C$  is blocked in  $F$  by  $l$  if

- either  $l$  is *pure* in  $F$  or
- for every clause  $\{\neg l\} \cup D$  in  $F$  the resolvent  $C \cup D$  is a *tautology*.

Removal of an arbitrary blocked clause **preserves satisfiability**.

Blocked clause elimination (BCE) has a unique fixpoint.

## Example

$$F := (a \vee b) \wedge (a \vee \neg b \vee \neg c) \wedge (\neg a \vee c)$$

First clause is not blocked, second is blocked by both  $a$  and  $\neg c$ , third is blocked by  $c$ .

# Recap

## Today

- Unit Propagation
- Clause Forgetting
- Modern Branching Heuristics
- Preprocessing
  - Subsumption
  - Self-Subsuming Resolution
  - Bounded Variable Elimination
  - Blocked Clause Elimination