CME364a: Convex Optimization I
Sales Planning Problem

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1. Introduction

In this report, I present one approach to optimize sales strategy by using convex optimization. In section 2, I give a brief description of what convex optimization is like, followed by an explanation of the sales planning problem in section 3. From sections 4 to 7, I discuss various problem settings of the sales planning problem and solve them with several assumptions about its functions and variables. As I basically develop my model by adding constraints and assumptions in the later section, I recommend reading this report sequentially. An overall conclusion is given in section 8.

2. Convex Optimization Problem

A convex optimization problem is a particular type of optimization problem that has several properties such as every local minimum is a global minimum; the optimal set is convex; if the objective function is strictly convex, then the problem has a single optimal point. A convex optimization problem can be formulated as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \cdots, m \\ & Ax = b \end{array}$$

where the objective f_0 is convex, the inequality constraints $f_1, \cdots, f_m$ are convex, and the equality constraints $Ax = b$ are affine. Another way to describe a convex problem is to write the problem as maximizing a concave function since it is equivalent to minimizing the concave objective times -1, which is convex. Because of some nice properties that this type of problem has, convex optimization problems are relatively easy to solve, and actually useful solvers have been developed. In this report, I will use CVXPY for solving sales planning problems.

3. Sales Planning Problem

In traditional brokerage firms, sales representatives are selling financial products to their clients. The goal of sales planning is to allocate business resources, the time each sales reps spend on selling each financial product, in particular, to maximize the profit. In a very simple setting where sales reps sell only stocks and bonds, we can see that there may exist an optimal balance between the times spent on selling stocks and bonds. This is because in an extreme case where all sales reps work on selling stocks, one could earn more brokerage fees by selling bonds because it's less competitive. Then, the optimization comes in.

I define functions f_s and f_b that fed sales efforts as an input and output the amount of stock sales and bond sales, respectively. As a natural assumption, the more the firm put their sales efforts into a certain product, the more difficult it becomes to obtain an additional order. Reflecting this, f_s and f_b should be non-decreasing (i.e. the first-order derivative is nonnegative) and the second-order derivative is nonpositive. Thus, these are concave functions. To compute the profit, I assume the brokerage fees are proportional to the amount of stocks and bonds sold. Let r_s and r_b be the brokerage fee rate of stocks and bonds. Then, we can describe the profit p as

$$p = f_s(E_s)r_s + f_b(E_b)r_b$$

where E_s and E_b denote the sales efforts for stocks and bonds. As both f_s and f_b are concave, maximizing p over E_s and E_b is a convex optimization problem.

4. Simple Setting: Two Products Model

In this section, I present a very simple example of the sales planning problem.

Example 4

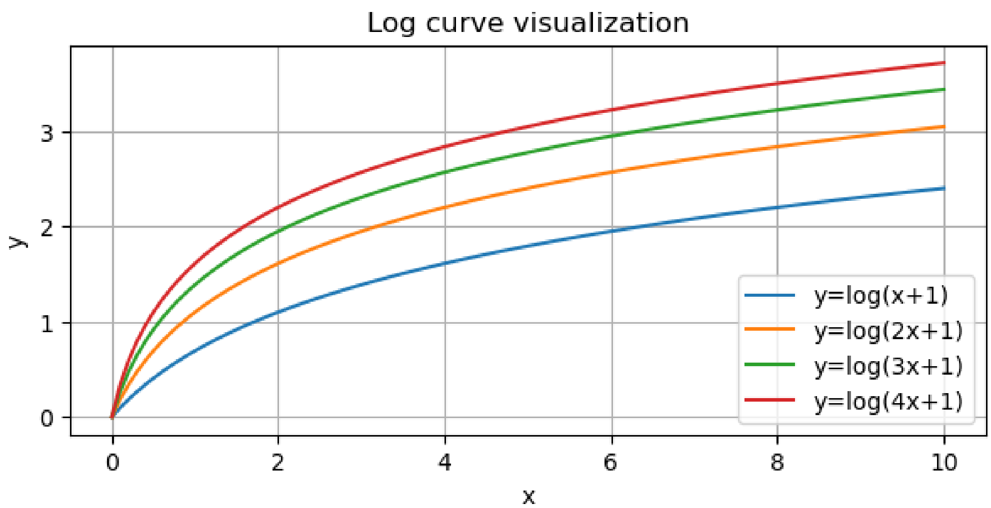
Let T be the total amount of time that the firm can allocate, and T_s, T_s be the time for working on selling stocks and bonds, respectively. As we are assuming there are only two products, stocks and bonds, we have $T = T_s + T_b$. To bridge the time spent on the sales activities and the sales efforts E_s and E_b, I use the following formula in this example.

$$E_s = 3T_s, \quad E_b = 2T_b$$

The constants 3 and 2 are arbitrary, but one way to understand this is that it describes sales efficiency. Later on, I modify these constants to model different types of sales reps. For instance, those who are good at selling stocks have E_s with a larger coefficient. Also, let's suppose f_s and f_b are given as

$$f_s = \log(E_s + 1), \quad f_b = \log(E_b + 1)$$

In practice, we may empirically approximate the relationship between the amount of sales and the amount of time spent from the past sales activity data.



Now, we can formulate the convex optimization problem.

$$\begin{array}{ll} \text{maximize} & \log(3T_s + 1) \cdot r_s + \log(2T_b + 1) \cdot r_b \\ \text{subject to} & T_s + T_b = T \\ & T_s, T_b \geq 0 \end{array}$$

Note that this is actually a convex problem. This is because the objective, the sum of two concave functions (log is concave), is concave which we want to maximize. Also, the constraints are affine. Throughout this report, I set the brokerage fee as

$$r_s = 0.005, \quad r_b = 0.008$$

Also, I set the total amount of time as

$$T = 10$$

By solving this problem, we get the maximum profit $p^* = 0.03335$ with $T_s = 3.833$ and $T_b = 6.167$.

Implication

First, as we intuitively expected, there exists an optimal balance in time allocation. Secondly, in the real business situation, the majority of stocks that the clients buy are from the stock exchange. The brokerage firms are simply placing an order on the stock exchange. So, under the assumption of no liquidity issue in the stock exchange, the amount of stocks that the firm can sell to the clients is unlimited. On the other hand, the majority of bonds that they sell are newly issued bonds, and they obtain a fee not from the investors but from the issuer which is proportional to the amount of bonds they underwrite. So, given the total time for sales activities, we can determine the optimal amount of bonds that the firm should underwrite to maximize their profit. In this example, this amount is given by $f_b^* = 2.59$. Underwriting more or less will decrease the profit.

5. Two Types of Sales Reps

As I mention briefly in the previous section, we can model different sales staff having different skill sets by modifying the sales effort functions E_s and E_b. To keep things simple, I use two pairs of sales effort functions, $(E_s^{(s)}, E_b^{(s)})$ and $(E_s^{(b)}, E_b^{(b)})$. One way to understand them is to think that some sales reps are better at selling stocks than bonds. In other words, these people have a high skill in selling stocks. So, these stock sales reps' effort function, which we assume linear in time spent again, should be $E_s^{(s)} = c_s^{(s)} T_s^{(s)}$ and $E_b^{(s)} = c_b^{(s)} T_b^{(s)}$ with $c_s^{(s)} > c_b^{(s)}$. Because of these constants, stock sales reps can convert their working time into sales effort for stocks more efficiently, and, as a consequence, we can expect great sales outcomes for stocks from them. Similarly, the sales effort functions for the bond sales reps should be $E_s^{(b)} = c_s^{(b)} T_s^{(b)}$ and $E_b^{(b)} = c_b^{(b)} T_b^{(b)}$ with $c_s^{(b)} < c_b^{(b)}$. Note that I use superscripts to distinguish variables and functions for stock sales reps and bond sales reps.

Example 5.1

So, in this example, I will define two types of sales effort functions and see what the optimal time allocation is like. To embody different sales staff, I will use

$$\begin{cases} E_s^{(s)} = 4T_s^{(s)}, & E_b^{(s)} = T_b^{(s)} \\ E_s^{(b)} = 2T_s^{(b)}, & E_b^{(s)} = 3T_b^{(b)} \end{cases}$$

Comparing $E_s^{(s)}$ and $E_b^{(s)}$, we can see this sales staff has higher productivity in stock sales, and a similar argument can be made for $E_s^{(b)}$ and $E_b^{(b)}$ as well. Then, using the same sales functions f_s and f_b as in example 4, we can formulate this problem as

$$\begin{aligned} \text{maximize} \quad & \log(4T_s^{(s)} + 2T_s^{(b)} + 1) \cdot r_s + \log(T_b^{(s)} + 3T_b^{(b)} + 1) \cdot r_b \\ \text{subject to} \quad & T_s^{(s)} + T_b^{(s)} + T_s^{(b)} + T_b^{(b)} = T \\ & T_s^{(s)}, T_b^{(s)}, T_s^{(b)}, T_b^{(b)} \geq 0 \end{aligned}$$

Again, this is a convex optimization problem and thus, solvable with CVXPY. By solving this with $T = 10$, $r_s = 0.005$, and $r_b = 0.008$, we obtain the maximum profit $p^* = 0.03773$ with $T_s^{(s)} = 3.821$, $T_b^{(s)} = 0.0$, $T_s^{(b)} = 0.0$, and $T_b^{(b)} = 6.179$.

Implication

One way to understand this result is that, given sales functions and other constants, we can determine the optimal HR strategy. In concrete terms, if we see $T = 10$ as that this brokerage firm hires 10 sales reps, they should hire 4 people proficient in stock sales and 6 people having profound knowledge in bond sales to maximize their profit. On top of that, we want them to work only on what they are good at doing as the outcome of $T_b^{(s)} = 0.0$ and $T_s^{(b)} = 0.0$ indicates. To maximize profit, workers have to be a specialist but not a generalist. This *do what you can do well* result has a sense of *Ricardo's comparative advantage theory* in economics. To see this more precisely, in example 5.2, I manipulate the sales effort function.

Example 5.2

As we see in example 5.1, we can determine the optimal HR strategy. However, in real business situations, optimizing the sales staff portfolio by firing existing staff and hiring new ones may not be easily done for reasons like labor contracts, laws, hiring costs, etc. Reflecting this, we add new constraints to the previous example such that

$$T_s^{(s)} + T_b^{(s)} = T_{\text{total}}^{(s)}, \quad T_s^{(b)} + T_b^{(b)} = T_{\text{total}}^{(b)}$$

We have $T = T_{\text{total}}^{(s)} + T_{\text{total}}^{(b)}$ sales resource but the resource for the stock sales staff and the bond sales staff is fixed. Intuitively, if the firm is hiring T sales reps, $T_{\text{total}}^{(s)}$ of them is good at selling stocks and the other $T_{\text{total}}^{(b)}$ are skilled in bond selling. Also, to show the similarity to the *Ricardo's comparative advantage theory*, I set the sales effort function as

$$\begin{cases} E_s^{(s)} = 2T_s^{(s)}, & E_b^{(s)} = T_b^{(s)} \\ E_s^{(b)} = 3T_s^{(b)}, & E_b^{(s)} = 4T_b^{(b)} \end{cases}$$

$E_s^{(s)}$ has higher coefficient than $E_b^{(s)}$ to model these people are relatively good at selling stocks, and $E_b^{(s)}$ has a higher coefficient that $E_s^{(b)}$. However, unlike the previous example, we have $E_s^{(s)}$ having lower coefficient than $E_s^{(b)}$. Namely, no matter which product to promote, stock or bond, bond staff do well than stock staff. So, there is no reason to hire stock staff but we have a constraint of $T_{\text{total}}^{(s)}$. Under this setting, we want to best allocate the time to maximize the profit. It can be formulated as

$$\begin{aligned} \text{maximize} \quad & \log(2T_s^{(s)} + 3T_s^{(b)} + 1) \cdot r_s + \log(T_b^{(s)} + 4T_b^{(b)} + 1) \cdot r_b \\ \text{subject to} \quad & T_s^{(s)} + T_b^{(s)} = T_{\text{total}}^{(s)} \\ & T_s^{(b)} + T_b^{(b)} = T_{\text{total}}^{(b)} \\ & T_{\text{total}}^{(s)} + T_{\text{total}}^{(b)} = T \\ & T_s^{(s)}, T_b^{(s)}, T_s^{(b)}, T_b^{(b)} \geq 0 \end{aligned}$$

Solving this problem with $T_{\text{total}}^{(s)} = 6$ and $T_{\text{total}}^{(b)} = 4$ yields the optimal profit of 0.03549 with $T_s^{(s)} = 6.0$, $T_b^{(s)} = 0.0$, $T_s^{(b)} = 0.0$, and $T_b^{(b)} = 4.0$. Note that the optimal profit is not comparable to the one in example 5.1 because we are using different sales effort functions here.

Implication

Again, we get a suggestion that people should *do what you can do well.* The *Ricardo's comparative advantage theory* claims that countries can benefit from international trade by specializing in the production of goods for which they have a relatively lower opportunity cost in production even if they do not have an absolute advantage in the production of any particular good. For example, in the early 19th century, a mutual benefit would be realized between China and the United Kingdom from China specializing in the production of porcelain and the United Kingdom concentrating on machine parts even if the United Kingdom would have higher productivity in porcelain. Similarly, here, even though stock sales reps have no absolute advantage in stock promoting, letting them concentrate on stock sales and letting bond staff sell bonds turns out to be the best strategy. Thus, what matters is not the absolute advantage but the relative strength.

6. Managing Sales Balance Across Various Products

So far, we have seen how to maximize profit under the given constraints. However, the optimal sales strategy that leads you to the best possible profit may not be the best in the long run. For example, if the firm focuses heavily on selling bonds, and if the stock performance is brilliant in the following years, the clients might not want to hear their sales reps anymore. Another example would be that the firm cares about the balance of the total assets of clients in custody because it is an important factor for their future business, so they may want to control it like the asset managers balance the risk and return of their portfolio. So, in this section, I introduce a ratio R_{BS} such that

$$R_{\mathrm{BS}} = \frac{f_b(E_b)}{f_s(E_s)}$$

This BS ratio represents the balance between bond sales amount and stock sales amount. Thus, the firm can control the allocation of their clients' assets in custody through this ratio.

Example 6

In example 5.1, the optimal BS ratio that maximizes the profit is $R_{BS} = 1.605$. Let's assume that the firm wants to have this ratio $R_{BS} = 1.010$ and the functions and the variables are the same as in example 5.1. Then, writing this problem down straightforwardly yields

$$\begin{array}{ll} \text{maximize} & \log(4T_s^{(s)} + 2T_s^{(b)} + 1) \cdot r_s + \log(T_b^{(s)} + 3T_b^{(b)} + 1) \cdot r_b \\ \text{subject to} & T_s^{(s)} + T_b^{(s)} + T_s^{(b)} + T_b^{(b)} = T \\ & T_s^{(s)}, T_b^{(s)}, T_s^{(b)}, T_b^{(b)} \geq 0 \\ & \dfrac{\log(T_b^{(s)} + 3T_b^{(b)} + 1)}{\log(4T_s^{(s)} + 2T_s^{(b)} + 1)} = 1.01 \end{array}$$

However, the third constraint does not satisfy the property of the convex optimization problem that the equality constraints must be affine. To address this issue, I introduce new variables and transform this problem into a convex problem that is equivalent to the problem above. Let S_{amount} and B_{amount} be the variables that represent the amount of stock sales and bond sales, respectively, such that

$$S_{\mathrm{amount}} = \log(4T_s^{(s)} + 2T_s^{(b)} + 1), \quad B_{\mathrm{amount}} = \log(T_b^{(s)} + 3T_b^{(b)} + 1)$$

using them, the original problem can be rewritten as

$$\begin{array}{rl}
\text{maximize} & S_{\text{amount}} \cdot r_s + B_{\text{amount}} \cdot r_b \\
\text{subject to} & T_s^{(s)} + T_b^{(s)} + T_s^{(b)} + T_b^{(b)} = T \\
& T_s^{(s)}, T_b^{(s)}, T_s^{(b)}, T_b^{(b)} \geq 0 \\
& B_{\text{amount}} = R_{\text{BS}} \cdot S_{\text{amount}} \\
& S_{\text{amount}} \leq \log(4T_s^{(s)} + 2T_s^{(b)} + 1) \\
& B_{\text{amount}} \leq \log(T_b^{(s)} + 3T_b^{(b)} + 1)
\end{array}$$

The fourth and fifth constraints are obtained by relaxing the problem. We need equality holds for both, but with equality, the constraint must be affine. So, equality is replaced with inequalities. Let the objective be f_0. Then, $\frac{\mathrm{d}f_0}{\mathrm{d}S_{\text{amount}}} = r_s > 0$ and $\frac{\mathrm{d}f_0}{\mathrm{d}B_{\text{amount}}} = r_b > 0$. So, to maximize f_0, both S_{amount} and B_{amount} get as large as possible under the constraints, resulting in satisfying the equalities. Therefore, the rewritten convex problem is equivalent to the originally formulated problem. By solving this, we obtain the optimal profit of 0.03769 with $T_s^{(s)} = 4.211$, $T_b^{(s)} = 0.0$, $T_s^{(b)} = 0.0$, and $T_b^{(b)} = 5.789$.

Implication

Here, since we are using the same sales effort function as in example 5.1, both results are comparable. In example 5.1, the optimal profit is 0.03773, but it is decreased to 0.03769 this time. This is the consequence of the R_{BS} being introduced, which we have $R_{\text{BS}} = 1.065$ in the example 5.1 while this is set to $R_{\text{BS}} = 1.010$ here. Another observation is that we have the *do what you can do well* outcome in this example again. Under the constraints of this problem, the HR people have to hire not generalists but specialists. Lastly, I introduced the new variables S_{amount} and B_{amount} to convert the problem into the convex optimization settings. So far, we assume the functions $f_s(E_s)$ and $f_b(E_b)$ are the form of log, but the same or similar tricks can be used for other concave functions as well.

7. Optimizing Over a Group of Generalists

The optimized time allocation from the previous sections suggests that people had better be a specialist and focus on what they have a relative strength for. In reality, however, sales staff are often in charge of clients being assigned to them, and the financial products that they have to cover have a wide range from stocks and bonds to trust funds, moreover, even from tax consulting to inheritance. One of the advantages of this model is that sales reps can do their business based on strong ties with the clients. They can build a personal connection with the clients, understand each client very well, and then, they can provide financial advice that aligns with the client's personal goal. So, in this context, our model might be oversimplified because our sales effort functions only take into account the amount of time the sales reps work. Nevertheless, it would be intriguing to see what happens in this generalist setting.

Example 7

To model this setting, we assume each sales representative holds its own sales functions $f_s(E_s)$ and $f_b(E_b)$ while we aggregate the amount of time for each product and feed it to the sales functions in the previous sections. We consider two types of sales reps model as before. To formulate this problem which requires several steps to find the optimum, we first see $T = 10$ as the firm hires T sales staff and each of them works 1 time. Given the same fixed working time, the optimal time allocation of the stock sales staff must be the same because other conditions, such as sales effort functions and brokerage fees, are the same. Let $\bar{T}_s^{(s)}$ and $\bar{T}_b^{(s)}$ be the time allocation for each stock sales staff. Then, we need to solve

$$\begin{aligned} \text{maximize} \quad & \frac{1}{T}\log(4\bar{T}_s^{(s)} + 1) \cdot r_s + \frac{1}{T}\log(\bar{T}_b^{(s)} + 1) \cdot r_b \\ \text{subject to} \quad & \bar{T}_s^{(s)} + \bar{T}_b^{(s)} = 1 \\ & \bar{T}_s^{(s)}, \bar{T}_b^{(s)} \geq 0 \end{aligned}$$

The factor of $\frac{1}{T}$ in the objective corresponds to the setting that each sales staff is in charge of clients. For example, suppose we have $1,000$ clients. In the previous sections, all $T = 10$ sales staff have an access to all $1,000$ clients, but this time, as each sales staff is in charge of 100 clients, the sales functions should be divided by T. By solving this, we obtain that the max profit for each stock sales staff is 0.00088 with $\bar{T}_s^{(s)} = 0.6154$ and $\bar{T}_b^{(s)} = 0.3846$. We can do the same computation for the bond sales staff as well by introducing $\bar{T}_s^{(b)}$ and $\bar{T}_b^{(b)}$ that are their time allocation. By solving

$$\begin{aligned} \text{maximize} \quad & \frac{1}{T}\log(2\bar{T}_s^{(b)} + 1) \cdot r_s + \frac{1}{T}\log(2\bar{T}_b^{(b)} + 1) \cdot r_b \\ \text{subject to} \quad & \bar{T}_s^{(b)} + \bar{T}_b^{(b)} = 1 \\ & \bar{T}_s^{(b)}, \bar{T}_b^{(b)} \geq 0 \end{aligned}$$

we can get the optimal profit of 0.00115 with $\bar{T}_s^{(b)} = 0.2051$ and $\bar{T}_b^{(b)} = 0.7949$. With these unit max profits of both stock and bond sales reps, our optimization problem can be reduced to how many units of them we need to deploy to maximize the entire profit. Clearly, if we don't pose additional constraints, hiring bond people to the maximum and not hiring stock people is the best because the unit profit is bigger for bond sales staff. So, in this example, we consider the sales balance of $R_{\text{BS}} = 1.01$, which is equal to the one in example 6 so that we can compare the results. Let $T_{\text{all}}^{(s)}$ and $T_{\text{all}}^{(b)}$ be the amount of units the firm hires stock people and bond people. Namely, we have $T_{\text{all}}^{(s)} = T_s^{(s)} + T_b^{(s)}$ and $T_{\text{all}}^{(b)} = T_s^{(b)} + T_b^{(b)}$. Then, the convex optimization problem is

$$
\begin{array}{rl}
\text{maximize} & 0.00088 \cdot T_{\text{all}}^{(s)} + 0.00115 \cdot T_{\text{all}}^{(b)} \\
\text{subject to} & B_{\text{amount}} = R_{\text{BS}} \cdot S_{\text{amount}} \\
& T_{\text{all}}^{(s)} + T_{\text{all}}^{(b)} = T \\
& T_{\text{all}}^{(s)}, T_{\text{all}}^{(b)} \geq 0 \\
& S_{\text{amount}} = \dfrac{T_{\text{all}}^{(s)}}{T} \log(4\bar{T}_s^{(s)} + 1) + \dfrac{T_{\text{all}}^{(b)}}{T} \log(2\bar{T}_s^{(b)} + 1) \\
& B_{\text{amount}} = \dfrac{T_{\text{all}}^{(s)}}{T} \log(\bar{T}_b^{(s)} + 1) + \dfrac{T_{\text{all}}^{(b)}}{T} \log(3\bar{T}_b^{(b)} + 1)
\end{array}
$$

Here, as $\bar{T}_s^{(s)}$, $\bar{T}_b^{(s)}$, $\bar{T}_s^{(b)}$, and $\bar{T}_b^{(b)}$ are not variables but are already computed in the previous steps, the log in the forth and fifth constraints are just constants. Thus, this is a convex optimization problem, and solving this yields the optimal profit of 0.01018 with $T_s^{(s)} = 2.980$, $T_b^{(s)} = 1.863$, $T_s^{(b)} = 1.058$, and $T_b^{(b)} = 4.099$.

Implication

Same as the previous settings, solving the optimization problem above provides the single optimal HR strategy of $T_{\text{all}}^{(s)} = 4.843$ and $T_{\text{all}}^{(b)} = 5.157$. The result of this example is comparable to the one in example 6 as we use the same functions and variables. The best possible profit is dropped from 0.03769 in example 6 to 0.01018 here. It suggests that people should not be a generalist but a specialist to attain better productivity. For example, sales staff who are proficient in equity sales spare $\frac{T_b^{(s)}}{T_{\text{all}}^{(s)}} = 0.384$ of their resource on bond sales, and this is the consequence of their being able to access only the clients assigned to them. However, it would be going too much to conclude that this generalist approach is inferior to the specialist approach because we have a caveat that the problem settings may oversimplify what is happening in real. There are many crucial features that our future model has to take in such as a generalist approach can build a better relationship between clients and sales reps.

8. Conclusion

So far, I discuss what a convex optimization problem is like, and what the sales planning problem is like, and show several problem settings with concrete examples. In all the examples, the emphasis is on the fact that we can obtain the global optimum so long as sales functions, which are given as $f_s(E_s)$ and $f_b(E_b)$ in examples, are concave. This is consistent with our intuition that the more we sell a product, the harder it gets to sell more, indicating that the sales functions are non-decreasing, and the first derivative of them is non-increasing. In the examples above, the number of financial products is set to 2, and the number of types of sales reps is also set to 2 for simplicity. However, we can easily extend our model to 3 or more asset classes as well as 3 or more sales staff abilities. In section 6, I introduce the notion of the BS ratio. We could add more constraints to better capture real business while keeping the optimization problem feasible. Though it is not always guaranteed, using a trick of introducing new variables would help formulate the setting into a convex optimization problem. I believe this convex optimization model can be a powerful tool to analyze sales activities in the past and build a strong strategy for the future.

In [1]:
```
import numpy as np
import cvxpy as cp
```

In [2]:
```
# Setting the brokerage fee rate.
stockfee = 0.005
bondfee = 0.008
```

In [3]:
```
# Example 4
T = 10

_T = cp.Variable(2)
objective = cp.Maximize(cp.log(3*_T[0]+1)*stockfee + cp.log(2*_T[1]+1)*bondfee)
constraints = [cp.sum(_T) == T,
               _T >= 0]
profit = cp.Problem(objective, constraints).solve()

print("The optimal profit: " + str(round(profit, 5)))
print("The optimal time allocation: [stock, bond] = [" \
      + str(round(_T.value[0],3)) + ", " + str(round(_T.value[1],3)) + "]")
print("The optimal amount of bond to underwrite: " + str(round(np.log(2*_T.value[1]+1),3)))
```

```
The optimal profit: 0.03335
The optimal time allocation: [stock, bond] = [3.833, 6.167]
The optimal amount of bond to underwrite: 2.59
```

In [4]:

```python
# Example 5.1
T = 10

Ts, Tb = cp.Variable(2), cp.Variable(2)
objective = cp.Maximize(cp.log(4*Ts[0]+2*Tb[0]+1)*stockfee + cp.log(Ts[1]+3*Tb[1]+1)*bondfee)
constraints = [cp.sum(Ts) + cp.sum(Tb) == T,
               Ts >= 0,
               Tb >= 0]
profit = cp.Problem(objective, constraints).solve()

print("The optimal profit: " + str(round(profit, 5)))
print("The optimal time allocation:")
print("   stock sales: [stock, bond] = [" + str(round(Ts.value[0],3)) + ", " + str(round(Ts.value[1],3)) + "]")
print("   bond sales: [stock, bond] = [" + str(round(Tb.value[0],3)) + ", " + str(round(Tb.value[1],3)) + "]")
bs_ratio_opt = np.log(Ts.value[1]+3*Tb.value[1]+1) / np.log(4*Ts.value[0]+2*Tb.value[0]+1)
print("BS ratio: " + str(round(bs_ratio_opt,3)))
```

```
The optimal profit: 0.03773
The optimal time allocation:
   stock sales: [stock, bond] = [3.821, 0.0]
   bond sales: [stock, bond] = [0.0, 6.179]
BS ratio: 1.065
```

In [5]:
```python
# Ricardian theory of comparative advantage
T = 10
Ts_total = 6
Tb_total = T - Ts_total

Ts, Tb = cp.Variable(2), cp.Variable(2)
objective = cp.Maximize(cp.log(4*Ts[0]+2*Tb[0]+1)*stockfee + cp.log(Ts[1]+3*Tb[1]+1)*bondfee)
constraints = [cp.sum(Ts) == Ts_total,
               cp.sum(Tb) == Tb_total,
               Ts >= 0,
               Tb >= 0]
profit = cp.Problem(objective, constraints).solve()

print("The optimal profit: " + str(round(profit, 5)))
print("The optimal time allocation:")
print("   stock sales: [stock, bond] = [" + str(round(Ts.value[0],3)) + ", " + str(round(Ts.value[1],3)) + "]")
print("   bond sales: [stock, bond] = [" + str(round(Tb.value[0],3)) + ", " + str(round(Tb.value[1],3)) + "]")
print("BS ratio: " + str(round(np.log(Ts.value[1]+3*Tb.value[1]+1) / np.log(4*Ts.value[0]+2*Tb.value[0]+1),3)))
```

```
The optimal profit: 0.03661
The optimal time allocation:
   stock sales: [stock, bond] = [6.0, 0.0]
   bond sales: [stock, bond] = [0.0, 4.0]
BS ratio: 0.797
```

In [6]:
```
# Example 5.2
# Ricardian theory of comparative advantage
T = 10
Ts_total = 6
Tb_total = T - Ts_total

Ts, Tb = cp.Variable(2), cp.Variable(2)
objective = cp.Maximize(cp.log(2*Ts[0]+3*Tb[0]+1)*stockfee + cp.log(Ts[1]+4*Tb[1]+1)*bondfee)
constraints = [cp.sum(Ts) == Ts_total,
               cp.sum(Tb) == Tb_total,
               Ts >= 0,
               Tb >= 0]
profit = cp.Problem(objective, constraints).solve()

print("The optimal profit: " + str(round(profit, 5)))
print("The optimal time allocation:")
print("   stock sales: [stock, bond] = [" + str(round(Ts.value[0],3)) + ", " + str(round(Ts.value[1],3)) + "]")
print("   bond sales: [stock, bond] = [" + str(round(Tb.value[0],3)) + ", " + str(round(Tb.value[1],3)) + "]")
print("BS ratio: " + str(round(np.log(Ts.value[1]+3*Tb.value[1]+1) / np.log(4*Ts.value[0]+2*Tb.value[0]+1),3)))
```

```
The optimal profit: 0.03549
The optimal time allocation:
   stock sales: [stock, bond] = [6.0, 0.0]
   bond sales: [stock, bond] = [0.0, 4.0]
BS ratio: 0.797
```

In [7]:

```python
# Example 6
bs_ratio_target = 1.01
T = 10

Ts, Tb = cp.Variable(2), cp.Variable(2)
Samt, Bamt = cp.Variable(), cp.Variable()
objective = cp.Maximize(Samt*stockfee + Bamt*bondfee)
constraints = [cp.sum(Ts) + cp.sum(Tb) == T,
               Ts >= 0,
               Tb >= 0,
               Bamt == bs_ratio_target*Samt,
               Bamt <= cp.log(Ts[1]+3*Tb[1]+1),
               Samt <= cp.log(4*Ts[0]+2*Tb[0]+1)]
profit = cp.Problem(objective, constraints).solve()

print("The optimal profit: " + str(round(profit, 5)))
print("The optimal time allocation:")
print("   stock sales: [stock, bond] = [" + str(round(Ts.value[0],3)) + ", " + str(round(Ts.value[1],3)) + "]")
print("   bond sales: [stock, bond] = [" + str(round(Tb.value[0],3)) + ", " + str(round(Tb.value[1],3)) + "]")
print("BS ratio: " + str(round(np.log(Ts.value[1]+3*Tb.value[1]+1) / np.log(4*Ts.value[0]+2*Tb.value[0]+1),3)))
```

```
The optimal profit: 0.03769
The optimal time allocation:
   stock sales: [stock, bond] = [4.211, 0.0]
   bond sales: [stock, bond] = [0.0, 5.789]
BS ratio: 1.01
```

In [8]:
```python
# Example 7
bs_ratio_target = 1.01
T = 10

t_stock_unit = cp.Variable(2)
objective = cp.Maximize(1/T*cp.log(4*t_stock_unit[0]+1)*stockfee + 1/T*cp.log(t_stock_unit[1]+1)*bondfee)
constraints = [t_stock_unit[0] + t_stock_unit[1] == 1,
               t_stock_unit >= 0]
unit_profit_s = cp.Problem(objective, constraints).solve()

t_bond_unit = cp.Variable(2)
objective = cp.Maximize(1/T*cp.log(2*t_bond_unit[0]+1)*stockfee + 1/T*cp.log(3*t_bond_unit[1]+1)*bondfee)
constraints = [t_bond_unit[0] + t_bond_unit[1] == 1,
               t_bond_unit >= 0]
unit_profit_b = cp.Problem(objective, constraints).solve()

Samt, Bamt = cp.Variable(), cp.Variable()
Ts_all, Tb_all = cp.Variable(), cp.Variable()
objective = cp.Maximize(Ts_all * unit_profit_s + Tb_all * unit_profit_b)
constraints = [Bamt == bs_ratio_target*Samt,
               Ts_all + Tb_all == T,
               Ts_all >= 0,
               Tb_all >= 0,
               Samt == Ts_all/T*cp.log(4*t_stock_unit.value[0]+1) + Tb_all/T*cp.log(2*t_bond_unit.value[0]+1),
               Bamt == Ts_all/T*cp.log(t_stock_unit.value[1]+1) + Tb_all/T*cp.log(3*t_bond_unit.value[1]+1)]
profit = cp.Problem(objective, constraints).solve()

print("The optimal profit: " + str(round(profit, 5)))
print("The optimal time allocation:")
print("   stock sales: [stock, bond] = [" \
      + str(round(t_stock_unit.value[0]*Ts_all.value,3)) + ", " \
      + str(round(t_stock_unit.value[1]*Ts_all.value,3)) + "]")
print("   bond sales: [stock, bond] = [" \
      + str(round(t_bond_unit.value[0]*Tb_all.value,3)) + ", " \
      + str(round(t_bond_unit.value[1]*Tb_all.value,3)) + "]")
print("BS ratio: " + str(round(Bamt.value / Samt.value, 3)))
```

```
The optimal profit: 0.01018
The optimal time allocation:
   stock sales: [stock, bond] = [2.98, 1.863]
```

```
   bond sales: [stock, bond] = [1.058, 4.099]
BS ratio: 1.01
```