Information Theoretic Alternative to Signal Generation with an Application to Principled Probability Extremization

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Abstract

Typically randomness in scientific estimation is assumed to arise from unmeasured or uncontrolled factors. This paper presents an information theoretic framework that introduces an alternative source of randomness. Our framework, which is particularly appropriate for subjective response data, is illustrated on probability aggregation. The probabilities are given by a group of experts who forecast whether an event will occur or not. Our aggregator uses the distribution of information among the experts and depends on easily interpretable parameters. Even though shifting the average probability forecasts closer to its nearest extreme, known as *extremizing*, is known to yield improved forecast performance, it is much less understood when and how much the average forecast should be extremized. By assuming a simplified information structure in our model, the amount of extremizing is studied closely under different values of the parameters. This leads to novel observations and a more principled understanding of extremization.

1 Introduction

Experts are often required to form estimates under incomplete information. These estimates can be made in several ways. Currently the two main approaches classify the estimate either as *generated* or *interpreted* (Hong and Page [2009]). The estimate is called generated if the expert is assumed to sample the estimate from a given probability distribution. For instance, estimating the length of an object with a ruler can be assumed to be equivalent to drawing a value from a symmetric distribution, such as the Gaussian distribution, centered at the true length. An estimate is said to be interpreted if the expert first filters reality into set of categories and then makes an estimate by applying active cognitive effort to these categories. For instance, the score given by an Olympic judge on a figure skating performance can be considered interpreted. The judge has his personal set of criteria (categories) that ultimately decides the score.

The interpreted framework is clearly a more realistic model for subjective response data. Given that such data is very common in real-world applications, such as product reviews, online auctions, and voting, this framework is a great advance in understanding the gap between experimental and real-world results. Unfortunately, it is too general to be useful in developing novel methodology. First, interpretations differ between individuals and arise from a complex cognitive process that can be very difficult to estimate. Second, it is unclear how to construct the model to incorporate all the information that is used in making these interpretations.

The first contribution of this paper is to introduce a novel framework for signal generation. This framework, that is called *information theoretic*, is based on the distribution of information among the experts. Each expert gives an estimate solely based on his personal information set. The information set includes any internal factors, such as personal preferences and interpretation criteria, and external factors, such as knowledge of the target event, that affect the estimation process. Therefore two overlapping information sets are assumed to produce positively correlated estimates. This framework, which is a good compromise between the psychological and analytical models, is much more amenable for development of future methodology.

The second contribution of this paper is to illustrate the information theoretic framework on probability aggregation. Combining multiple probability forecasts is an important problem with many applications including medical diagnosis (Pepe [2003], Wilson et al. [1998]), political and socio-economic foresight (Tetlock [2005]), and meteorology (Baars and Mass [2005], Sanders [1963], Vislocky and Fritsch [1995]). There is strong empirical evidence that bringing together the strengths of different experts by combining their probability forecasts into a single consensus, known as the *crowd belief*, improves predictive performance. Recent developments suggest that shifting the average probability closer to its nearest extreme (0.0 or 1.0), known as *extremizing*, yields even further improvements in forecasting performance. For instance, Satopää et al. [2014] uses a linear regression model in the logit-space to derive an extremizing aggregator that performs well on real-world data. Ranjan and Gneiting [2010] propose transforming the average probability with the CDF of a beta distribution. If both the shape and scale of this beta distribution are equal and constrained to be at least 1.0, the aggregator extremizes and has some attractive theoretical properties (Wallsten and Diederich [2001]). Baron et al. [2013] provide two intuitive justifications for extremizing.

Many of the current extremizing aggregators, however, are overly simplistic or too detached from the psychology literature to provide the researcher any insight beyond the aggregate probability. It is mainly for this reason that it is still not well-understood when and how much the average probability should be extremized. Therefore it is necessary to learn the amount of extremizing from a separate training dataset. Furthermore, many of these aggregators assume that the individual probability forecasts are generated estimates. As was illustrated earlier, it is unlikely that this framework is appropriate for probability aggregation.

These shortcomings can be remedied by developing an aggregator based on our information theoretic framework. Under this model the average forecast is always extremized, and the amount of extremization is available in a closed-form. Given that this form depends on three intuitive parameters (the number of experts, the amount of information known by an expert, and the amount of information shared by two experts), it allows us to investigate when and how much extremization should be performed.

This paper is structured as follows. The first section introduces our information theoretic framework and compares it with the generated and interpreted frameworks. The framework is then illustrated on probability aggregation. After deriving a closed-form expression for the amount of extremization, this form is analyzed in full generality and under a compound symmetric information structure. This simplified structure allows us to discuss and understand extremization in terms of a few intuitive parameters. The paper concludes with a discussion of model limitations and future directions.

2   Information Theoretic Framework

This section illustrates some of the similarities and differences between our information theoretic framework and the classical data generative process from regression analysis. This comparison is by no means comprehensive as regression analysis encompasses a very vast literature in statistics. The main difference is in the way the two frameworks incorporate randomness.
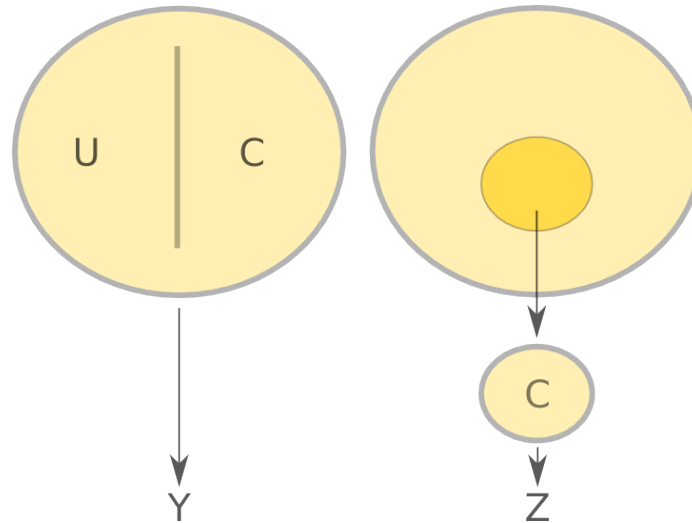

Figure 1: Comparison of the physical and information theoretic frameworks.

To guide our comparison, Figure 1 illustrates both setups with the regression framework on the left and the information theoretic framework on the right. The two large circles represent the set of relevant factors, or simply the universe. This is the same under both frameworks. Consider first the

regression framework on the left side. From the experimenter's point of view, the universe is divided into controlled (C) and uncontrolled factors (U) that affect his measurement (Y). The uncontrolled factors are assumed to have a random effect, typically with mean of zero, on the measurement. Therefore any repeated measurement under the same set of controlled factors leads to a slightly different answer.

The right side of Figure 1 depicts our information theoretic framework. Under this framework the data are generated by first picking a random subset of factors from the universe. This subset is then assumed to form its own sub-universe, where all factors are controlled. Repeating the measurement in this sub-universe always leads to the same answer (Z). Therefore, while in the regression framework any differences between two measurements is assumed to arise from the uncontrolled factors, these differences in the information theoretic framework arise due to different sets of sub-universes. In both cases the differences are random.

The need for our information theoretic framework is becomes clear by considering subjective responses. For instance, the response variable in the study may be the subject's personal belief about the closing price of a particular capital stock. The person in this case forms the sub-universe depicted by the small circle in Figure 1. He holds some subset of the information that make up the universe. He uses his information efficiently to produce an answer that is his *personal truth*. There is no noise in his answer. Therefore, if he is asked the same question immediately after, he gives the same response. If, on the other hand, he is asked the same question a day later, he may give a different answer because his information set may have changed by then.

It is unclear how subjective responses can arise from the regression framework because under this setup exact measurements are not possible. Furthermore, repeated measurements cannot be identical. Instead they differ by a random amount that is often assumed to have a mean of zero. Therefore averaging can help to smooth the data and reveal a better picture of the underlying truth. But as has been shown in previous literature, mere averaging does not lead to optimal probability aggregation. The average must be extremized. Therefore the regression context does not appear suitable for analyzing probability forecasts. The next section introduces a information theoretic model for analyzing probability forecasts.

3   Model for Probability Forecasts

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event $A \in \mathcal{F}$ to be forecasted. Expert j knows information $\mathcal{F}_j \subseteq \mathcal{F}$ and forecasts $p_j = \mathbb{P}(A|\mathcal{F}_j)$. The best in-principle forecast given the knowledge of the N forecasters is $P(A|\mathcal{F}')$, where $\mathcal{F}' = \mathcal{F}_1 \cup \mathcal{F}_2 \cup .. \cup \mathcal{F}_N$. As in this work we do not allow the experts to exchange information with each other, the best aggregate probability is given by $\mathbb{P}(A|p_1, p_2, \ldots, p_N)$.

For illustrative purposes, we first assume only two experts and then generalize the model to

N experts. Under our model the event A is determined by a pool of white noise. Experts 1 and 2 see respective δ_1 and δ_2 portions of the noise. These portions form their information sets. The overlap in their information sets is a fixed share ρ of what is seen by either expert. This can be made more precise by letting $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we define a white noise process indexed by the unit interval S. A white noise process is a Gaussian process $\{X_B\}$ indexed by Borel measurable subsets B. We endow the unit interval with the uniform measure, μ. This gives the white noise process a covariance structure of $Cov(X_B, X_{B'}) = \mu(B \cap B') = |B \cap B'|$, i.e. the length of the intersection. The target event is defined as $A = \{X_S > 0\}$. Let $I_1, I_2 \subseteq S$ be the information sets observed by experts 1 and 2, respectively. Thus,

$$\mu(I_1) = |I_1| = \delta_1$$
$$\mu(I_2) = |I_2| = \delta_2$$
$$\mu(I_1 \cap I_2) = |I_1 \cap I_2| = \rho$$

Call X_{I_j} the probit forecast of the jth expert. If Φ denotes the standard normal CDF, then

$$p_j = \mathbb{P}(A|\mathcal{F}_{I_j}) = \Phi(X_{I_j})$$

for $j = 1, 2$, and the aggregator is given by $\mathbb{P}(X_S > 0 | p_1, p_2)$.
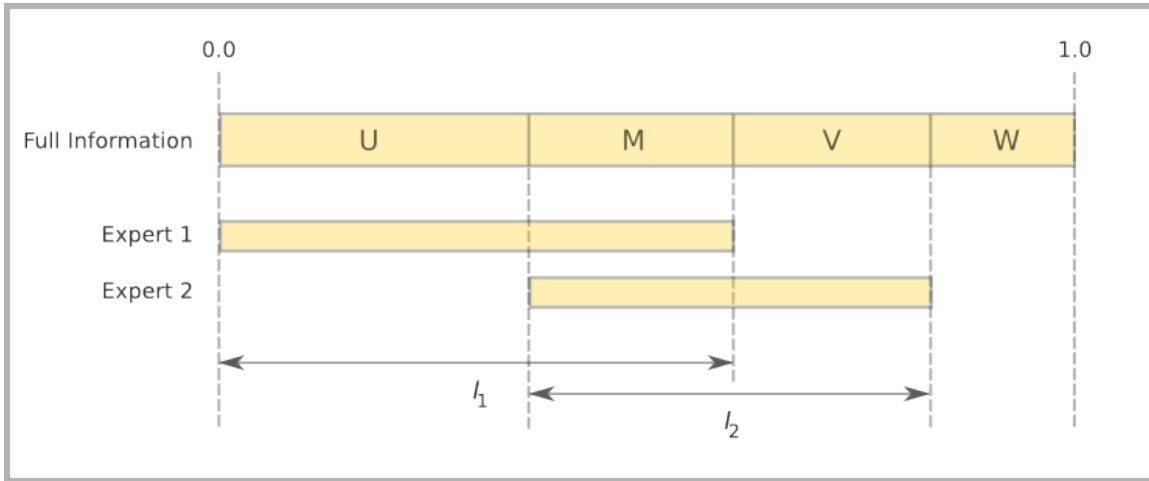

Figure 2: Illustration of the model with two experts.

Figure 2 illustrates the model with $N = 2$. The Gaussian process has been partitioned into four parts based on the information sets I_1 and I_2:

$$U = X_{I_1/I_2}$$
$$M = X_{I_1 \cap I_2}$$
$$V = X_{I_2/I_1}$$

$$W = X_{(I_1 \cup I_2)^c}$$

Then $X_{I_1} = U + M$, $X_{I_2} = M + V$, and $X_S = U + M + V + W$, where U, V, M, W are independent Gaussians with respective variances $\delta_1 - \rho$, $\delta_2 - \rho$, ρ, $1 + \rho - \delta_2 - \delta_3$. This gives (X_S, X_{I_1}, X_{I_2}) a multivariate normal distribution. More specifically, we have

$$\begin{pmatrix} X_S \\ X_{I_1} \\ X_{I_2} \end{pmatrix} \sim \mathcal{N} \left(\mathbf{0}, \begin{pmatrix} 1 & \delta_1 & \delta_2 \\ \delta_1 & \delta_1 & \rho \\ \delta_2 & \rho & \delta_2 \end{pmatrix} \right) \tag{1}$$

Figure 3: Illustration of the model with N experts.

Consider now N experts. Let $|I_j| = \delta_j$ be the amount of information known by the jth expert, and $|I_i \cap I_j| = \rho_{ij}$ be the information overlap between the ith and jth experts. Then expression (1) generalizes to the vector $(X_S, X_{I_1}, X_{I_2}, \ldots, X_{I_N})$. This gives us

$$\begin{pmatrix} X_S \\ X_{I_1} \\ \vdots \\ X_{I_N} \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mathbf{0}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & \delta_1 & \delta_2 & \cdots & \delta_N \\ \hline \delta_1 & \delta_1 & \rho_{1,2} & \cdots & \rho_{1,N} \\ \delta_2 & \rho_{2,1} & \delta_2 & \cdots & \rho_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_N & \rho_{N,1} & \rho_{N,2} & \cdots & \delta_N \end{array} \right) \right)$$

This is illustrated in Figure 3. It is important to notice that the I_j does not have to be contiguous segment of the unit interval. The sub-matrix Σ_{22} fully describes the information structure among the experts. This matrix has some technical conditions such as symmetry and non-singularity. In addition, Σ_{22} must describe a coherent information structure. The matrix Σ_{22} is coherent if and only if its information can be transformed into a diagram such as the one depicted by Figure 3.

4   Extremization

Let $\boldsymbol{X}$ be a column vector of length N such that $X_j = X_{I_j}$. If Σ_{22} is a coherent overlap structure such that Σ_{22}^{-1} exists, then $X_S|\boldsymbol{X} \sim \mathcal{N}(\bar{\mu}, \bar{\Sigma})$, where

$$\bar{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}_2) = \Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{X} \tag{2}$$

and

$$\bar{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{3}$$

See Result 5.2.10 on p. 156 in Ravishanker and Dey [2001] for the formulas of a conditional multivariate normal distribution. The aggregator then becomes

$$\mathbb{P}\left(A\Big|\boldsymbol{X}\right) = \mathbb{P}\left(X_S > 0\Big|\boldsymbol{X}\right) = \Phi\left(\frac{\Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{X}}{\sqrt{1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}}\right)$$

Let α represents the amount of extremization that is performed for the average probit forecast. If $\bar{X}$ denotes the sample average, then

$$\alpha\bar{X} = \frac{\Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{X}}{\sqrt{1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}} \qquad \Leftrightarrow \qquad \alpha = \frac{N\Sigma_{12}\Sigma_{22}^{-1}\boldsymbol{X}}{(\mathbf{1}_N'\boldsymbol{X})\sqrt{1 - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}} \tag{4}$$

Even though this expression assumes no structure on Σ_{22} and depends on $N + \frac{N(N-1)}{2}$ unknown parameters, it can be used to gain insight on the behavior of the extremizing constant, α. One useful development is to determine the amount of information in $\boldsymbol{X}$.

Lemma 4.1. *If δ_0 denotes the information in $\boldsymbol{X}$, then*

$$\alpha = \frac{1}{\sqrt{1 - \delta_0}} \qquad \Leftrightarrow \qquad \delta_0 = 1 - \alpha^{-2}$$

Proof. Let α_N denote the extremizing constant for $\boldsymbol{X}$. Consider a single expert whose probit forecast is $\bar{X}$. Denote the size of his information set by δ_0. The extremizing constant for his forecast, as is given by (4), simplifies to

$$\alpha_1 = \frac{1}{\sqrt{1 - \delta_0}}$$

Setting $\alpha_1 = \alpha_N$ gives us the final result.

$\square$

Based on Lemma 4.1 there is a monotonic and positive relationship between α and δ_0. This means that the more the sample average is extremized the more information its corresponding $\boldsymbol{X}$

contains, and *vice versa*. Lemma 4.1 is interesting for two reasons: (a) it allows the researcher to use black-box models from existing literature to determine the extremizing constant and then use it to analyze the amount of information in X, and (b) it allows us to easily show that $\alpha \geq 1$ under any information structure.

Theorem 4.2. *Under the model described in Section 3, the extremizing factor, α, is always greater or equal to 1. This means that the average probit forecast is always extremized.*

Proof. By Lemma 4.1 we have that

$$\alpha = \frac{1}{\sqrt{1 - \delta_0}}$$

Given that $\delta_0 \in [0, 1]$, it follows that $\alpha \in [1, \infty)$.

$\square$

To continue our analysis of extremization, it is necessary to reduce the number of degrees of freedom by assuming a simpler form for the overlap structure Σ_{22}.

4.1   Compound Symmetric Information Structure

In this section we assume that any two experts know and share the same amount of information. This gives us the compound symmetric overlap structure.

$$
\begin{pmatrix} S \\ X_1 \\ \vdots \\ X_N \end{pmatrix} \sim \mathcal{N} \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \mathbf{0}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = \left(\begin{array}{c|cccc} 1 & \delta & \delta & \dots & \delta \\ \hline \delta & \delta & \rho\delta & \dots & \rho\delta \\ \delta & \rho\delta & \delta & \dots & \rho\delta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta & \rho\delta & \rho\delta & \dots & \delta \end{array} \right) \right),
$$

where $\delta \in [0, 1]$ and $\rho \in \left[\max\left\{ \frac{N - \delta^{-1}}{N - 1}, 0 \right\}, 1 \right]$. The lower bound for ρ becomes necessary when $\delta > 1/N$ because then overlap is unavoidable. This minimum can be computed by assuming $\delta > 1/N$ and letting the shared information be the same for all N experts. That is, $|I_i \cap I_j| = |I| = \rho\delta$ for all $i \neq j$. The minimum sharing occurs, when $\rho\delta + N(\delta - \delta\rho) = 1$, which gives us the lower bound. The quantity $\rho\delta + N(\delta - \delta\rho)$ also describes the maximum coverage of the N experts. That is, $\rho\delta + N(\delta - \delta\rho) = \max |I_1 \cup I_2 \cup \dots \cup I_N|$.

Note that this model constrains the marginal distribution of $p_j = \Phi(X_{I_j})$ to be uniform on $[0, 1]$ when $\delta_j = 1$. To see this, recall that if $X_{I_j} \sim \mathcal{N}(0, 1)$, then $\Phi(X_{I_j})$ is uniform on $[0, 1]$. If this does not hold empirically, it is a sign that the model cannot be correct on the micro-level. If X_{I_j} appears more (respectively less) concentrated about 0.5, then the model can be adjusted by changing δ_j to a smaller fraction.

Notice that Σ_{22} can be written as $\Sigma_{22} = I_N(\delta - \rho\delta) + J_{N \times N}\rho\delta$. Therefore its inverse is $\Sigma_{22}^{-1} = I_N \left(\frac{1}{\delta - \rho\delta} \right) - J_{N \times N} \frac{\rho\delta}{(\delta - \rho\delta)((\delta - \rho\delta) + N\rho\delta)}$ (see the supplementary material for Dobbin and Simon [2005]). Applying this to equations (2) and (3) gives us the conditional mean

$$\bar{\mu} = \frac{1}{(N-1)\rho + 1} \sum_{j=1}^{N} X_j$$

and variance

$$\bar{\Sigma} = 1 - \frac{\delta N}{(N-1)\rho + 1}$$

The aggregation rule then becomes

$$\mathbb{P}\left(X_S > 0 \middle| \boldsymbol{X} \right) = \Phi \left(\frac{\frac{1}{(N-1)\rho + 1} \sum_{j=1}^{N} X_{I_j}}{\sqrt{1 - \frac{N\delta}{(N-1)\rho + 1}}} \right)$$

From this aggregator we obtain an expression for the amount of extremization under the compound symmetric information structure.
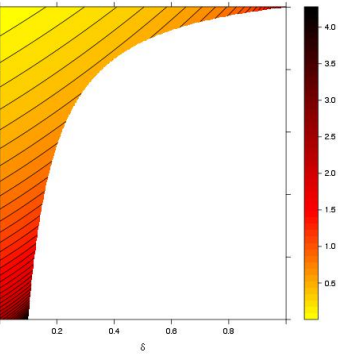
$$\alpha = \frac{\frac{N}{(N-1)\rho + 1}}{\sqrt{1 - \frac{N\delta}{(N-1)\rho + 1}}} \tag{5}$$

Unlike (4), the extremizing constant under the compound symmetric information structure does not depend on the forecasts, $\boldsymbol{X}$. As the term inside the square-root must be non-negative, we have another technical restriction on ρ. That is, in addition to $\rho \in \left[\max \left\{ \frac{N - \delta^{-1}}{N-1}, 0 \right\}, 1 \right]$, we require

$$\rho \geq \frac{N\delta - 1}{N - 1}$$

Notice, however, that $N\delta - 1 > N - \delta^{-1}$ only when $\delta < 1/N$. But when $\delta < 1/N$, both $N\delta - 1$ and $N - \delta^{-1}$ are negative. Therefore this technical condition is redundant and can be ignored.

Expression (5) is particularly convenient because it only depends on three intuitive parameters. Therefore it can be examined graphically. Figures 4 to 6 describe the amount of log-extremization, $\log(\alpha)$, under different values of ρ, δ, and N. By Theorem 4.2 the amount of extremizing is always greater or equal to 1.0. Notice that most extremization occurs when $\delta = 1.0$ and $\rho = 1$, or when $\delta = 1/N$ and $\rho = 0$. In the first case, all N experts know whether the event A materializes or not. In the second case, each expert holds an independent set of information such that the group knows all the information. Such a group of experts can re-construct X_S by simply adding up their individual probit forecasts. This means that aggregation becomes voting: if the sum of the probit forecasts is above 0, the event A materializes; else it does not. Therefore in the real-world voting can be expected to work well when the voters form a very knowledgable and diverse group of people.
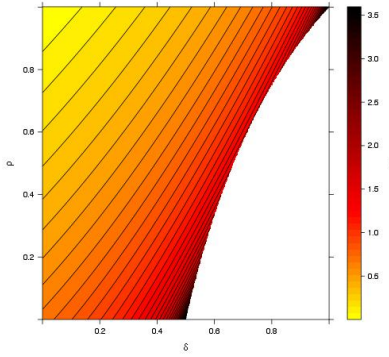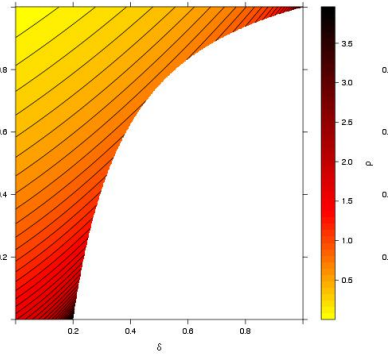
Figure 4: N = 2   Figure 5: N = 5   Figure 6: N = 10

As we move from these two extreme points towards the upper left corner, where $\delta = 0.0$ and $\rho = 1.0$, the amount of extremizing decreases monotonically to 1.0. This trend can be deduced directly from Lemma 4.1. The decrease in the amount of information in X is caused by a combination of (i) decrease in the amount of information that each individual expert holds and (ii) increase in the amount of shared information. Therefore the more knowledgable and diverse the group of experts is, the more their average probit forecast should be extremized.

Figures 4 to 6 also make it clear that the feasible set of (δ, ρ)-values becomes smaller as N increases. This limitation arises from assuming a compound symmetric overlap structure. Having many experts, each with a considerable amount of information, simply leads to unavoidable overlap in the information sets.

5   Conclusion

References

Jeffrey A Baars and Clifford F Mass. Performance of national weather service forecasts compared to operational, consensus, and weighted model output statistics. *Weather and Forecasting*, 20(6): 1034–1047, 2005.

J. Baron, L. H. Ungar, B. A. Mellers, and P. E. Tetlock. Two reasons to make aggregated probability forecasts more extreme. Manuscript submitted for publication (A copy can be requested by emailing Lyle Ungar at ungar@cis.upenn.edu), 2013.

William H Batchelder, Alex Strashny, and A Kimball Romney. Cultural consensus theory: Aggregating continuous responses in a finite interval. In *Advances in Social Computing*, pages 98–107. Springer, 2010.

Robert T Clemen and Robert L Winkler. Aggregating probability distributions. *Advances in Decision Analysis*, pages 154–176, 2007.

Kevin Dobbin and Richard Simon. Sample size determination in microarray experiments for class comparison and prognostic classification. *Biostatistics*, 6(1):27–38, 2005.

C. Genest and J. V. Zidek. Combining probability distributions: A critique and an annotated bibliography. *Statistical Science*, 1(1):114–148, 1986.

Lu Hong and Scott Page. Interpreted and generated signals. *Journal of Economic Theory*, 144(5): 2174–2196, 2009.

Margaret Sullivan Pepe. *The Statistical Evaluation of Medical Tests for Classification and Prediction*. Oxford University Press Oxford, 2003.

Cristina Primo, Christopher AT Ferro, Ian T Jolliffe, and David B Stephenson. Calibration of probabilistic forecasts of binary events. *Monthly Weather Review*, 137(3):1142–1149, 2009.

R. Ranjan and T. Gneiting. Combining probability forecasts. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 72:71–91, 2010.

Nalini Ravishanker and Dipak K Dey. *A first course in linear model theory*. CRC Press, 2001.

Frederick Sanders. On subjective probability forecasting. *Journal of Applied Meteorology*, 2(2): 191–201, 1963.

V. A. Satopää, J. Baron, D. P. Foster, B. A. Mellers, P. E. Tetlock, and L. H. Ungar. Combining multiple probability predictions using a simple logit model. *International Journal of Forecasting*, 30(2):344–356, 2014.

Philip E Tetlock. *Expert Political Judgment: How Good Is It? How Can We Know?* Princeton University Press, 2005.

Robert L Vislocky and J Michael Fritsch. Improved model output statistics forecasts through model consensus. *Bulletin of the American Meteorological Society*, 76(7):1157–1164, 1995.

T. S. Wallsten and A. Diederich. Understanding pooled subjective probability estimates. *Mathematical Social Sciences*, 18:1–18, 2001.

Thomas S. Wallsten, David V. Budescu, and Ido Erev. Evaluating and combining subjective probability estimates. *Journal of Behavioral Decision Making*, 10:243–268, 1997.

Peter WF Wilson, Ralph B DAgostino, Daniel Levy, Albert M Belanger, Halit Silbershatz, and William B Kannel. Prediction of coronary heart disease using risk factor categories. *Circulation*, 97(18):1837–1847, 1998.